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Numerical Studies of the Generalized l_1 Greedy Algorithm for Sparse Signals

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ABSTRACT

The generalized l_1 greedy algorithm was recently introduced and used to reconstruct medical images in computerized tomography in the compressed sensing framework via total variation minimization. Experimental results showed that this algorithm is superior to the reweighted l_1-minimization and l_1 greedy algorithms in reconstructing these medical images. In this paper the effectiveness of the generalized l_1 greedy algorithm in finding random sparse signals from underdetermined linear systems is investigated. A series of numerical experiments demonstrate that the generalized l_1 greedy algorithm is superior to the reweighted l_1-minimization and l_1 greedy algorithms in the successful recovery of randomly generated Gaussian sparse signals from data generated by Gaussian random matrices. In particular, the generalized l_1 greedy algorithm performs extraordinarily well in recovering random sparse signals with nonzero small entries. The stability of the generalized l_1 greedy algorithm with respect to its parameters and the impact of noise on the recovery of Gaussian sparse signals are also studied.

Keywords: Compressed Sensing; Gaussian Sparse Signals; l_1-Minimization; Reweighted l_1-Minimization; l_1; Greedy Algorithm Generalized l_1 Greedy Algorithm

1. Introduction

In signal processing one wants to reconstruct a signal from highly incomplete sets of linear measurements of the signal, that is, the number of measurements is much smaller than the dimension of the signal. More precisely, assuming $A \in \mathbb{R}^{m\times n}$ with $m \ll n$, one wants to reconstruct an unknown signal $x_0 \in \mathbb{R}^n$ from a set of m measurements $b = Ax_0$. This requires one to solve the system of linear equations

$$Ax = b \tag{1}$$

to determine the solution that is exactly equal to x_0. Since system (1) is consistent and underdetermined, it has infinitely many solutions making it difficult to find the correct solution x_0. In many actual applications, such as image reconstruction and decoding, however, the signal one wants to reconstruct is known to be sparse (or nearly sparse) in the sense that its coefficients in some orthonormal basis are mostly zero (or approximately zero). The theory of compressed sensing [1-5] reveals that signals that have sparse representations can be reconstructed with high precision from far fewer measurements than the dimension of the signal itself. In fact, if the columns of A are chosen from a suitable distribution and the signal is sufficiently sparse, then the signal can be exactly recovered by solving the following standard l_1-norm minimization problem:

$$\min_{x\in\mathbb{R}^n} \|x\|_1 \quad \text{subject to} \quad Ax = b \tag{2}$$

where $\|x\|_1 = \sum_{i=1}^{n} |x_i|$. This optimization problem of a convex objection function can be solved effectively and it has broad applications [6-10]. But the iterative l_1-minimization method has a shortcoming in finding the sparsest solution. Since the larger entries of x in each iteration skew the l_1-norm, they are more heavily penalized in the l_1-minimization process. To address this imbalance, weighted algorithms were introduced to reduce the influence of the larger entries. Two major algorithms designed for this purpose are the reweighted l_1-minimization and l_1 greedy algorithms [11,12].

Suppose that $\{x^k\}$ is the sequence of vectors gener-

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ated by l_1-minimization. In the k-th iteration of the reweighted l_1-minimization method [11], one minimizes $\|W^k x\|_1$ instead of $\|x\|_1$ in (2), where

$$W^k = \operatorname{diag}\left\{w_1^k, \cdots, w_n^k\right\}$$
$$w_i^k = \frac{1}{\varepsilon + \left|x_i^{k-1}\right|}, \ \varepsilon > 0, \ i = 1, \cdots, n \tag{3}$$

Observe that the weights in (3) are roughly inversely proportional to the sizes of the entries of the previous iterate x^{k-1}. So the larger entries are weighted down to rectify their undue influence in the next iteration of the l_1-minimization process. Numerical experiments [11] have indicated that the reweighted l_1-minimization recovers random sparse signals with a much higher probability than the standard l_1-minimization in (2). The reweighted l_1-minimization algorithm has been extensively studied in recent years. The l_q-minimization problem, $0 < q \leq 1$, was discussed and implemented using the reweighted l_1-minimization scheme [13,14]. A two-step reweighted l_1-minimization was introduced to improve the recovery of sparse signals [15], and a reweighted l_1-minimization for a nonuniform sparsity model was proposed [16]. The performance of the reweighted l_1-minimization with noisy data was also rigorously analyzed [17], and some convergence conditions of reweighted l_1-minimization for a special family of measurement matrices were studied [18].

In the l_1 greedy algorithm [12], instead of using variable weights as in (3) the weights are set to a fixed small constant δ for entries whose magnitude is above a certain threshold and to 1 for the other entries. This threshold is lowered after each iteration so that more and more large entries are weighted down by δ in each subsequent iteration step. More precisely, the weights w_i^k in the k-th iteration of the l_1 greedy algorithm are defined by

$$w_i^k = \begin{cases} \delta, & \text{for } \left|x_i^{k-1}\right| \geq \beta^k M \\ 1, & \text{otherwise} \end{cases}, \ i = 1, \cdots, n$$

where $M = \|x^0\|_\infty = \max_{1 \leq i \leq n} \left\{\left|x_i^0\right|\right\}$, x^0 is generated by the standard l_1-minimization, $\beta \in (0,1)$ and $\delta \in (0, 0.001]$. Numerical experiments showed that the l_1 greedy algorithm outperforms both the unweighted and reweighted l_1-minimization algorithms in recovering random sparse signals [12,19].

A generalized l_1 greedy algorithm in the compressed sensing framework was recently introduced by the authors of [20]. The new algorithm not only incorporates the threshold feature of the l_1 greedy algorithm to counteract the influence of large entries but also assigns significantly large weights to the smallest nonzero entries to speed up the identification of nonzero entries. Moreover, in contrast to the l_1 greedy algorithm where the remaining entries are assigned a neutral weight, the remaining entries in the generalized l_1 greedy algorithm receive weight roughly inversely proportional to their magnitudes as in the reweighted l_1-minimization algorithm. Thus the generalized l_1 greedy algorithm not only incorporates features of both the l_1 greedy and reweighted l_1-minimization algorithms but also enhances the impact of the small entries in the l_1-minimization process.

Generalized l_1 greedy algorithm

1) Generate x^0 by the reweighted l_1-minimization;

2) For $k = 1$ to $k_{\max}$;

a) Update the weight matrix W^k; where

$$w_i^k = \begin{cases} \delta, & \text{for } \left|x_i^{k-1}\right| \geq \beta M s^{k-1} \\ \gamma, & \text{for } \left|x_i^{k-1}\right| < \alpha M s^{k-1} \\ \dfrac{1}{\varepsilon + \left|x_i^{k-1}\right|}, & \text{otherwise} \end{cases}$$
$$M = \|x^0\|_\infty, \ O \leq \alpha \leq \beta \leq 1, \gamma \geq 1,$$
$$\delta \in (0, 0.001], \ s \in (0,1], \ \varepsilon > 0$$

b) Solve weighted l_1-minimization problem:

$$x^k = \arg \min_{x \in \mathbb{R}^n} \|W^k x\|_1 \text{ subject to } Ax = b$$

c) Return if a stopping criterion is met.

In [20] the generalized l_1 greedy algorithm was applied to the problem of reconstructing essentially piecewise constant medical images in computerized tomography (CT) in the compressed sensing framework via total variation minimization. Tested with the Shepp-Logan phantom and a real cardiac CT image, the generalized l_1 greedy algorithm was shown to perform better than the reweighted l_1-minimization and l_1 greedy algorithms. In particular, it was observed that in the context of reconstructing these two images the generalized l_1 greedy algorithm was superior to the others at distinguishing small gradients. However, to show that the generalized l_1 greedy algorithm is truly superior to the other two algorithms at detecting small entries in general we should compare the performance of the three algorithms in recovering random sparse signals. So in this paper, following [11,12], we present a series of rigorous numerical studies of the performance of the generalized l_1 greedy algorithm in the general setting of Gaussian random matrices A and random sparse signals x. The rest of this paper is organized as follows. In Section 2, the relative frequencies of successful recovery of random Gaussian sparse signals for the reweighted l_1-minimization, l_1 greedy, and generalized l_1 greedy algorithms are compared. Section 3 presents the stability of the generalized l_1 greedy algorithm with respect to its parameters. Section 4 studies the performance of the generalized l_1 greedy algorithm on noisy data. Section 5 shows that the

generalized l_1 greedy algorithm is better at detecting smaller entries in the general setting than the other two algorithms. Section 6 concludes with a brief summary of the generalized l_1 greedy algorithm and the results.

2. Relative Frequency of Success in Recovering Gaussian Sparse Signals

In our first experiment we want to determine how well each of the three algorithms can recover random Gaussian sparse signals from random Gaussian measurements $b = Ax$. Following the same approach taken in [11], we implement each of the three algorithms in MATLAB and invoke the *l1eq-pd* solver from the l_1-MAGIC software package developed by E. Candes and J. Romberg (available at www.l1-magic.org). We set $m = 128$ and $n = 256$. For each trial, a random matrix $A \in \mathbb{R}^{m\times n}$ with i.i.d. Gaussian entries is selected and its columns are normalized. A random k-sparse signal $x_0 \in \mathbb{R}^n$ is also selected in such a way that the k nonzero positions are randomly distributed and the nonzero components satisfy the standard Gaussian distribution $N(0,1)$. We run 150 trials for each sparsity level k between 50 and 90. The total number of iterations (excluding the initial l_1-minimization step) for each of the three algorithms is set to 16. For the generalized l_1 greedy algorithm we start with 4 iterations of the reweighted l_1-minimization. The criterion for successful recovery for all three algorithms is set to

$$\|x - x_0\|_\infty \le 0.001,$$

where x is the reconstruction of x_0 by the algorithm. The parameters chosen for the three algorithms are listed as follows:

1) In the reweighted l_1-minimization: $\varepsilon = 0.1$.

2) In the l_1 greedy algorithm: $\beta = 0.8$; $\delta = 0.001$; others are the same as in 1).

3) In the generalized l_1 greedy algorithm: $\alpha = 0.25$; $s = 0.8$; $\gamma = 40$; others are the same as in 2).

The settings in this section will be used throughout the paper unless changes are explicitly stated otherwise.

The output of this experiment is presented in **Figure 1**. As one can see from the graph, for a fixed sparsity level k the probability of successful recovery of a k-sparse signal by the generalized l_1 greedy algorithm is higher than in the both cases of the reweighted l_1-minimization and $l1$ greedy algorithms. On average, the l_1 greedy algorithm and the generalized l_1 greedy algorithm recover about 14% and 18% more entries than the reweighted l_1-minimization method, respectively, for $50 \le k \le 90$. Furthermore, on average, the generalized l_1 greedy algorithm recovers about 6% more entries than the l_1 greedy algorithm.

3. Influence of the Parameters on Reconstruction Success

An empirical analysis of the reweighted l_1-minimization

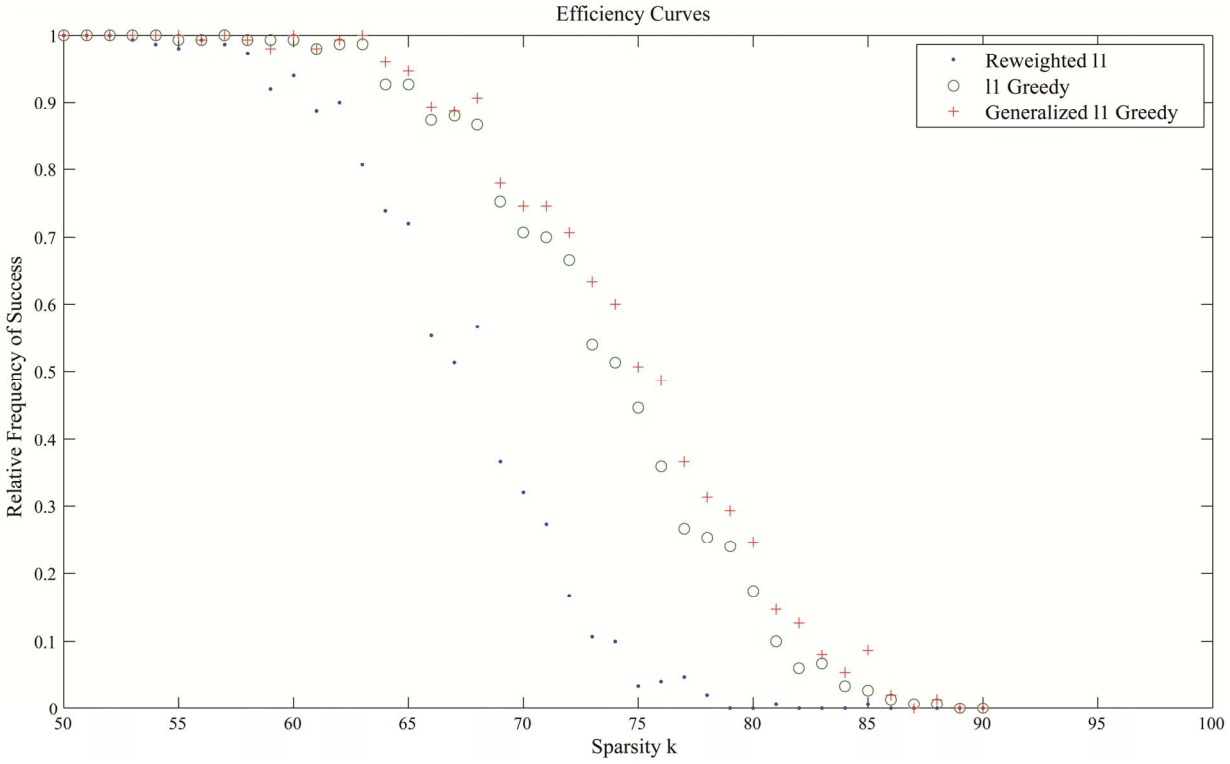

Figure 1. Improvements in recovering sparse solutions from the generalized l_1 greedy algorithm when compared to the reweighted l_1-minimization and l_1 greedy algorithms.

algorithm determined that the algorithm is robust with respect to ε (chosen from a suitable range) and that much of the improvement in recovery comes from the first few reweighting iterations [11]. We performed a similar analysis of the generalized l_1 greedy algorithm, and our results indicate that the algorithm is stable with respect to each of its parameters (within a certain range of values). We illustrate this behavior with the parameter α using the same settings as in Section 2 for the remaining parameters. From **Figure 2** one can see that the algorithm is fairly stable for the following values of α: 0.2; 0.3; 0.4. Our experimental results also show that the algorithm is very robust with respect to γ for $\gamma \geq 10$ and fairly robust with respect to s and β for values between 0.7 and 0.9. It is also evident from **Figure 3** that the number of iterations $k_{\max}$ of the generalized l_1 greedy algorithm has minimal affect on the performance of the algorithm when $k_{\max} \geq 10$. So in practice only a few iterations are needed to achieve the best performance of the generalized l_1 greedy algorithm.

4. Influence of Noise on Reconstruction Success

In real life applications measured data are often corrupted by a small amount of noise. Thus one needs to recover the original signal x_0 from noisy data

$$b^* = Ax_0 + e;$$

where $e \in \mathbb{R}^m$ is an unknown noise term. The signal-to-noise ratio (SNR) in dB is defined by

$$\text{SNR} = 20\log_{10}\frac{\|b\|_2}{\|b - b^*\|_2},$$

where $b = Ax_0$ is noise-free data. In this section we show how white Gaussian noise at SNR levels 40 dB and 60 dB, respectively, affect the performance of the generalized l_1 greedy algorithm. We also compare the performance of the reweighted l_1-minimization, l_1 greedy, and generalized l_1 greedy algorithms on noisy data with an SNR of 60 dB. As in [12] the precision of recovery is set according to the noise level. More precisely, the criterion for successful recovery are taken to be $\|x - x_0\|_\infty \leq 0.001$ and $\|x - x_0\|_\infty \leq 0.002$ for noisy data with an SNR of 40 dB and with an SNR of 60 dB, respectively. All the other settings are the same as in Section 2. **Figure 4** shows that the performance of the generalized l_1 greedy algorithm is very robust with respect to noise at an SNR level 60 dB and fairly robust with respect to noise at an SNR level 40 dB for $30 \leq k \leq 90$. **Figure 5** compares the performance of the three algorithms on noisy data with an SNR of 60 dB. Clearly, the generalized l_1 greedy algorithms outperform the other two algorithms. Moreover, for noisy data with an SNR of 60 dB, on average, the generalized l_1 greedy algorithm recovers about 17% more entries than the reweighted l_1-minimization algorithm and about 5% more entries than the l_1 greedy algorithm for $50 \leq k \leq 90$.

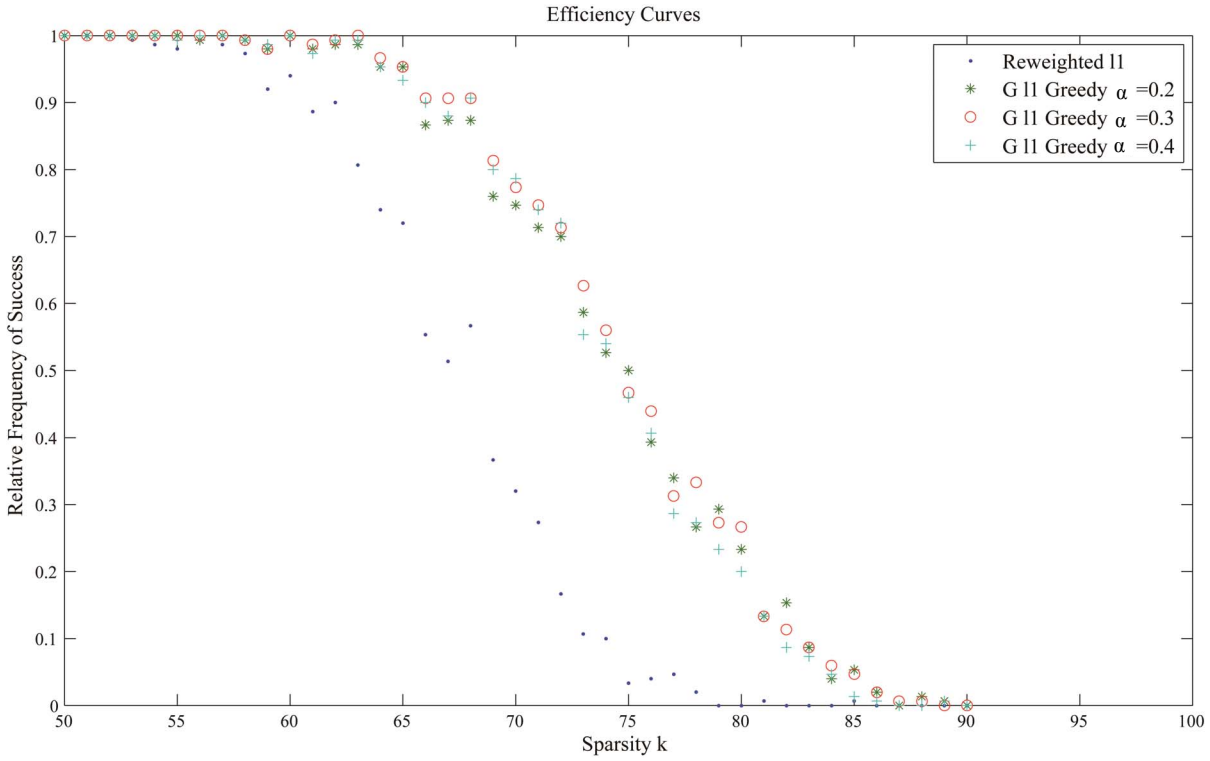

Figure 2. Stability of the generalized l_1 greedy algorithm with respect to the parameter α.

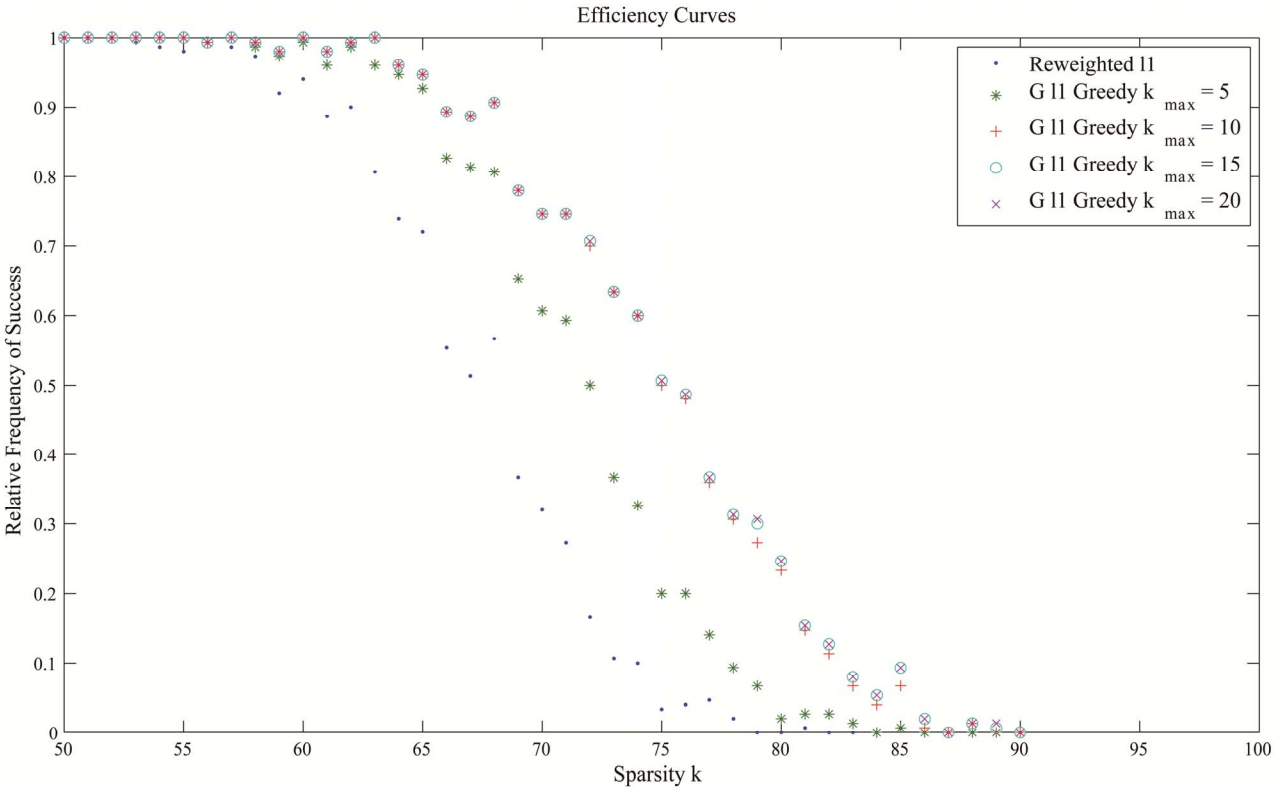

Figure 3. Effect of the number of iterations on the performance of the generalized l_1 greedy algorithm.

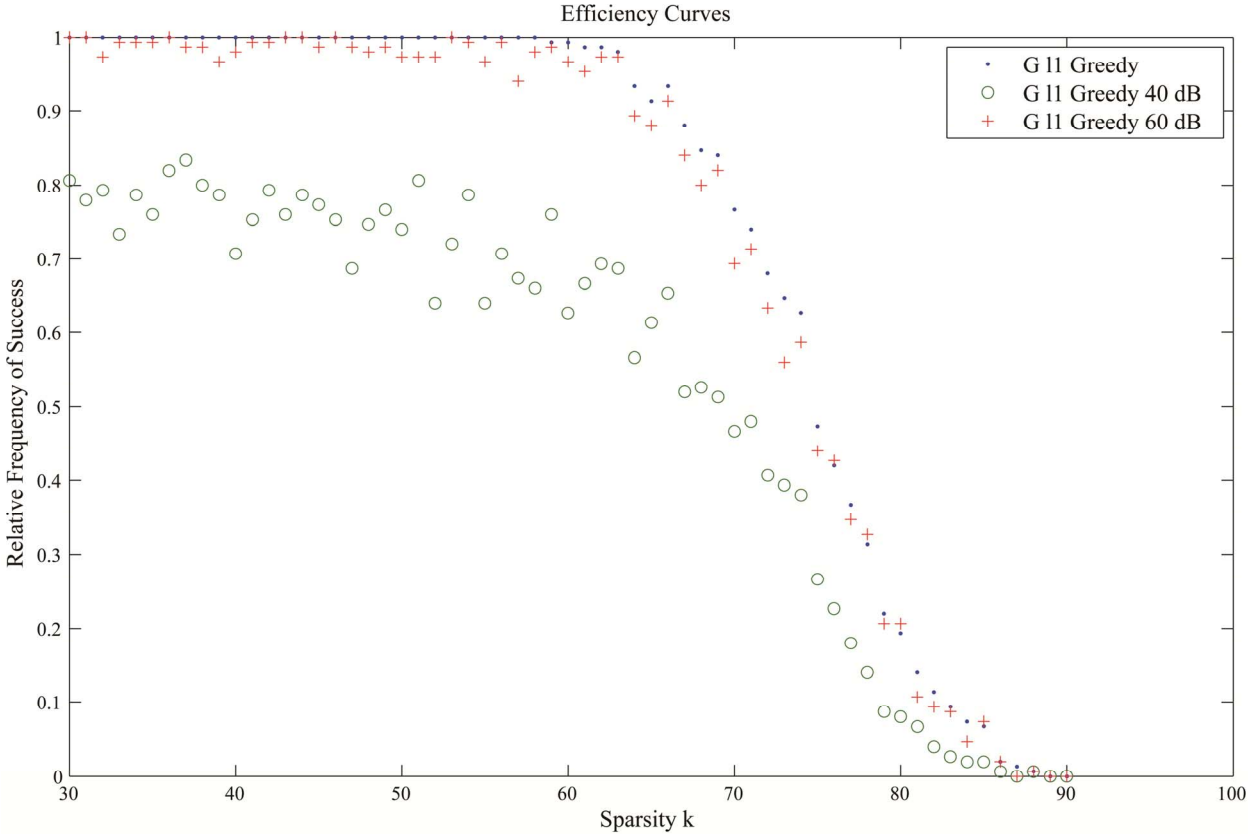

Figure 4. Performance of the generalized l_1 greedy algorithm with noisy data with an SNR of 40 dB and 60 dB, respectively.

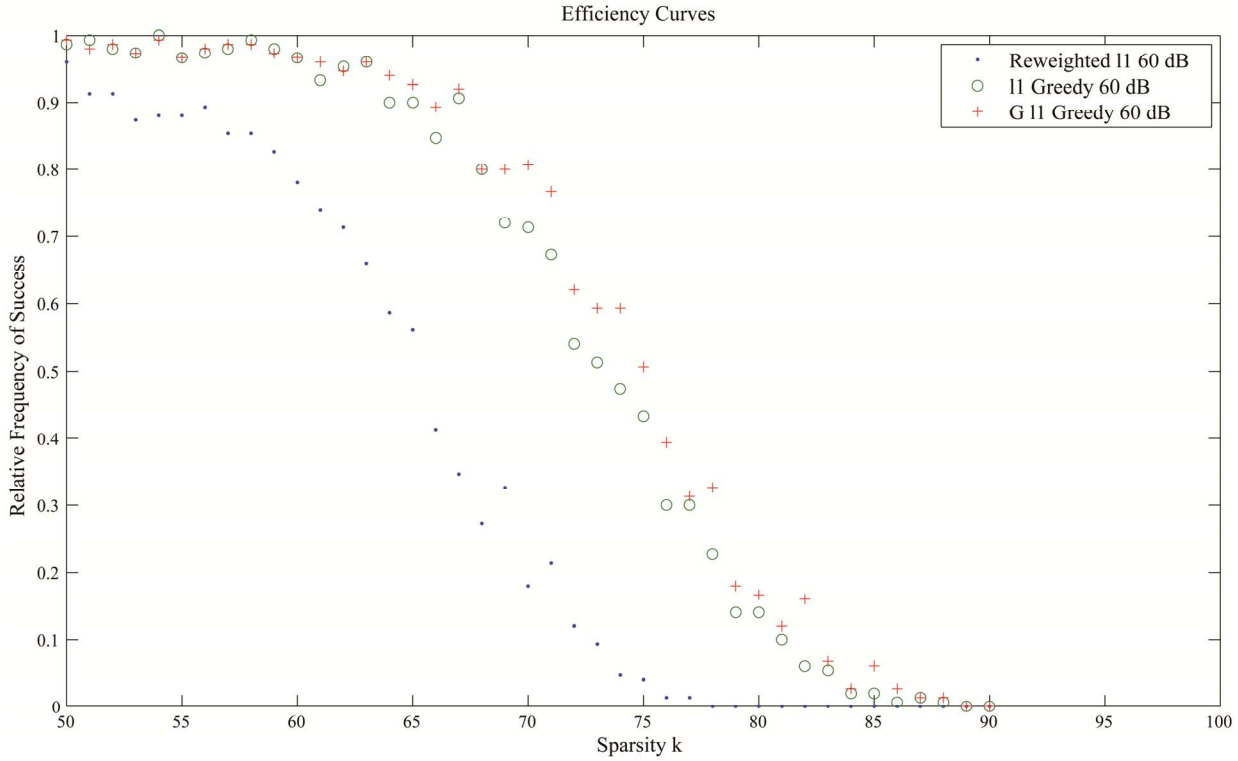

Figure 5. Improvements in recovering sparse solutions with noisy data with an SNR of 60 dB from the generalized l_1 greedy algorithm when compared to the reweighted l_1-minimization and l_1 greedy algorithms.

5. Reconstruction of Sparse Signals Containing Nonzero Small Entries

It is known that the l_1 greedy algorithm outperforms the reweighted l_1 minimization algorithm in finding spare signals [12,19]. However, the reweighted l_1-minimization algorithm was designed to help speed up the detection of small entries [11]. The generalized l_1 greedy algorithm of [20], which incorporates features of both algorithms, should have the performance advantage of the l_1 greedy algorithm while enhancing the power of the reweighted l_1-minimization algorithm in detecting small entries. In fact, the generalized l_1 greedy algorithm appears to be superior to the other two algorithms in distinguishing small gradients in the task of reconstructing images via total variation minimization [20]. In this section we want to see how well the generalized l_1 greedy algorithm would perform in recuperating random sparse signals with a guaranteed percentage of small entries. More precisely, in our last experiment we want to determine the extent to which the ratio of very small entries in the sparse signals affects the probability of successful recovery by each of the algorithms under consideration. The entries of the sparse signal in each trial are obtained from a mixed Gaussian distribution as follows: a random 30% of the entries are generated using a Gaussian distribution with mean 0 and standard deviation 0.01 while the remaining 70% of the entries are generated using the standard Gaussian distribution $N(0,1)$. We need to fine tune the generalized l_1 greedy algorithm to make it most efficient at detecting the small entries in the range we set. Experimental trials show that setting $\alpha = 0.01$ results in the best performance. The values of the other parameters are left unchanged. We then run 150 trials for each sparsity level k, $50 \le k \le 105$, and set the criterion for successful recovery to $\|x - x_0\|_\infty \le 0.001$. As one can see from **Figure 6**, the generalized l_1 greedy algorithm vastly outperforms both the reweighted l_1-minimization and the l_1 greedy algorithms in recovering sparse solutions containing a few nonzero small entries. Moreover, on average, the generalized l_1 greedy algorithm recovers 32% more entries than the reweighted l_1-minimization algorithm and 11% more entries than the l_1 greedy algorithm for $50 \le k \le 105$.

6. Conclusion

Our statistical experiments indicate that the generalized l_1 greedy algorithm outperforms the reweighted l_1-minimization and l_1 greedy algorithms in recovering random sparse signals from random Gaussian measurements. In fact, the generalized l_1 greedy algorithm recovers more entries than the other two algorithms. Moreover, the performance of the algorithm is robust with respect to its parameters and to noisy data at different noise levels. Finally, the generalized l_1 greedy algorithm performs

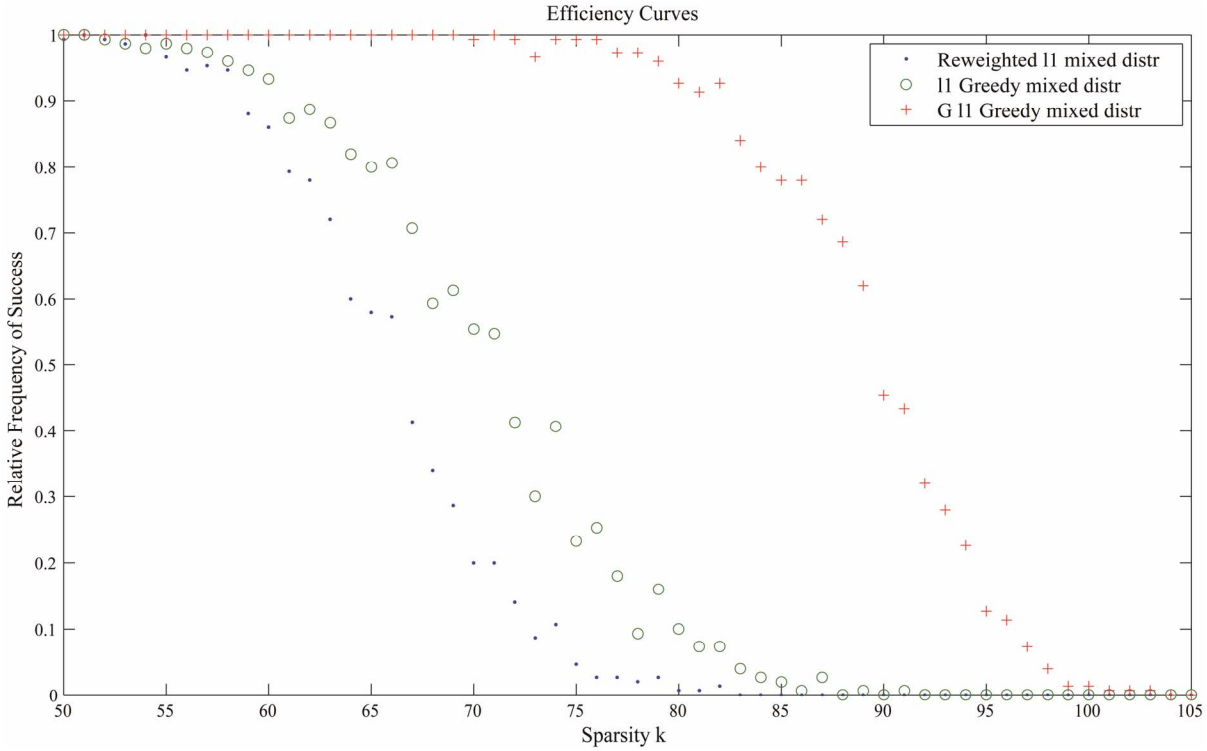

Figure 6. Improvements in recovering sparse solutions with a mixed Gaussian distribution from the generalized l_1 greedy algorithm when compared to the reweighted l_1-minimization and l_1 greedy algorithms.

extremely well in detecting small entries of unknown sparse signals thereby dramatically speeding up their recovery via l_1-minimization. It is expected that more details of signals could be recovered by using the generalized l_1 greedy algorithm without extra cost.

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