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Sze-Man Ngai

Georgia Southern University, smngai@georgiasouthern.edu

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Eigenvalue Estimates of Laplacians defined by fractal measures

Sze-Man Ngai

Georgia Southern University and Hunan Normal University

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Joint with [Da-Wen Deng](#) (= [Tai-Man Tang](#))

Definition of Dirichlet Laplacian: weak formulation

Assumptions throughout this talk:

- (1) $\Omega \subseteq \mathbb{R}^n$ bounded, open, connected.
- (2) $\mu =$ regular Borel probability measure on \mathbb{R}^n , $\text{supp}(\mu) \subseteq \overline{\Omega}$, $\mu(\Omega) > 0$.

Poincaré type inequality:

(PI) \exists constant $C > 0$ s.t.

$$\int_{\Omega} |u|^2 d\mu \leq C \int_{\Omega} |\nabla u|^2 dx \quad \forall u \in C_c^\infty(\Omega). \quad (1.1)$$

Elements in the same equivalence class of $H_0^1(\Omega)$ can belong to different equivalence classes of $L^2(\Omega, \mu)$. (PI) \Rightarrow each equivalence class of $u \in H_0^1(\Omega)$ contains a unique $\bar{u} \in L^2(\Omega, \mu)$ satisfying (1.1).

Define $\iota : H_0^1(\Omega) \rightarrow L^2(\Omega, \mu)$, $\iota(u) = \bar{u}$. Let $\mathcal{N} := \ker \iota$. Then $\iota : \mathcal{N}^\perp \hookrightarrow L^2(\Omega, \mu)$.

Define a nonnegative bilinear form $\mathcal{E}(\cdot, \cdot)$ on $L^2(\Omega, \mu)$ by

$$\mathcal{E}(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx,$$

with $\text{Dom}(\mathcal{E}) = \mathcal{N}^\perp$ (more precisely $\iota(\mathcal{N}^\perp)$).

Facts: Assume (PI).

- (a) Then \mathcal{E} is closed. Denote the corresponding self-adjoint operator by: Δ_μ , *Dirichlet Laplacian* with respect to μ .
- (b) Let $u \in H_0^1(\Omega)$ and $f \in L^2(\Omega, \mu)$. TFAE:
- (i) $u \in \text{Dom}(\Delta_\mu)$ and $\Delta_\mu u = f$;
 - (ii) $\Delta u = f d\mu$ as distributions.

$\Delta_\mu(u) = u_{tt}$ models a (nonhomogeneous) vibrating string or membrane with mass distribution μ .

A sufficient condition for (PI)

Definition: The *lower L^∞ -dimension* of μ :

$$\underline{\dim}_\infty(\mu) = \lim_{\delta \rightarrow 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln \delta}.$$

where the supremum is taken over all $x \in \text{supp}(\mu)$.

Based mainly on a result of [Maz'ja](#), we obtained:

Theorem

(*Hu-Lau-N., 06*) Assume that $\underline{\dim}_\infty(\mu) > d - 2$.

- (a) (PI) holds.
- (b) \exists an orthonormal basis $\{u_n\}_{n=1}^\infty$ of $L^2(\Omega, \mu)$ consisting of eigenfunctions of $-\Delta_\mu$.
- (c) The eigenvalues of $\{\lambda_n\}_{n=1}^\infty$ satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \quad \text{with} \quad \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

Motivations:

- 1) Estimate the Poincaré constant $= \frac{1}{\lambda_1^\mu}$.
- 2) Study existence of spectral gaps.
- 3) Study of bounds for eigenfunctions.

Classical results:

- **Faber-Krahn, 23, 25:** Let $\Omega \subset \mathbb{R}^n$ and $\text{Vol}(\Omega) = \text{Vol}(B_r(0))$.

$$\lambda_1(\Omega) \geq \lambda_1(B_r(0)).$$

Not true for manifolds!



Figure: E. Calabi: dumbbell M homeomorphic to S^2 , $\lambda_1 \rightarrow 0$ as radius of connecting pipe (of fixed length) $\rightarrow 0$.

- **Cheeger, 70:** M be a compact Riemannian manifold.

$$h_D(M) := \inf \left\{ \frac{\text{Vol}(\partial U)}{\text{Vol}(U)} : U \subset\subset M \right\}. \text{ Then}$$

$$\lambda_1 \geq \frac{1}{4} h_D^2(M).$$

Theorem 1. *Let $\Omega = (0, 1)$. Then $\lambda_1^\mu \geq \pi$.*

We improve this theorem for the **infinite Bernoulli convolutions**:

$$S_1(x) = rx, \quad S_2(x) = rx + 1 - r, \quad 0 < r < 1,$$

Let $\mu = \mu_{r,p}$ be the associated self-similar measure:

$$\mu = p\mu \circ S_1^{-1} + (1 - p)\mu \circ S_2^{-1}, \quad 0 < p < 1.$$

$\text{supp}\mu \subseteq [0, 1]$.

Proposition 2. Let μ be an *infinite Bernoulli convolution on $[0, 1]$ with $p = 1/2$* . Then $\lambda_1^\mu > \pi$. In fact,

- (a) for $r \in (0, 1/2]$ (μ is a Cantor-type or Lebesgue), $\lambda_1^\mu \geq 4/r$;
- (b) for $r \in (1/2, 2/3]$, $\lambda_1^\mu \geq 24/7 \approx 3.428$;
- (c) for $r \in (2/3, 1)$, $\lambda_1^\mu \geq 3.2$.

Definition

μ on $\Omega \subseteq \mathbb{R}^n$ is **upper s -regular** if $\exists c_1 > 0$ s.t.

$$\mu(F) \leq c_1 |F|^s \quad \text{for all } \mu \text{ measurable subsets } F \subseteq \Omega.$$

Theorem 3. Let $\Omega \subset \mathbb{R}^2$. Suppose that μ is **upper s -regular** for some $s \geq 1$. Let $\beta = c_1 |\Omega|^{s-1} / 2$ with c_1 being the constant above. Then

$$\lambda_1^\mu \geq \frac{\sqrt{\lambda_1}}{2\beta}.$$

where λ_1 is the first eigenvalue of $-\Delta$.

$\Omega \subset \mathbb{R}^n$ and μ absolutely continuous. By using Cheeger's argument:

Theorem 4. Let $\Omega \subseteq \mathbb{R}^n$, μ be *absolutely continuous* w.r.t. Lebesgue measure with a bounded density ρ , and

$$\beta := \inf_{U \subseteq \Omega} \frac{1}{\mu(U)} \int_{\partial U} \rho \, d\mathcal{H}^{n-1}.$$

Then $\lambda_1^\mu \geq \beta^2 / (4\|\rho\|_{L^\infty(dx)})$.

Upper estimates

It is not clear how to obtain upper estimates in general.

We can obtain upper estimates for infinite Bernoulli convolution

- 1) Cantor type $r < 1/2$, and
 - 2) golden ratio $(\sqrt{5} - 1)/2$.
- Erdős, 39: μ is singular (also true for other Pisot numbers).
 - L^q -spectrum, multifractal formalism, and dimension (Lau-N., 98, 99, Feng 05).
 - Spectral dimension (N., 11).
 - Wave equation (Chan-Teplyaev-N., to appear).

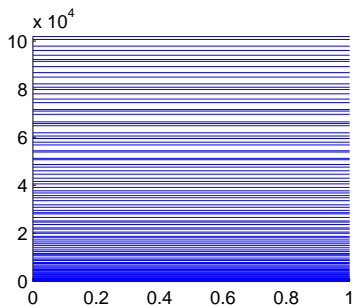


Figure: First 100 Dirichlet eigenvalues for the Bernoulli convolution associated with the golden ratio. (Chen-N., 10)

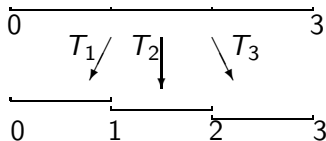


Second-order identities by [Strichartz-Taylor-Zhang, 95](#):

$$T_1(x) = S_1 S_1(x) = r^2 x,$$

$$T_2(x) = S_1 S_2 S_2(x) = S_2 S_1 S_1(x) = r^3 x + r^2,$$

$$T_3(x) = S_2 S_2(x) = r^2 x + r.$$



For any Borel subset $A \subseteq [0, 1]$,

$$\begin{bmatrix} \mu(T_1 T_j A) \\ \mu(T_2 T_j A) \\ \mu(T_3 T_j A) \end{bmatrix} = M_j \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \quad j = 1, 2, 3,$$

where M_1, M_2, M_3 are, respectively,

$$\begin{bmatrix} p^2 & 0 & 0 \\ (1-p)p^2 & (1-p)p & 0 \\ 0 & 1-p & 0 \end{bmatrix}, \begin{bmatrix} 0 & p^2 & 0 \\ 0 & (1-p)p & 0 \\ 0 & (1-p)^2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & p & 0 \\ 0 & (1-p)p & (1-p)^2 p \\ 0 & 0 & (1-p)^2 \end{bmatrix}.$$

Let $J = j_1 \cdots j_m$, $j_i \in \{1, 2, 3\}$. Then

$$\mu(T_J A) = c_J \begin{bmatrix} \mu(T_1 A) \\ \mu(T_2 A) \\ \mu(T_3 A) \end{bmatrix}, \quad \text{where } c_J = \mathbf{e}_{j_1} M_{j_2} \cdots M_{j_m} = (c_J^1, c_J^2, c_J^3).$$

Define

$$w_J^* := \mu(T_J[0, 1]) = \frac{1}{1 - p + p^2} c_J \begin{bmatrix} p^2 \\ p(1 - p) \\ (1 - p)^2 \end{bmatrix}.$$

Theorem 5. Let $\mu = \mu_{r,p}$ be an infinite Bernoulli convolution and f_m be a piecewise linear approximate first eigenfunction obtained by the finite element method.

(a) If $0 < r \leq 1/2$, then

$$\lambda_1^{\mu_{r,p}} \leq \frac{\sum_{J \in \mathcal{J}_m} (f_m(T_J(1)) - f_m(T_J(0)))^2 / (T_J(1) - T_J(0))}{\sum_{J \in \mathcal{J}_m} w_J \min\{f_m(T_J(1))^2, f_m(T_J(0))^2\}},$$

for all $m \geq 1$. For the standard Cantor measure, we have

$$12 \leq \lambda_1^{\mu_{1/3,1/2}} \leq 14.3865.$$

(c.f.: numerical approximation by FEM with $m = 7$ is 14.4353...)

(b) If $r = (\sqrt{5} - 1)/2$, the reciprocal of the golden ratio, then

$$\lambda_1^{\mu_{r,p}} \leq \frac{\sum_{J \in \mathcal{J}_m} (f_m(T_J(1)) - f_m(T_J(0)))^2 / (T_J(1) - T_J(0))}{\sum_{J \in \mathcal{J}_m} w_J^* \min\{f_m(T_J(1))^2, f_m(T_J(0))^2\}}$$

for all $m \geq 1$. In particular, if $p = 1/2$, we have

$$6.33437 \leq \lambda_1^{\mu_{r,1/2}} \leq 8.05171.$$

(c.f.: numerical approximation by FEM with $m = 7$ is
8.03475...)

Preparations for proofs.

Definition

Let $\Omega \subseteq \mathbb{R}^n$ be open (*not necessarily bounded*), $\emptyset \neq K \subset \Omega$ compact. Define the **1-capacity** of K relative to Ω :

$$\text{Cap}_1(K, \Omega) := \inf \left\{ \int_{\Omega} |\nabla \psi| \, dx : \psi \in C_c^\infty(\Omega), \psi \geq 1 \text{ on } K \right\}.$$

Examples:

- 1) $K \subset \mathbb{R}$ compact $\Leftrightarrow \text{Cap}_1(K, \mathbb{R}) = 2$.
- 2) $\text{Cap}_1(K, \mathbb{R}^n) = 0 \Leftrightarrow \mathcal{H}^{n-1}(K) = 0$.

Proposition

(Maz'ja, Sobolev spaces) Let $\Omega \subseteq \mathbb{R}^n$. If \exists a constant $\beta > 0$ s.t. \forall compact $F \subset \Omega$,

$$\mu(F) \leq \beta \operatorname{Cap}_1(F, \Omega),$$

then

$$\int_{\Omega} |u| d\mu \leq \beta \int_{\Omega} |\nabla u| dx, \forall u \in C_c^\infty(\Omega),$$

Bound of μ -measure by 1-capacity \longrightarrow 1-Poincaré inequality.

We also need Rayleigh's formula:

$$\lambda_1^\mu = \inf_{f \in \operatorname{Dom} \mathcal{E}} \frac{\int_{\Omega} (f')^2 dx}{\int_{\Omega} f^2 d\mu}.$$

Proof of Theorem 1:

Step 1. By Rayleigh's formula for λ_1 ,

$$\lambda_1 \leq \frac{\int_0^1 (f')^2 dx}{\int_0^1 f^2 dx}, \quad \lambda_1^\mu = \frac{\int_0^1 (f')^2 dx}{\int_0^1 f^2 d\mu}.$$

Hence

$$\frac{\lambda_1^\mu}{\lambda_1} \geq \frac{\int_0^1 f^2 dx}{\int_0^1 f^2 d\mu}. \quad (3.2)$$

Step 2. For $F \subset \Omega$, $\mu(F) \leq 1$ and $\text{Cap}_1(F, \mathbb{R}) = 2$ implies

$$\mu(F) \leq \text{Cap}_1(F, \mathbb{R})/2.$$

Thus by the Proposition, For all $u \in H_0^1(\Omega)$,

$$\int_0^1 |u| d\mu \leq \frac{1}{2} \int_0^1 |u'| dx \quad \forall u \in C_c^\infty(\Omega). \quad (3.3)$$

For $u = [\bar{u}] \in H_0^1(\Omega)$, take $\{u_n\} \in C_c^\infty(\Omega)$ that converges to \bar{u} simultaneously in $H_0^1(\Omega)$ and $L^2(\Omega, \mu)$. Then $\mu(\Omega) < \infty$ implies that $u_n \rightarrow \bar{u}$ in $L^1(\Omega, \mu)$ and $\nabla u_n \rightarrow \nabla \bar{u}$ in $L^1(\Omega, dx)$. Taking limit.

Step 3. (Cheeger's argument: **Convert L^1 -estimate to L^2 -estimate.**) Taking $u = f^2$, we get

$$2 \int_0^1 f^2 d\mu \leq \int_0^1 2|f||f'| dx \leq 2 \left(\int_0^1 f^2 dx \right)^{1/2} \left(\int_0^1 (f')^2 dx \right)^{1/2}.$$

Hence

$$\lambda_1^\mu \int_0^1 f^2 d\mu = \int_0^1 (f')^2 dx \geq \frac{(\int_0^1 f^2 d\mu)^2}{\int_0^1 f^2 dx},$$

implying

$$\lambda_1^\mu \geq \frac{\int_0^1 f^2 d\mu}{\int_0^1 f^2 dx}. \quad (3.4)$$

Combining (3.2) and (3.4) gives $(\lambda_1^\mu)^2 \geq \lambda_1 = \pi^2$.

Prove of Theorem 3 is similar.

Step 1: Use Rayleigh formula for λ_1 .

Step 2: Prove that $\mu(F) \leq \beta \text{Cap}_1(F; \Omega)$ (needs upper s -regularity)
and thus by Proposition

$$\int_{\Omega} |u| d\mu \leq \beta \int_{\Omega} |\nabla u| dx.$$

Step 3: Cheeger's argument.

Step 2: Claim: μ upper s -regular $s \geq 1$ (i.e., $\mu(F) \leq c_1|F|^s$).
Then for all compact $F \subset \Omega$,

$$\mu(F) \leq \beta \text{Cap}_1(F, \Omega) \quad \forall \text{ compact } F \subset \Omega,$$

where $\beta = c_1|\Omega|^{s-1}/2$.

Reason: Suppose $F \subset \Omega$ compact and connected. F a singleton: then $\mu(F) = 0$ and conclusion holds trivially for any β . F not a singleton: its connectedness implies that $\text{Cap}_1(F; \Omega) > 0$. Let ball $B_{|F|}$ contain F . Then $|F|^s = \pi^{-s/2} \mathcal{L}^2(B_{|F|})^s/2$. Notice:

$$\text{Cap}_1(F; \Omega) \geq \text{Cap}_1(F; \mathbb{R}^2) = \text{Cap}_1(\text{co}(F); \mathbb{R}^2) \geq 2|F|.$$

Hence

$$\frac{|F|^s}{\text{Cap}_1(F; \Omega)} \leq \frac{\pi^{-s/2} (\mathcal{L}^2(B_{|F|}))^{s/2}}{2|F|} = \frac{|F|^{s-1}}{2} \leq \frac{|\Omega|^{s-1}}{2}.$$

Together with upper s -regularity, we get

$$\frac{\mu(F)}{\text{Cap}_1(F; \Omega)} \leq \frac{C|\Omega|^{s-1}}{2}.$$

For general compact sets, approximate from outside by the union of countably many compact connected sets.

Main ideas in the proof of Proposition 2: (Use some results in [Bird-Teplyaev-N. 2003](#))

$$\mu = \frac{1}{2}\mu \circ S_1^{-1} + \frac{1}{2}\mu \circ S_2^{-1}.$$

Let u be an λ_1^μ -eigenfunction corresponding to with $u'(0) = 1$.
Known: u and u' are continuous and

$$u'(x) = u'(0) - \lambda_1^\mu \int_0^x u(y) d\mu(y).$$

Also, $u'(1/2) = 0$ by symmetry.

Hence

$$\lambda_1^\mu = \left(\int_0^{1/2} u(y) d\mu(y) \right)^{-1} = \left(\int_0^{1-r} u(y) d\mu(y) + \int_{1-r}^{1/2} u(y) d\mu(y) \right)^{-1}$$

- 1) Use concavity of u and $u'(0) = 1$ to get $u(y) \leq y \forall y \in [0, 1]$.
- 2) Fact: $\int_0^1 y d\mu(y) = 1/2$.
- 3) Self-similar identity.

Proof of Theorem 4. Step 1. Suppose $\exists \beta > 0$ s.t.

$$\beta \int_{\Omega} |\phi| d\mu \leq \int_{\Omega} |\nabla \phi| d\mu \quad \forall \phi \in C_c^\infty(\Omega) \text{ (hence } H_0^1). \quad (3.5)$$

Then $\lambda_1^\mu \geq \frac{\beta^2}{4\|\rho\|_\infty}$.

Take $\phi = f^2$, f a λ_1^μ -eigenfunction. Assume $\phi \in H_0^1(\Omega)$; otherwise, take truncations of f and then take limit.

$$\beta \int_{\Omega} f^2 d\mu \leq 2 \int_{\Omega} |f| |\nabla f| d\mu \leq 2 \|f\|_{L^2(\mu)} \|\nabla f\|_{L^2(\mu)},$$

which implies that

$$\frac{\beta^2}{4} \int_{\Omega} f^2 d\mu \leq \int_{\Omega} |\nabla f|^2 \rho dx \leq \|\rho\|_{\infty} \int_{\Omega} |\nabla f|^2 dx.$$

Thus,

$$\frac{\beta^2}{4\|\rho\|_{\infty}} \int_{\Omega} f^2 d\mu \leq \int_{\Omega} |\nabla f|^2 dx.$$

Notice that $\lambda_1^{\mu} \int_{\Omega} f^2 d\mu = \int_{\Omega} |\nabla f|^2 dx$.

Step 2. Proof of (3.5). Using the coarea formula,
Let ψ be a nonnegative Borel measurable function on Ω and let $u \in C^{0,1}(\Omega)$. Then

$$\int_{\Omega} \psi(x) |\nabla u(x)| \, dx = \int_{-\infty}^{\infty} \left(\int_{\{|\psi|=t\}} \psi(x) \, d\mathcal{H}^{n-1}(x) \right) dt.$$

In its simplest form $\psi \equiv 1$, $n = 2$, $u =$ “2-dim tent function”, the formula says the area of a disk can be evaluated by integrating the circumferences of circles making up the disk.

$$\begin{aligned}\int_{\Omega} |\nabla\phi| d\mu &= \int_{\Omega} |\nabla\phi|\rho dx = \int_0^{\infty} \left(\int_{|\phi|=t} \rho d\mathcal{H}^{n-1} \right) dt \\ &= \int_0^{\infty} \left(\frac{1}{\mu\{|\phi| \geq t\}} \int_{\{|\phi|=t\}} \rho d\mathcal{H}^{n-1} \right) \mu\{|\phi| \geq t\} dt \\ &\geq \int_0^{\infty} \left(\frac{1}{\mu\{|\phi| \geq t\}} \int_{\partial\{|\phi| \geq t\}} \rho d\mathcal{H}^{n-1} \right) \mu\{|\phi| \geq t\} dt \\ &\geq \left(\inf_{U \subseteq \Omega} \frac{1}{\mu(U)} \int_{\partial U} \rho d\mathcal{H}^{n-1} \right) \int_0^{\infty} \mu\{|\phi| \geq t\} dt \\ &= \beta \int_{\Omega} |\phi| d\mu\end{aligned}$$

Theorem 4 proved.

Proof of Theorem 5.

- We choose any positive numerical first eigenfunction f_m .
- f_m is piecewise linear and $\int_0^1 (f'_m)^2 dx$ can be evaluated exactly.
- For $\int_0^1 f_m^2 d\mu$, multiply the measure of an interval between two consecutive nodes by the minimum value of f_m on that interval to get a lower bound $s(f_m)$.
- Use Rayleigh's formula

$$\lambda_1^\mu = \min_{u \in \text{Dom}(\mathcal{E})} \frac{\int_0^1 (u')^2 dx}{\int_0^1 u^2 d\mu} \leq \frac{\int_0^1 (f'_m)^2 dx}{\int_0^1 f_m^2 d\mu} \leq \frac{\int_0^1 (f'_m)^2 dx}{s(f_m)}.$$

Problems for further study:

- 1) Other eigenvalues λ_n^μ , $n \geq 2$.
- 2) Spectral gap conjecture.
- 3) Relationship between eigenvalues and geometry of the set and measure-theoretic properties of μ .

Thank you!