The Analysis of Restrained Purlins Using Generalised Beam Theory

C. Jiang

Philip Leach

J. Michael Davies

Twelfth International Specialty Conference on Cold-Formed Steel Structures
St. Louis, Missouri, U.S.A., October 18-19, 1994

The Analysis of Restrained Purlins using Generalised Beam Theory

by J M Davies[1], C Jiang[1] and P Leach[1]

1.0 Introduction

A paper presented at the Eleventh Speciality Conference in 1992[(1)] introduced the Generalized Beam Theory (GBT) and illustrated its use. It was used in first-order analyses to calculate the stress distribution in a cross section takeing account of cross section distortion, and in second-order bifurcation problems to calculate the critical buckling load of a free cross section subject to axial load. Subsequent papers[(2,3)] have given more detailed information on the basis of GBT and used its second-order facilities to investigate the buckling of sections under both uniform and non-uniform bending moment.

This paper extends the use of GBT to consider the behaviour of a cross section which is elastically restrained continuously along its length. A typical application of this facility is in the analysis of a purlin which receives both lateral and torsional restraint from the sheeting which it supports. The paper illustrates how the basic equations of GBT can be used to calculate the buckling load of an elastically restrained cross section taking account of interaction between the different buckling modes. Using this estimate of the buckling load, an assessment of the collapse load of a restrained section can be made using the interaction formulae of Eurocode 3[(4)] to allow for both buckling and yielding.

2.0 The basic equations of Generalised Beam Theory

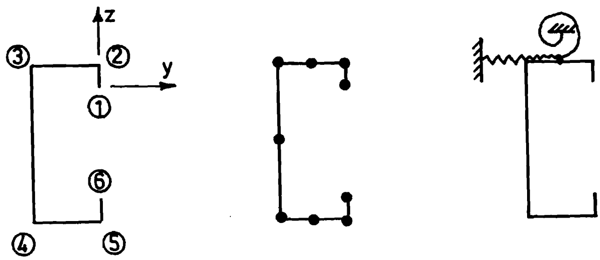
Analysis using GBT is carried out in two parts. In the first part, generalised section properties are evaluated which may include second-order terms. The second part of GBT then uses these properties in a global analysis which takes account of the loads and boundary conditions. Continuous elastic restraints are taken into account in the first part of GBT, the second part remaining essentially unchanged.

The essential concept of GBT is the separation of the behaviour of a prismatic member into a series of orthogonal displacement modes. It is one of the strengths of the procedures arising from GBT that these modes may be then considered separately or in any combination in order to investigate different aspects of structural response.

The number of displacement modes available in the analysis of any given cross-section is related to the number of nodes. In general, these are the natural nodes at the extremities of an open cross-section and at the fold lines as shown in Fig.1(a). In addition, in order to allow local buckling in second-order analyses, intermediate nodes may be introduced, as shown in Fig.1(b). For the analyses described in this paper, any *face element* may be restrained either lateral or torsionally or both as shown in Fig.1(c). As purlins are generally sufficiently stocky for elastic local buckling to

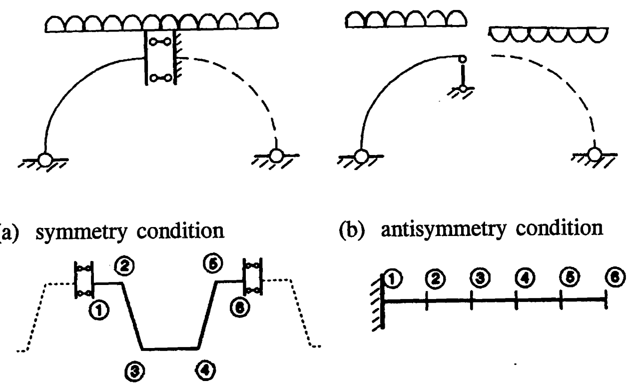
[1] Telford Research Institute, University of Salford, England

be insignificant, and as we are particularly concerned with the interaction of lateral torsional buckling and cross-section distortion, it has not been considered necessary to introduce intermediate nodes into the analyses considered in this paper. It may be noted that GBT also allows the complete restraint of any natural node as shown in Fig.2.



(a) natural nodes (b) intermediate nodes (c) elastic restraints

Fig1. Nodes and restraints in GBT analysis



(c) periodic construction (d) rigid restraint

Fig.2 Further examples of restrained cross-sections

As shown in Reference 1, the following equation is valid for all first and second-order problems:

$$\sum_{k=1}^{n} \Big[E\,{}^{k}C\,{}^{k}V'''' - G\,{}^{k}D\,{}^{k}V'' + {}^{k}B\,{}^{k}V + \sum_{i,j} {}^{ijk}\kappa_{\sigma}\left({}^{i}W\,{}^{j}V'\right)' + {}^{ijk}\kappa_{\tau}\left({}^{i}W''\,{}^{j}V + 2\,{}^{i}W'\,{}^{j}V'\right) = {}^{k}q\Big] \qquad (1)$$

where:

${}^{k}B$	=	Stiffness in mode k with respect to transverse bending (distortion)
${}^{k}C$	=	Stiffness in mode k with respect to longitudinal strains
${}^{k}D$	=	Torsional stiffness in mode k
E	=	Young's modulus
G	-	Shear modulus
${}^{ijk}\kappa_{\sigma}$	-	Second-order terms resulting from longitudinal stress
${}^{ijk}\kappa_{\tau}$	-	Second-order terms resulting from shear stress
${}^{i}W$	-	Stress resultant in mode i
${}^{j}V$	-	Non-dimensional shape function for mode j
${}^{k}q$	-	Applied distributed load in mode 'k'
n	-	Number of nodes in the cross section (number of plate elements = n-1)

In the above family of equations, the unknown values are the non-dimensional shape functions, ${}^{j}V$ which may be considered to be generalised displacements. For first order problems, the kappa terms are zero and these equations are uncoupled and reduce to the standard equation for 'beam on elastic foundation' problems for which several alternative methods of solution are available. For second-order problems, the equations become coupled and it is necessary to use numerical methods of solution. The finite difference method has proved to be the most appropriate to date.

3.0 Elastic Torsional and Translational Restraints

The calculation of the cross sectional properties for an unrestrained cross section has been fully described in a previous paper(5). The method used to derive the properties consisted of calculating the virtual work done by the cross section when displacing in any mode, and equating this to the virtual work done by the load. In order to introduce lateral or torsional restraints to the section, it is therefore simply necessary to evaluate the virtual work done by these restraints, and to add this to the work done by the section, before solving the energy equation. The following sections describe the calculation of these additional terms.

3.1 Work Done by a Torsional Restraint

The notation for displacements in the plane of the cross-section is given in Fig.3 and, with this notation, the work W_{θ} done by any torsional restraints to a short length dx of a section is given by:

$$W_\theta = \sum_{i=1}^{n-1} f_{\theta,i}\, p_{\theta,i}\, dx = F_\theta P_\theta \qquad (2)$$

Where F_θ is the matrix of plate rotations and P_θ is the matrix of the forces induced by these displacements in each of the (potentially) 'n-1' rotational springs attached to the section.
The latter term can be replaced by:

$$P_\theta = C_\theta F_\theta^T \qquad (3)$$

where C_θ is the matrix of torsional spring stiffnesses restraining the cross section. In all cases the only non zero terms are on the leading diagonal and, in the most common case of a single torsional restraint on the section, this matrix consists of a single number with all other entries zero.

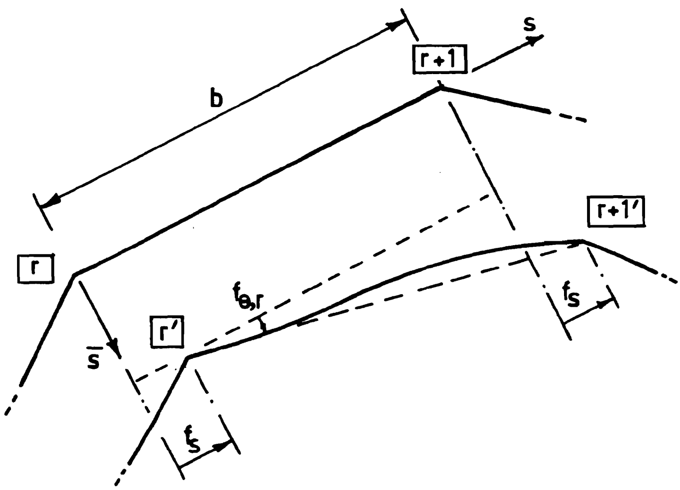


Fig.3 **Notation for the displacements of plate elements in the cross-section**

Combining equations (2) and (3) gives:

$$W_\theta = F_\theta C_\theta F_\theta^T \qquad (4)$$

This term should be taken into account by adding it to the virtual work of the transverse bending moments calculated according to Reference 5.

3.2 Work Done by a Translational Restraint

The work W_δ done by any translational restraints to a short length of section is given by:

$$W_\delta = \sum_{i=1}^{n-1} f_{s,i} p_{s,i} dx = F_s P_s \tag{5}$$

where F_s is the matrix of plate displacements (around the cross-section) and P_s is the matrix of forces induced by these displacements in each of the (potentially) 'n-1' translational springs attached to the section. The latter term can be replaced by:

$$P_s = C_s F_s^T \tag{6}$$

where C_δ is the matrix of translational spring stiffnesses restraining the cross section. In all cases the only non zero terms are on the leading diagonal, and in the most common case of a single translational restraint on the section, this matrix consists of a single number with all other entries zero.

Combining equations (5) and (6) gives:

$$W_\delta = F_s C_s F_s^T \tag{7}$$

This term should also be taken into account by adding it to the virtual work of the transverse bending moments calculated according to Reference 5. The assembly of the virtual work matrices, the orthogonalisation of the deformation modes and the extraction of the rigid-body modes is otherwise unchanged, leading to the section properties ${}^kC$, ${}^kD$ and ${}^kB$ and the κ-terms in equation (1) as well as other information about the properties of the cross-section.

4. Solution of the Second Order Equation of Generalised beam Theory

The solution of equation (1) requires the use of numerical techniques such as the finite difference method or the finite element method. In this paper the finite difference method has been exclusively used although the finite element method may in some cases provide a more elegant solution.

Equations (1) can be rewritten in finite difference form [6] using the following substitutions:

$$\begin{aligned} V(i)'' &= \frac{V(i-1) - 2V(i) + V(i+1)}{dx^2} \\ V(i)'''' &= \frac{V(i-2) - 4V(i-1) + 6V(i) - 4V(i+1) + V(i+2)}{dx^4} \end{aligned} \tag{8}$$

If the member is divided into 'm-1' slices along its length, this gives 'm' finite difference equations for each of the 'n' modes considered. In order to solve these (n x m) equations, the boundary conditions must be introduced. The boundary conditions can all be stated in terms of the displacement ${}^kV$ or its derivatives (i.e. ${}^kV' = d^kV/dz$ or ${}^kV'' = d^{2k}V/dz^2$ etc). Most cases of support condition can be covered by the following four boundary conditions:

- pinned end: $^kV_i = 0$, $^kV_i{}'' = 0$ i.e. $^kV_{i-1} = {}^kV_{i+1}$

- fixed end: $^kV_i = 0$, $^kV_i{}' = 0$ i.e. $^kV_{i-1} = -{}^kV_{i+1}$

- clamped end: $^kV_i{}' = 0$, $^kV_i{}''' = 0$ i.e. $^kV_{i-1} = {}^kV_{i+1}$ and $^kV_{i-2} = {}^kV_{i+2}$

-free end: $^kV_i{}'' = 0$, $^kV_i{}''' = 0$ i.e. $^kV_{i-1} = 2{}^kV_i + {}^kV_{i+1}$ and $^kV_{i-2} = 4{}^kV_i - 4{}^kV_{i+1} + 4{}^kV_{i+2}$

It may be noted that a more sophisticated form of the finite difference method and its boundary conditions is given in Reference (2). This latter method allows a reduction in the number of finite difference slices for a given accuracy.

For second-order bifurcation problems, the applied load in the direction of the displacement is zero, so that the right hand side of equation (1) becomes zero. This equation can then be re-arranged to become:

$$\sum_{k=1}^{n}\left[{}^kF = \sum_{j=1}^{n}\sum_{i=1}^{n} {}^{ijk}S \right] \quad \textbf{(9)}$$

where:

$$^kF = E\,{}^kC\,{}^kV'''' - G\,{}^kD V'' + {}^kB\,{}^kV \quad \textbf{(10)}$$

$$^{ijk}S = -\,{}^{ijk}\kappa_\sigma({}^iW\,{}^jV)'' - {}^{ijk}\kappa_\tau({}^iW''\,{}^jV + 2\,{}^iW'\,{}^jV') \quad \textbf{(11)}$$

These equations can be applied to any second-order bifurcation problem.

5.0 Solution of the Buckling Equation

For second order problems, the buckling load coefficient λ can be obtained by solving the following equations:

$$\sum_{k=1}^{n}({}^kF - \lambda\,{}^{ijk}S)\,{}^kV = 0 \quad \textbf{(12)}$$

where S and F are the matrices defined in section 4.

For a free section, if only a single mode k is taken into account in equation (12), the critical load of this single mode can be obtained by solving the finite difference form of a single differential equation. If modes 1 to 4 are taken into account (i.e. $k = 1,4$) in equation (12), the critical loads calculated are those related to rigid body displacements. If all 'n' modes are taken into account in equation (12), the critical loads calculated are those related to interactive buckling of the section allowing for cross section distortion in addition to rigid body displacements. Similarly, by taking a single value of 'i' in matrix S, the critical loads under a single load (e.g. axial load or major axis bending moment) can be obtained.

For a member with arbitrary boundary conditions and loading, a computer is needed to solve the above eigenvalue problem. In this work, the Jacobi method has been used to solve equation (12) but, in general, any standard eigenvalue routine is suitable.

For a section which is restrained along its length, the number of rigid body modes is usually decreased to either 2 or 3. The reason for this reduction is that if a section is torsionally restrained along one element, the torsional mode which would be possible in a free section becomes mixed with the distortional modes. Similarly if the section is laterally restrained along one element, the lateral buckling mode which would be possible in a free section is also mixed with the distortional modes.

6.0 Comparison of GBT Analyses with Full Scale Tests

In the development of their new Multibeam III range of purlins and sheeting rails, a comprehensive series of full scale single span tests were carried out by Ward Building Systems in the UK. The tests were carried in a vacuum loading rig and, in some of the tests, the test specimens were inverted in order to simulate wind suction with the free flange in compression. A range of purlin sections with different sheeting types were considered. In some tests the top flange of the purlin was connected directly to the sheeting and in others a layer of insulation was introduced between the sheeting and purlin. The tests considered here were all on a single simply-supported span subject to wind suction loading. Critical buckling analyses of all the purlins were carried out using the GBT method described above.

In the analyses, the boundary conditions assumed were simply supported with regard to rigid body modes and fixed with respect to the higher-order distortional modes. The load was assumed to be applied through the shear centre of the cross section. A rigid translational restraint was assumed to the top flange of the section and an elastic rotational restraint was also introduced in order to simulate the restraining effects of the sheeting on the purlin. The values of the rotational restraints c_θ were derived from small scale 'F' tests as recommended in Eurocode 3 Part 1.3

GBT analysis gives rise to the elastic buckling load M_{cr} and, in order to introduce a yield criterion into the analysis, the equations of Eurocode 3 Annex 1.3 (Clause 6.1) were adopted, with an imperfection factor α_{LT} equal to 0.21 giving the failure moment M_f in the tables of results which follow. This is in accordance with the trend in current design standards to express all cases of interaction between buckling and yielding by a "Perry-Robertson" type equation in which the imperfection factor is chosen to give safe results. Thus:

$$M_f = \chi_{LT} M_y \tag{13}$$

where χ_{LT} = the reduction factor for lateral torsional buckling
M_y = the yield moment of the cross-section

$$\chi_{LT} = \frac{1}{\phi_{LT} + \left[\phi_{LT}^2 - \bar{\lambda}_{LT}^2\right]^{0.5}} \quad \text{but } \chi_{LT} \leq 1 \tag{14}$$

$$\phi_{LT} = 0.5\left[1 + 0.21\left(\bar{\lambda}_{LT} - 0.2\right) + \bar{\lambda}_{LT}^2\right] \tag{15}$$

$$\bar{\lambda}_{LT} = \left[\frac{M_y}{M_{cr}}\right]^{0.5} \tag{16}$$

The cross-section of the Multibeam Mark III purlin is shown in Fig.4 and Table 1 below gives the dimensions of the purlins which were tested and then compared to the GBT analysis. The results of these analyses are then shown in Table 2.

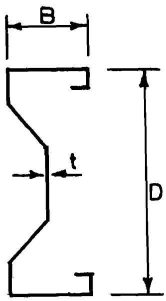


Fig.4 **Cross-section of the Multibeam Mark III purlin**

Section	Depth D (mm)	Breadth B (mm)	Thickness t (mm)
2070	145	57.56	1.228
2071	145	57.32	1.232
3062	175	57.53	1.226
3063	175	60.48	1.224
2316	145	48.43	1.235
2317	145	48.64	1.234

Table 1. **Dimensions of the tested purlins**

Section	c_θ (kNm/m /radian)	Span L (m)	Yield Strength (N/mm^2)	M_{cr} (kNm)	M_f (kNm)	M_{test} (kNm)	M_f/ M_{test}
2070	0.810	6.0	409.60	6.07	4.33	4.33	1.000
2071	0.763	6.0	406.07	5.07	4.27	4.33	0.986
3062	1.102	6.0	421.00	6.67	5.15	5.32	0.968
3063	1.141	6.0	424.98	6.76	5.27	5.32	0.991
2316	0.695	6.0	411.86	4.72	3.56	3.96	0.899
2317	0.707	6.0	410.00	4.73	3.56	3.96	0.899

Table 2 Comparison of GBT and test results

It can be seen that the complex shape of Multibeam Mark III includes 10 folds and this gives rise to 12 natural nodes and 12 modes of deformation in the GBT analysis. These are 4 rigid body modes and 8 different modes of distortion and all of these were included in the analyses. In the unrestrained section, the behaviour is dominated by modes 3 and 4 (lateral torsional buckling) and the distortional modes play no part unless the span is unrealistically small. However, this situation may change when restraints are introduced and distortion of the cross-section occurs over the whole range of spans.

It can be seen from Table 2 that in all cases the theoretical analysis offers a conservative estimate of the failure load and that the prediction is within 10% of the test load.

7.0 Further development of the use of GBT for purlin design

GBT has an extremely useful second-order form whereby solutions to a range of bifurcation problems may be obtained simply and conveniently. This simplified form is applicable to cases where the applied load causing buckling is either an axial load or a uniform bending moment and the displacement function of each of the active modes is a half sine wave. This includes many problems of practical significance.

The theory for this application of GBT has been given in References 1 and 3.

Here, the applied load is a uniform bending moment about the major axis, which is an approximation to the bending moment arising from a uniformly distributed load. The half sine wave is a very good approximation to the deflected shape of a simply-supported purlin undergoing lateral torsional buckling. For the purposes of this study, a particular case of the Multibeam Mark III shown in Fig.4 will be considered with D = 175mm, B = 60mm and t = 0.198mm. As before, when unrestrained, this purlin has 10 folds and therefore 12 natural buckling modes which are orthogonal to each other. Four of these are the basic rigid-body modes (axial, bending about the two principal axes and torsion) and the remaining eight are distortional.

It appears that none of the 8 distortional modes has independent significance but that for short wavelength buckling, several of them may combine with the lateral and torsional modes to form a composite distortional mode which, for the typical purlin size being considered, has a wavelength of about 50cm. The buckling stresses for some critical mode combinations as a function of the buckling length is shown in Fig.5.

An important point to note is that, once the buckling half wave length exceeds a certain value (about 160 cm in Fig.5), the distortional modes cease to have any influence on the lateral torsional buckling stress which may be calculated with excellent accuracy by considering only the rigid body modes 3 and 4.

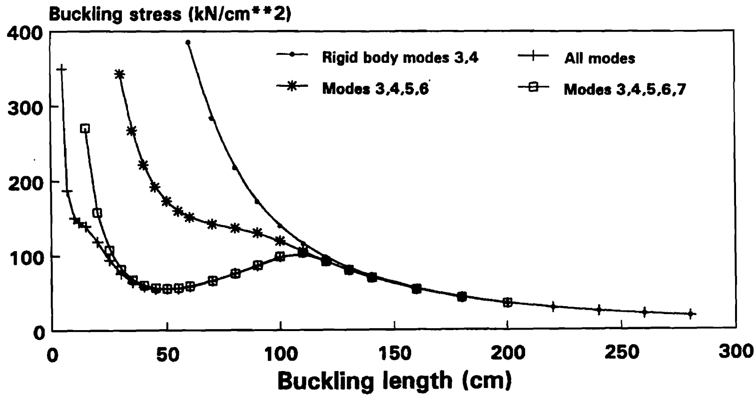


Fig.5 Buckling stress as a function of buckling half wavelength - unrestrained section

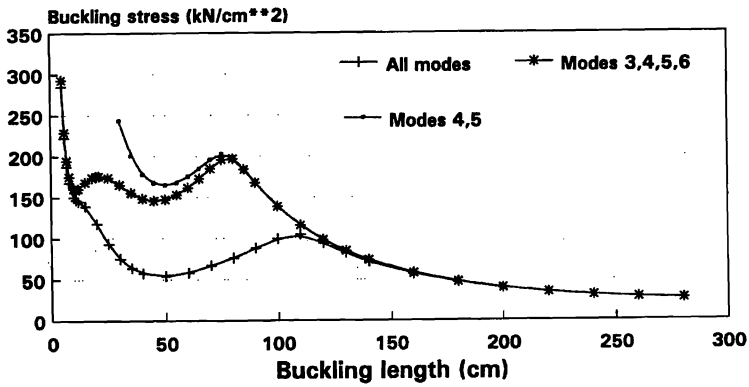


Fig.6 Buckling stress as a function of half wavelength - lateral and torsional restraints

In addition to the unrestrained cross-section considered in Fig.5, similar studies were carried out with the upper flange of the cross section torsionally restrained (c_θ = 1.0 kNm/m/radian), fully restrained laterally and with a combination of lateral and torsional restraint. Fig.6 shows the results obtained for the combined case. For short wavelength buckling, some of the combination cases are different from those for the unrestrained purlin but the result when all of the modes are considered is surprisingly similar in both shape and magnitude. Equally important is the fact that once a similar critical buckling length has been exceeded, the behaviour is again entirely a form of lateral torsional buckling (with some distortion) given by a combination of the two dominant modes.

What the vertical scale of Fig.6 conceals, however, is the very significant effect of the lateral and torsional restraints at realistic buckling lengths. Table 3 summarises the results obtained from the four alternative systems of restraint at the typical span of 6 metres. The results given in this table are precisely the same whether the two dominant modes or all modes are considered.

Evidently, it is very important to consider the *combination* of lateral and torsional restraint and, when this is done, very significant increases in buckling stress are obtained. Furthermore, accurate values of these enhanced buckling stresses are obtained by consideration of the two dominant modes. Once the enhanced section properties have been obtained for a given purlin and restraint system using the first part of GBT, the second part is a simple explicit calculation that can be done "on the back of an envelope". The use of GBT in this context is being studied further and will be the subject of another technical paper in due course.

Restraints	Critical moment (kNm)	Buckling stress (N/mm^2)
Unrestrained section	1.78	47.6
Torsional restraint c_θ = 1.0	6.22	166
Full lateral restraint	1.85	49.4
Lateral and torsional restraint	14.9	398

Table 3 Effect of different restraints on a simply-supported purlin of 6 metres span

8.0 Conclusions

This paper has illustrated the procedure for the calculation of the critical buckling load of any restrained cross section with arbitrary boundary conditions and loading. The Generalised Beam Theory (GBT) method which was used distinguishes between a free cross section and a restrained cross section in the calculation of the cross section properties. The fundamental equation of GBT is unchanged.

In order to confirm the analysis technique, a series of tests were carried out on the most common restrained cross section used in practice, namely that of a purlin subject to uplift load in which the tension flange is restrained laterally and torsionally by the sheeting.

By combining a yield criterion with the buckling analysis, a theoretical failure load can be calculated which can then be compared to test results. The comparisons showed that the theoretical predictions were all accurate to within 10% and in all cases the method of analysis proved to be safe. This type of calculation assumes considerable practical importance when a particular family of purlins is to be used with a range of cladding systems all providing different amounts of restraint to the purlins.

The procedure used in the main part of the paper is capable of considerable simplification as illustrated in section 7. Once the enhanced section properties have been determined, the global analysis for buckling stress becomes trivial. The practical implications of this are still being explored.

9.0 References

[1] J M Davies and P Leach "Some applications of Generalised Beam Theory" Proc 11th Int. Speciality Conf. on Cold-Formed Steel Structs., St. Louis, Missouri, October 1992, pp479-501.

[2] J M Davies and P Leach "First order Generalised Beam Theory", Jnl. of Constructional Steel Research, Special Issue on Cold-formed Steel Structures, 1994.

[3] J M Davies, P Leach and D Heinz "Second order Generalised Beam Theory", Jnl. of Constructional Steel Research, Special Issue on Cold-formed Steel Structures, 1994.

[4] Eurocode 3 "The Design of Steel Structures: Part 1.1, General rules and rules for buildings", BS DD ENV 1993-1-1, British Standards Institution, 1992.

[5] P Leach "The Calculation of Modal Cross Section Properties for use in the Generalized Beam Theory" Thin Walled Structures - To be Published.

[6] P Leach "The Generalized Beam Theory with Finite Difference Applications", PhD Thesis, University of Salford, 1989.