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Proximity of Contraflexure to Plate Buckling Load

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SUMMARY

This paper presents a quantitative investigation of the correspondence between critical load of plates and contraflexure load on a load-deflection diagram. Their proximity is shown to be imperfection-sensitive. Alternative procedures not requiring prior knowledge of the initial plate imperfections are suggested. Examples are given.

1. INTRODUCTION

The continuing practice to meet the scarcity of structural raw materials by the use of high-strength thin-walled cold-formed steel sections brings in its wake the enhanced need to be able to design such structures adequately against buckling failure. It is not so long ago that the only type of buckling behavior to which the majority of engineers were introduced was simple Euler-strut buckling characterized by a linearized elastic analysis leading to an eigenvalue problem in load-deflection space with neutral equlibrium at large deflections; (see fig. 1a). However, during the last few decades, there has been a considerable increase in nonlinear structural stability analysis(1),(2), leading to general formulations(3) and

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(1)Superscripts in parenthesis refer to references in section 7.

theories[4] of nonlinear elastic stability. If the fundamental equilibrium path exhibits a local maximum, snap-type of buckling occurs (as shown in fig. 1b); if, on the other hand, a branch-point or intersection-point occurs on the equilibrium-path, the postbuckling behavior has been classified by Thompson[5] in accordance with the secondary path's initial slope and curvature (as shown in figs. 1c - e). The heavy lines in figs. 1a - e represent the idealized load-deflection behavior of mathematical fictions referred to as "perfect" structures, (continuous = stable, dashed = unstable); the light lines represent the behavior of real so-called "imperfect" structures, as classified by Roorda[6].

Cold-formed steel structures frequently can be considered from their behavior as assemblages of plates. This is a fortunate fact inasmuch as it is well known that plates exhibit elastic stable-symmetric postbuckling behavior so that there may be a considerable reserve of strength above the critical load P_c (≡ the elastic first eigenvalue[+]). Of course, the ultimate plate strength will depend on the inelastic behavior; nevertheless P_c remains one of the fundamental parameters[7]. Unhappily it is an elusive quantity from a physical viewpoint, even for plates; in many cases, because of mathematical complications, one still determines P_c by resorting to experimental testing.

A review and qualitative evaluation of experimental techniques for determining P_c for plates was presented by Vann & Sehested[8] at the Second Specialty Conference On Cold-formed Steel Structures. One of the best-known such techniques is the graph proposed by Southwell[9]

[+] A separate list of definitions of all symbols will be found in section 8.

in 1932 for the case of columns. In 1936 Timoshenko[10] observed that "it is advantageous to apply the method (for the case of plates)" but in fact many experimenters have had difficulty in applying the technique to such structural elements[11]. Spencer and Walker[11], by considering the effect of practical boundary conditions realizable experimentally and by using the postbuckling equations for plates, have shown quantitatively that the simple Southwell Plot may not be reliable for either columns or plates.

Another widely-used technique for evaluating the P_c of plates experimentally is the "inflexion-point method" which uses the proximity of P_c to P_{CF} (the latter being defined as the load corresponding to the point of contraflexure on a graph of compressive load vs. lateral deflection; see fig. 2). The technique was mentioned (but apparently not used) by Hoff[12] reporting in 1948 and by Coan[13] in 1951. It has been used extensively by Schlack[14] and by Schmied et al[15]. Vann and Sehested report[8] in 1973 that "as yet, no analytical study has been presented concerning the accuracy of the inflexion-point method", and they conclude from a qualitative study (a) that "of the three lateral deflection techniques discussed, the inflexion-point method appears to give the best value of (P_c) " and (b) "with the possible exception of the inflexion-point method, all of the methods considered for evaluating (P_c) experimentally tend to decrease in accuracy as the imperfection amplitude increases."

The present paper analyzes the accuracy of the contraflexure (≡ inflexion-point) method quantitatively on the basis of a series approximation to the postbuckling equation for plates, and it compares the method with that of the Spencer Plot.

2. PRACTICAL TEST BOUNDARY-CONDITIONS

It is worthwhile to consider the background of the experimental results from which one tries to infer the various plate buckling characteristics such as critical load, etc. The facility with which simple standard boundary conditions can be formulated, bears no relationship to the effort of translating such idealizations into practical designs. It is very difficult indeed to get reliable buckling data, and perhaps even more difficult to interpret such data accurately. Engineers conceive a structure in terms of its components and thus there is a tendency to design experiments on structural components: struts, plates, etc. The buckling behavior of these components depends markedly on the boundary conditions but, for columns and even more so for plates, it turns out to be virtually impossible to realize in practice the simple idealized boundary conditions which it is customary to assume on paper. The compromises which it is then necessary to make manifest themselves as imperfections which, not uncommonly, owing to imperfection-sensitivity, cause inaccuracies and scatter of the postbuckling data(3),(6).

Suppose an attempt is made to reproduce a boundary-condition as apparently simple as "simply-supported" along a load-bearing edge of a test plate; figs. 3a - d shows four of the solutions which have been tried. One of them is here discussed in detail; the effects introduced by the others are of course slightly different but can be characterized generally as imperfections.

Fig. 3a shows a simple male knife-edge on the test-specimen loaded by a simple female knife-edge on the bearing-plate. For a start, this introduces an unknown error into the effective length of the plate, particularly for thick plates, the more so if one has to allow for the possibility of large plate rotations so that the male wedge-angle must be fairly acute and the female wedge-angle fairly obtuse. For the case of columns this has been investigated by Hayashi et al [16]. Errors probably of a more serious kind would be introduced by the inaccuracies of the machining of the (male and female) knife-edges particularly for very thin plates of thickness $h \leqslant$ say 1 mm. Such errors will be of three kinds: Firstly, any amounts by which the average position of the knife-edges are off-centre will be "seen" by the experimental results as an imperfection of the load-eccentricity type. Secondly, if either the male or the female knife-edge is not perfectly straight (as viewed in the loading-direction), the loaded edge of the specimen will be bent which may give rise to cylindrical-panel behavior, and the amount of such bending (and hence of such behavior) may change during the course of the experiment. Thirdly, (and this problem may also occur with the other designs of fig. 3 ,) if the clearance between the two knife-edges varies longitudinally, the specimen will experience a form of patch-loading [12]. Finally, the sharper the knife-edges have been machined, the sooner they are likely to become blunted during an experiment; the specimen would then tend to behave as if it experiences partial rotational restraint at the loaded edges.

Perhaps one of the best simulations to simply-supported boundary-conditions is shown in fig. 3d. This was proposed originally by Barlow [17] for columns, and has been developed by Coan [13] and by Walker [18] for plates. The disadvantage of this solution is twofold: Firstly the manufacture of the slotted roller bearings is difficult and expensive; secondly, for small plate specimens, the set-up does not necessarily give sufficient longitudinal freedom for differential rotation owing to the finite length of the bearings.

Clearly then the entire attempt to reproduce idealized boundary conditions for buckling tests in the laboratory is fraught with difficulties. The diametrically opposite approach is to test plates as they are actually used in engineering practice, and then to attempt to analyze the boundary-conditions. For example, one can test box columns[15]. If the box columns have a square cross-section, and if the deflection of the plates under compressive load is such as to keep the corners square but rotated, then it is reasonable to assume that each buckled plate of the box can be represented by the simply-supported plate approximation. In order to analyze such tests one needs also to assume that each side of the box carries one quarter of the total applied load.

Thus it becomes abundantly clear that it is essential to calibrate whatever experimental set-up is to be used for a buckling test. One of the prime parameters for such a calibration is P_c. If P_{CF} can be used as a measure of P_c then it is necessary to know

(a) how accurately they correspond to one another,

(b) whether the error can be estimated, and

(c) under what conditions, if any, this method of measuring P_c offers advantages over other methods.

3. PRACTICAL DESIGN EQUATIONS FOR THE LATERAL DEFLECTION OF COMPRESSED PLATES

The large-deflection behavior of thin, isotropic, elastic, initially-flat plates under the action of in-plane forces can be described by the Karman equations

$$\frac{D}{h} \nabla^4 w = L(w,F) \tag{1}$$

$$\frac{1}{E} \nabla^4 F = -\tfrac{1}{2} L(w,w) \tag{2}$$

$$L(w,F) \equiv w,_{xx} F,_{yy} + w,_{yy} F,_{xx} - 2 w,_{xy} F,_{xy} \tag{3}$$

in which $D \equiv E h^3 (1 - \nu^2)^{-1} / 12$, $E \equiv$ Young's modulus, $h \equiv$ plate thickness, $F(x,y) \equiv$ Airy stress function of generalized plane stress in the plate, $\nu \equiv$ Poisson's ratio, $w(x,y) \equiv$ net lateral plate deflection, and x and y are orthogonal coordinates in the plane of the undeflected plate boundary. These equations, together with the plate boundary conditions, form a highly nonlinear boundary-value problem which, though it may present a stimulating challenge to the research worker, does not have a form suitable for a design office.

If w is written as a series sum of the eigen-modes of eq (1),

$$w = \sum_{m=1}^{\infty} w_m f_m(x,y) \tag{4}$$

where $w_m \equiv$ amplitudes of eigenmodes $f_m(x,y)$

it is possible to generate a perturbation scheme to obtain explicit series that form approximate solutions to the large-deflection equations (1) and (2). If the eigenvalues are well-separated, as is often the case for practical plate geometries and loading, the perturbation procedure is rapidly convergent and the number of terms required in the series

$$w_m = \sum_k a_{m(k)} s^k \tag{5}$$

where $a_{m(k)}$ are coefficients and s is the perturbation parameter, is generally small. Stein[19] noted that $s = (P/P_c - 1)^{\frac{1}{2}}$ is a good perturbation parameter for the perfect-plate postbuckling path and this was subsequently used by Walker[20].

For many practical cases involving rectangular plates, the fundamental mode w_1 turns out to be predominant and thus to be approximated by the maximum total deflection W. Also, the symmetry of plate postbuckling (see fig. 2) requires k in the expansion (5) to be odd, whence finally the perfect-plate load-deflection relationship takes the form

$$\frac{W}{h} = A_1 (P/P_c - 1)^{\frac{1}{2}} + A_3 (P/P_c - 1)^{3/2} + \ldots \tag{6}$$

If now one considers the practical real plate, which has an initial deflection $w_o(x,y)$ at zero load, the Karman equations assume the form

$$\frac{D}{h} \bar{\nabla}^4 (w_T - w_o) = L(w_T,F) \tag{7}$$

$$\frac{1}{E} \bar{\nabla}^4 F = - L(w,w) + L(w_o,w_o) \tag{8}$$

where the subscript T indicates total deflection from the xy-plane. The solution of these equations may be presented as an expansion of $((W_T/h)^2 - (W_o/h)^2)^{\frac{1}{2}}$ where W_o is the maximum imperfection amplitude of the plate in the same form as the fundamental eigenmode. For moderate W_o/h, Williams and Walker[21] have proposed the perturbation parameter

$$\phi = (P/P_c - 1 + W_o/W_T)^{\frac{1}{2}} \tag{9}$$

whence

$$((W_T/h)^2 - (W_o/h)^2)^{\frac{1}{2}} = A_1 \phi + A_3 \phi^3 + \ldots \tag{10}$$

This expression appears to be valid over a wide range of loading conditions, boundary conditions, and aspect ratios for rectangular plates. The load-deflection curves for these numerous cases have been calculated by numerical solution of the boundary-value problem using a finite-difference dynamic-relaxation program, and subsequently the coefficients A_1 and A_3 have been found by curve fitting[21]. It is worthwhile emphasizing that, for moderate imperfections, the coefficients A_1 and A_3 in eq (9) are the same as the ones in eq (6), so that the results of the perfect-plate analysis can be used for the imperfect plate. Thus, provided A_1 and A_3 are known, eq (10) represents a very powerful design tool. A partial list of A_1 and A_3 from ref. 21 is given in Table 1.

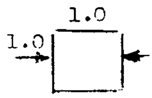
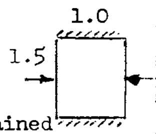
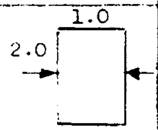
Boundary Conditions	Aspect Ratio	Lateral Edges Stress-free		Lateral Edges Straight	
		A_1	A_3	A_1	A_3
simply-supported all round 	1.0 : 1	2.157	0.010	1.840	-0.259
loaded edges simply-supported; lateral edges rotationally restrained 	1.5 : 1	1.846	-0.101	1.812	-0.097
simply-supported all round 	2.0 : 1	1.430	0.203	1.408	0.168

Table 1: Partial List Of Coefficients A_1 And A_3

4. QUANTITATIVE ANALYSIS OF PROXIMITY OF LOAD-DEFLECTION-CONTRAFLEXURE TO CRITICAL LOAD OF PLATES.

The perturbation solution eq (10) enables postbuckling problems of plates to be discussed quantitatively to a high degree of approximation. It is clear from Table 1 that, in general, the absolute value of the perturbation coefficient A_3 is an order of magnitude smaller than that of A_1. In view of the rapid convergence of the series one feels justified in truncating the right side of eq (10) to two terms only to obtain

$$\left((W_T/h)^2 - (W_o/h)^2 \right)^{\frac{1}{2}} = A_1 \phi + A_3 \phi^3 \tag{11}$$

Now
$$W_T = W + W_o \tag{12}$$
so that differentiation with respect to W is the same as with respect to W_T.

From the square of eq (9)

$$(\phi^2)_{,W} = P_{,W}/P_c - W_o/W_T^{\,2} \tag{13}$$

$$(\phi^2)_{,WW} = P_{,WW}/P_c + 2\, W_o/W_T^{\,3} \tag{14}$$

where $_{,W}$ denotes d/dW

By squaring eq (11), repeated differentiation, and use of eqs (13) and (14), it can be shown that

$$\frac{P_{,W}}{P_c} = \frac{W_o}{W_T^{\,2}} + \frac{2\,W_T}{(A_1\,h)^2} \cdot \frac{1}{1 + 4(A_3/A_1)(\phi^2) + 3(A_3/A_1)^2(\phi^2)^2} \tag{15}$$

and

$$\frac{P_{,WW}}{P_c} = \frac{-2W_o}{W_T^3} + \frac{2/(A_1h)^2 - 4(A_3/A_1)(P_{,W}/P_c - W_o/W_T^{\,2})^2\,(1 + 3/2\,(A_3/A_1)(\phi^2))}{1 + 4(A_3/A_1)(\phi^2) + 3\,(A_3/A_1)^2\,(\phi^2)^2} \tag{16}$$

At contraflexure $P,_{WW} = 0$. Using the subscript CF for the values of the variables at contraflexure, eqs (15) and (16) give

$$P,_{W(CF)}/P_c = W_o/W_{CF}^2 + \frac{2\,W_{CF}}{(A_1h)^2} \cdot \frac{1}{1 + 4(A_3/A_1)(\phi_{CF}^2) + 3(A_3/A_1)^2(\phi_{CF}^2)^2} \quad (17)$$

and

$$2\,(W_o/W_{CF}^3)\,(1 + 4(A_3/A_1)(\phi_{CF}^2) + 3(A_3/A_1)^2(\phi_{CF}^2)^2) = $$
$$= 2/(A_1h)^2 - 4(A_3/A_1)(P,_{W(CF)}/P_c - W_o/W_{CF}^2)^2\,(1 + 3/2\,(A_3/A_1)\phi_{CF}^2 \quad (18)$$

By considering $A_3/A_1 \ll 1$, eq (18) yields the useful approximation

$$W_o/W_{CF} \cong (W_{CF}/(A_1h))^2 \quad (19)$$

which, when used in eq (11), gives

$$P_{CF}/P_c \cong 1 - (W_o/h)^2/A_1^2 \quad (20)$$

Similarly, by using eq (17) to calculate an approximation to $P,_{W(CF)}$ and reversion of eq (11), it can be shown that, to a higher degree of approximation,

$$P_{CF}/P_c = 1 - (W_o/h)^2/A_1^2 + 10\,(W_o/W_{CF})^2\,A_3/A_1 \quad (21)$$

The proximity of P_{CF} to P_c can now be calculated provided that A_1, A_3, W_o/h, and W_o/W_{CF} are known for the test being analyzed. The second correction term on the right side of eq (21) can be approximated by use of eq (19) to give

$$10\,(W_o/W_{CF})^2\,A_3/A_1 \cong 10\,(W_o/(A_1h))^{4/3}\,A_3/A_1 \quad (22)$$

However, in practice there would be no need to use this approximation because $W_{CF} = W_o$ + net deflection at contraflexure, the latter being determined at the same time as P_{CF}.

It is interesting to note that, if A_1 and A_3 are of like sign, the two correction terms on the right side of eq (21) tend to cancel each other. On the other hand, as can be seen from Table 1, it is precisely for the practically important case of the uniaxially-loaded, simply-supported, square plate with lateral edges held straight that A_1 and A_3 are of opposite sign.

Tables 2 and 3 present numerical values of the two correction terms over a practical range of parameters A_1 and W_o/h, and $|A_3/A_1|$ and W_o/W_{CF}.

A_1 \ W_o/h	0.10	0.50	1.00
1.2	0.0069	0.174	0.694
1.3	0.0059	0.148	0.592
1.4	0.0051	0.128	0.510
1.5	0.0044	0.111	0.444
1.6	0.0039	0.098	0.391
1.7	0.0035	0.087	0.346
1.8	0.0031	0.077	0.309
1.9	0.0028	0.069	0.277
2.0	0.0025	0.063	0.250
2.1	0.0023	0.057	0.227
2.2	0.0021	0.052	0.207

Table 2: Values of $(W_o/h)^2/A_1^{\,2}$

$\lvert A_3/A_1\rvert$ \ W_o/W_{CF}	0.10	0.20	0.40
0.05	0.005	0.020	0.080
0.10	0.010	0.040	0.160
0.15	0.015	0.060	0.240
0.20	0.020	0.080	0.320
0.25	0.025	0.100	0.400
0.30	0.030	0.120	0.480
0.35	0.035	0.140	0.560
0.40	0.040	0.160	0.640

Table 3: Values of $10(W_o/W_{CF})^2(A_3/A_1)$

EXAMPLE

The example chosen is for the case of a simply-supported square plate, uniaxially-loaded, with lateral edges stress-free, this example being among the most favorable to the Contraflexure Method. For this case, from Table 1, $A_1 = 2.16$ and $A_3 = 0.01$.

From a finite-difference program run at University College London[11] to study plate postbuckling behavior, for this case:

For $W_o/h = 0.05$, there results $W_o/W_{CF} = 0.083$;
hence $P_{CF}/P_c = 1 - 0.00054 + 0.00091 = \underline{1.0005}$

For $W_o/h = 0.50$, there results $W_o/W_{CF} = 0.36$;
hence $P_{CF}/P_c = 1 - 0.054 + 0.006 = \underline{0.95}$

5. EVALUATION OF CONTRAFLEXURE METHOD AND COMPARISON WITH SPENCER PLOT

The effect (on the proximity of P_{CF} to P_c) of introducing the second perturbation constant A_3 has been shown in eq (21) in terms of the ratio W_o/W_{CF} . For the present discussion it will suffice to introduce the approximation (22) into eq (21) to obtain

$$P_{CF}/P_c \cong 1 - (W_o/(A_1 h))^2 + 10\, A_3/A_1 (W_o/(A_1 h))^{4/3} \qquad (23)$$

Therefore the sensitivity of the contraflexure-proximity P_{CF}/P_c to the relative imperfection W_o/h is given by

$$\frac{\partial(P_{CF}/P_c)}{\partial(W_o/h)} \cong -\frac{2}{A_1^2}\frac{W_o}{h} + \frac{40}{3}\frac{A_3}{A_1^2} (W_o/(A_1 h))^{1/3} \qquad (24)$$

Similarly the sensitivity of the contraflexure-proximity to the perturbation constants A_1 and A_3 is given by

$$\frac{\partial (P_{CF}/P_c)}{\partial A_1} \cong + \frac{2}{A_1^3} (W_o/h)^2 - \frac{70}{3} \frac{A_3}{A_1^2} (W_o/(A_1 h))^{4/3} \qquad (25)$$

$$\frac{\partial (P_{CF}/P_c)}{\partial A_3} \cong \frac{10}{A_1} (W_o/(A_1 h))^{4/3} \qquad (26)$$

With the aid of eqs (23) to (26) a quantitative appraisal of the contraflexure method can now be attempted. First of all the question arises: how accurately can P_{CF} be located from experimental data? Certainly this problem appears to have been solved successfully in the careful experiments reported by Schlack[14] who, however, was fortunate in this respect in dealing with very small imperfections. Studies by the authors on computer-generated data indicate that the inflexion point is reasonably easy to find for $W_o/h < 0.1$, increasingly difficult to find for $0.1 < W_o/h < 0.5$, and almost impossible for $W_o/h > 0.5$. For practical data with scatter, it is probably necessary to use some least-squares approach to improve on the technique of locating P_{CF} with a French curve. However, if one already goes to the expense and trouble of running some least-squares computer program, then why bother with P_{CF} at all when it would be possible to use the program to find the empirical constants in eq (11) directly ?

Once P_{CF} has been determined by the experimenter (and sometimes this can indeed be done by inspection), eq (23) shows how P_{CF}/P_c depends on W_o/h. It is clear from Tables 2 and 3 that the necessary corrections are not always negligible. How then is one to measure W_o ? It is possible to develop apparatus[22] to measure the initial lack of flatness;

however, W_o should be thought of not as the true physical imperfection but as an empirical constant which will make eq (11) fit the data. The writers have conducted extensive numerical analyses of their own plate-deflection data and of the data published in the literature, by means of the nonlinear least-squares programs[23] developed at the Numerical Optimisation Centre, Hatfield Polytechnic, to find these empirical constants from the point-of-view of Curve Fitting. If, for example, one takes Schlack's data, ref. 14, Table 1, $\gamma = 0.0$, then, for the results of three of Schlack's runs (which plot so closely together on a superimposed load-deflection graph that it is almost impossible to distinguish between the three sets of points) W_o varies approximately 25%. And as for the uncertainty of boundary conditions discussed in Section 2, it is clear that the effect on the post-critical curvature A_1 might approach 20% .

If the values of the coefficients A_1 and A_3 from Table 1, for the simply-supported, uniaxially-loaded, square plate with $W_o/h = 0.5$ are introduced into eqs (24), (25) and (26), the following gradients are obtained for the sensitivity of P_{CF}/P_c with resp ct to the empirical constants:

$$\frac{\partial(P_{CF}/P_c)}{\partial(W_o/h)} = \begin{cases} -0.20 & \text{for stress-free lateral edges} \\ -0.96 & \text{for straight lateral edges} \end{cases}$$

$$\frac{\partial(P_{CF}/P_c)}{\partial A_1} = \begin{cases} 0.04 & \text{for stress-free lateral edges} \\ 0.40 & \text{for straight lateral edges} \end{cases}$$

$$\frac{\partial(P_{CF}/P_c)}{\partial A_3} = \begin{cases} 0.66 & \text{for stress-free lateral edges} \\ 0.96 & \text{for straight lateral edges} \end{cases}$$

Clearly then it is necessary to have some information on the order of magnitude of the imperfection before a decision can be made as to whether or not P_{CF} is a good measure of P_C .

Finally a brief comparison will be sketched with the results of the Spencer plot[11]. This graphical procedure starts with eq (10) with the right side truncated to one term only, leaving three empirical constants (A_1 , P_c , W_o) unknown. By means of the pivot point concept[11],[24], one of these can be eliminated; the Spencer plot eliminates A_1 (thus tending to eliminate the effect of boundary conditions provided they remain constant during the test), and then predicts that there is a linear relationship between two functions H_1 and H_2 of (P,W) such that a graph of H_1 vs. H_2 will give a straight line whose slope and intercept are measures respectively of P_c and W_o . The elimination of A_1 does not of course eliminate the effect of the boundary conditions on P_c . The restrictions on the technique are that the imperfection must be small compared to the maximum reliable deflection data, and it checks on this assumption because it finds a measure of W_o .

For the data shown in the example at the end of section 4, the Spencer plot gave an accuracy of 2% .

For Schlack's data, ref. 14, Table 1, $\gamma = 0.0$, average of three runs, Schlack found P_{CF} = 2890 lbs., the Southwell plot is so nonlinear that it yields nothing[11], and the Spencer plot found P_c = 2850 lbs and W_o = 0.03" . Unfortunately such apparently good correspondence between P_{CF} and P_c may hide unsuspected experimental errors. If, for example, one examines the experimental data of some of the great pioneer work on plate buckling done by the Hoff group at the Polytechnic Institute of Brooklyn, e.g: Coan's specimen 28A, (ref. 13, fig. 4), estimated W_o/h = 0.05, one must agree that P_{CF} occurs close to P_c . The Southwell plot is not only curved but its slope changes sign. But the

Spencer plot (confined to $P/P_c < 2.0$ so that any possibility of plasticity effects is ruled out), see fig. 4 and ref. 11 for method and notation, shows up two, distinct, sharply-defined, separate, linear portions giving normalized critical loads of about 0.97 and 1.11 respectively. This could have been caused by a change of boundary conditions during the test. Even if one questions the pivot point concept(24) on which the Spencer plot is based and prefers some statistical-based numerical analysis, the former procedure gives a quick, direct, visual idea of where to start looking for a change in boundary conditions.

If a change of boundary conditions during a test is not sudden but gradual, numerical procedures may not be capable of analyzing the phenomenon by the application of statistical theory. During a post-buckling test conducted at University College London on a simply-supported, square, uniaxially-loaded, spring-steel plate, P_{CF} occured undoubtedly at about 12 kN. A point-by-point analysis of the data(11) using three pivot points revealed that, as the test was progressing, P_c was effectively changing from 6 to 14 kN; i.e: the boundary conditions were gradually changing as the load was applied.

It would appear that the pivot point concept(24) is capable of being extended to measure also the post-critical curvature, again by a graphical procedure. Work is currently proceeding on this.

6. CONCLUSIONS

The contraflexure method for plates can be used with very smooth data to give a quick approximate guide to the critical load. It has been shown here that if the intrinsic imperfections in the plate and in the test boundary conditions are small, the error incurred by the use of the inflexion point will consequently be very small. However, as the imperfection magnitudes increase, the error becomes greater and may become unacceptably large. Formulations are presented in this paper from which, with a knowledge of the imperfection and of the perfect-plate postbuckling characteristics, the magnitude of the error can be calculated. However, an important feature is that if information regarding the imperfection and boundary-condition parameters is required, a better approach would be to use a variation of the Southwell method known as the Spencer plot.

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8. NOTATION

$a_{m(k)}$	=	coefficient of s^k in the expansion of w_m
A_i	=	i'th coefficient in the series-expansion of W
c	=	(as a subscript) pertaining to conditions at P_c
CF	=	(as a subscript) pertaining to conditions at P_{CF}
D	=	$E\,h^3 / (12 (1 - \nu^2))$
E	=	Young's modulus
f_m	=	m'th eigenmode of w
F	=	Airy generalized plane stress function
ϕ	=	$(P/P_c - 1 + W_o/W_T)^{\frac{1}{2}}$
γ	=	Schlack's perforated-plate diameter-ratio
h	=	plate thickness
L	=	Wolmir operator
ν	=	Poisson's ratio
o	=	(as a subscript) pertaining to initial conditions particularly imperfections
P	=	total compressive edge load on a plate
P_c	=	critical load ≡ lowest eigenvalue load
P_{CF}	=	contraflexure load ≡ load at which contraflexure on a load: deflection diagram occurs
s	=	perturbation parameter
T	=	(as a subscript) refers to "total" usually deflection, as distinct from "net" deflections (unsubscripted).
w	=	net plate deflection as a function of (x,y)
w_m	=	amplitude of the m'th eigenmode of w
w_o	=	initial plate imperfection as a function of (x,y)
w_T	=	$w + w_o$ = "total" plate deflection as a function of (x,y)
W	=	central deflection = value of w(x,y) at the centre of the plate
W_o	=	central initial imperfection = $w_o(x,y)$ at the centre of the plate
W_T	=	$W + W_o$ = total central deflection = $w_T(x,y)$ at plate centre
x,y	=	orthogonal coordinates in plane of undeflected plate boundary

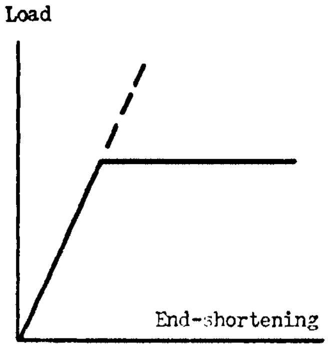


fig. 1a
Euler-strut buckling

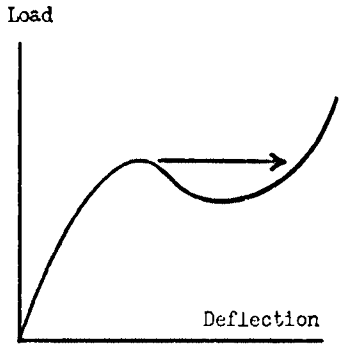


fig. 1b
Snap buckling for an arch

P.P. = primary path

S.P. = secondary path

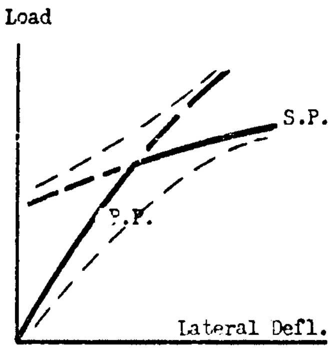


fig. 1c
Asymmetric Bifurcation

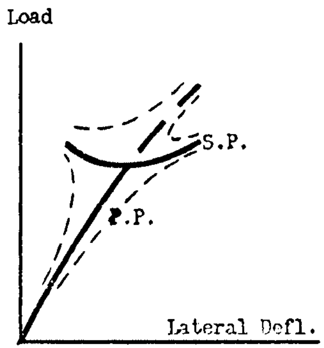


fig. 1d
stable-symmetric Bifurcation

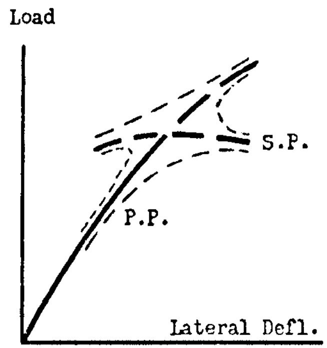


fig. 1e
Unstable-symmetric Bifurcation

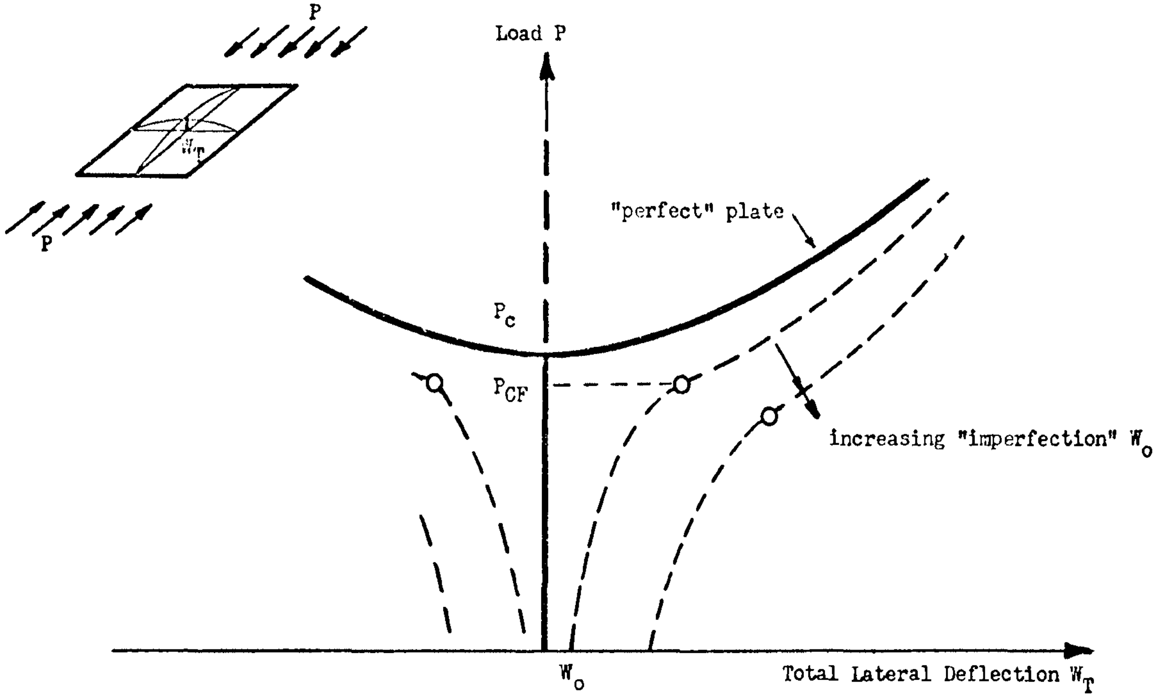


fig. 2 Load-deflection for plates

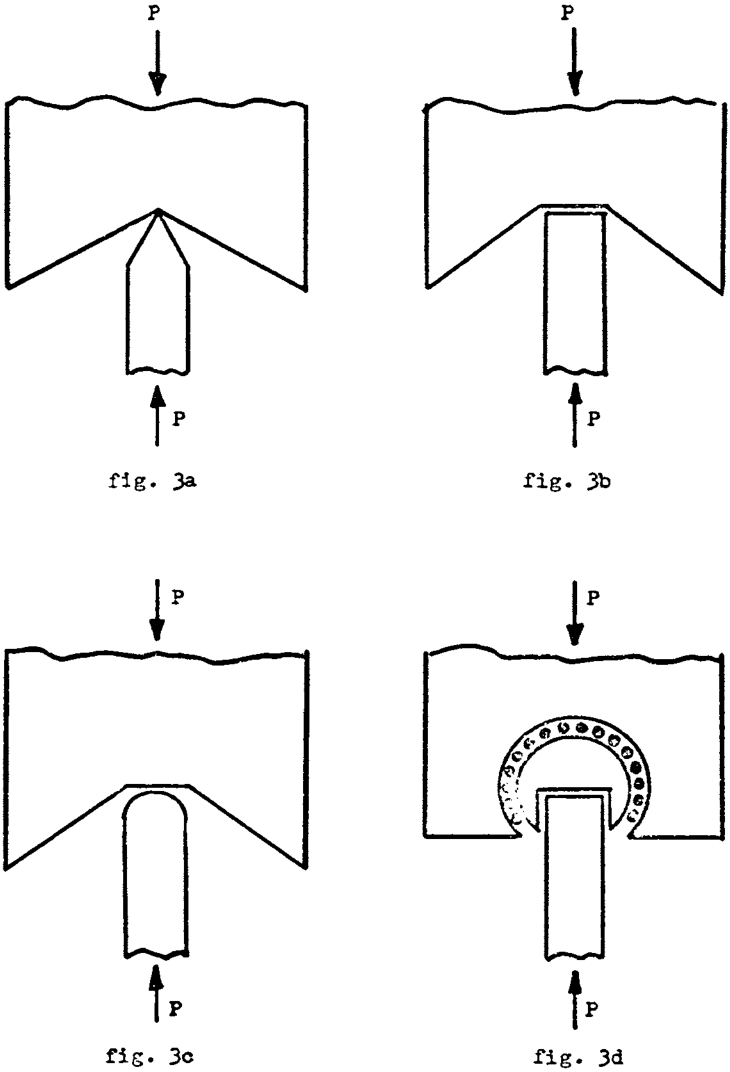
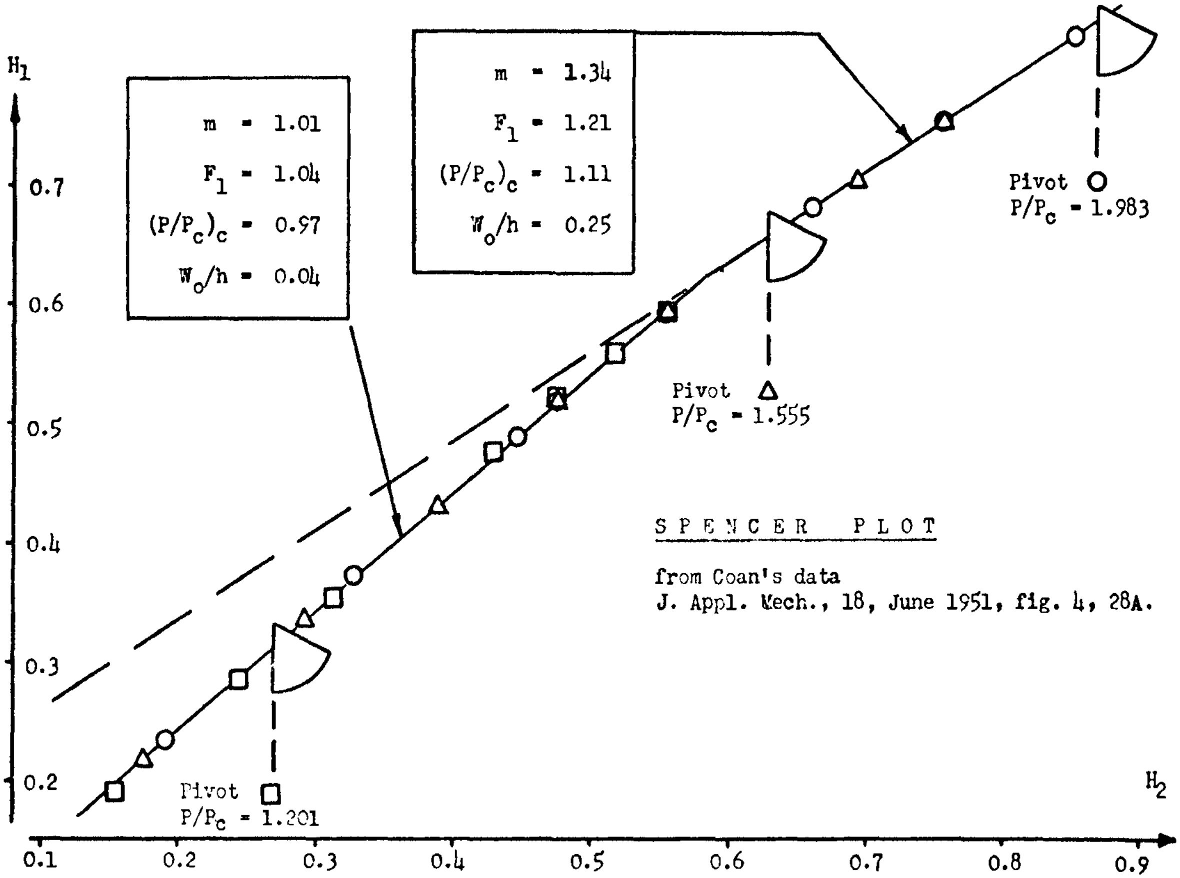


fig. 3 Load-bearing boundaries for plates



H1
H2
m = 1.01
F1 = 1.04
(P/Pc)c = 0.97
Wo/h = 0.04
m = 1.34
F1 = 1.21
(P/Pc)c = 1.11
Wo/h = 0.25
Pivot □
P/Pc = 1.201
Pivot △
P/Pc = 1.555
Pivot ○
P/Pc = 1.983
SPENCER PLOT
from Coan's data
J. Appl. Mech., 18, June 1951, fig. 4, 28A.
0.2
0.3
0.4
0.5
0.6
0.7
0.1
0.8
0.9

fig. 4