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INTRODUCTION

Circular rings as structural elements find many applications in the field of engineering design. One of the most common usages, however, is as a stiffening element for externally loaded cylindrical shells. Used in this capacity, a primary consideration with respect to the design of the ring is its buckling strength. The buckling is assumed to occur in the plane of the ring and results from a uniform pressure directed radially toward the center of the ring.

A common fabrication technique for circular rings consists of press forming or rolling a long straight bar of the desired cross section to a required radius and completing the ring with a single butt weld. Since this process is generally accomplished without any heat treatment, significant residual stresses are induced in the ring. These residual stresses will necessarily affect the in-service performance of the ring.

Work relating to the effects of residual stresses on buckling has been reported in the literature for structures other than the cold-formed rings. Osgood (9)[3] derived a general expression for the buckling load of a column containing residual stresses through the use of the Tangent Modulus Theory for inelastic columns as introduced by Shanley (10). Beedle, Tall, Yang, Johnston, and Huber (1,7,14) correlated this same approach to data resulting from buckling tests of built-up columns and wide flange sections that contained residual stresses. The source of the residual stresses in these studies was the cooling of the respective sections after the hot rolling process. In a later study, Gjelsvik and Bodner (5) used the classical Reduced Modulus Theory to derive an expression for the buckling of an elastic-plastic cylinder subjected to external pressure and heating from one surface. The resulting stress distributions were derived with both the thick walled and the thin walled theories, while the buckling problem was formulated on the basis of the thin walled theory only. Finally, in a recent study, Tao and Gjelsvik (11) considered the stability of a heated elastic-plastic column and concluded that column behavior as affected by residual stresses is most realistically predicted with the use of the Tangent Modulus Theory.

The purpose of the present investigation is to evaluate the buckling strength of thin cold-formed circular rings on the basis of their particular residual stress distribution. A stability criteria is derived from a consideration of the total potential energy of the ring system. Throughout this discussion the ring's material is assumed to be elastic-perfectly plastic.

STRESS DISTRIBUTION

The inital state of stress and the prebuckled state of stress for a ring, which has been cold-formed into its final configuration, is established in the following sections.

Initial Residual Stresses

The mechanics of cold-forming a ring involves two basic events. The flat bar from which the ring is to be fabricated must first be bent or rolled plastically to some curvature greater than that finally desired. During this bending or rolling process, external force is being applied which must be in equilibrium with the internal stresses. The second phase occurs when the external force is removed. At this time the ring springs back elastically to the finally desired curvature. This sequence is illustrated for a segment of a ring in Fig. 1, and may be expressed mathematically by

$$\rho_F = \rho_o + \Delta\rho_c - \Delta\rho_s \tag{1}$$

in which ρ_F = final curvature; ρ_o = initial curvature; $\Delta\rho_c$ = curvature change to constrained position; and $\Delta\rho_s$ = curvature change during springback.

The assumption that plane sections remain plane during bending leads to the expression

$$\rho = \epsilon/c \tag{2}$$

in which ϵ = extreme fiber strain; and $c = \frac{t}{2}$, the half thickness of the ring. Substitution of Eq. 2 into Eq. 1 yields, for zero initial curvature,

$$\rho_F = (\epsilon_c - \epsilon_s)/c \tag{3}$$

where ϵ_c and ϵ_s are extreme fiber strains in constrained position and during springback, respectively. With $\rho_F = 1/R$, where R is the final radius of the ring, Eq. 3 may be solved for the strain in the constrained position to yield

$$\epsilon_c = c/R + \epsilon_s \tag{4}$$

The state of stress in the constrained position is shown in Fig. 2a in which the boundary between the elastic and plastic regions in the cross section is defined by y_1, measured from the neutral axis, and is given by

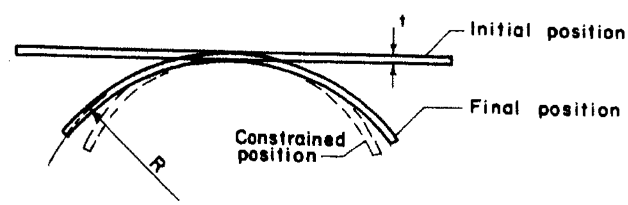

Fig. 1 - Ring Forming Procedure

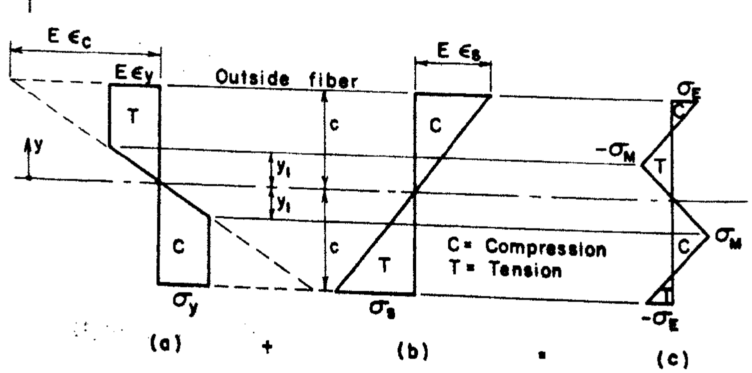

Fig. 2 - Ring Residual Stresses: (a) Constrained; (b) Springback; and (c) Final

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[3] Numerals in parentheses refer to corresponding items in Appendix I. - References.

$$y_1 = \epsilon_y c/\epsilon_c \quad (5)$$

where ϵ_y = strain at initial yield. Substituting from Eq. 4 into Eq. 5 for ϵ_c and multiplying both the numerator and the denominator by E gives

$$y_1 = \frac{\sigma_y c}{E\ c/R + \sigma_s} \quad (6)$$

in which σ_s = extreme fiber stress during springback; σ_y = yield stress of material; and E = Young's modulus. The moment existing in the ring of unit width in the constrained position is given by

$$M = \sigma_y \left[c^2 - \frac{y_1^2}{3}\right] \quad (7)$$

Upon release of the constraining moment the ring springs back elastically producing the stress diagram shown in Fig. 2b where the extreme fiber stresses are given by

$$\sigma_s = 6\ M/t^2 = 3\ M/2c^2$$

into which a substitution of the value of M obtained from Eq. 7 yields

$$\sigma_s = \frac{\sigma_y}{2}\,[3 - (y_1/c)^2] \quad (8)$$

For a ring of radius R, the two unknowns, σ_s and y_1, are obtained by solving Eqs. 6 and 8 simultaneously by trial and error procedure.

The residual stress distribution is found by combining Figs. 2a and 2b to yield Fig. 2c. The stresses at the extreme fiber and at y_1 are defined as σ_E and σ_M, respectively and may be expressed as

$$\left.\begin{aligned} &\sigma_E = \sigma_s - \sigma_y \\ \text{and} \quad &\sigma_M = \sigma_y - (y_1/c)\sigma_s \end{aligned}\right\} \quad (9)$$

Prebuckled State of Stress

For the case of an elastic thin ring free of residual stresses

$$\left.\begin{aligned} &\sigma_\phi = \frac{N_\phi}{t} = \frac{pR}{t} \\ \text{and} \quad &\epsilon_\phi = \frac{pR}{Et} \end{aligned}\right\} \quad (10)$$

in which σ_ϕ = circumferential stress; ϵ_ϕ = circumferential strain; N_ϕ = circumferential stress resultant; and p = the uniform external pressure. The second of these equations yields the prebuckled radial displacement w_o given by

$$w_o = \frac{pR^2}{Et} = \frac{\sigma_\phi R}{E} \quad (11)$$

Rings with some residual stress pattern lead to the following elastic-plastic cases:

a) Elastic Case -- For a ring with a residual stress distribution of the form shown in Fig. 2c, the response remains linear elastic until yield develops at some fiber in the cross section. The limit of this case is determined by that pressure which causes the stress at $y = -y_1$ to reach σ_y or

$$\sigma_{\phi 1} = \sigma_y - \sigma_M$$

Substitution of σ_M from Eq. 9 yields

$$\sigma_{\phi 1} = (y_1/c)\sigma_s \quad (12)$$

$$\text{and} \quad p_1 = 2\ y_1\ \sigma_s/R \quad (13)$$

where $\sigma_{\phi 1}$ and p_1 are, respectively, the limit stress and the limit pressure for Case a). The residual stresses, the change in the stress due to pressure, and the final stresses for all $p \leq p_1$ are shown in Fig. 3.

b) Elastic-Plastic Case: One Plastic Zone -- In this case, $p > p_1$ and $(\sigma_E + \sigma_\phi) < \sigma_y$, for which the stress distribution is shown in Figure 4. Distances y_2 and y_3 defining the limits of the plastic zone, may be derived as functions of the stresses σ_ϕ, σ_s and σ_y, and the distances y_1 and c in the form

$$y_2 = y_1 \left[2 - \frac{\sigma_\phi - (y_1/c)\sigma_s}{\sigma_y - (y_1/c)\sigma_s}\right] \quad (14)$$

$$\text{and} \quad y_3 = c\ [1 - \sigma_\phi/\sigma_s] \quad (15)$$

in which the following restrictions must apply, i.e., $[0 < y_2 < 2y_1]$ and $[0 < y_3 < (c - y_1)]$.

The upper limit of Case b) is determined by that pressure and in turn by that σ_ϕ which brings the outside fiber stress to the yield limit, i.e., $\sigma_E + \sigma_{\phi 2} = \sigma_y$ and together with Eq. 9 gives

$$\sigma_{\phi 2} = 2\sigma_y - \sigma_s \quad (16)$$

in which $\sigma_{\phi 2}$ = limit stress for Case b). With this value of $\sigma_{\phi 2}$, Eqs. 14 and 15 define the upper limits of the stress diagram in this case as

$$y_{22} = y_1 \left[1 - \frac{\sigma_y - \sigma_s}{\sigma_y - (y_1/c)\sigma_s}\right] \quad (17)$$

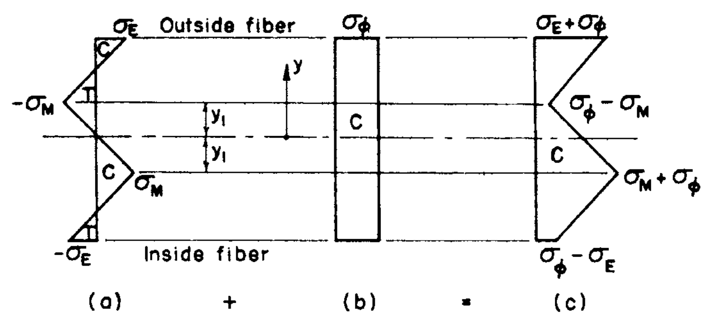

Fig. 3 - Stresses -- Case a) (a) Residual, (b) Change and (c) Final

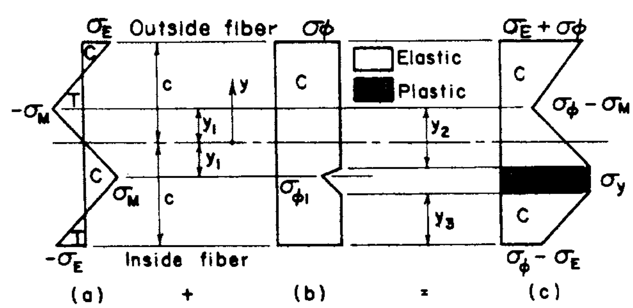

Fig. 4 - Stresses -- Case b) (a) Residual, (b) Change and (c) Final

$$\text{and} \quad y_{32} = 2c\ [1 - \sigma_y/\sigma_s] \quad (17)$$

where y_{22} = upper limit of y_2 and y_{32} = upper limit of y_3 for Case b).

The total stress resultant and the pressure at this upper limit, as a function of y_{22} and y_{32}, are given by

$$\left.\begin{aligned} &N_{\phi 2} = t\ \sigma_{\phi 2} - \left\{\tfrac{1}{2}\,[c + y_1 - y_{32} - y_{22}][2\sigma_y - \sigma_s(1 + y_1/c)]\right\} \\ \text{and} \quad &p_2 = N_{\phi 2}/R \end{aligned}\right. \quad (18)$$

and the corresponding N_ϕ for $p_1 < p < p_2$ by

$$N_\phi = t\ \sigma_\phi - \left\{\tfrac{1}{2}\,[c + y_1 - y_2 - y_3][\sigma_\phi - (y_1/c)\sigma_s]\right\} \quad (19)$$

In Eq. 19, t, c, y_1 and σ_s are established values; and y_2 and y_3 are secondary unknowns expressable as functions of the primary unknown

σ_ϕ by Eqs. 14 and 15. Writing Eq. 10 with all of the appropriate substitutions for N_ϕ, y_2 and y_3 yields a quadratic function in σ_ϕ of the form

$$\sigma_\phi^2(-a_2/2) + \sigma_\phi(t + a\, a_2/2 - a_3/2) + (a\, a_3/2 - pR) = 0$$

which may further be written as

$$a_4\sigma_\phi^2 + a_5\sigma_\phi + a_6 = 0 \qquad (20)$$

with the constants defined as follows:

$a = (y_1/c)\sigma_s$ $\qquad$ $a_1 = y_1/(\sigma_y - a)$

$a_2 = a_1 + c/\sigma_s$ $\qquad$ $a_3 = -(aa_1 + y_1)$

$a_4 = a_2/2$ $\qquad$ $a_5 = t + aa_2/2 - a_3/2$

and $\quad a_6 = a\, a_3/2 - pR$

Equation 20 may be solved directly for the unknown, σ_ϕ.

c) <u>Elastic-Plastic Case: Two Plastic Zones</u> - In this case $p > p_2$ and $(\sigma_\phi - \sigma_M) < \sigma_y$. The stress distribution for this case, in which two plastic regions are separated by an elastic region, is shown in Fig. 5. The distances y_2 and y_3 in this case are again given by Eqs. 14 and 15. In a similar manner, the distance y_4 may be expressed as

$$y_4 = (c/\sigma_s)(2\sigma_y - \sigma_\phi) - y_1 \qquad (21)$$

in which the following restrictions must apply, i.e., $[0 < y_4 < (c-y_1)]$.

The upper limit for Case c) is that pressure which will cause full yielding of the cross section, i.e.,

$$\text{and} \quad \left.\begin{aligned} N_{\phi 3} &= \sigma_y t \\ p_3 &= N_{\phi 3}/R \end{aligned}\right\} \qquad (22)$$

where $N_{\phi 3}$ = limiting stress resultant of Case c); and p_3 = the corresponding limiting pressure.

The stress condition, σ_ϕ, for this case may be defined in a similar manner as for case b) to yield an expression for the stress resultant, N_ϕ, analogous to Eq. 19, i.e.,

$$N_\phi = t\,\sigma_y - \tfrac{1}{2}\{[y_2+y_4][2\sigma_y - a - \sigma_\phi] + y_3[\sigma_s - \sigma_\phi]\} \qquad (23)$$

Again, a quadratic function in σ_ϕ results from the appropriate substitutions for y_2, y_3 and y_4 in Eq. 23 to yield

$$\sigma_\phi^2\,(a_8 + c/\sigma_s) + \sigma_\phi(-a_7 - a_8a_9 - 2c) + (a_7a_9 + c\,\sigma_s + 2pR - 2t\sigma_y) = 0$$

which may be reduced to the form

$$a_{10}\sigma_\phi^2 + a_{11}\,\sigma_\phi + a_{12} = 0 \qquad (24)$$

where $\quad a_7 = [y_1 + aa_1 + t\,\sigma_y/\sigma_s]$ $\qquad$ $a_8 = [a_1 - \sigma_\phi]$

$a_9 = [2\sigma_y - a]$ $\qquad$ $a_{10} = [a_8 + c/\sigma_s]$

$a_{11} = -[a_7 + a_8a_9 + 2c]$ $\qquad$ $a_{12} = [a_7a_9 + c\sigma_s + 2(pR - t\sigma_y)]$

The complete stress distribution in the ring as a function of the external pressure, p, is thus described by three Eqs. 10, 20, and 24, corresponding to the extent of yielding developed in the thickness of the ring.

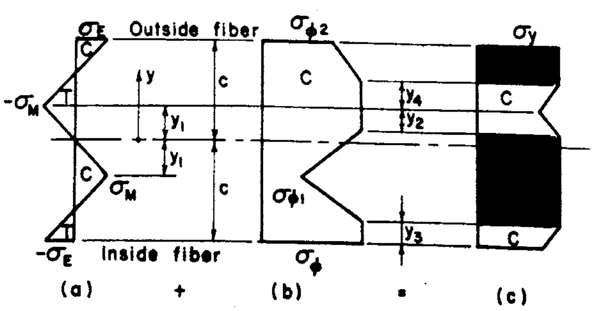

Fig. 5 - Stresses -- Case III (a) Residual, (b) Change and (c) Final

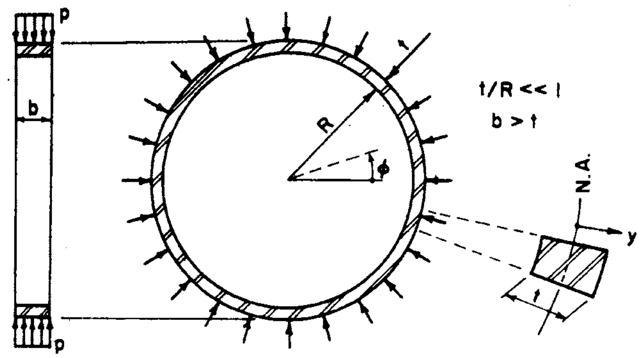

Fig. 6 - Ring Geometry

STABILITY CRITERION

The equilibrium and the stability of any continuous structural system may be determined through an investigation into the nature of the total change in the potential energy resulting from an infinitesimally small displacement from any arbitrary configuration which satisfies the physical constraints (boundary conditions) of the system. For conservative systems, the total potential energy is expressed as (8)

$$V = U + \Omega \qquad (25)$$

in which V = total potential energy; U = internal or strain energy; and Ω = potential energy of external forces.

Within the framework of the calculus of variations, a change in the total potential energy is constructed by replacing the displacement parameters implicit to Eq. 25 with an arbitrary admissible displacment function plus some infinitesimal functional displacement. The change thus achieved is written as

$$\Delta V = \delta V + \frac{1}{2!}\delta^2 V + \frac{1}{3!}\delta^3 V + \ldots \qquad (26)$$

where ΔV = the change in the total potential energy; δV = that part of the change which is linear in the infinitesimal displacement parameter; $\delta^2 V$ = that part of the change which is quadratic in the infinitesimal displacement parameter; and so on.

The "Theorem of Stationary Potential Energy" states that the amount of the total potential energy, V, remains stationary when a structure is displaced from its equilibrium position to an infinitesimally near configuration. This can be expressed as

$$\delta V = \delta U + \delta\Omega = \delta(U + \Omega) = 0 \qquad (27)$$

in which again δV is that portion of ΔV which is linear in the infinitesimal displacement paramters. For any structure, there exist three possible conditions of equilibrium state; namely, stable, neutral and unstable. The equilibrium condition is determined by the sign of the change in the total potential energy, ΔV. In particular, the conditions of stable, neutral and unstable equilibrium are associated with positive, zero, and negative values of ΔV. The sign of ΔV is determined uniquely by the sign of $\delta^2 V$, the quadratic portion of ΔV. This dependency on the sign results from the fact that $\delta V = 0$ and that the absolute value of $\delta^2 V$ is greater than two times the absolute value of the sum of all other terms of the total variation, i.e.,

$$\left|\delta^2 V\right| > 2\left|R_3\right|$$

where

$$R_3 = \frac{1}{3!}\,\delta^3 V + \frac{1}{4!}\delta^4 V + \ldots$$

The term $\delta^2 V$ is a quadratic form and its character must be evaluated on the basis of the algebraic theory of quadratic forms (8). Accordingly, the stability load of a structure will be defined as that load for which the second variation of the potential energy, $\delta^2 V$, changes its character from positive definite to negative definite, negative semidefinite or indefinite.

RING WITHOUT RESIDUAL STRESSES

The structural system to be considered is shown in Fig. 6. For the y and ϕ coordinate system shown in the figure, w and v are defined as the radial and the tangential displacements, respectively.

Potential Energy Expression.

The strain energy of the ring is given by

$$U = \frac{EbR}{2} \iint \epsilon_\varphi^2 \, d\varphi \, dy \tag{28}$$

in which b = width of the ring; and ϵ_φ = the strain in the tangential direction. Expressing the strain, ϵ_φ as

$$\epsilon_\varphi = \epsilon + y\,\kappa \tag{29}$$

where ϵ = strain at the neutral axis; κ = curvature change; and y = distance from the neutral axis; and substituting into Eq. 28 yields after simplification

$$U = \frac{EbR}{2}\left[\int_0^{2\pi} t\,\epsilon^2 \, d\varphi + \int_0^{2\pi} \frac{t^3}{12}\kappa^2 \, d\varphi\right] \tag{30}$$

It is convenient at this time to introduce a notation which will be useful at a later stage when the effect of residual stresses in the ring will be considered. The strain energy function may be written as

$$U = EbR/2 \int_0^{2\pi} \left\{A\,\epsilon^2 + B\epsilon\kappa + C\kappa^2 + D\,\epsilon + F\kappa + G\right\} d\varphi \tag{31}$$

The constants, A, B, C, D, F, and G are a direct result of the integration of a generalized energy expression in terms of ϵ and κ with respect to the radial coordinate, y. For the ring problem this integration results in the values

$$A = t,\; B = 0,\; C = t^3/12 \text{ and } D = F = G = 0$$

The next step in the strain energy formulation is the introduction of the desired set of kinematic relations. For a ring, several sets of kinematic relations are available. Using Donnell equations (4)

$$\left.\begin{aligned} \epsilon &= \frac{v' - w}{R} + \frac{1}{2}\left(\frac{w'}{R}\right)^2 \\ \text{and} \qquad \kappa &= \frac{w''}{R^2} \end{aligned}\right\} \tag{32}$$

into Eq. 30 yields the expression for the strain energy

$$U = \frac{EbR}{2}\int_0^{2\pi}\left\{t\left[\frac{v'-w}{R} + \frac{1}{2}\left(\frac{w'}{R}\right)^2\right]^2 + \frac{t^3}{12}\left[\frac{w''}{R^2}\right]^2\right\} d\varphi \tag{33}$$

in which $(\;)' = d(\;)/d\varphi$.

The expression for the potential energy of external loads used in the ring problem is based on the Donnell assumption that the magnitude and direction of the loading remain constant during deformation. On this basis, the potential energy of external loads, i.e., pressure p, may be expressed as a linear function by

$$\Omega = R\int_0^{2\pi} - pw \, d\varphi \tag{34}$$

The total potential energy is then given by the sum of Eqs. 33 and 34 to yield

$$V = \frac{EbR}{2}\int_0^{2\pi}\left\{t\left[\frac{v'-w}{R} + \frac{1}{2}\left(\frac{w'}{R}\right)^2\right]^2 + \frac{t^3}{12}\left[\frac{w''}{R^2}\right]^2 - \frac{2\,p\,w}{Eb}\right\} d\varphi \tag{35}$$

Second Variation

To construct the second variation of the potential energy it is necessary to replace the displacement terms v and w in Eq. 35 with $(v_o + v_1)$ and $(w_o + w_1)$, respectively; in which v_o and w_o correspond to the equilibrium configuration of the unbuckled ring, and v_1 and w_1 are infinitesimal incremental displacements from that equilibrium position. This replacement yields the total change in the potential energy; and that part of the total change which is quadratic in the incremental displacements is the desired variation, which is given by

$$\frac{1}{2}\delta^2 V = \frac{EbR}{2}\int_0^{2\pi}\left\{\frac{t}{R^2}\left[(v_1' - w_1)^2 + \frac{1}{R}w_o\,(w_1')^2\right] + \frac{t^3}{12R^4}\left[(w_1'')^2\right]\right\} d\varphi \tag{36}$$

Limit of Positive Definiteness - Stability Equations

The criterion for the limit of positive-definiteness of a functional such as that in Eq. 36 is attributed to Trefftz (13). It consists of determining that value of pressure, p, for which $\delta^2 V$ is stationary in some particular v_1 and w_1. To establish the desired relationship it is then necessary to apply the criterion $\delta\,(\delta^2 V) = 0$. This is equivalent to the requirement that the variables involved satisfy the appropriate Euler equations arising from the calculus of variations.

For a functional such as that in Eq. 36 with two dependent and one independent variables, the Euler equations are of the form

$$\left.\begin{aligned} &\frac{\partial F}{\partial v_1} - \frac{d}{d\varphi}\left(\frac{\partial F}{\partial v_1'}\right) = 0 \\ &\frac{\partial F}{\partial w_1} - \frac{d}{d\varphi}\left(\frac{\partial F}{\partial w_1'}\right) + \frac{d^2}{d\varphi^2}\left(\frac{\partial F}{\partial w_1''}\right) = 0 \end{aligned}\right\} \tag{37}$$

in which F is the expression within the brackets {} in Eq. 36. Performing the above operations yields the stability equations

$$\left.\begin{aligned} &(v_1' - w_1')' = 0 \\ \text{and} \qquad &(v_1' - w_1) + \left(\frac{w_o}{R}\right) w'' - \left(\frac{t^2}{12R^2}\right) w_1^{IV} = 0 \end{aligned}\right\} \tag{38}$$

Critical Buckling Load

Assuming the displacements v_1 and w_1 in the post buckled ring as $v_1 = S\sin(n\varphi)$ and $w_1 = T\cos(n\varphi)$ where S and T are arbitrary constants and n is the buckling mode, i.e., n = 2, 3, 4 ...; Eq. 38 leads to the stability equations

$$\left.\begin{aligned} &\sin(n\varphi)\,[S\,(-n^2) + T\,(n)] = 0 \\ &\cos(n\varphi)\,[S\,(n) + T\,(-1 - n^2\,w_o/R - n^4 t^2/12R^2)] = 0 \end{aligned}\right\} \tag{39}$$

The non-trivial solution of these equations is determined by setting the determinant of the coefficients of S and T equal to zero. Expanding the determinant and noting that $w_o = -pR^2/Et$ leads to the value of the critical buckling pressure

$$p_{cr} = \frac{n^2\,E(t^3/12)}{R^3} \tag{40}$$

This equation varies slightly from the classical buckling solution for the ring problem in which the buckling mode factor, n^2, is replaced by $(n^2 - 1)$. This modification results directly from the use of the Donnell kinematic relations (Eq. 32) which are simplified strain-displacement relations based upon the assumption of a large number of circumferential waves. The buckling pressure given by Eq. 40 for $n = 2$ is obtained by multiplying with a factor of 0.75. The results due to the Donnell equations are generally less than 10 per cent in error for rings that buckle with three circumferential waves (4). Therefore, for the lowest buckling mode possible for the ring, i.e., for $n = 2$, the critical pressure is given by

$$p_{cr} = 3\, EI/R^3 \qquad (41)$$

where $I = t^3/12$.

RING WITH RESIDUAL STRESSES

Structural systems with plastic deformations are non-conservative and, therefore, must be analyzed by developing a modified potential energy function which includes the irreversible energy expended in plastic deformations.

The general form of the modified potential energy function is given as (6)

$$V = U + \Omega + U^* \qquad (42)$$

in which V = total potential energy; U = elastic strain energy; Ω = potential energy of external loads; and U^* = plastic strain energy.

Depending on the R/t ratio of a given ring, buckling may occur in any one of the three cases discussed earlier. For large values of the R/t ratio the ring buckles as in Case a) and the entire section remains elastic. In this case, the strain energy is given by Eq. 28, the total potential energy remains independent of the residual stress pattern and the buckling load is given by Eq. 41. However, for rings which do not buckle in accordance with Case a), the strain energy and hence the total potential energy, become functions of the residual stress in the ring. The amount of strain energy stored per unit volume at a point, which is elastic before the application of pressure p and plastic afterwards, is given by the shaded area in Fig. 7 and can be expressed for a volume V as

$$U^* = \frac{1}{2}\int_V (\sigma_y - \sigma_R)(2\epsilon_\phi + \epsilon_R - \epsilon_y)\, dV \qquad (43)$$

in which σ_R = residual stress, ϵ_ϕ = additional strain, ϵ_R = residual strain and ϵ_y = yield strain.

It is obvious that the extent of yielding in a ring is dependent upon the magnitude of the external pressure. In addition, the constants A, B, C, D, F, and G in the generalized energy expression, Eq. 31, become functions of the residual stress pattern. The determination of the buckling pressure then becomes a trial and error procedure. The desired solution is achieved when the pressure assumed to establish a

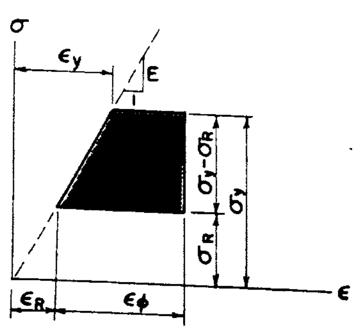

Fig. 7 - Stress - Strain Diagram -- Elastic - Plastic Fibers

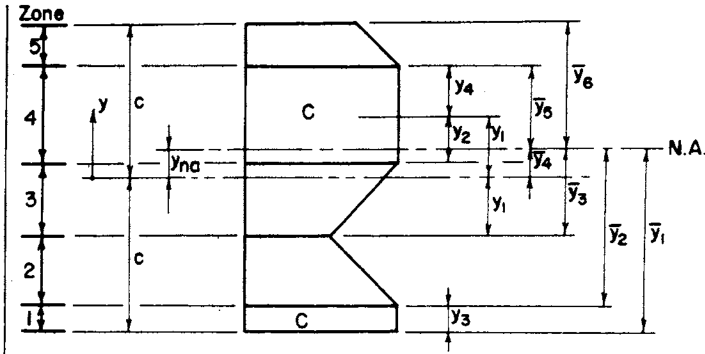

Fig. 8 - Stresses in Ring Cross Section -- Elastic and Plastic Zones

particular set of constants A, B, etc. leads to a value of the critical buckling pressure equal to this assumed value.

The fundamental step in the above process is the determination of a set of coefficients and the associated buckling equation as a function of a given external pressure. This procedure may be generalized for the three cases by using the stress block for Case c), as shown in Fig. 5b, along with judicious choice of the respective limits on distances y_2, y_3 and y_4 of Fig. 5. The essential elements of the stress block of Case c), insofar as the buckling problem is concerned, are given in Fig. 8 in which the distance y_{na} locates that particular fiber for which no extensional strain results from the bending action in the buckling process. For the elastic-perfectly plastic material, the only portion of the cross section available to resist the bending action during buckling is the remaining elastic portion (9). This leads to the location of the neutral axis which is given by

$$y_{na} = (1/t_e)\left[y_4\,(y_1 + y_4/2) + y_2(y_1 - y_2/2) + y_3(y_3/2 - c)\right] \qquad (44)$$

where $t_e = y_2 + y_3 + y_4$. In general, the other distances in Fig. 8 are defined by $\bar{y} = y - y_{na}$ which yields

$$\left.\begin{aligned} \bar{y}_1 &= -c - y_{na} & \qquad \bar{y}_4 &= y_1 - y_2 - y_{na} \\ \bar{y}_2 &= \bar{y}_1 + y_3 & \bar{y}_5 &= y_1 + y_4 - y_{na} \\ \bar{y}_3 &= -\bar{y}_1 - y_{na} & \bar{y}_6 &= c - y_{na} \end{aligned}\right\} \qquad (45)$$

Potential Energy Function

The strain energy of the elastic portions of a cross section is given by Eq. 28 which for Fig. 8 becomes

$$U_i = \frac{EbR}{2}\int_0^{2\pi}\int_{\bar{y}_i}^{\bar{y}_{i+1}} \epsilon_\varphi^2\, d\varphi\, d\bar{y} \qquad (46)$$

in which i is the subscript denoting the zone under consideration. Substituting for ϵ_ϕ from Eq. 29 into Eq. 46 leads to the generalized form

$$U_i = \int_0^{2\pi}\left\{A_i\, \epsilon^2 + B_i\, \epsilon\, \kappa + C_i\, \kappa^2\right\} d\varphi \qquad (47)$$

in which $A_i = \dot{\bar{y}}_{i+1,\,i}$; $B_i = \ddot{\bar{y}}_{i+1,i}$; $C_i = \dddot{\bar{y}}_{i+1,i}$ and the dot quantities are defined by

$$\left.\begin{aligned} \dot{\bar{y}}_{i+1,i} &= \bar{y}_{i+1} - \bar{y}_i \\ \ddot{\bar{y}}_{i+1,i} &= (\bar{y}_{i+1})^2 - (\bar{y}_i)^2 \\ \text{and}\quad \dddot{\bar{y}}_{i+1,i} &= \tfrac{1}{3}\,[(\bar{y}_{i+1})^3 - (\bar{y}_i)^3] \end{aligned}\right\} \qquad (48)$$

The generalized constants introduced in Eq. 47 are all seen to arise directly from integration within each zone on the parameter $\bar{y}$.

The strain energy of the plastic portions is obtained from Eq. 43, in which substitutions of σ_y, σ_R, and ϵ_φ and integration within appropriate limits yields

$$U_i^* = \int_0^{2\pi} \left\{ D_i\, \epsilon + F_i K + G_i \right\} d\varphi \tag{49}$$

where $D_i = 2\left[\dot{\bar{y}}_{i+1,i}\,(\epsilon_y - \hat{b}_i/E) - \ddot{\bar{y}}_{i+1,i}\,(m_i/2E)\right]$

$$F_i = \left[\dot{\bar{y}}_{i+1,i}\,(\epsilon_y - \hat{b}_i/E) - 2\,\ddot{\bar{y}}_{i+1,i}\,(m_i/E)\right]$$

and G_i = a constant

The constants m_i and $\hat{b}_i$ define the residual stresses in zone i according to the relation

$$\sigma_{Ri} = m_i\,\bar{y} + \hat{b}_i$$

The value of each constant is evaluated on the basis of Fig. 2c and Eq. 9.

With the potential energy of the external loads and the kinematic relations defined by Eqs. 34 and 32, respectively, the total potential energy function is constructed from Eq. 42 in the following form

$$V = \frac{EbR}{2}\int_0^{2\pi} \left\{ A\left[\frac{v'-w}{R} + \frac{1}{2}\left(\frac{w'}{R}\right)^2\right]^2 + C\left[\frac{w''}{R^2}\right]^2 + D\left[\frac{v'-w}{R} + \frac{1}{2}\left(\frac{w'}{R}\right)^2\right] + F\left[\frac{w''}{R^2}\right] + G - 2pw/Eb \right\} d\varphi \tag{50}$$

where $A = \Sigma A_i$, $C = \Sigma C_i$, etc.

Equilibrium Equations

The most direct way to obtain the equilibrium equations is to minimize the expression in Eq. 50 with respect to both of the displacement parameters v and w through the Euler equations given in Eq. 37 which leads to

$$\left(\frac{2A\epsilon}{R} + \frac{D}{R}\right)' = 0 \tag{51}$$

and $$-\left(\frac{2A\epsilon}{R} + \frac{D}{R} + \frac{2p}{Eb}\right) - \left\{(w')\left(\frac{2A\epsilon}{R} + \frac{D}{R}\right)\right\}' + \left(\frac{2Cw''}{R^2} + \frac{F}{R^2}\right)'' = 0 \tag{52}$$

The braced $\{\,\}$ term in Eq. 52 represents the derivative of a product. Since the derivative of the second term thereof is equal to zero as per Eq. 51, the derivative of the entire product becomes

$$\frac{d\{\,\}}{dx} = [w'']\left[\frac{2A\epsilon}{R} + \frac{D}{R}\right]$$

and Eq. 52 reduces to the form

$$-\left[\frac{2A\epsilon}{R} + \frac{D}{R} + \frac{2p}{Eb}\right] - \left[\frac{2A\epsilon}{R} + \frac{D}{R}\right]\left[w''\right] + \frac{2\,C\,w^{IV}}{R^2} = 0 \tag{53}$$

Of particular interest is the solution of the above equation for the prebuckled axi-symmetric case. For this condition all derivatives of the radial displacement, w, are equal to zero and the hoop strain ϵ equals w_o/R. Eq. 53 then reduces to

$$-\frac{2p}{Eb} = \frac{2\,A\,w_o}{R^2} + \frac{D}{R} \tag{54}$$

Critical Buckling Pressure -- P_{cr}

The expression for the second variation of the potential energy function, Eq. 50, containing only terms which are quadratic in the infinitesimal post buckling displacements, v_1 and w_1, is given as

$$\frac{1}{2}\,\delta^2 V = \frac{EbR}{2}\int_0^{2\pi}\left\{\frac{A}{2R^2}\left[(v_1' - w_1)^2 + \frac{1}{2}\,w_o\,(w_1')^2\right] + \frac{B}{R^3}\left[w_1''\,(v_1' - w_1)\right] + \frac{C}{R^4}\left[(w_1'')^2\right] + \frac{D}{2R^2}\left[(w_1')^2\right]\right\} d\varphi \tag{55}$$

As before, the limit of positive definiteness is determined by the application of the Euler equations, Eq. 37, the the braced $\{\,\}$ quantities in Eq. 55 to yield the stability equations

$$\left.\begin{aligned} -\frac{2A}{R^2}(v_1'' - w_1') - \frac{B}{R^3}(w_1''') &= 0 \\ -\frac{2A}{R^2}(v_1' - w_1) - \frac{B}{R^3}(w_1'') - \left(\frac{2A\,w_o}{R^3} + \frac{D}{R^2}\right)w_1'' + \frac{B}{R^3}(v_1''' - w_1'') + \frac{2C}{R^4}w_1^{IV} &= 0 \end{aligned}\right\} \tag{56}$$

The non-trivial solution of these equations is obtained in a similar manner as described before for rings with no residual stresses. The value of B in Eq. 56 is always zero, based on the definition of the neutral-axis given in Eq. 44. This fact and the substitution of Eq. 54 in the above mentioned procedure yields

$$P_{cr} = n^2\,E\,I_e/R^3 \tag{57}$$

in which I_e = moment of inertia of the elastic portion of the ring cross-section. Finally, the above equation is reduced to the form of Eq. 41 by replacing n^2 with n^2-1 and using the buckling mode corresponding to n=2 to yield

$$P_{cr} = 3\,E\,I_e/R^3 \tag{58}$$

RESULTS

Computer Program

As mentioned earlier, the solution for the buckling load of the cold-formed ring is obtained by a trial and error procedure. The buckling strength of the ring becomes a function of its physical dimensions, the elastic modulus and the yield strength of the material. In order to compute the buckling strength over a wide range of these parameters, a computer program was developed.

The basic logic flowchart of the program used is given in Fig. 9. As indicated in the flowchart, the basic elements of the program are the four subroutines, ESTABLISH, CRITPRES, INTERPLT, and SIGMAPHI, and the input-output features.

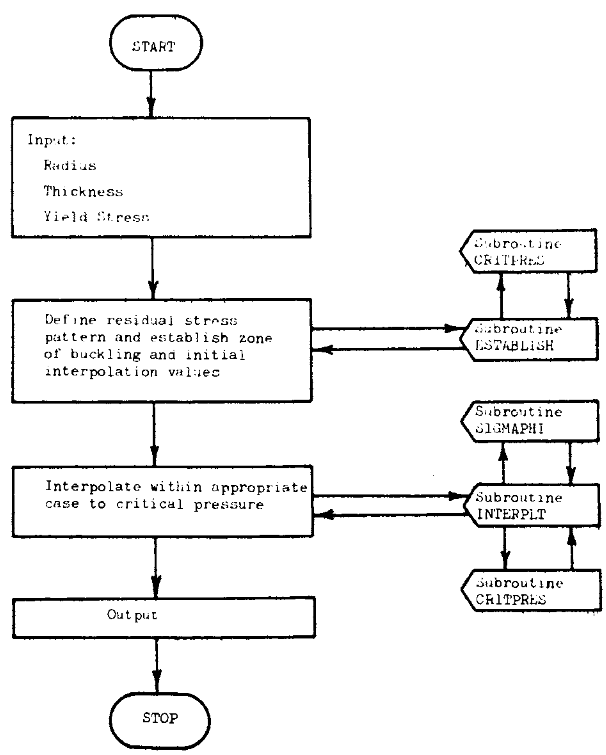

Fig. 9 - Logic Flowchart of the Computer Program

The first subroutine indicated, ESTABLISH, has several functions. It defines the initial residual stress pattern based on the geometry and the yield stress. Further, it establishes the particular case for which buckling occurs in the ring. This is accomplished by solving for the critical buckling load with CRITPRES at the respective limits of each case as given by Eqs. 13, 18 and 22. Buckling occurs in that case for which P_{cr} (L.L.) > P_{LL} and P_{cr}(U.L.) < P_{UL} where P_{cr}(L.L.) = the critical pressure as a function of the lower limit pressure; P_{LL} = lower limit pressure; P_{cr}(U.L) = the critical pressure as a function of the upper limit pressure; and P_{UL} = upper limit pressure. This provides the required starting values for the interpolation process within the appropriate case.

On the basis of the ring geometry, material properties and the value of σ_ϕ evaluated by Eqs. 10, 12, 16, 20 and 24, subroutine CRITPRES solves for the critical buckling pressure. The major steps in the computation are: (1) the determination of the elastic-plastic interfaces, Eqs. 14, 15 and 21; (2) the calculation of the neutral axis and the corresponding $\overline{y}$ dimensions, Eqs. 44 and 45; (3) the evaluation of the constants, Eq. 50; and (4) the computation of the critical pressure.

The function of subroutine INTERPLT is to interpolate within the appropriate zone until the computed buckling pressure matches the initially assumed pressure. The interpolation technique is a variation of the "Secant Method" known as the "Method of False Position" (3). In general, this interpolation routine needed fewer than five iterations to achieve convergence to within one-tenth of one per cent.

Finally, the purpose of subroutine SIGMAPHI is to evaluate the value of σ_ϕ from either of the Eqs. 10, 20, or 24.

The required input for the program are the radius and thickness of the ring and the yield stress of the material. The Young's Modulus is defined within the program as 30×10^6 psi.

Numerical Data

Given in Fig. 10 are the curves representing the buckling strengths of rings as computed on the basis of the theory presented herein. The

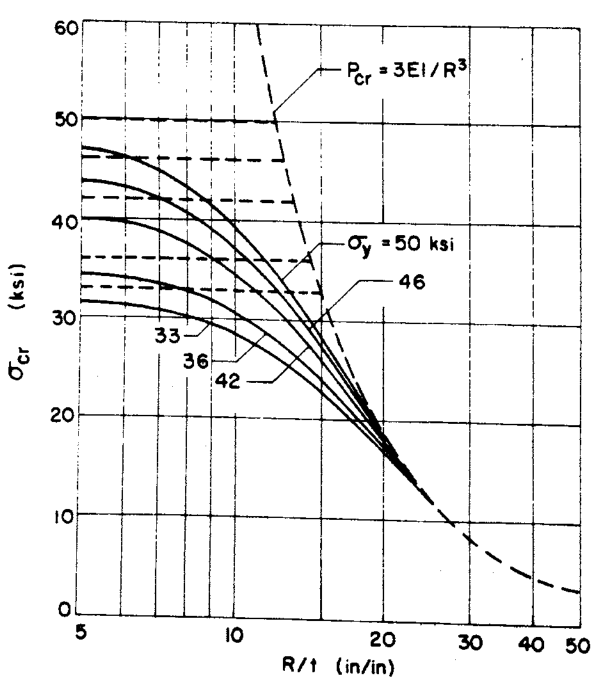

Fig. 10 - Buckling Stresses of Cold - Formed Cylindrical Rings

R/t ratios are presented on a logrithmic scale and cover a range of five to fifty. Separate curves are drawn for each of the commonly specified yield strengths of steel. In all cases, the ordinate σ_{cr} is defined on the basis of the nominal hoop stress in the ring as computed by the thin-ring hoop stress formula $\sigma_\varphi = pR/t$. The dashed curve to the right represents the elastic buckling curve for the ring.

CONCLUSIONS

From the results in Fig. 10, it is evident that the residual stresses developed during the rolling process of a cold-formed circular ring account for a significant reduction in the buckling strength. However, in the evaluation and the application of these results, a particular limitation of the theory as presented must be considered.

All of the stress distributions have been developed on the basis of the thin ring theory, i.e., that the applied external pressure results in a uniform stress distribution through the thickness. With reference to the theory of thick rings, the per cent error in the extreme fiber stresses due to the assumption of uniform stress distribution ranges from 2 to 5 per cent for R/t ratios of 20 to 10. For an R/t value of 15, for which the effect of the residual stress is greatest, the maximum error in the prebuckled state of stress due to the above assumption is of the order of 3 per cent. In addition, the following conclusions can be made:

1. The buckling strength of a cold-formed ring is unaffected by the resulting presence of residual stresses for R/t ratios of 25 or greater.
2. The maximum reduction in the buckling strength occurs in the R/t range of 12 to 15 and is of the order of magnitude of thirty-five per cent.
3. The theory presented is sufficiently accurate for engineering or design purposes to evaluate the buckling strengths of cold-formed rings with an R/t ratio of 10 or greater.

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APPENDIX II - NOTATION

A, B, C, D, F, G,	= constants of integration
A_e	= area of elastic portion of cross-section
b	= width of ring
$\hat{b}$	= residual stress constant
c	= half-thickness of ring
E	= modulus of elasticity
I	= moment of inertia
I_e	= moment of intertia of elastic portion
M	= constraining moment
m	= slope of residual stress distribution
N_Φ	= normal stress resultant
$N_{\Phi i}$	= limiting stress resultant for Case i)
n	= buckling mode
p	= radial pressure
p_{cr}	= critical buckling pressure
p_i	= limiting pressure for Case i)
R	= radius of ring
S, T	= arbitrary constants
t	= thickness of ring
te	= thickness of elastic portion of ring
U	= elastic strain energy
U*	= plastic strain energy
V	= total potential energy
v, v_o, v_1	= displacement in circumferential direction, its value in equilibrium configuration and a small increment
w, w_o, w_1	= displacement in radial direction, its value in equilibrium configuration and a small increment
y	= radial coordinate
y_i	= value of y at point i
y_{ij}	= upper limit of y_i for Case j)
$\bar{y}_i$	= distance of point i from neutral axis
y_{na}	= location of neutral axis
y', y'', y^*	= intermediate variables
ΔV	= change in total potential energy
$\Delta\rho_c$	= curvature change to constrained position
$\Delta\rho_s$	= curvature change during springback
ϵ	= strain at neutral axis
$\epsilon_c, \epsilon_s, \epsilon_y, \epsilon_\Phi$	= strains in constrained position, springback, at initial yield and in circumferential direction, respectively.
κ	= curvature change
Ω	= work energy
Φ	= circumferential coordinate
ρ, ρ_F, ρ_o	= curvature, its final and initial values
$\sigma_y, \sigma_\Phi, \sigma_R$	= yield, circumferential and residual stress, respectively
σ_E, σ_M	= residual stress at extreme fiber and at y_i, respectively
σ_s	= extreme fiber stress during springback