1-2009

Complete Identification of Permissible Sampling Rates for First-Order Sampling of Multi-Band Bandpass Signals

Yan Wu
Georgia Southern University

Daniel F. Linder
Georgia Southern University, dflinder@georgiasouthern.edu

Complete Identification of Permissible Sampling Rates for the First-Order Sampling of Multi-Band Bandpass Signals

YAN WU AND DANIEL. F. LINDER
Department of Mathematical Sciences
Georgia Southern University
Statesboro, GA 30460
USA
yan@georgiasouthern.edu

Abstract: - The first-order sampling of multi-band bandpass signals with arbitrary band positions is considered in this paper. Gaps between the spectral sub-bands are utilized to achieve lower sampling rates than the Nyquist. The lowest possible sampling rate along with other permissible sampling rates is identified via a unique partition of the frequency axis. With the complete identification of all the permissible sampling rates, a necessary and sufficient sampling theorem for multi-band bandpass signals is presented in terms of a series of csinc-interpolators.

Key-Words: - Multi-band bandpass signals, Sampling theorem, First-order sampling, Nyquist rate, Permissible sampling rates, Aliasing, csinc-interpolation

1 Introduction

Sampling theory plays an essential role in the advancement of digital signal processing (DSP). Reliable sampling of an analog signal is crucial for the successive phases of DSP. In this paper, we will identify the optimal and all other permissible sampling rates for the first-order sampling of multi-band bandpass signals without any restrictions on the band positions. It is well-known that the Nyquist rate for the first-order sampling of a bandpass signal is twice the bandwidth if the lower frequency (or upper frequency) of the passband is an integral multiple of the bandwidth of the signal [1, 2]. Kohlenberg studied the problem of exact interpolation of band-limited functions in [1], in which interpolation formulas for lowpass and bandpass signals are presented, respectively, for the spectrum of a multiply-periodic, amplitude modulated sequence of pulses. For the case of bandpass signals, Kohlenberg restricted the sampling density at exactly twice the bandwidth (*2W*) of the signal, which is the theoretical minimum sampling rate for a bandpass signal. He further revealed that it is possible for the first-order sampling of a bandpass signal with the prescribed sampling rate only if the lower frequency (or upper frequency) is an integer multiple of the bandwidth of the signal. Otherwise, one may use the second-order sampling method proposed in the paper, i.e. an interpolation formula built on two special groups of sampling points. The spacing between the points within each group is at $1/W$, and the points from two groups are interlaced with respect to each other. A setback of Kohlenberg's method is that any imperfection of the implementation will cause aliasing due to the stern precision of sampling. Kohlenberg's method was later improved in [3] through the addition of a guard-band in between the shifted spectrum components, which allows flexibility in the implementation. Vaughan, *et. al.* [4] studied several bandpass sampling methods from the perspectives of band position, noise, and parameter sensitivity. Marvasti [5] proposed a general interpolation formula based on non-uniform sampling of a bandpass signal, where the positions of the samples satisfy a nonlinear functional equation. The effective sampling rate can be as low as half of the Nyquist rate; however, the restriction on the sampling positions could take toll on the engineering implementation. Wen, *et. al.* [6] employed a so-called periodically Nonuniform sampling method, considered as a higher-order sampling method, to a subband in a bandlimited signal with large bandwidth. Lower sampling rate is achieved with respect to the narrow subband of interest. Sampling theory is closely tied up with the interpolation and extrapolation of signals. In [7], a series of subband predictive filter banks are proposed to accurately extrapolate wide-band bandlimited signals. In particular, a wide-band signal is decomposed into narrow passbands (in addition to the baseband), and a predictive filter is designed for each passband. All the predictive filters are adopting the same sampling frequency, i.e. twice the highest frequency of the original signal. However, since the bandwidth of the passbands is narrow, it is shown in this paper that much lower sampling rate can be achieved while maintaining the validity of prediction. In this paper, we focus on the permissible sampling rates for multi-band (two or more isolated subbands) bandpass signals. Detailed discussions on the first-order sampling of single-band bandpass signals can be found in [8]. The interpolation formula for bandpass signals in [8] was extended to one that adapts to both low pass and bandpass signals [9]. A simple band-embedding approach is introduced by Brown [2] to restore the integral positioning constraints so that the sampling rate can be *2W*, where *W* is the bandwidth of the bandpass signal with a guard-band. Brown's approach may lead to over-sampling if the bandwidth of the signal is slightly mismatched with the cutoff frequency. And, it is almost impossible to implement Brown's method for the multi-band bandpass signals. A spectrum arrangement approach for the sampled signal is proposed in [10], where the original spectrum is extracted from the replicas of the original signal under normal or inverse placement of the spectrum. The authors consider both the right-shift and left-shift of the spectrum to obtain the permissible sampling frequencies. However, the effectiveness of the spectral placement approach is unknown for multi-band bandpass signals. A formal proof of the optimal and permissible sampling rates for first-order sampling of single-band bandpass signal is presented in [11]. The proof makes use of the symmetry of the passband and its mirror image, and the permissible sampling rates proposed in the paper are incomplete as the sampling rates are calculated as integral multiple of the optimal sampling rate. Consequently, the sampling theorem in [11] is only a sufficient theorem, just like the Shannon's sampling theorem. As a matter of fact, most known results on the sampling of bandpass signals in the literature are for single-band bandpass signals, as mentioned above. It is perceivable that the complexity of first-order sampling of bandpass signals increases dramatically if the signal has multiple passbands, unless the signal is treated as a lowpass signal or a single-band bandpass signal. In this paper, we focus on multi-band bandpass signals in the most general setting. There are no restrictions on the band positions. The complete set of sampling rates for the first-order sampling of bandpass signals is identified via a unique partition of the frequency axis. As a result, we present a necessary and sufficient sampling theorem for bandpass signals. Moreover, it will be seen that, for most bandpass signals, the permissible sampling rates constitute continuous intervals, which allows substantial flexibility for engineering implementation so that imperfections will not cause contamination in the samples.

A multi-band bandpass signal can be conveniently represented as a function $f(t)$ whose Fourier transform is compactly supported over a union of disjoint intervals (over the frequency axis). Introduce

$$\Phi_I = \bigcup_{i=1}^{n} [\omega_{l_i}, \omega_{u_i}], \ \ 0 < \omega_{l_1} < \omega_{u_1} < \omega_{l_2} < \omega_{u_2} < \ldots < \omega_{l_n} < \omega_{u_n} \tag{1}$$

and

$$\Phi_{-I} = \bigcup_{i=1}^{n} [-\omega_{u_i}, -\omega_{l_i}] \tag{2}$$

A multi-band bandpass signal satisfies the following condition

$$F(\omega) = 0 \ \text{ if } \ \omega \notin \Phi_{-I} \cup \Phi_I, \tag{3}$$

where $F(\omega)$ is the Fourier transform of $f(t)$. The magnitude plot for a three-band bandpass signal in the frequency domain is given in Fig. 1. Classical results can be applied [11] if these passbands are bundled together to be treated as a single-band band pass signal. However, lower sampling rates can be achieved if the gaps between the passbands are large enough to allow replicas of the spectrum to occupy the space in between without intersecting the original spectrum. It is the purpose of this paper to completely identify all the permissible sampling rates for multi-band bandpass signals by utilizing the gaps between the subbands. Closed form formulas for computing the sampling rates are provided. All results are rigorously proved.

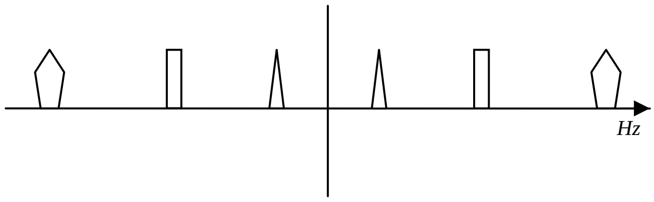

Fig. 1 A three-band bandpass signal

The rest of the paper is organized as follows. In Section 2, we will explore a two-band model without any restrictions on the band positions for both bands (recall that most known results are on single-band bandpass signals, and the second band is simply a mirror image (on the negative side of the frequency axis) of the first band (on the positive side of the frequency axis)). As a result of sampling, the original spectrum, such as the one in Fig.1, shifts to the right and to the left along the frequency axis at the stepsize of ω_s, the sampling frequency. Of course, the only criterion for a permissible ω_s is that the replicas of the original spectrum do not intersect any passbands along the path (anti-aliasing). Even though the complexity of bands' interactions grows dramatically as the number of bands increases, one can always reduce the problem into a two-band setting. That is why a great deal of effort is dedicated to the two-band case. After the groundwork, we apply the results to bandpass signals with arbitrary number of bands in Section 3, including a sampling theorem for multi-band bandpass signals via a series of csinc-interpolators [9]. In section 4, a fast algorithm for symbolic computations of intervals is introduced. This algorithm makes it possible to implement the mathematical formulation in real design. There are a number of ways of measure the complexity of an algorithm. Tarek [12] gave a systematic tutorial for the time and space complexity analysis of algorithms, which is adopted in the complexity analysis of the proposed fast algorithm in this paper. Because of requirement of strenuous set operations for multi-band signals, further improvement of the algorithm can be explored based on the algorithms proposed by Guo [13] and implemented in Java. Closing remarks and further discussions are found in the Conclusion section.

2 Permissible Stepsizes for Two Disjoint Intervals

The interactions between two passbands as they shift horizontally can be treated as those of two disjoint intervals, such as the intervals shown in Fig. 2. Without loss of generality, we only consider the right-shift of the intervals as the results are identical to those of left-shift. In this section, we will establish a complete set of permissible stepsizes for an arbitrary two-interval system. To set the stage, we first introduce the following settings:

(i) $I^{m\Delta}_{[p,q]} = [p + m\Delta, q + m\Delta]$, where $\Delta \in R^+$, $m \in Z^+$, and $I^0_{[p,q]} = I_{[p,q]} = [p,q]$.

(ii) In this section, it is always assumed that the intervals $[a,b]$ and $[d,c]$ are disjoint, i.e. $d < c < a < b$, see Fig. 2.

(iii) The stepsize Δ is feasible for two intervals $[a,b]$ and $[d,c]$ if Δ satisfies the conditions:

$$I^{m\Delta}_{[d,c]} \cap I_{[d,c]} = \phi \text{ and } I^{m\Delta}_{[d,c]} \cap I_{[a,b]} = \phi,\ m \in Z^+. \tag{4}$$

From the sampling point of view, the stepsize Δ is still considered feasible even if $I^{m\Delta}_{[d,c]}$ intersects $[a,b]$ or $[d,c]$ at the end points. Therefore, a weaker condition than (4) is allowed for the definition of a feasible stepsize, as follows

$$I^{m\Delta}_{[d,c]} \cap I_{[d,c]} \subset \{d,c\} \text{ and } I^{m\Delta}_{[d,c]} \cap I_{[a,b]} \subset \{a,b\}, m \in Z^+ \tag{5}$$

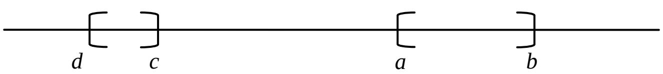

Fig. 2 Two disjoint intervals $[d,c]$ and $[a,b]$

We first present and prove, selectively, a series of lemmas. These lemmas will be used in the proof of the main theorem.

Lemma 2.1 If $\Delta \geq (b-d)$, then the stepsize Δ is feasible, i.e. condition (4) or (5) is satisfied.

This result is analogous to the Nyquist rate.

Lemma 2.2 If $\exists m \in Z^{+}$ such that $c + m\Delta \leq a$ and $d+(m+1)\Delta \geq b$, then $I_{[d,c]}^{n\Delta} \cap I_{[a,b]}$ satisfies (4) or (5) for all non-negative integer n .

Proof: The proof is straightforward based on the four cases for n: (i) $n<m$, (ii) $n=m$, (iii) $n=m+1$, and (iv) $n>m+1$.

Lemma 2.3 If $I_{[d,c]}^{m\Delta} \cap I_{[a,b]} = \phi$ for any $m \in Z^{+}$, then $I_{[d,c]}^{m\Delta} \cap I_{[a,b]}^{n\Delta} = \phi$ for all $m,n \in Z^{+}$.

Proof: Assume there exist $k, p \in Z^{+}$ with $k > p$ such that $I_{[d,c]}^{k\Delta} \cap I_{[a,b]}^{p\Delta} \neq \phi$. Consider the following possible cases as a result of the assumption:

(i) $d + k\Delta \leq a + p\Delta < c + k\Delta \leq b + p\Delta$

Subtracting $p\,\Delta$ from each of the four expressions in the inequalities to get the following inequalities

$$d+(k-p)\Delta \leq a < c+(k-p)\Delta \leq b,$$

which contradicts $I_{[d,c]}^{m\Delta} \cap I_{[a,b]} = \phi$. Similar approach can be applied to prove that each of the following cases will reach a contradiction to $I_{[d,c]}^{m\Delta} \cap I_{[a,b]} = \phi$,

(ii) $d + k\Delta \leq a + p\Delta < b + p\Delta \leq c + k\Delta$

(iii) $a + p\Delta < d + k\Delta \leq b + p\Delta \leq c + k\Delta$

(iv) $a + p\Delta \leq d + k\Delta < c + k\Delta \leq b + p\Delta$

Therefore, the result $I_{[d,c]}^{m\Delta} \cap I_{[a,b]}^{n\Delta} = \phi$ holds for all $m,n \in Z^{+}$.

Lemma 2.4 Let $[d,c]$ and $[a,b]$ be disjoint intervals and let $\Delta \geq (c-d)$ and $\Delta \geq (b-a)$. If there exists $m \in Z^{+}$ such that $c+m\Delta \leq a$ and $d+(m+1)\Delta \geq b$, then Δ is a feasible step size.

The proof is immediate with Lemmas 2.1-2.3. The following lemmas will allow us to partition the positive real line (for the stepsize) by using a set of points calculated from the integer $\left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$, where $\lfloor\ \rfloor$ is the floor function. This value can be thought of as the capacity of the gap between the two intervals $[a,b]$ and $[d,c]$ to accommodate the interval-shifting through the gap without causing intersections. Using this value, we can partition the positive side of the real line, which yields the feasible stepsizes.

Lemma 2.5 If $t = \left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$, then $\frac{b-d}{t+1} \leq \frac{a-c}{t}$.

Proof: From the definition of the floor function, $t \leq \frac{a-c}{(b-a)+(c-d)}$. Because the quantity is clearly positive, we have $\frac{1}{t} \geq \frac{(b-a)+(c-d)}{a-c} = \frac{b-d}{a-c} - 1$, implying that $\frac{b-d}{t+1} \leq \frac{a-c}{t}$.

Lemma 2.6 Let $t = \left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$, then $\frac{b-d}{t+1} \geq b-a$ and $\frac{b-d}{t+1} \geq c-d$.

Proof: The above inequalities can be obtained as follows,

$$\frac{b-d}{t+1} \geq \frac{b-d}{\frac{a-c}{(b-a)+(c-d)}+1} = (b-a)+(c-d).$$

Lemma 2.7 Let $t = \left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$, then

$$\frac{b-d}{t+1-n} \leq \frac{a-c}{t-n}, \quad n = 0,1,\ldots,t-1.$$

Proof: With $n = 0,1,\ldots,t-1$, and the fact that $t \leq \frac{a-c}{(b-a)+(c-d)}$, one has the following inequality $t-n \le \dfrac{a-c}{(b-a)+(c-d)}$. With some manipulations, it can be shown that $\dfrac{1}{t-n} \ge \dfrac{b-d}{a-c} - 1$, which implies that $\dfrac{t+1-n}{t-n} \ge \dfrac{b-d}{a-c}$, hence the inequality in Lemma 2.7 holds.

The result below can be deduced from Lemma 2.7.

$$\left[\frac{b-d}{t+1-i},\frac{a-c}{t-i}\right]\cap\left[\frac{b-d}{t+1-j},\frac{a-c}{t-j}\right]=\phi,\ i\neq j,\ i,j=0,1,\dots,t-1. \quad (6)$$

It will be shown later that the disjoint intervals in (6) contain the feasible stepsizes to prevent intersection between two intervals when one is shifted at the stepsize toward the other.

Lemma 2.8 Let $t=\left\lfloor \dfrac{a-c}{(b-a)+(c-d)} \right\rfloor$, then

$$\frac{b-d}{t+1+n} > \frac{a-c}{t+n}, \quad n=1,2,\dots$$

Proof: Since $n \ge 1$, one has $t+n > \dfrac{a-c}{(b-a)+(c-d)}$, which leads to $\dfrac{1}{t+n} < \dfrac{(b-a)+(c-d)}{a-c} = \dfrac{b-d}{a-c} - 1$, hence the inequality in Lemma 2.8.

Lemma 2.9 Let $t=\left\lfloor \dfrac{a-c}{(b-a)+(c-d)} \right\rfloor$, then

$$\bigcup_{n=1}^{\infty}\left(\frac{a-c}{t+n},\frac{b-d}{t+n}\right)=\left(0,\frac{b-d}{t+1}\right).$$

Proof: For any $x \in \bigcup_{n=1}^{\infty}\left(\dfrac{a-c}{t+n},\dfrac{b-d}{t+n}\right)$, then there exists $k \in N$ (set of natural numbers), such that $x \in \left(\dfrac{a-c}{t+k},\dfrac{b-d}{t+k}\right)$. Hence, the inequalities $0 < \dfrac{a-c}{t+k} < x < \dfrac{b-d}{t+k} < \dfrac{b-d}{t+1}$, which shows that $x \in \left(0,\dfrac{b-d}{t+1}\right)$. Therefore, $\bigcup_{n=1}^{\infty}\left(\dfrac{a-c}{t+n},\dfrac{b-d}{t+n}\right) \subseteq \left(0,\dfrac{b-d}{t+1}\right)$. In what follows, we will establish $\bigcup_{n=1}^{\infty}\left(\dfrac{a-c}{t+n},\dfrac{b-d}{t+n}\right) \supseteq \left(0,\dfrac{b-d}{t+1}\right)$. For any $y \in \left(0,\dfrac{b-d}{t+1}\right)$, there exists $\varepsilon > 0$, such that $y > \varepsilon$. Since $\dfrac{a-c}{t+n} \to 0$ as $n \to \infty$, with the same ε, there exists $L > 0$ such that $\dfrac{a-c}{t+n} < \varepsilon$ whenever $n > L$. Therefore, $\dfrac{a-c}{t+n} < y$. Based on the Well-Ordering property of real numbers, there exists smallest positive integer M such that $\dfrac{a-c}{t+M} < y \le \dfrac{a-c}{t+M-1}$. If $M > 1$, based on Lemma 2.8, one has $\dfrac{a-c}{t+M-1} < \dfrac{b-d}{t+M}$. This inequality leads to $\dfrac{a-c}{t+M} < y < \dfrac{b-d}{t+M}$, i.e. $y \in \left(\dfrac{a-c}{t+M},\dfrac{b-d}{t+M}\right)$; if $M = 1$, with $y \in \left(0,\dfrac{b-d}{t+1}\right)$, one has $\dfrac{a-c}{t+1} < y < \dfrac{b-d}{t+1}$, i.e. $y \in \left(\dfrac{a-c}{t+1},\dfrac{b-d}{t+1}\right)$. Therefore, $y \in \bigcup_{n=1}^{\infty}\left(\dfrac{a-c}{t+n},\dfrac{b-d}{t+n}\right)$. This proves $\bigcup_{n=1}^{\infty}\left(\dfrac{a-c}{t+n},\dfrac{b-d}{t+n}\right) \supseteq \left(0,\dfrac{b-d}{t+1}\right)$. Combined with the earlier result $\bigcup_{n=1}^{\infty}\left(\dfrac{a-c}{t+n},\dfrac{b-d}{t+n}\right) \subseteq \left(0,\dfrac{b-d}{t+1}\right)$, it is shown that $\bigcup_{n=1}^{\infty}\left(\dfrac{a-c}{t+n},\dfrac{b-d}{t+n}\right) = \left(0,\dfrac{b-d}{t+1}\right)$.

Lemma 2.10 Let $t=\left\lfloor \dfrac{a-c}{(b-a)+(c-d)} \right\rfloor$ and $\Delta_x \in \left(0,\dfrac{b-d}{t+1}\right)$, then, Δ_x is not a feasible stepsize for the intervals $[a,b]$ and $[d,c]$.

Proof: Since $\Delta_x \in \left(0,\dfrac{b-d}{t+1}\right)$, from Lemma 2.9 $\Delta_x \in \bigcup_{n=1}^{\infty}\left(\dfrac{a-c}{t+n},\dfrac{b-d}{t+n}\right)$. Therefore, $\Delta_x \in \left(\dfrac{a-c}{t+k},\dfrac{b-d}{t+k}\right)$ for some integer $k \ \ge\ 1$. This leads to the inequalities $c+(t+k)\Delta_x > a$ and $d+(t+k)\Delta_x < b$. We shall discuss based on the following two cases:

(i) If $c+(t+k)\Delta_x \ge b$, then $d+(t+k)\Delta_x < b \le c+(t+k)\Delta_x$. Thus, the shifted interval $\left(d+(t+k)\Delta_x, c+(t+k)\Delta_x\right)$ intersects with the interval (a,b).

(ii) If $c+(t+k)\Delta_x < b$, then $a < c+(t+k)\Delta_x < b$, implying that the shifted

interval $\left(d+(t+k)\Delta_x, c+(t+k)\Delta_x\right)$ intersects with the interval (a,b).
Therefore, Δ_x violates condition (4) defined in Section 2, that is Δ_x is not a feasible stepsize for the intervals $[a,b]$ and $[d,c]$.

Lemma 2.11 Let $\Delta_x \in \bigcup_{n=0}^{t-1}\left(\frac{a-c}{t-n}, \frac{b-d}{t-n}\right)$ and $t=\left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$, then Δ_x is not a feasible step size for the intervals $[a,b]$ and $[d,c]$.

The proof is similar to that of Lemma 2.10.

Lemma 2.12 Let
$\Delta_f \in \bigcup_{n=0}^{t-1}\left[\frac{b-d}{t-n+1}, \frac{a-c}{t-n}\right] \bigcup [b-d,\infty)$ and $t=\left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$, then Δ_f is a feasible stepsize for the intervals $[a,b]$ and $[d,c]$.

Proof: It is obvious that Δ_f is a feasible stepsize if $\Delta_f \in [b-d,\infty)$. Thus, we only consider $\Delta_f \in \bigcup_{n=0}^{t-1}\left[\frac{b-d}{t-n+1}, \frac{a-c}{t-n}\right]$. Then $\Delta_f \in \left[\frac{b-d}{t-k+1}, \frac{a-c}{t-k}\right]$ for some $k=0,1,\ldots,t-1$, or $\frac{b-d}{t-k+1} \le \Delta_f \le \frac{a-c}{t-k}$, implying the inequality $c+(t-k)\Delta_f \le a$ and $d+(t-k+1)\Delta_f \ge b$. According to Lemma 2.4, Δ_f is a feasible stepsize for the intervals $[a,b]$ and $[d,c]$.

Lemma 2.12 also reveals that $\frac{b-d}{t+1}$ is the smallest feasible stepsize for the intervals $[a,b]$ and $[d,c]$.

Lemma 2.13 Let $t=\left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$, then

$$\bigcup_{n=0}^{k}\left[\frac{b-d}{t+1-n}, \frac{a-c}{t-n}\right] \cup \left(\frac{a-c}{t-n}, \frac{b-d}{t-n}\right) = \left[\frac{b-d}{t+1}, \frac{b-d}{t-k}\right), k=0,1,\ldots,t-1. \tag{7}$$

Proof: The proof is done by mathematical induction.

(i)If $k=0$, it is obvious that $\left[\frac{b-d}{t+1}, \frac{a-c}{t}\right] \cup \left(\frac{a-c}{t}, \frac{b-d}{t}\right) = \left[\frac{b-d}{t+1}, \frac{b-d}{t}\right)$.

(ii) Assume (7) is true for $k-1$, then

$$\bigcup_{n=0}^{k}\left[\frac{b-d}{t+1-n}, \frac{a-c}{t-n}\right] \cup \left(\frac{a-c}{t-n}, \frac{b-d}{t-n}\right)$$
$$= \bigcup_{n=0}^{k-1}\left[\frac{b-d}{t+1-n}, \frac{a-c}{t-n}\right] \cup \left(\frac{a-c}{t-n}, \frac{b-d}{t-n}\right) \cup \left[\frac{b-d}{t+1-k}, \frac{a-c}{t-k}\right] \cup \left(\frac{a-c}{t-k}, \frac{b-d}{t-k}\right)$$
$$= \left[\frac{b-d}{t+1}, \frac{b-d}{t-(k-1)}\right) \cup \left[\frac{b-d}{t+1-k}, \frac{a-c}{t-k}\right] \cup \left(\frac{a-c}{t-k}, \frac{b-d}{t-k}\right)$$
$$= \left[\frac{b-d}{t+1}, \frac{a-c}{t-k}\right] \cup \left(\frac{a-c}{t-k}, \frac{b-d}{t-k}\right)$$
$$= \left[\frac{b-d}{t+1}, \frac{b-d}{t-k}\right).$$

Lemma 2.14 Let $t=\left\lfloor \frac{a-c}{(b-a)+(c-d)} \right\rfloor$, then

$$\bigcup_{n=0}^{t-1}\left[\frac{b-d}{t+1-n}, \frac{a-c}{t-n}\right] \cup \left(\frac{a-c}{t-n}, \frac{b-d}{t-n}\right) = \left[\frac{b-d}{t+1}, b-d\right) \tag{8}$$

The identity (8) is a direct result of Lemma 2.13.
The preceding lemmas allow us to decompose the positive half of the real number line into subintervals from which feasible non-feasible step sizes for the interaction between two disjoint intervals are clearly indicated. Lemma 2.10 shows that no feasible step sizes exist in the interval $\left(0, \frac{b-d}{t+1}\right)$. Lemma 2.14 implies that the interval $\left[\frac{b-d}{t+1}, b-d\right)$ contains all feasible step sizes (less than $b-d$). Finally, all real numbers in the interval $[b-d,\infty)$, are feasible stepsizes. It is illustrated in Fig. 3 how the positive real line (frequency axis) is decomposed into feasible and non-feasible regions.

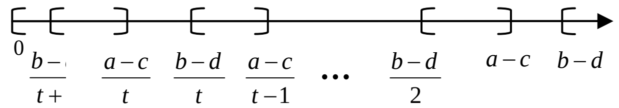

Fig. 3 Partition of the number line with the feasible and non-feasible intervals

Now, we are ready to present the feasible stepsize theorem for two arbitrary disjoint intervals followed by a more general result for any number of disjoint intervals.

Theorem 2.1 The stepsize Δ_f is a feasible solution for the intervals $[d,c]$ and $[a,b]$ if and only if $\Delta_f \in \bigcup_{n=0}^{t-1}\left[\frac{b-d}{t+1-n},\frac{a-c}{t-n}\right]\cup[b-d,\infty)$, where $t=\left\lfloor\frac{a-c}{(b-a)+(c-d)}\right\rfloor$.

Proof: Since the interval $\left(0,\frac{b-d}{t+1}\right)$ contains no feasible step sizes by Lemma 2.10, and the intervals $\bigcup_{n=0}^{t-1}\left(\frac{a-c}{t-n},\frac{b-d}{t-n}\right)$ contain none of the feasible stepsizes by Lemma 2.11, and Lemmas 2.12 and 2.14 conclude that the only feasible stepsizes belong to the intervals $\bigcup_{n=0}^{t-1}\left[\frac{b-d}{t+1-n},\frac{a-c}{t-n}\right]\cup[b-d,\infty)$.

3 Sampling Theorem for Multi-Band Bandpass Signals

We are ready to extend Theorem 2.1 to the most general case. The next theorem gives a closed form formula that identifies all feasible stepsizes for arbitrary number of disjoint intervals. To set the stage, we introduce some symbols and set notations on the operations of intervals to extend the result in Theorem 2.1. Let $\alpha_i,\beta_i\in\mathbb{R}, i=1,2,...,n,$ and $0\le\alpha_1<\beta_1<\alpha_2<\beta_2<...<\alpha_n<\beta_n$ so that the set $\Omega=\{[\alpha_i,\beta_i], i=1,2,...,n\}$ consists of disjoint intervals. Furthermore, introduce a new set S obtained from a cross product between Ω and itself, i.e.

$$S=\Omega\times\Omega=\left\{\{[\alpha_i,\beta_i],[\alpha_j,\beta_j]\}, i<j,\ i,j\in\{1,2,...,n\}\right\} \tag{9}$$

Clearly, the cardinality of set S is $\binom{n}{2}$, i.e. $|S|=\binom{n}{2}=\frac{n(n-1)}{2}$.

Theorem 3.1 Let $\Omega=\{[\alpha_i,\beta_i], i=1,2,...,n\}$, where $\alpha_i,\beta_i\in\mathbb{R}, i=1,2,...,n,$ and $0\le\alpha_1<\beta_1<\alpha_2<\beta_2<...<\alpha_n<\beta_n$. The set S is defined by (9). An arbitrary element of S is given by

$$\{[d_k,c_k],[a_k,b_k]\}\in S,\ d_k<c_k<a_k<b_k,\ k=1,2,...,\binom{n}{2}$$

The step size Δ_f is a feasible solution for Ω if and only if

$$\Delta_f\in\bigcap_{k=1}^{\binom{n}{2}}\left(\bigcup_{m=0}^{t_k-1}\left[\frac{b_k-d_k}{t_k+1-m},\frac{a_k-c_k}{t_k-m}\right]\cup[b_k-d_k,\infty)\right), \tag{10}$$

where $t_k=\left\lfloor\frac{a_k-c_k}{(b_k-a_k)+(c_k-d_k)}\right\rfloor, k=1,2,...,\binom{n}{2}$.

The proof of Theorem 3.1 is carried out by arguing that the feasible solution for all the intervals in Ω is the intersection of the feasible solution intervals for each pair of intervals from set S.

In order to connect the results obtained so far from set theory to a new sampling theorem for multi-band passband signals, we first derive the impulse response of an ideal multi-band bandpass filter. Introduce an ideal bandpass filter $\Gamma(\omega)$ as

$$\Gamma(\omega)=\begin{cases}1 & \text{if } \omega\in\Phi_I\\ 0 & \text{otherwise}\end{cases} \tag{11}$$

where Φ_I is defined in (1), and Φ_{-I} is defined in (2), which will be used in Theorem 3.2. The impulse response of the ideal bandpass filter (11) is derived as

$$\gamma(t)=\sum_{i=1}^{n}\frac{\sigma_i}{\pi}\mathrm{csinc}_{[\omega_i,\sigma_i]}(t) \tag{12}$$

where

$$\mathrm{csinc}_{[\omega_i,\sigma_i]}(t)=\frac{\cos(\omega_i t)\sin(\sigma_i t/2)}{\sigma_i t/2},$$

$\omega_i=\frac{\omega_{l_i}+\omega_{u_i}}{2}$, and $\sigma_i=\omega_{u_i}-\omega_{l_i}$. The lower and upper cutoff frequencies, ω_{l_i} and ω_{u_i}, are also defined in (1). The role of this ideal bandpass filter is to extract the original spectra of a multi-band bandpass signal from the replicas of the sampled signal. This is done via a discrete convolution between the samples of the signal and the digital filter (12). The necessary and sufficient sampling interval T (anti-aliasing) for such a bandpass signal is the reciprocal of Δ_f, which is calculated from (10) in Theorem 3.1, where the intervals are the compact supports of the passbands in the frequency domain. We proceed with a new sampling theorem for multi-band passband signals as follows.

Theorem 3.2 Suppose a signal $f(t)$ is bandpassed to Φ_I. Let $\Omega = \Phi_{-I} \cup \Phi_I$ and $S = \Omega \times \Omega$. Let ω_s and T be the sampling frequency and sampling interval, respectively, and $\omega_s = \dfrac{2\pi}{T}$. Then, the original signal $f(t)$ can be completely determined from its samples $f(lT)$ via the interpolation formula

$$f(t) = \sum_{l=-\infty}^{\infty} \sum_{i=1}^{n} \frac{2\sigma_i}{\omega_s} f(lT) \mathrm{csinc}_{[\omega_i, \sigma_i]}(t - lT) \quad (13)$$

if the sampling frequency ω_s satisfies the following feasibility condition,

$$\omega_s \in \bigcap_{k=1}^{\binom{n}{2}} \left(\bigcup_{m=0}^{t_k - 1} \left[\frac{b_k - d_k}{t_k + 1 - m}, \frac{a_k - c_k}{t_k - m} \right] \cup [b_k - d_k, \infty) \right) \quad (14)$$

where $\{[d_k, c_k], [a_k, b_k]\} \in S$, and

$$t_k = \left\lfloor \frac{a_k - c_k}{(b_k - a_k) + (c_k - d_k)} \right\rfloor, k = 1, 2, \ldots, \binom{n}{2}.$$

Proof: Introduce an impulse train modulated by the samples $f(lT)$ of the signal $f(t)$:

$$f_\delta(t) = \sum_{l=-\infty}^{\infty} Tf(lT)\delta(t - lT).$$

According to Poisson formula, the Fourier transform of $f_\delta(t)$ is given by

$$F_\delta(\omega) = \sum_{l=-\infty}^{\infty} Tf(lT) e^{-jlT\omega} = \sum_{l=-\infty}^{\infty} F(\omega + l\omega_s) \quad (15)$$

where j is the imaginary unit, and $F(\omega)$ is the Fourier transform of $f(t)$. The spectrum of $f(t)$ can be recovered from (15) by applying a matching ideal bandpass filter (11) to $F_\delta(\omega)$ as follows:

$$F(\omega) = \Gamma(\omega) F_\delta(\omega) = \Gamma(\omega) \sum_{l=-\infty}^{\infty} F(\omega + l\omega_s). \quad (16)$$

This is true because none of the spectra $F(\omega + l\omega_s)$, $l \neq 0$, overlap with $F(\omega)$ at the sampling rate ω_s given by (14), which is guaranteed by Theorem 3.1. Therefore, taking inverse Laplace transform of (16) results in the following

$$\begin{aligned} f(t) &= \gamma(t) * f_\delta(t) = \sum_{l=-\infty}^{\infty} Tf(lT)\gamma(t - lT) \\ &= \sum_{l=-\infty}^{\infty} \sum_{i=1}^{n} f(lT) \frac{T\sigma_i}{\pi} \mathrm{csinc}_{[\omega_i, \sigma_i]}(t - lT) \\ &= \sum_{l=-\infty}^{\infty} \sum_{i=1}^{n} \frac{2\sigma_i}{\omega_s} f(lT) \mathrm{csinc}_{[\omega_i, \sigma_i]}(t - lT). \end{aligned}$$

■

As an illustrative example, consider a bandpass signal with the following band positions $[150kHz, 160kHz] \cup [800kHz, 830kHz]$. If the signal is taken as a bandpass signal with a single passband over $[150kHz, 830kHz]$, only the classical Nyquist rate (for lowpass signals) is applicable, which is calculated as $1660kHz$. However, with the proposed algorithm, the lowest necessary and sufficient sampling rate is calculated as $99kHz$. The difference is significant. Furthermore, there are other permissible sampling rates for this signal that can be computed directly from (14), for instance, any number in the interval $[198kHz, 200kHz]$ represents a feasible sampling rate the above bandpass signal.

4 Fast Algorithm for Intervals

It is observed from (14) that the interval(s) for feasible sampling rates are calculated from the intersections of different groups of intervals. The number of intervals could be large in (14), hence increase the amount of computation significantly. In this section, we present a fast algorithm for computing the intersections among given intervals. This algorithm takes advantage of the facts that the set of all feasible intervals for each pair of bands is a union of

disjoint intervals and the feasible intervals are ordered along the number line (frequency axis).
We present the algorithm in the form of a pseudo-code as follows:

Initialization:

Let each of the sets $I_{1,n_1}, I_{2,n_2}, ..., I_{k,n_k}$ be a union of closed disjoint intervals and the intervals in each set are ordered, and I_{i,n_i} is the i th set containing n_i such intervals. Thus,

$$
\begin{aligned}
I_{1,n_1} &= [a_{1,1}, b_{1,1}] \cup [a_{1,2}, b_{1,2}] \cup ... \cup [a_{1,n_1}, b_{1,n_1}] \\
I_{2,n_2} &= [a_{2,1}, b_{2,1}] \cup [a_{2,2}, b_{2,2}] \cup ... \cup [a_{2,n_2}, b_{2,n_2}] \\
&\vdots \\
I_{k,n_k} &= [a_{k,1}, b_{k,1}] \cup [a_{k,2}, b_{k,2}] \cup ... \cup [a_{k,n_k}, b_{k,n_k}]
\end{aligned}
$$

Step 1.

:if one of the unions is empty, then stop.

:find the maximum value of the left endpoint of the first interval in each union call it $L_{\max}$,

:find the minimum value of the right endpoint of the first interval in each union call it $R_{\min}$.

Step 2.

:If $L_{\max} \le R_{\min}$, then $[L_{\max}, R_{\min}]$ is an intersection and is stored in F. Remove the interval that $R_{\min}$ occurred in from the union it was contained in, and go to Step 1.

:Else remove the interval that $R_{\min}$ occurred in from the union it was contained in and go to Step 1.

The above selection process can be carried out by dropping the intervals from the right side with some minor modifications of the above algorithm.
In order to estimate the complexity of the algorithm, let $C\{I_{1,n_1}, I_{2,n_2}, ..., I_{k,n_k}\}$ represent the number of symbolic operations on the intervals required for the selection process, an upper bound (worst case scenario) on the complexity of this algorithm is given as follows

$$C\{I_{1,n_1}, I_{2,n_2}, ..., I_{k,n_k}\} \le (2k-1)\left[\sum_{i=1}^{k}(n_i - 1) + 1\right] \tag{17}$$

A program is made in MATLAB to realize this algorithm. In most experiment, the number of operations is much less than the upper bound in (10) because some I_{i,n_i} runs out quickly during the process.

5 Conclusion

It is investigated in this paper that, under the cost-effective first-order sampling regime, sampling rates that are lower than the Nyquist are achievable if the gaps between the subbands are sufficiently large to accommodate intermediate shifting of the spectrum. Closed form formulas are obtained for calculating all feasible sampling rates based on the locations of the subbands. All results are proved rigorously. The main formula (10) or (14) can be implemented via computer programming such as in MATLAB.
It is well-known that theoretical minimum sampling rate, such as the Nyquist, is susceptible to any engineering imperfection in implementation since the margin of error is zero. In this paper, the lowest sampling rate is identified as the left end-point of the solution intervals by (14). However, more importantly, all other permissible sampling rates are also implied by (14). The proposed sampling algorithm allows much more flexibility for implementation, hence, more practical in design.

References:

[1] A. Kohlenberg, Exact Interpolation of Band-Limited Functions, *J. Appl. Phys.*, Vol 24, 1953, pp 1432-1436.

[2] J. L. Brown, First-Order Sampling of Bandpass Signals-A New Approach, *IEEE Trans. Info. Theory*, Vol IT-26, 1980, pp 613-615.

[3] Meng Xiangwei, A Discussion of Second Order Sampling of Bandpass Signal, *Proc. 4th International Conference on Signal Processing*, Vol 1, 1998, pp 51-52.

[4] R. G. Vaughan, N. L. Scott, and D. R. White, The Theory of Bandpass Sampling, *IEEE Trans. Signal Processing*, Vol 39, 1991, pp1973-1984.

[5] F. Marvasti, Nonuniform Sampling Theorems for Bandpass Signals at or Below the Nyquist Density, *IEEE Trans. Signal Processing*, Vol 44, 1996, pp 572-576.
[6] Y. Wen, J. Wen, and P. Li, On Sampling a Subband of a Bandpass Signal by Periodically Nonuniform Sampling, *Proc. IEEE International Conference on Acoustics, Speech, and Signal Processing*, Vol 4, 2005, pp 225-228.
[7] Y. Wu, On the Design of an Array of Subband Predictive Filters for Bandlimited Signals, *WSEAS Trans. Signal Processing*, Vol 2, No. 11, 2006, pp1441-1447.
[8] J. G. Proakis and D. G. Manolakis, *Digital Signal Processing, Principles, Algorithms, and Applications*, Prentice Hall, New Jersey, 1996.
[9] Y. Wu, A Universal Interpolative Filter for Lowpass and Bandpass Signals-csinc Interpolator, *Digital Signal Processing*, Vol 15, 2005, pp 425-436.
[10] H. Liu and X. Zhou, Spectrum Arrangement and the Generalized Bandpass Signal Direct Sampling Theorem, *Proc. 3rd International Conference on Signal Processing*, Vol 1, 1996, pp 28-31.
[11] Y. Wu, A Proof on the Minimum and Permissible Sampling Rates for the First-Order Sampling of Bandpass Signals, *Digital Signal Processing*, Vol 17, 2007, pp 848-854.
[12] A. Tarek, A Generalized Set Theoretic Approach for Time and Space Complexity Analysis of Algorithms and Functions, *WSEAS Trans. Math.*, Vol 6, No. 1, 2007, pp 60-68.
[13] W. Guo, Flexible Selection of Output Format for Sets in Java Collections: Algorithms and Their Complexity and Reusability, *WSEAS Trans. Math.*, Vol 6, No. 2, 2006, pp 309-315.