GBT-based Distortional Buckling Formulae for Thin-walled Channel Columns and Beams

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ABSTRACT

After a brief outline of the Generalized Beam Theory (GBT) fundamentals and linear stability analysis procedure, the main concepts and steps involved in the derivation of GBT-based fully analytical formulae are described and discussed. Such formulae provide distortional bifurcation stress estimates in cold-formed steel channel columns and beams with arbitrary sloping single-lip edge stiffeners and pinned/free-to-warp or fixed/warping-free end sections. The application of the proposed formulae is illustrated in detail and, in order to assess their accuracy and validity, results concerning several specific channel columns and beams are presented. In particular, the GBT-based analytical estimates are compared with exact numerical results and, whenever possible, also with values yielded by the formulae developed by Lau & Hancock, Hancock and Schafer.

1. INTRODUCTION

The slender thin-walled open cross-sections displayed by most cold-formed steel members make them highly susceptible to *local* buckling, *i.e.*, buckling phenomena involving only in-plane cross-section deformations, as the member axis remains undeformed. It is still possible to make a distinction between local-plate buckling (plate bending *with no* fold line motions) and distortional buckling (plate bending *and* fold line motions). Figure 1(b) depicts the distortional buckling mode (DM) configuration exhibited by lipped channel columns and beams. Since the application of all available code rules[3], concerning the cold-formed steel member distortional buckling behavior, requires a previous determination of the corresponding bifurcation stress values, it is essential for designers to possess accurate and easy-to-use tools to perform this task. In spite of the growing availability of user-friendly finite strip and/or finite element computer programs, (approximate) analytical formulae are still sought and regarded as the most popular and efficient design aids.

Experimental tests and numerical simulations, performed in either columns or beams undergoing distortional buckling, have provided clear evidence that (i) only the web exhibits relevant flexural

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[3] The Australian/New Zealand code (AS/NZS 4600) was the first to incorporate specific and rational provisions intended for to the design against distortional buckling. A very thorough report recently made available by Schafer (2000) indicates that the North American (AISI) Specification will soon follow this trend.

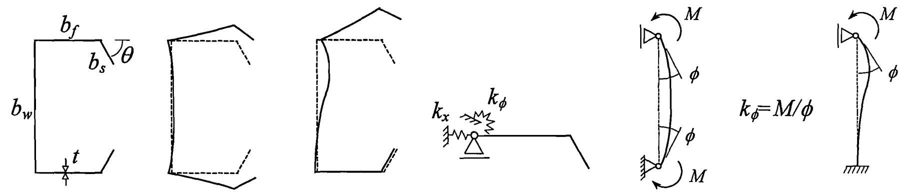

Fig. 1 – Lipped channel: (a) geometry, (b) column and beam DM configurations, (c) structural model and (d) column and beam web deformed configurations

deformations and (ii) the compressed flange-stiffener assemblies remain practically undeformed, as they simply experience "rigid-body" rotation about the web-flange longitudinal edge (see figure 1(b)). Based on such evidence, Lau & Hancock (1987) unveiled the structural similarity existing between (i) the thin-walled member distortional buckling behavior and (ii) the flexural-torsional buckling behavior of the (fictitious) structural model shown in figure 1(c), consisting of an uniformly compressed "flange-stiffener strut", elastically supported at the web-flange edge. The flexural-torsional buckling behavior of this structural model involves the "strut cross-section" rotation (compressed flange-stiffener assembly) about the supported edge (web-flange fold line) and the distributed elastic rotational spring (stiffness k_ϕ) stands for the restraining effect provided by the member web. Obviously, such restraining effect differs in columns (both flange-stiffener assemblies are compressed and the web exhibits single-curvature bending – see figure 1(d_1)) and beams (only one compressed flange-stiffener assembly and double-curvature bending in the web – see figure 1(d_2)). In addition, the compressed flange-stiffener assemblies are assumed free to translate horizontally, which amounts to neglecting the translational spring stiffness k_x, shown in figure 1(c). Based on the flexural-torsional instability of the above strut model, Lau & Hancock (1987) and Hancock (1997) developed approximate analytical formulae[1] which are, at present, routinely employed to obtain column and beam distortional bifurcation stress estimates.

Although the approach outlined in the previous paragraph (i) is reasonably easy-to-use and (ii) provides quite accurate distortional bifurcation stress (σ_{dist}) values for commonly used cross-section dimensions, it is possible to identify a number of limitations, namely:

(i) The evaluation of σ_{dist} involves a two-step (iterative) procedure.

(ii) It was developed exclusively in the context of members with pinned and free-to-warp end sections, which means that it cannot be readily applicable to other support conditions.

(iii) Does not yield accurate results for members displaying very slender webs, *i.e.*, members for which distortional buckling is triggered (governed) by the flexural instability of the web (*i.e.*, high b_w/b_f values), as recently pointed out by Schafer (2000). In fact, for high b_w/b_f values, the rotational stiffness k_ϕ becomes *negative* and the method ceases to yield meaningful results[2]. It is worth mentioning that, in order to overcome this limitation, relatively minor alterations to Lau & Hancock's approach were proposed by Davies & Jiang (1996a, 1996b), both for columns and beams.

[1] Analytical *procedures* is probably a better way to describe this approach.

[2] In order to overcome this limitation, Hancock *et al.* (1996) proposed a modified procedure for flexural members.

Also motivated by the inability of Lau & Hancock's approach to handle the situations in which $k_\phi < 0$, Schafer (1997) developed a slightly different methodology to calculate σ_{dist} estimates, applicable to either columns or beams. The novel idea behind such methodology, which still employs the structural model depicted in figure 1(c), consists of expressing k_ϕ explicitly in terms of the bifurcation stress, by including the web stiffness reduction due to geometrical effects. This idea makes it possible to obtain, after some simplifications[1], an analytical expression which (i) provides *directly* the σ_{dist} estimate (no iteration required) and (ii) is also valid when $k_\phi < 0$.

On the other hand, the Generalized Beam Theory (GBT – Schardt, 1989, and Davies *et al.*, 1994), has also been employed to develop an expression to evaluate σ_{dist} in thin-walled columns with pinned and free-to-warp end sections (Davies & Jiang, 1998). However, since such an expression requires the knowledge of a "distortional cross-section geometrical property", which can only be determined through an involved numerical procedure (first order GBT), it seems fair to argue that its "analytical character" is quite debatable. The objective of this work is to present the derivation and illustrate the application of a number of fully analytical GBT-based formulae to estimate distortional bifurcation stresses in cold-formed steel channel columns (uniformly compressed members) and beams (members under major axis bending) with arbitrarily inclined single-lip edge stiffeners (see figure 1(a)), including hat-sections ($\theta = -90°$). Two support conditions are dealt with, namely members with end sections (i) pinned and free-to-warp or (ii) fixed and warping-free. In order to (i) assess the accuracy and validity and (ii) show the potential of the derived analytical expressions, several numerical results are presented and discussed. In particular, the analytical estimates provided by the GBT-based formulae are compared with the (i) results of exact linear stability analyses and, for some pinned/free-to-warp members only, also with (ii) the values yielded by the formulae developed by Lau & Hancock (1987) and Schafer (1997).

2. BRIEF GBT OUTLINE

The Generalized Beam Theory (GBT) was first developed by Schardt (1989), in the context of isotropic thin-walled members, and later extensively employed by Davies *et al.* (1994, 1998) to investigate the stability behavior of cold-formed steel structural elements. It has been shown to provide a general and unified approach to obtain accurate and clarifying solutions for a wide range of structural problems, namely buckling (bifurcation) problems. In fact, by decomposing the member deformed configuration (buckling mode shape) into a linear combination of cross-section *deformation modes* (including local modes, which involve using folded-plate theory) GBT offers unique possibilities, which complement and compete with the use of powerful numerical techniques, such as the finite element or finite strip methods. The distortional buckling formulae developed here provide a perfect illustration of this statement.

It is beyond the scope of this paper to present a detailed and/or complete account of the second order GBT formulation, which can be found elsewhere (Davies *et al.*, 1994, and Silvestre & Camotim, 2002). Instead, only the concepts, expressions and procedures more closely related to the derivation of the distortional buckling formulae will be very briefly overviewed. Then, let us start by considering the system of GBT equilibrium equations

[1] Although Schafer's methodology is applicable to a wider range of cross-sections, the (unavoidable) simplifications involved are also responsible for a (small) decrease in accuracy, with respect to Lau & Hancock's approach.

$$EC_\eta \phi_{\eta,xxxx} - GD_\eta \phi_{\eta,xx} + B_\eta \phi_\eta + W_p (X_{p\eta} \phi_{\eta,xx} + X_{p\eta\xi} \phi_{\xi,xx}) = 0$$
$$EC_\xi \phi_{\xi,xxxx} - GD_\xi \phi_{\xi,xx} + B_\xi \phi_\xi + W_p (X_{p\eta\xi} \phi_{\eta,xx} + X_{p\xi} \phi_{\xi,xx}) = 0 \quad , (1)$$

which describes the structural behavior associated to member deformed configurations combining *two* distinct cross-section deformation modes, here designated as modes η and ξ. In equations (1), (i) x is the member axial coordinate, (ii) $\phi_\eta(x)$ and $\phi_\xi(x)$ are the two "modal amplitude functions", defined along the member length, (iii) E and G are Young's and shear moduli and (iv) $(\cdot)_{,x} \equiv d(\cdot)/dx$. Moreover, notice that, in each equation, (i) the first three terms correspond to the member linear behaviour and (ii) the last term, related to the second order effects, accounts for all the interaction between in-plane stresses and out-plane deformations. Finally, the components C_k (generalized warping constant), D_k (generalized torsion constant), B_k (generalized transverse bending stiffness) and X_{pjk} (geometric stiffness related to the applied stress resultant W_p) are given by ($k = \eta, \xi$)

$$C_k = t \int_S u_k^2 ds \qquad D_k = \frac{t^3}{3} \int_S w_{k,s}^2 ds \qquad B_k = K \int_S w_{k,ss}^2 ds \qquad (2)$$

$$X_{p\eta} = \int_S \frac{u_p t}{C_p} (v_\eta^2 + w_\eta^2) ds \qquad X_{p\eta\xi} = \int_S \frac{u_p t}{C_p} (v_\eta v_\xi + w_\eta w_\xi) ds \qquad X_{p\xi} = \int_S \frac{u_p t}{C_p} (v_\xi^2 + w_\xi^2) ds \quad , (3)$$

where (i) t is the wall thickness, (ii) K is the wall bending stiffness ($K = Et^3/12(1-\nu^2)$), (iii) s is the coordinate along the cross-section mid-line, (iv) u and v are membrane displacement components along x and s, respectively, and (v) w is the flexural displacement component. In *columns*, the stability analysis requires considering $u_p = 1$ (*uniform* normal stress diagram), which leads to $W_p = P$ (compressive axial force) and $C_p = A$ (cross-section area). In *beams*, on the other hand, it is necessary to consider $u_p = z$ (*linear* normal stress diagram) and one has $W_p = M$ (*major* axis bending moment) and $C_p = I$ (major axis moment of inertia).

It is also important to address the GBT modal decomposition of the column and beam *distortional* buckling mode (*DM*) shapes, already presented in figure 1(b):

(i) In *columns*, the *DM* shape coincides with the symmetric distortional (*SD*) deformation mode shown in figure 2(a).
(ii) In *beams*, the *DM* shape combines the two deformation modes depicted in figure 2(b), namely (ii_1) a symmetric (*SD*) and (ii_2) an anti-symmetric (*AD*) distortional modes.

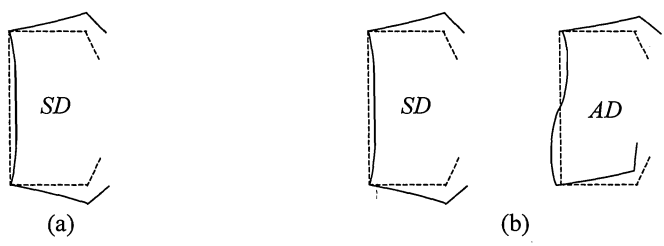

Fig. 2 – Most relevant GBT deformation modes for distortional buckling: (a) columns (b) beams

3. LINEAR STABILITY ANALYSIS

Once (exact or approximate) analytical expressions to describe the amplitude functions of the two deformation modes participating in the buckling mode ($\phi_\eta(x)$ and $\phi_\xi(x)$) are known, it is possible to obtain formulae which provide bifurcation stress estimates. In order to achieve this goal, let us write $\phi_\eta(x)$ and $\phi_\xi(x)$ in the form

$$\phi_\eta(x) = a_\eta \bar{\phi}(x) \qquad \phi_\xi(x) = a_\xi \bar{\phi}(x) \quad , \quad (4)$$

where (i) a_η and a_ξ are the deformation mode magnitudes and (ii) $\bar{\phi}(x)$ is a shape function which describes, either exactly or approximately, their (identical) longitudinal variation. Introducing these expressions in equations (1) and applying Galerkin's method[1], one is led to the eigenvalue problem defined by

$$\begin{bmatrix} K_\eta - W_b X_\eta & -W_b X_{\eta\xi} \\ -W_b X_{\eta\xi} & K_\xi - W_b X_\xi \end{bmatrix} \begin{Bmatrix} a_\eta \\ a_\xi \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad , \quad (5)$$

where (i) the eigenvalues are the applied stress resultants (W_b), (ii) the eigenvector components are the deformation mode magnitudes ($\{a\}$) and (iii) one has ($i = \eta, \xi$)

$$K_i = EC_i \frac{\int_0^L \bar{\phi}_{,xx}^2 dx}{\int_0^L \bar{\phi}_{,x}^2 dx} + GD_i + B_i \frac{\int_0^L \bar{\phi}^2 dx}{\int_0^L \bar{\phi}_{,x}^2 dx} \quad . \quad (6)$$

Since all shape functions ($\bar{\phi}(x)$) employed in this work are trigonometric (*i.e.*, periodic with n half-waves), a change-of-variable procedure is required in order to obtain analytical expressions providing the "critical length" value (L_{cr}), *i.e.*, the member length associated to a minimum bifurcation stress. Incorporating such change-of-variable, defined by $y = \pi x/L$, in (6) leads to

$$K_i = EC_i \left(\frac{\pi}{L}\right)^2 \mu_C + GD_i + B_i \left(\frac{L}{\pi}\right)^2 \mu_B \qquad \mu_B = \frac{\int_0^\pi \bar{\phi}^2 dy}{\int_0^\pi \bar{\phi}_{,y}^2 dy} \qquad \mu_C = \frac{\int_0^\pi \bar{\phi}_{,yy}^2 dy}{\int_0^\pi \bar{\phi}_{,y}^2 dy} \quad . \quad (7)$$

It is still worth noticing that, if more than two deformation modes are considered and the steps just described are followed, one is led to the solution of an eigenvalue problem analogous to (5) but of an higher dimension[2]. Since, in general, such problems cannot be solved analytically, it is not possible to derive formulae to estimate bifurcation stresses and critical bifurcation lengths and one must resort to numerical solutions. Finally, it is important to realize that the parameters μ_B and μ_C do not depend on the particular combination of deformation modes chosen to participate in the buckling mode. In fact, these two parameter values are only influenced by the amplitude function $\phi(y)$ characteristics, *i.e.*, the member end support conditions half-wave number (n).

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1 Naturally, $\bar{\phi}(x)$ must satisfy all the boundary conditions concerning modes η and ξ (assumed here to be identical).

2 The decision concerning how many and which deformation modes to consider depends on an *a priori* knowledge of their relevance and depends on several factors, namely on the member cross-section shape and dimensions.

3.1 Buckling Involving a Single Deformation Mode

Let us consider a situation in which the member buckling mode shape coincides with a GBT deformation mode, generally designated as "mode η". Then, solving the eigenvalue problem defined by the system (5) simply consists of annulling its first diagonal component, which leads to

$$W_b = \frac{K_\eta}{X_\eta} \quad , \quad (8)$$

a general expression providing, in terms of the member length L, the stress resultant value associated to *bifurcation in mode η*. In order to obtain the *critical length* value L_{cr}, yielding the minimum bifurcation stress value ($W_{b.min}$), one just needs to find the relevant root of $dW_b/dL=0$, which is given by the formula

$$L_{cr} = \pi \sqrt[4]{\frac{EC_\eta}{B_\eta}} \sqrt[4]{\frac{\mu_C}{\mu_B}} \quad (9)$$

and corresponds to

$$W_{b.min} = \frac{2\sqrt{EC_\eta B_\eta}\sqrt{\mu_C \mu_B} + GD_\eta}{X_\eta} \quad . \quad (10)$$

The above expressions can be readily applied to obtain L_{cr} and $W_{b.min}$ estimates concerning the distortional buckling of *columns*, which stems from the fact that, as mentioned earlier, this buckling mode coincides with the *SD* deformation mode (see figure 2(a)). In fact, it suffices to determine the C_η, B_η, D_η and X_η values for $\eta \equiv SD$.

On the other hand, expressions (9) and (10) do not apply to the distortional stability of beams, as the buckling mode combines the *SD* and *AD* deformation modes (see figures 1(b$_2$) and 2(b)).

3.2 Buckling Involving Two Deformation Modes

When the member buckling mode shape can be obtained by combining two GBT deformation modes (say modes η and ξ), it is still possible to solve analytically the eigenvalue problem defined by (5). Then, the stress resultant value associated to *bifurcation in a combination of modes η and ξ* can be estimated, again in terms of the member length L, by means of

$$W_b = \frac{K_\eta X_\xi + K_\xi X_\eta - \sqrt{(K_\eta X_\xi - K_\xi X_\eta)^2 + 4K_\eta K_\xi X_{\eta\xi}^2}}{2(X_\eta X_\xi - X_{\eta\xi}^2)} \quad . \quad (11)$$

In general, a highly non linear relation exists between W_b and L, which makes it impossible to find the relevant root of $dW_b/dL=0$ analytically. Obviously, this fact precludes the derivation of general expressions to estimate L_{cr} and $W_{b.min}$. However, for the specific buckling phenomenon dealt with in this work (beam distortional buckling), it is still possible to obtain reasonably simple and accurate formulae to determine the above quantities. Such task is performed next.

After assuming $\eta \equiv SD$ and $\xi \equiv AD$ (see figure 2(b)), one realizes that the geometric stiffness matrix is *not full* ($X_{SD}=0$, $X_{AD}=0$ and $X_{SD.AD}\neq 0$, *i.e.*, null diagonal components), a feature which makes the analytical determination of L_{cr} and $W_{b.min}$ feasible. In fact, for this particular case, expression (11) can be considerably simplified, to read

$$W_{b.dist}^{beam} = \frac{\sqrt{K_{SD}K_{AD}}}{X_{SD.AD}} \quad , \quad (12)$$

and the condition $dW_b/dL=0$ corresponds to the fourth order polynomial ($Z=L^2/\pi^2$)

$$2B_{SD}B_{AD}\mu_B^2 Z^4 + G\mu_B(B_{SD}D_{AD}+B_{AD}D_{SD})Z^3 - EG\mu_C(C_{SD}D_{AD}+C_{AD}D_{SD})Z - 2E^2C_{SD}C_{AD}\mu_C^2 = 0 \ , \quad (13)$$

the relevant root of which (Z_{dist}) yields the distortional critical length value $L_{dist}=\pi\sqrt{Z_{dist}}$. However, after a successful attempt to keep the procedure fully analytical, it was found that the expression

$$L_{dist} = \pi \sqrt[8]{\frac{E^2C_{SD}C_{AD}}{B_{SD}B_{AD}}}\sqrt[4]{\frac{\mu_C}{\mu_B}} \quad . \quad (14)$$

estimates L_{dist} quite accurately (the errors never exceed 1%). Then, taking into account that

$$\begin{aligned} K_{SD} &= C_{SD}\sqrt{E\mu_B\mu_C}\sqrt[4]{\frac{B_{SD}B_{AD}}{C_{SD}C_{AD}}} + GD_{SD} + B_{SD}\sqrt{E\mu_B\mu_C}\sqrt[4]{\frac{C_{SD}C_{AD}}{B_{SD}B_{AD}}} \\ K_{AD} &= C_{AD}\sqrt{E\mu_B\mu_C}\sqrt[4]{\frac{B_{SD}B_{AD}}{C_{SD}C_{AD}}} + GD_{AD} + B_{AD}\sqrt{E\mu_B\mu_C}\sqrt[4]{\frac{C_{SD}C_{AD}}{B_{SD}B_{AD}}} \end{aligned} \quad . \quad (15)$$

and introducing the (exact or approximate) value of L_{dist} in (12), one is led to an expression which provides the *beam* minimum distortional bifurcation stress resultant (bending moment).

4. DISTORTIONAL BUCKLING FORMULAE

Since the application of the distortional buckling formulae developed in the previous section depends on the knowledge of (i) the GBT cross-section modal mechanical (C_η, B_η and D_η) and geometrical (X_η) properties and (ii) the member end support conditions (μ_B and μ_C), it is now necessary to provide expressions which yield the values of such quantities, in terms of the cross-section dimensions (b_w, b_f, b_s, θ and t) and steel elastic constants (E and ν). The expressions presented next were derived by means of the symbolic manipulation program MAPLE (WMS, 2001) and enable the use of distortional buckling formulae related to the following problems:

(i) Column distortional buckling ($W_{dist}\equiv P_{dist}$).
(ii) Beam distortional buckling ($W_{dist}\equiv M_{dist}$).

4.1 End Support Conditions

As mentioned earlier, members with two different end support conditions are considered in this work, namely members with (i) pinned and free-to-warp and (ii) fixed and warping-free end sections. In the first case, the amplitude functions adopted are $\bar{\phi}(y)=sin(ny)$, which constitute *exact solutions* of equations (1) and lead to (see (7))

$$\mu_B=\frac{1}{n^2} \qquad \mu_C=n^2 \quad . \quad (16)$$

In the second case, the amplitude functions adopted are $\bar{\phi}(y)=sin(ny)sin(y)$, which satisfy all the member end support conditions and were successful used by Bradford & Azhari (1995), in the context of plate linear stability finite strip analyses. Such functions constitute *approximate solutions* of equations (1) and their introduction in (7) yields

$$\mu_B=\frac{3\ (if\,n=1)\ or\ 2\ (if\,n\geq 2)}{(n-1)^2+(n+1)^2} \qquad \mu_C=\frac{(n-1)^4+(n+1)^4}{(n-1)^2+(n+1)^2} \quad . \quad (17)$$

4.2 Cross-Section Modal Properties

The derivation of the formulae providing the cross-section modal properties requires a sequential procedure, which will be described here for the general case of channels with arbitrarily inclined sloping edge stiffeners (see figure 1(a)). Moreover, two particular sets of formulae, applicable to the commonly used (i) channels with orthogonal lips ($\theta=90°$) and (ii) hat-sections ($\theta=-90°$), are included in an annex, presented at the end of the paper. Let us now turn our attention to the above mentioned sequential procedure, which involves the following steps and expressions:

(I) Determination of *geometrical and mechanical parameters* (α_1, α_2, β_1, β_2, K, A and I)

$$\alpha_1=\frac{b_f}{b_w} \qquad \alpha_2=\frac{b_s}{b_w} \qquad \beta_1=\alpha_1\tan\theta \qquad \beta_2=\alpha_2\sin\theta$$

$$K=\frac{Et^3}{12(1-\nu^2)} \qquad A=b_w t(1+2\alpha_1+2\alpha_2) \qquad I=\frac{b_w^3 t}{12}[1+6\alpha_1+2\alpha_2(3-6\beta_2+4\beta_2^2)]$$

(II) Determination of *nodal warping function values* (u_1 and u_2)

(II.1) *SD mode* (columns and beams)

$$u_{1.SD}=\frac{\alpha_2}{\gamma_{0.SD}}[\beta_1\alpha_1+\beta_2(2\alpha_1+\alpha_2)] \qquad u_{2.SD}=-\frac{\alpha_2}{\gamma_{0.SD}}[\beta_1(2\alpha_1+3)+2\beta_2(\alpha_1+1)]$$

$$\gamma_{0.SD}=\beta_1[\alpha_1(\alpha_1+2)+\alpha_2(2\alpha_1+3)]+\alpha_2\beta_2(\alpha_1+1)$$

(II.2) *AD mode* (beams)

$$u_{1.AD}=\frac{\alpha_2}{\gamma_{0.AD}}\{\beta_1[3\alpha_1+4\beta_2(5\alpha_1+3\alpha_2)]+3\beta_2(2\alpha_1+\alpha_2)\}$$

$$u_{2.AD}=-\frac{\alpha_2}{\gamma_{0.AD}}\{\beta_1[3(2\alpha_1+1)+4\beta_2(4\alpha_1+1)]+2\beta_2(3\alpha_1+1)\}$$

$$\gamma_{0.AD} = \beta_1[\alpha_1(3\alpha_1+2)+3\alpha_2(2\alpha_1+1)+2\alpha_2\beta_2(4\alpha_1+1)]+\alpha_2\beta_2(3\alpha_1+1)$$

(III) Determination of *nodal transverse bending moments* (m) and *wall element chord rotations* (φ_0, φ_1 and φ_2,) and *displacements* (w_0, w_1 and w_2) – *SD* and *AD* modes

$$m = -\frac{6K}{b_w^3}\cdot\frac{\beta_1(u_2-1+\gamma_2)+\beta_2(u_2-u_1)}{\alpha_1\beta_1\beta_2(2\alpha_1+\gamma_1)} \qquad \varphi_0 = 2\gamma_4\frac{u_2-u_1}{\alpha_1 b_w^2}$$

$$\varphi_1 = -\frac{\beta_1(u_2-1-\gamma_3)+\beta_2(u_2-u_1)}{\alpha_1\beta_1\beta_2 b_w^2} \qquad \varphi_2 = m\frac{\alpha_1 b_w}{6K}+\varphi_1 \qquad w_0 = \gamma_5\frac{u_1-u_2}{\alpha_1 b_w}$$

$$w_1 = \frac{\beta_1(u_2-1+\gamma_3)+\beta_2(u_2-u_1)}{2\beta_1\beta_2 b_w} \qquad w_2 = -\frac{\alpha_2 b_w}{2}\varphi_2 + \frac{\alpha_1^2\beta_2(u_2-1)+\alpha_2^2\beta_1(u_2-u_1)}{\alpha_1\alpha_2\beta_1\beta_2 b_w}$$

Mode	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6
SD	*3*	*0*	*0*	*0*	*1*	*15*
AD	*1*	$2\beta_2(u_2-2u_1)$	$2\beta_2 u_1$	*1*	*0*	*1*

(IV) Determination of *cross-section modal mechanical properties* (C_η, B_η and D_η): C_{SD}, B_{SD} and D_{SD} (columns) and C_{SD}–C_{AD}, B_{SD}–B_{AD} and D_{SD}–D_{AD} (beams)

$$C = \frac{b_w t}{3}[2\alpha_2(u_2^2+u_2+1)+2\alpha_1(u_2^2+u_2u_1+u_1^2)+\gamma_1 u_1^2] \qquad B = \frac{m^2 b_w(2\alpha_1+\gamma_1)}{3K}$$

$$D = \frac{b_w t^3}{3}(2\alpha_2\varphi_2^2+2\alpha_1\varphi_1^2+\varphi_0^2)+\begin{cases} 0 & columns \\ \dfrac{b_w^3 t^3 m^2}{540K^2}(8\alpha_1^3+\gamma_6) & beams\end{cases}$$

(V) Determination of *cross-section modal geometrical properties* (X_η)

(V.1) X_{SD} (columns)

$$X_{SD} = \frac{t}{A}\left[\frac{X_1}{\alpha_1\alpha_2 b_w}+\frac{X_2 b_w^3}{7560K^2}\right]$$

$$X_1 = 2\alpha_1(u_2-1)^2+2\alpha_2(u_2-u_1)^2+\alpha_1\alpha_2 b_w^2(2\alpha_2 w_2^2+2\alpha_1 w_1^2+w_0^2)$$

$$X_2 = m^2 b_w^2(32\alpha_1^5+63)+42m\varphi_1 K b_w\alpha_1^4+1260K[m(w_0+\alpha_1^3 w_1)+K(\alpha_2^3\varphi_2^2+\alpha_1^3\varphi_1^2)]$$

(V.2) $X_{SD.AD}$ (beams)

$$X_{SD.AD} = \frac{t}{I}\left[\frac{X_1}{\alpha_1\alpha_2}+\frac{b_w^3(X_2+X_3+X_4)}{15120K^2}\right]$$

$$X_1 = \alpha_1(1-\beta_2)(u_{2.SD}-1)(u_{2.AD}-1)+\alpha_2(u_{2.SD}-u_{1.SD})(u_{2.AD}-u_{1.AD})+$$
$$+\alpha_1\alpha_2 b_w^2[\alpha_1 w_{1.SD}w_{1.AD}+\alpha_2 w_{2.SD}w_{2.AD}(1-\beta_2)]$$

$$X_2 = m_{SD}m_{AD}b_w^3(32\alpha_1^5 + 3) + 1260K^2b_w[\alpha_1^3\varphi_{1.SD}\varphi_{1.AD} + \alpha_2^3\varphi_{2.SD}\varphi_{2.AD}(1-\beta_2)]$$

$$X_3 = 21Kb_w^2[-3m_{SD}\varphi_{0.AD} + \alpha_1^4(m_{SD}\varphi_{1.AD} + m_{AD}\varphi_{1.SD})] +$$
$$+ 42Kb_w[m_{AD}w_{0.SD} + 15\alpha_1^3(m_{SD}w_{1.AD} + m_{AD}w_{1.SD})]$$

$$X_4 = 1260K^2[-w_{0.SD}\varphi_{0.AD} + 2\alpha_2^2\beta_2(w_{2.AD}\varphi_{2.SD} + w_{2.SD}.\varphi_{2.AD})]$$

5. ILLUSTRATION AND VALIDATION

In order to illustrate the application and validate the proposed distortional buckling formulae, the following numerical results are presented next:

(i) First, for a specific cross-section geometry, the steps involved in the use of all the formulae derived here are reported in great detail. In particular, (i_1) the results presented concern the four kinds of members dealt with in this work (pinned and fixed columns and beams) and (i_2) all the numerical values involved in the application of each formula are provided.

(ii) Then, for columns and beams displaying several combinations of cross-section geometry and end support conditions, the critical bifurcation stresses are evaluated and compared with (ii_1) exact numerical GBT results, yielded by analyses accounting for all deformation modes, and, when possible, also with (ii_2) values provided by the formulae developed by Lau & Hancock, Hancock and Schafer (only applicable to pinned/free-to-warp members).

Finally, some differences between the nature of the distortional bifurcation stress values presented for pinned/free-to-warp and fixed/warping-free columns and beams should be pointed out:

(i) All the values concerning pinned/free-to-warp members are *minimum values*, *i.e.*, they (i_1) are associated to the member critical length L_{dist}, yielded by expression (9) (columns) or (14) (beams), and (i_2) correspond to a single-wave critical buckling mode ($n=1$).

(ii) In fixed/warping-free members, since the critical buckling mode often exhibits more than one half-wave, it makes no sense to talk about "*the* minimum bifurcation stress value". In fact, there exists one such value per half-wave number and, moreover, it may happen that neither of them is associated with the critical stress of any specific member. Therefore, it is only meaningful to estimate distortional bifurcation stresses for (ii_1) a member with a given *length* (L) and (ii_2) a buckling mode exhibiting a given *half-wave number* (n)[1].

5.1 Illustrative Example

The cross-section geometry selected to illustrate the application of the derived formulae is defined by $b_w=120\ mm$, $b_f=60\ mm$, $b_s=15\ mm$, $t=1.5\ mm$ and $\theta=45°$ and the cold-formed steel properties adopted are $E=200\ GPa$ and $\nu=0.3$. First of all, it is necessary to compute the cross-section modal properties, using the expressions presented in section 4.2. Their values are:

(I) *Geometrical and mechanical parameters*:

$\alpha_1 = 0.5$ $\qquad\alpha_2 = 0.125$ $\qquad\beta_1 = 0.5$ $\qquad\beta_2 = 0.0884$

[1] The half-wave number leading to the *minimum* bifurcation stress value varies with the member geometry.

$K = 61813\ Nmm$ $\qquad A = 405\ mm^2$ $\qquad I = 999050\ mm^4$

(II) *Nodal warping values, transverse moments and chord rotations/displacements*

SD mode

$u_{1.SD} = 0.0490\ mm$ $\qquad u_{2.SD} = -0.3176\ mm$ $\qquad m_{SD} = 1.6784\ Nmm/mm^2$

$\varphi_{0.SD} = 0$ $\qquad \varphi_{1.SD} = 0.00217\ rad/mm$ $\qquad \varphi_{2.SD} = 0.00244\ rad/mm$

$w_{0.SD} = 0.0061\ mm/mm$ $\qquad w_{1.SD} = -0.0652\ mm/mm$ $\qquad w_{2.SD} = -0.1148\ mm/mm$

AD mode

$u_{1.AD} = 0.1484\ mm$ $\qquad u_{2.AD} = -0.3788\ mm$ $\qquad m_{AD} = 3.8645\ Nmm/mm^2$

$\varphi_{0.AD} = -0.00015\ rad/mm$ $\qquad \varphi_{1.AD} = 0.00235\ rad/mm$ $\qquad \varphi_{2.AD} = 0.00298\ rad/mm$

$w_{0.AD} = 0$ $\qquad w_{1.AD} = -0.0682\ mm/mm$ $\qquad w_{2.AD} = -0.1267\ mm/mm$

(III) *Cross-section mechanical and geometrical properties* (*SD* and *AD* modes)

$C_{SD} = 17.443\ mm^4$ $\qquad C_{AD} = 19.3505\ mm^4$ $\qquad B_{SD} = 0.00729\ N/mm^2$

$B_{AD} = 0.01933\ N/mm^2$ $\qquad D_{SD}^{col} = 0.00084\ mm^2$ $\qquad D_{SD}^{beam} = 0.00097\ mm^2$

$D_{AD} = 0.00114\ mm^2$ $\qquad X_{SD} = 0.005416$ $\qquad X_{SD.AD} = 0.000115\ mm^{-1}$

Moreover, the expressions presented in section 4.1 yield the values of the parameters related to the member end support conditions. For instance, if $n=1$, they read:

$$\mu_B^{pin} = 1 \qquad \mu_C^{pin} = 1 \qquad\qquad \mu_B^{fix} = 0.75 \qquad \mu_C^{fix} = 4$$

Finally, introducing the above results in the expressions presented in sections 3.1 and 3.2, one obtains estimates for the (i) critical lengths and bifurcation stress resultants (pinned/free-to-warp column and beam) and (ii) bifurcation stress resultants for given member length and number of buckling mode half-waves ($L=1000\ mm$ and $n=1$–3, in this particular case). Such estimates are presented next, together with the exact GBT values (indicated between square brackets):

(i) Pinned/free-to-warp column: $L_{dist}=465\ mm$ [*460*] and $P_{dist}=70.8\ kN$ [*71.7*].

(ii) Pinned/free-to-warp beam: $L_{dist}=417\ mm$ [*420*] and $M_{dist}=4351\ kNmm$ [*4369*].

(iii) Fixed/warping-free column: $P_{dist}=141$ ($n=1$); *93* ($n=2$); *114 kN* ($n=3$) [*91* – $n=2$].

(iv) Fixed/warping-free beam: $M_{dist}=9945$ ($n=1$); *5494* ($n=2$); *6114 kNmm* ($n=3$) [*5435* – $n=2$].

5.2 Comparison with Other Available Formulae and Exact (GBT) Values

Tables 1 and 2 show, respectively for pinned/free-to-warp columns and beams, the minimum flange[1] distortional bifurcation stress values (in *MPa*) concerning several cross-section geometries and for $E=200\ GPa$, $\nu=0.3$, $t=1\ mm$. Such values were obtained from (i) the formulae proposed in this work and, for comparison and validation purposes, also from (ii) the formulae developed by Lau & Hancock (1987), Hancock (1997) and Schafer (1997) and (iii) exact GBT analyses.

[1] Notice that, in the case of hat-section ($\theta=-90°$) beams, the flange stress *is not* the maximum applied stress.

The observation of the results displayed in tables 1 and 2 leads to the following conclusions:

(i) For the cross-section geometries dealt with, the GBT-based formulae consistently yield quite accurate estimates for both columns and beams. In fact, the average and standard deviation values of the ratio $\sigma_{dist}/\sigma_{d.ex}$ read (i$_1$) *1.01* and *0.040*, for columns, and (i$_2$) *1.02* and *0.033*, for beams. Moreover, with a single exception, the error never exceeds 7%.

(ii) Both for columns and beams, the GBT-based results always compare quite favourably with the values provided by the other formulae. However, it should be mentioned that, for members with predominantly slender webs ($b_w/b_f > 2$), the estimates yielded by the GBT-based and Schafer formulae display a similar accuracy and both of them tend to be more precise than the ones due to Lau & Hancock or Hancock.

(iii) For beams, while the formulae developed by Hancock and Schafer yield quite a significant number of rather unconservative estimates, the GBT-based estimates never overestimate σ_{dist} by more than 7% and only four of them fall outside the 5% error range.

(iv) Unlike the GBT-based formulae, the other formulae are not able to distinguish between channel and hat-section members with the same cross-section dimensions. This limitation becomes quite severe in the case of beams, as shown by the results displayed in table 2.

Table 1. Minimum distortional bifurcation stresses for pinned/free-to-warp columns

				Exact	*Lau & Hancock*		*Schafer*		*GBT*	
θ	b_w	b_f	b_s	$\sigma_{d.ex}$	σ_{dist}	$\sigma_{dist}/\sigma_{d.ex}$	σ_{dist}	$\sigma_{dist}/\sigma_{d.ex}$	σ_{dist}	$\sigma_{dist}/\sigma_{d.ex}$
90°	90	30	5	156	163	1.05	153	0.98	164	1.05
		60		81	81	1.00	82	1.02	79	0.98
		90		40	41	1.05	44	1.12	39	0.99
		30	10	243	284	1.17	270	1.11	269	1.10
		60		145	141	0.98	152	1.05	146	1.01
		90		77	75	0.99	86	1.12	78	1.01
45°		18	5	118	3	0.02	109	0.93	116	0.98
		30		125	133	1.06	118	0.94	124	1.00
		60		64	64	1.01	65	1.02	60	0.95
		90		32	34	1.07	36	1.13	30	0.97
		18	10	153	150	0.98	177	1.15	165	1.07
		30		178	201	1.13	184	1.04	184	1.03
		60		103	102	0.99	108	1.04	102	0.99
		90		55	56	1.01	62	1.12	56	1.00
-90°		30	5	170	163	0.96	153	0.90	181	1.06
		60		82	81	0.99	82	1.01	80	0.98
		90		40	41	1.04	44	1.12	40	0.99
		30	10	316	284	0.90	270	0.85	339	1.07
		60		151	141	0.94	152	1.01	151	1.01
		90		77	75	0.98	86	1.11	78	1.01
					Mean	*1.02*[1]	*Mean*	*1.04*	*Mean*	*1.01*
					Sd.dev.	*0.064*[1]	*Sd.dev.*	*0.086*	*Sd.dev.*	*0.040*

[1] This value does not include the "meaningless" $\sigma_{dist}/\sigma_{d.ex}=0.02$ estimate, which stems from the very slender web ($b_w/b_f=5$) – the web rotational stiffness becomes negative and the Lau & Hancock formulae should not be applied.

Table 2. Minimum (flange) distortional bifurcation stresses for pinned/free-to-warp beams

				Exact	*Hancock*		*Schafer*		*GBT*	
θ	b_w	b_f	b_s	$\sigma_{d.ex}$	σ_{dist}	$\sigma_{dist}/\sigma_{d.ex}$	σ_{dist}	$\sigma_{dist}/\sigma_{d.ex}$	σ_{dist}	$\sigma_{dist}/\sigma_{d.ex}$
90°	90	30	5	340	340	1.00	330	0.97	339	1.00
		60		103	115	1.12	113	1.10	105	1.02
		90		47	57	1.21	56	1.20	48	1.02
		30	10	555	523	0.94	499	0.90	564	1.02
		60		198	207	1.04	205	1.03	208	1.05
		90		95	107	1.13	108	1.14	101	1.07
45°		18	5	460	453	0.99	445	0.97	437	0.95
		30		257	256	1.00	257	1.00	249	0.97
		60		79	90	1.14	90	1.14	79	1.00
		90		36	46	1.27	45	1.26	37	1.03
		18	10	514	537	1.05	495	0.96	505	0.98
		30		355	354	1.00	337	0.95	353	0.99
		60		136	147	1.08	144	1.06	140	1.03
		90		68	78	1.15	78	1.15	70	1.03
-90°		30	5	326	340	1.04	330	1.01	327	1.00
		60		98	115	1.18	113	1.16	100	1.02
		90		45	57	1.27	56	1.25	47	1.03
		30	10	468	523	1.12	499	1.07	496	1.06
		60		168	207	1.23	205	1.22	179	1.06
		90		83	107	1.29	108	1.30	89	1.07
					Mean	*1.11*	*Mean*	*1.09*	*Mean*	1.02
					Sd.dev.	*0.105*	*Sd.dev.*	*0.117*	*Sd.dev.*	0.033

Next, tables 3 and 4 show results dealing with fixed/warping-free columns and beams, again for several cross-section geometries and $E=200\ GPa$, $\nu=0.3$. As there are no distortional buckling formulae available for fixed/warping-free members, the GBT-based estimates are only compared with exact GBT values. However, for validation purposes, all the column dimensions considered here correspond to results reported by Lau (1988) and obtained by means of the spline finite strip method[1]. It is still worth mentioning that the estimates displayed in tables 3 and 4 correspond to the minimum value provided by the GBT-based formulae concerning $n=1, 2, 3, \ldots$. The buckling mode half-wave numbers (n) leading to such minimum values are indicated in the tables.

The observation of the results displayed in tables 3 and 4 leads to the following conclusions:

(i) The GBT-based formulae continue to provide quite accurate estimates, both for columns and beams. The average and standard deviation values of $\sigma_{dist}/\sigma_{d.ex}$ read now (i_1) *1.00* and *0.054*, for columns, and (i_2) *1.05* and *0.033*, for beams.

(ii) All the three "worst" predictions overestimate the exact values by 10-11% and correspond to the occurrence of a single-wave buckling mode ($n=1$). The results of the GBT exact analyses showed that this relative lack of accuracy stems from web or compressed flange

[1] The GBT exact and approximate results displayed were obtained after "replacing" the web-flange and flange-stiffener corners by larger width values (corner radius values were added to the width values). Although Lau assumed "fictitious" 45°-inclined corner finite strips, the exact GBT and spline finite strip results virtually coincide.

flexural deformation (local-plate mode) effects, which are not fully accounted for by the formulae. Such effects are less relevant for two or more half-wave bucking modes ($n \geq 2$).

(iii) Excluding the three cases discussed in the previous item, the estimates provided by the GBT-based formulae exhibit errors never exceeding 8%. Moreover, such estimates are mostly (iii$_1$) conservative, for the columns, and (iii$_2$) unconservative, for the beams.

Table 3. Critical distortional bifurcation stresses for fixed/warping-free columns

θ	b_w	b_f	b_s	t	L	$\sigma_{d.ex}$	n	σ_{dist}	$\sigma_{dist}/\sigma_{d.ex}$
90°	88.6	67.4	13.0	1.670	1640	300.3	3	315.5	1.05
					1900	294.6	4	308.3	1.05
	87.7	67.1	12.6	1.996	700	489.2	1	540.1	1.10
					1100	432.9	2	412.6	0.95
					1370	414	3	386.4	0.93
					1900	389.4	4	363.5	0.93
	85.6	63.0		2.394	800	629.1	2	599.4	0.95
					1100	561.9	3	547.5	0.97
					1500	523.1	4	506.4	0.97
-90°	89.2	77.2	12.9	1.666	1300	270.3	2	283.6	1.05
					1500	264.2	3	279.4	1.06
			12.6	1.976	800	425.8	1	442.5	1.04
					1300	350.4	2	336.8	0.96
	82.6	85.7	12.8	2.381	800	464.4	2	471.3	1.01
								Mean	*1.00*
								Sd.dev.	*0.054*

Table 4. Critical (flange) distortional bifurcation stresses for fixed/warping-free beams (t=*1 mm*)

θ	b_w	b_f	b_s	L	$\sigma_{d.ex}$	n	σ_{dist}	$\sigma_{dist}/\sigma_{d.ex}$
90°	90	30	10	700	690	2	703	1.02
				1000	624	3	647	1.04
		60		1100	249	2	261	1.05
				1500	224	3	239	1.07
		90		900	152	1	167	1.10
				1500	119	2	127	1.06
				2000	108	3	116	1.08
45°		30		600	436	2	434	1.00
				800	398	3	401	1.01
		60		1000	166	2	174	1.05
				1300	153	3	159	1.04
		90		800	103	1	115	1.11
				1300	84	2	87	1.04
				1700	76	3	80	1.05
							Mean	*1.05*
							Sd.dev.	*0.033*

6. CONCLUSION

After a brief overview of the Generalized Beam Theory (GBT) fundamentals and linear stability analysis procedure, the paper described and discussed the various steps involved in deriving fully analytical distortional buckling (approximate) formulae, in the context of cold-formed steel lipped channel members. An important feature of the proposed GBT-based formulae resides in the fact that they automatically incorporate folded-plate theory concepts and, therefore, directly (although partially) account for flexural deformation (local-plate mode) effects.

Fully analytical GBT-based formulae were developed, which provide distortional critical lengths and bifurcation stress resultant estimates for channel columns and beams with arbitrarily inclined single-lip edge stiffeners and pinned/free-to-warp or fixed/warping-free end sections (simpler formulae for orthogonal-lip channels and hat-sections were also presented in annex). It is worth mentioning that using the above formulae involves the sequential calculation of quantities, by means of analytical expressions which can be readily programmed even in a hand calculator.

The application of the proposed formulae was illustrated by means of a detailed analysis of a set of four identical columns/beams with pinned/free-to-warp or fixed/warping-free end sections. In addition, the accuracy/validity of the GBT-based estimates were assessed through comparisons with (i) exact GBT numerical results and, for pinned/free-to-warp members only, also with (ii) values yielded by formulae developed by Lau & Hancock, Hancock and Schafer. Columns and beams with several cross-section shapes (orthogonal and sloping-lip channels and hat-sections) and dimensions were considered and the results obtained led to the following main conclusions:

(i) The GBT-based formulae consistently yielded quite accurate estimates for pinned/free-to-warp and fixed/warping-free columns and beams. In fact, the four sets of $\sigma_{dist}/\sigma_{d.ex}$ values exhibited averages and standard deviations varying from *1.0* to *1.05* and *0.033* to *0.054*, respectively. Moreover, a large number of estimates fell inside the 5% error range.

(ii) For pinned/free-to-warp columns and beams, the GBT-based estimates always compared quite favourably with the values yielded by the formulae developed by Lau & Hancock, Hancock and Schafer.

(iii) For beams, either pinned/free-to-warp or fixed/warping-free, the GBT-based estimates were found to have a tendency to slightly overestimate the exact bifurcation stress values, as indicated by the $\sigma_{dist}/\sigma_{d.ex}$ average values of *1.02* and *1.05*. However, such values also exhibited a rather low scatter – the standard deviation value was *0.033* in both cases.

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8. ANNEX: Formulae for Orthogonal-Lip Channel and Hat-Section Members

In the case of orthogonal-lip channel ($\theta=+90°$) and hat-section ($\theta=-90°$) members, some of the expressions presented in section 4.2 can be simplified. In fact, noticing that (i) $\beta_1 \to \infty$ and that (ii) the top (bottom) sign corresponds to orthogonal-lip channels (hat-sections), one has:

$$\beta_2 = \pm\alpha_2 \qquad I = \frac{b_w^3 t}{12}[1+6\alpha_1+2\alpha_2(3\mp 6\alpha_2+4\alpha_2^2)]$$

$$u_{1.SD} = \frac{\alpha_1\alpha_2}{\gamma_{0.SD}} \qquad u_{2.SD} = -\frac{\alpha_2(2\alpha_1+3)}{\gamma_{0.SD}} \qquad \gamma_{0.SD} = \alpha_1(\alpha_1+2)+\alpha_2(2\alpha_1+3)$$

$$u_{1.AD} = \frac{3\alpha_1\alpha_2 \pm 4\alpha_2^2(5\alpha_1+3\alpha_2)}{\gamma_{0.AD}} \qquad u_{2.AD} = -\frac{3\alpha_2(2\alpha_1+1)\pm 4\alpha_2^2(4\alpha_1+1)}{\gamma_{0.AD}}$$

$$\gamma_{0.AD} = \alpha_1(3\alpha_1+2)+3\alpha_2(2\alpha_1+1)\pm 2\alpha_2^2(4\alpha_1+1)$$

$$m = \mp\frac{6K(u_2-1+\gamma_2)}{\alpha_1\alpha_2 b_w^3(2\alpha_1+\gamma_1)} \qquad \varphi_0 = 2\gamma_4\frac{u_2-u_1}{\alpha_1 b_w^2} \qquad \varphi_1 = \mp\frac{u_2-1-\gamma_3}{\alpha_1\alpha_2 b_w^2} \qquad \varphi_2 = m\frac{\alpha_1 b_w}{6K}+\varphi_1$$

$$w_0 = \gamma_5\frac{u_1-u_2}{\alpha_1 b_w} \qquad w_1 = \pm\frac{u_2-1+\gamma_3}{2\alpha_2 b_w} \qquad w_2 = -\frac{\alpha_2 b_w}{2}\varphi_2 \pm \frac{u_2-u_1}{\alpha_1 b_w}$$