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Some Applications of Generalized Beam Theory

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SOME APPLICATIONS OF GENERALIZED BEAM THEORY

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Introduction

Generalised Beam Theory (GBT) has been developed by Professor R Schardt and his colleagues at the University of Darmstadt in Germany. The definitive reference at the present time is a recent German text[1] which describes the first-order theory. For the analysis of cold-formed sections, second-order theory may be required and this is less well documented. This paper will attempt to describe the principles involved and illustrate them by means of some practical examples.

GBT unifies the conventional theories for the analysis of prismatic thin-walled structural members within a consistent notation. It then extends them into new territory. Conventional beam theory identifies four fundamental modes of deformation, namely extension, bending about two principal axes and torsion. These may be referred to as the "rigid-body" modes because they do not involve any distortion of the cross-section. Higher-order modes also exist but these involve cross-section distortion together with transverse bending.

In first-order theory, all modes are orthogonal. This means that they are uncoupled and can be considered separately before their effects are combined. In second-order theory, the modes may become coupled but their orthogonal nature ensures that the coupling is minimised so that important results can often be obtained by a trivial calculation involving a single mode.

GBT is a big subject with many ramifications and a full treatment is not possible within the confines of a single paper. No attempt will, therefore, be made to derive the basic equations and attention will be confined to explanation and application.

Modes of deformation and warping functions

A unifying feature of GBT is the "warping function" whereby each deformation mode 'k' is associated with a distribution of axial strain ${}^{k}\tilde{u}$. Thus, the first mode has a uniform distribution of axial strain over the cross-section and ${}^{1}\tilde{u} = -1$ for all the points 's' of the cross-section. The second and third modes are bending about the two principal axes and the associated warping functions are linear distributions of strain about two orthogonal axes through the centroid of the section. The fourth mode is torsion about the shear centre and here the term "warping" has its conventional meaning and the warping function is the sectorial coordinate which reflects the distribution of axial strain due to a bi-moment. For higher-order modes, the physical meaning of the warping function is less clearly visualised.

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From these warping functions, a number of other properties can be derived so that each mode k has associated with it:

- a warping function as already described
- a corresponding pattern of cross-sectional displacements
- (for modes 5 and above) a distribution of transverse bending stresses
- three section properties ${}^kC$, ${}^kD$ and ${}^kB$.

For the rigid body modes 1 to 4, the section properties are familiar, thus

${}^1C$	=	cross-sectional area;	${}^1D = {}^1B = 0$
${}^2C$	=	second moment of area about the first principal axis;	${}^2D = {}^2B = 0$
${}^3C$	=	second moment of area about the second principal axis;	${}^3D = {}^3B = 0$
${}^4C$	=	warping constant;	
${}^4D$	=	St Venant torsional constant;	${}^4B = 0$

For the higher order modes which involve cross-sectional distortion, all three section properties are, in general, non-zero.

Notation

The notation used in this paper for GBT follows that developed by Schardt(1). In general, terms will be defined as they are introduced but two points are worthy of particular note:

- a forward superscript is used to denote the mode number
- ~ over a symbol denotes a unit value of a quantity, e.g. a warping function or a related quantity derived from it.

Number of deformation modes

It is convenient to illustrate the principles of GBT with an example and a suitable cross-section for this purpose is shown in Fig. 1. This section has been used by Lau and Hancock(2) to illustrate the use of the finite strip method in investigating the elastic and inelastic buckling of columns. Later it will be convenient to compare some results obtained for this section using GBT with those obtained by Lau and Hancock.

The section shown in Fig. 1 has six nodes which, according to GBT, can "warp" independently. The warping functions are linear between the nodes and it follows that the section has six independent warping functions together with their associated deformation modes and section properties. Four of these are the rigid body modes shown in Fig. 2(a) and two are the cross-section distortion modes shown in Fig. 2(b). The full set of GBT properties for this example are tabulated in Fig. 3. This table may be interpreted as follows:

${}^k\tilde{u}$ = warping function defined at each node of the cross-section and assumed to be linear between the nodes. The remaining quantities are associated with a unit value of this warping function.

${}^kF\text{-}L$ = in-plane displacement of a face at its mid-point.

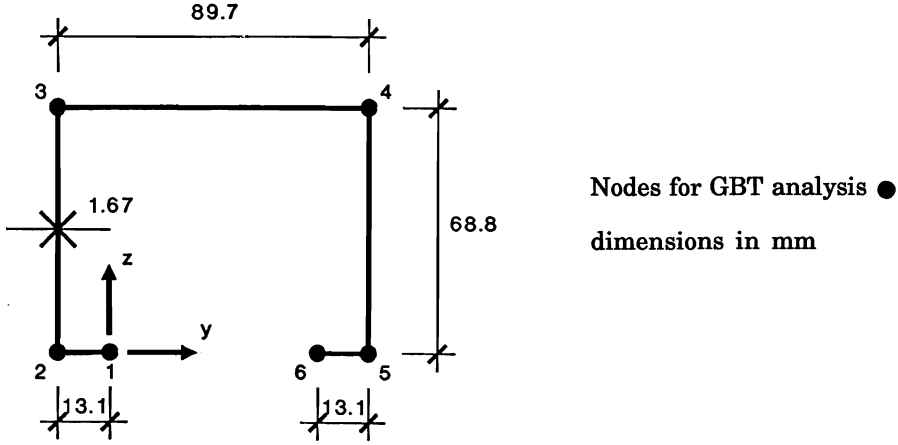


Fig. 1. Example for GBT analysis

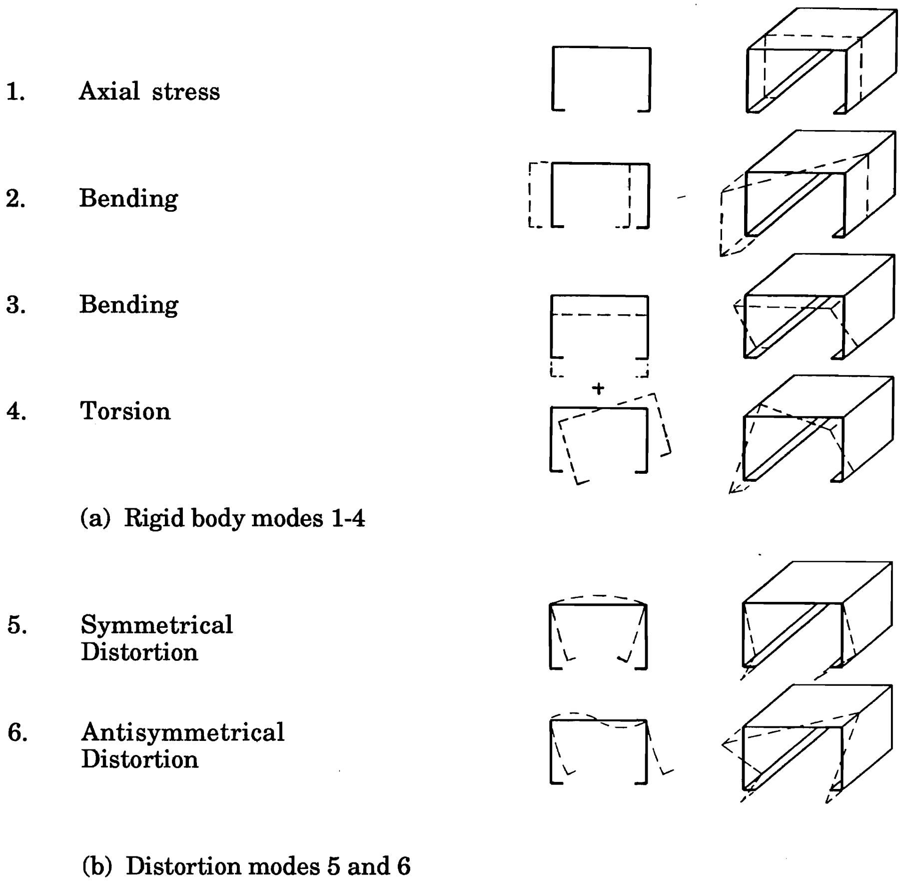


Fig. 2. Displaced shapes and warping functions

FIG 3. CROSS-SECTION VALUES FOR EXAMPLE

MODE k = 1

NODE	$\tilde{u}$	F-L	F-Q	F-θ	$\tilde{v}$	$\tilde{w}$	$\tilde{m}$	S/W'
1	-1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	-1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3	-1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
4	-1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
5	-1.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
6	-1.0000				0.0000	0.0000	0.0000	

C = 4.2335 D = 0.0000 B = 0.0000

MODE k = 2

NODE	$\tilde{u}$	F-L	F-Q	F-θ	$\tilde{v}$	$\tilde{w}$	$\tilde{m}$	S/W'
1	3.1750	-1.0000	0.0000	0.0000	-1.0000	0.0000	0.0000	0.0082
2	4.4850	0.0000	1.0000	0.0000	-1.0000	0.0000	0.0000	0.3743
3	4.4850	1.0000	0.0000	0.0000	-1.0000	0.0000	0.0000	1.0164
4	-4.4850	0.0000	-1.0000	0.0000	-1.0000	0.0000	0.0000	0.3743
5	-4.4850	-1.0000	0.0000	0.0000	-1.0000	0.0000	0.0000	0.0082
6	-3.1750				-1.0000	0.0000	0.0000	

C = 62.7540 D = 0.0000 B = 0.0000

MODE k = 3

NODE	$\tilde{u}$	F-L	F-Q	F-θ	$\tilde{v}$	$\tilde{w}$	$\tilde{m}$	S/W'
1	-4.3017	0.0000	-1.0000	0.0000	0.0000	1.0000	0.0000	-0.0214
2	-4.3017	-1.0000	0.0000	0.0000	0.0000	1.0000	0.0000	-0.4999
3	2.5783	0.0000	1.0000	0.0000	0.0000	1.0000	0.0000	0.0000
4	2.5783	1.0000	0.0000	0.0000	0.0000	1.0000	0.0000	0.4999
5	-4.3017	0.0000	-1.0000	0.0000	0.0000	1.0000	0.0000	0.0214
6	-4.3017				0.0000	1.0000	0.0000	

C = 28.8300 D = 0.0000 B = 0.0000

MODE k = 4

NODE	$\tilde{u}$	F-L	F-Q	F-θ	$\tilde{v}$	$\tilde{w}$	$\tilde{m}$	S/W'
1	28.8492	10.3510	3.8300	1.0000	10.3510	-3.1750	0.0000	0.0067
2	15.2894	4.4850	-6.9110	1.0000	10.3510	-4.4850	0.0000	0.1012
3	-15.5674	-3.4710	0.0000	1.0000	3.4710	-4.4850	0.0000	0.0134
4	15.5674	4.4850	6.9110	1.0000	3.4710	4.4850	0.0000	0.1012
5	-15.2894	10.3510	-3.8300	1.0000	10.3510	4.4850	0.0000	0.0067
6	-28.8492				10.3510	3.1750	0.0000	

C = 523.538 D = 0.039356 B = 0.0000

MODE k = 5

NODE	$\tilde{u}$	F-L	F-Q	F-θ	$\tilde{v}$	$\tilde{w}$	$\tilde{m}$	S/W'
1	1.0000	0.9448	-0.1455	0.1606	0.9448	0.2507	0.0000	0.5237
2	-0.2377	-0.0404	-0.4724	0.1373	0.9448	0.0404	0.0000	0.0026
3	0.0401	0.0000	0.0404	0.0000	0.0000	0.0404	0.1728	0.0000
4	0.0401	0.0404	-0.4724	-0.1373	0.0000	0.0404	0.1728	-0.0026
5	-0.2377	-0.9448	-0.1455	-0.1606	-0.9448	0.0404	0.0000	-0.5237
6	1.0000				-0.9448	0.2507	0.0000	

C = 0.16074 D = 0.00055128 B = 0.047461

MODE k = 6

NODE	$\tilde{u}$	F-L	F-Q	F-θ	$\tilde{v}$	$\tilde{w}$	$\tilde{m}$	S/W'
1	1.0000	1.0066	-0.2008	0.2017	1.0066	0.3329	0.0000	0.4320
2	-0.3186	-0.0687	-0.4862	0.1513	1.0066	0.0687	0.0000	-0.6687
3	0.1537	0.0343	0.0000	-0.0153	-0.0343	0.0687	0.3751	0.8765
4	-0.1537	-0.0687	0.4862	0.1513	-0.0343	-0.0687	-0.3751	-0.6687
5	0.3186	1.0066	0.2008	0.2017	1.0066	-0.0687	0.0000	0.4320
6	-1.0000				1.0066	-0.3329	0.0000	

C = 0.18592 D = 0.00068007 B = 0.12499

$^kF\text{-}Q$	=	displacement normal to the face at its mid-point
$^kF\text{-}\theta$	=	rotation of chord line of face
$^k\tilde{v}$, $^k\tilde{w}$	=	nodal displacements in horizontal and vertical directions
$^k\tilde{m}$	=	transverse bending moment at each node (distortional modes only)
^kS	=	shear force in each face (as a function of $^kW'$)

$$^kC = \int_A {}^k\tilde{u}^2\, dA + \frac{1}{K}\int_s {}^kF\text{-}Q^2\, ds \qquad \text{(warping resistance)} \qquad (1)$$

$$^kD = \frac{1}{3}\int_s {}^k\left(\frac{dF\text{-}Q}{ds}\right)^2 t^3 ds + \frac{2\vartheta K}{G}\int_s {}^kF\text{-}Q\ {}^k\left(\frac{d^2F\text{-}Q}{ds^2}\right) \mathrm{ds}$$

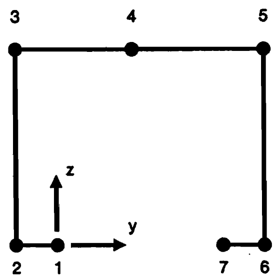
(torsional resistance) (2)

$$^kB = K\int_s {}^k\left(\frac{d^2F\text{-}Q}{ds^2}\right)^2 ds = \frac{1}{K}\int_s {}^k\tilde{m}^2\, ds \qquad \text{(transverse bending resistance)} \qquad (3)$$

with $K = \dfrac{E\pi^2}{12(1-\vartheta^2)}$ and ϑ = Poisson's ratio

It should be noted that all of the quantities tabulated for modes 1 to 4 are obtainable from the standard procedures of structural mechanics which are given in basic texts on the subject.

For second-order problems, it may be necessary to introduce intermediate nodes in order to allow local buckling of plate elements. Thus, if local buckling of the long side is to be considered, an additional node is inserted, as shown in Fig. 4(a) and an additional warping function is generated, as shown in Fig. 4(b).



(a) Addition of a mid-side node

MODE k = 7

NODE	u	F-L	F-Q	F-O	v	w	m	S/W'
1	1.0000	1.0026	-14.1458	21.5094	1.0026	28.2344	0.0000	0.0011
2	-0.3134	-0.0571	-0.5011	0.1458	1.0026	0.0571	0.0000	-0.0029
3	0.0797	0.0003	-94.8328	-42.3144	-0.0003	0.0571	158.926	-0.0019
4	0.0782	-0.0003	-94.8328	42.3144	0.0000	-189.723	-320.900	0.0019
5	0.0797	0.0571	-0.5011	-0.1458	0.0003	0.0571	158.926	0.0029
6	-0.3134	-1.0026	-14.1458	-21.5094	-1.0026	0.0571	0.0000	-0.0011
7	1.0000				-1.0026	28.2344	0.0000	

C = 70.986 D = 53.378 B = 40653.4

(b) corresponding additional warping function

Fig. 4. GBT model allowing local buckling of the longer plate element

The requirement that the warping functions should be orthogonal ensures that modes 1 to 6 remain unchanged by the insertion of an additional node or nodes.

Calculation of the warping functions and associated properties

For relatively simple problems, it is possible to calculate all of the properties in Fig. 3 by hand and explicit expressions have been given for some common shapes[1]. However, in general, it is simpler and more convenient to use standard computer software.

Fundamental equation for first-order theory

The basic equation of first-order GBT is

$$E\,{}^kC\,{}^kV'''' - G\,{}^kD\,{}^kV'' + {}^kB\,{}^kV = {}^kq \qquad (4)$$

In this equation, in addition to the section properties ${}^kC$, ${}^kD$ and ${}^kB$,

E = Young's Modulus

G = Shear Modulus

${}^kV$ = generalised deformation in mode k (e.g. horizontal and vertical displacement for modes 2 and 3, rotation for mode 4 etc)

${}^kq$ = distributed load applicable to mode k

and primes indicate differentiation with respect to z which lies along the length of the member.

It is easy to see that the differential equations for bending about the two principal axes and torsion are all special cases of equation (4).

Equivalence of GBT and conventional notations

The equivalence between the conventional and GBT notations extends across the whole spectrum of beam theory and progresses naturally from the familiar rigid-body modes into the higher-order modes using a unified notation. As a further example, the GBT equations for stresses and stress resultants are

$$^{k}W = -E\,^{k}C\,^{k}V'' = \int {}^{k}\sigma\,^{k}\tilde{u}\,dA \qquad (5)$$

where, in addition to quantities already defined:

^{k}W = stress resultant for mode k (e.g. bending moment for modes 2 and 3, bi-moment for mode 4, etc)

$^{k}\sigma$ = distribution of longitudinal stress for mode k

Equation (5) can be seen to embrace the familiar equations for stresses due to bending moment and bi-moment as special cases.

Solution of first-order problems using GBT

The use of GBT to obtain the stresses and deformations of a member under specified loading and support conditions requires three distinct steps. In the first step, only the cross-section is considered in order to obtain, for each mode k, the warping functions $^{k}\tilde{u}$ and the associated section properties, ^{k}C, ^{k}D and ^{k}B together with the other relevant information as shown in Fig. 3. In the second step, this basic information is used to obtain solutions of equation (4) for each mode taking account of the relevant loading and boundary conditions. The third step then involves combining the results of step 2 to calculate the required stresses and deflections.

Bearing in mind that the first step has already been discussed in sufficient detail for the purposes of this paper, attention now focuses on the second step, after which the final step is relatively trivial.

Equation (1) may be solved in a number of different ways depending on the loading and boundary conditions. It is, of course, identical in form to the differential equation for the displacement of an axially-loaded beam on an elastic foundation. In order to apply beam on elastic foundation solutions to GBT problems, the following substitutions must be made:

second moment of area	I	$\equiv$	^{k}C
axial load (tension positive)	N	$\equiv$	$G\,^{k}D$
foundation constant	k	$\equiv$	^{k}B
uniformly distributed load	q	$\equiv$	^{k}q

and, when the solution has been completed, the stress resultant

$M \equiv {}^kW$

It follows that the solutions for many relatively simple loading and support conditions are well documented[3]. Indeed, reference 3 is a whole book devoted to the topic and explicit solutions, such as those given therein, will provide an appropriate method for a wide range of practical problems.

Schardt and his associates[1] at Darmstadt have exclusively used the finite difference method for their solutions of the general case and, certainly, this approach has advantages for second-order analysis. However, for first-order analysis, the finite element method is demonstrably more efficient. The reason for this is that equation (4) falls into the class of equations for which the finite element solution is exact[4]. This means that, for many first-order problems, the computational requirements become almost trivial

Example of first-order Generalised Beam Theory

As a simple example, consider the cross-section of Fig. 1 simply supported over a span of 1.2 metres and subject to a single point load applied over one corner at the third point of the span as shown in Fig. 5.

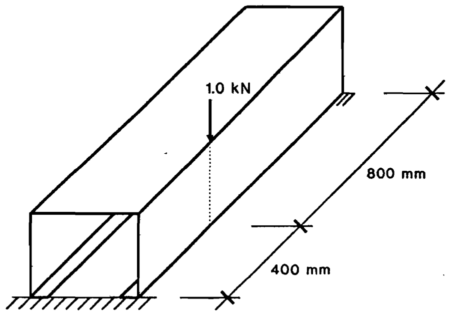


Fig. 5. Example of first-order GBT

It is first necessary to apportion the applied load between the various modes and this can best be done by considering the virtual work of the horizontal and vertical components of the nodal loads acting on the modal displacements ${}^k\tilde{V}$ and ${}^k\tilde{W}$, i.e.

$$ {}^kQ = \sum_{r=1}^{n+1} \left(Q_{y,r}\, {}^k\tilde{v}_r + Q_{z,r}\, {}^k w_r \right) \tag{6} $$

so that, noting that the load is applied at node 4 in the negative z direction,

${}^1Q$	=	${}^2Q$	=	0	
${}^3Q$	=	1.0			(bending about a horizontal axis)
${}^4Q$	=	-4.485			(torsion about the shear centre)
${}^5Q$	=	-0.0404			(symmetrical mode of distortion)
${}^6Q$	=	0.0687			(antisymmetrical mode of distortion)

Equation (4) has, therefore, to be solved four times although for mode 3 we have a simple case of bending for which the solution is trivial and for mode 4 we have a case of warping torsion for which alternative solutions exist[5]. Taking mode 6 as an example of the general case, it is necessary to solve the analogous beam on elastic foundation problem shown in Fig. 6(a) with the values.

$$E = 20000\ \text{kN/cm}^2$$

$$G = 7692.3\ \text{kN/cm}^2$$

$$I \equiv {}^6C = 0.1859\ \text{cm}^4$$

$$N \equiv G\,{}^6D = 7692.3 \times 0.0006801 = 5.2315\ \text{kN}$$

$$k \equiv {}^6B = 0.1250\ \text{kN/cm}^2$$

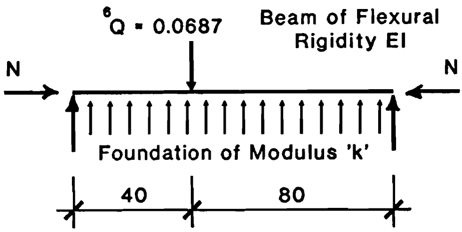
The finite element model shown in Fig. 6(b) has only four unknowns so that the calculation is almost trivial. The values of interest at the loaded section are:-

$$M \equiv {}^6W = -0.3003$$

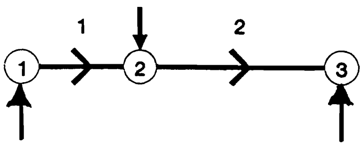
bending moment ≡ stress resultant

$$\delta \equiv {}^6V = -0.01418$$

deflection ≡ generalised deformation



(a) equivalent beam on elastic foundation



(b) finite element model (exact)

Fig. 6. Analytical model for mode 6

The actual stresses and deflections for this mode then follow, e.g.

$${}^6\sigma_1 = \frac{{}^6W\,{}^6\tilde{u}_1}{{}^6C} = \frac{0.3003 \times 1.0}{0.1859} = 1.615\ \text{kN/cm}^2$$

$${}^6v_1 = {}^6V\,{}^6\tilde{v}_1 = 0.01418 \times 1.0066 = 0.0143\ \text{cm}$$

$${}^6w_1 = {}^6V\,{}^6\tilde{w}_1 = 0.01418 \times .3329 = 0.0047\ \text{cm}$$

The full pattern results, when all modes are superposed, is given in Fig. 7.

It follows that Generalised Beam Theory provides a relatively simple and elegant solution for problems in which distortion of the cross-section cannot be

Mode	1	2	3	4	5	6
W	0	0	-26.67	-116.1	-0.2172	0.3003
V	0	0	-0.04933	-0.01172	-0.01762	0.01418

(a) Stress resultants (kN) and generalised deformation (cm)

Node	1	2	3	4	5	6
${}^3\sigma$	3.979	3.979	-2.385	-2.385	3.979	3.979
${}^4\sigma$	-6.395	-3.389	3.451	-3.451	3.389	6.395
${}^5\sigma$	-1.351	0.321	-0.054	-0.054	0.321	-1.351
${}^6\sigma$	1.615	-0.515	0.248	-0.248	0.515	-1.615
σ	-2.152	0.396	1.260	-6.138	8.204	7.408

(a) longitudinal stresses (kN/cm^2)

Node	1	2	3	4	5	6
${}^3v$	0	0	0	0	0	0
${}^4v$	-0.1213	-0.1213	-0.0407	-0.0407	-0.1213	-0.1213
${}^5v$	-0.0167	-0.0167	0	0	0.0167	0.0167
${}^6v$	0.0143	0.0143	-0.0005	-0.0005	0.0143	0.0143
v	-0.1237	-0.1237	-0.0412	-0.0412	-0.0903	-0.0903

(b) horizontal displacements (cm)

Node	1	2	3	4	5	6
${}^3w$	-0.0493	-0.0493	-0.0493	-0.0493	-0.0493	-0.0493
${}^4w$	0.0372	0.0525	0.0525	-0.0525	-0.0525	-0.0372
${}^5w$	-0.0044	-0.0007	-0.0007	-0.0007	-0.0007	-0.0044
${}^6w$	0.0047	0.0010	0.0010	-0.0010	-0.0010	-0.0047
w	-0.0118	0.0035	0.0035	-0.1035	-0.1035	-0.0956

(c) vertical displacements (cm)

Fig 7. Stress resultants, stresses and displacements below load neglected. However, it becomes even more powerful for buckling problems, as will now be shown.

Second-order Generalised Beam Theory

When second-order effects are included, a further group of terms is added to the basic equation of GBT. The usual form of the equation is then:

$$E^kC^kV'''' - G^kD^kV'' + {}^kB^kV + \sum_{i,j} {}^{ijk}\kappa\left({}^iW\,{}^jV'\right)' = {}^kq \tag{7}$$

In this equation,

- ${}^iW$ is the warping stress resultant in the i - th mode

- $${}^{ijk}\kappa = \frac{1}{{}^iC}\int_A {}^i\tilde{u}\left({}^jF\text{-}L\,{}^kF\text{-}L + {}^jF\text{-}Q\,{}^kF\text{-}Q\right) dA \tag{8}$$

is a three dimensional array of second-order terms which includes coupling terms so that the differential equations become linked and the individual modes are no longer independent.

It may be noted that, if in-plane shear strains are included in the analysis, the second-order terms are augmented to

$$\sum_{i,j} {}^{ijk}\kappa_\sigma\left({}^iW\,{}^jV'\right)' + {}^{ijk}\kappa_\tau\left({}^iW''\,{}^jV + 2\,{}^iW'\,{}^jV'\right) \tag{9}$$

where ${}^{ijk}\kappa_\tau$ is a further three dimensional array of coupling terms.

However, in the vast majority of practical cases, the additional shear terms have little influence on the results and, for the purpose of this paper, they will be neglected.

It may also be noted that for many bifurcation problems, including a number of the examples given later, a load is applied which is constant over the length of the member. This is the case when a column is subject to axial compressive load or when a beam buckles under uniform moment. Derivatives of ${}^iW$ are then zero, and the second-order terms simplify to

$${}^iW\sum_j {}^{ijk}\kappa\,{}^jV''$$

No attempt will be made here to explain the detailed derivation of ${}^{ijk}\kappa$, suffice to say that it generally requires the use of a computer and that the calculation has been programmed, as with all GBT calculations.

Simple bifurcation problems

The easiest way to illustrate the use of the augmented GBT equation (7) is by means of a simple example:

Consider the channel section shown in Figs 1 and 4 acting as a column subject to a uniform axial compression. In GBT terms, the applied load is the warping stress resultant ^1W. It is assumed that the column behaves as simply supported at its ends with respect to each buckling mode. As there is no load causing deformation prior to buckling, the right hand side term ^kq is zero and we have a bifurcation problem.

The cross-section has the seven nodes indicated in Fig. 4 so that generalized beam theory provides seven orthogonal modes of deformation. These modes are, of course, four rigid-body modes, two modes involving distortion of the cross-section and one local buckling mode. The relevant section properties are shown in Figs. 3 and 4.

Also necessary for the analysis of this section acting as an axially-loaded column is the array of second-order terms $^{1jk}\kappa$. This is shown in Fig. 8. The off-diagonal numerical values reflect the degree of coupling between the modes. For instance, it is immediately obvious that an analysis including only modes 2 and 3 would not be profitable because they would be uncoupled.

k	j						
	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0
2	0	-1	0	-6.049	0	-0.3087	0
3	0	0	-1	0	0.0994	0	-37.695
4	0	-6.049	0	-58.226	0	-2.557	0
5	0	0	0.0994	0	-0.2515	0	15.503
6	0	-0.3087	0	-2.557	0	-0.2543	0
7	0	0	-37.695	0	15.503	0	-6548.5

Fig.8 Values of ^{1jk}K

Consider the column buckling in a single mode k and assume that the buckled shape in this mode is given by

$$^kV = {}^ka\sin\frac{\pi x}{L}$$

Then we have a single governing equation and the only second-order term which appears in this equation is ${}^{1kk}\kappa$ so that, inserting ${}^{k}V$ and its derivatives in equation (2)

$$\left(E\,{}^{k}C\left(\frac{\pi}{L}\right)^{4} + G\,{}^{k}D\left(\frac{\pi}{L}\right)^{2} + {}^{k}B \;-\; {}^{1kk}\kappa\,{}^{1}W\left(\frac{\pi}{L}\right)^{2}\right){}^{k}a\sin\frac{\pi x}{L} = 0 \qquad (10)$$

For non-trivial solutions, the expression in brackets must be equal to zero so that:

$${}^{1}W = \left(E\,{}^{k}C\left(\frac{\pi}{L}\right)^{2} + G\,{}^{k}D + {}^{k}B\left(\frac{L}{\pi}\right)^{2}\right)\frac{1}{{}^{1kk}\kappa} \qquad (11)$$

This equation is valid for buckling in any individual mode. The modes given by k = 2 and k = 3 are Euler buckling about the principal axes and, for these modes, the only non-zero generalized warping is ${}^{k}C$. Furthermore,

$${}^{122}\kappa = {}^{133}\kappa = -1$$

so that for k = 2 or k = 3

$${}^{1}W = -\frac{\pi^{2}E\,{}^{k}C}{L^{2}} \quad \text{(tension positive)}$$

which is the well-known formula for global buckling about a principal axis.

The general solution for ${}^{1}W$ in equation (6) gives the critical axial load for buckling in mode k for any length L of the column. The relationship between ${}^{1}W$ and L may have one of two alternative shapes as shown in Fig. 4. If the mode is one of global buckling, the longer the column, the lower the buckling load as shown in Fig. 4(a). Conversely, if the mode is of a more local nature, there is a critical buckling length, as shown in Fig. 4(b) and long columns will buckle in a periodic mode with this wavelength.

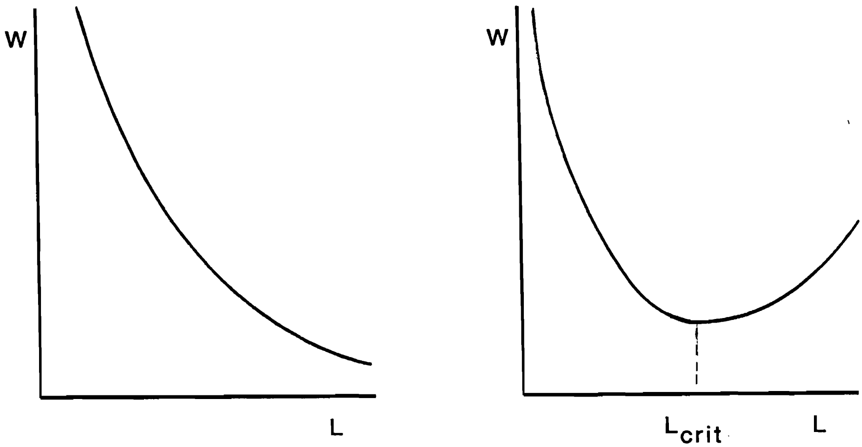
The calculation of the critical wavelength for local buckling follows directly from equation (6)

$$\frac{\partial\,{}^{1}W}{\partial(L/\pi)} = 0$$

i.e. $$-2E\,{}^{k}C\left(\frac{L}{\pi}\right)^{-3} + 2\,{}^{k}B\left(\frac{L}{\pi}\right) = 0$$

$$L_{crit} = \pi\sqrt[4]{\frac{E\,{}^{k}C}{{}^{k}B}} \qquad (12)$$

It follows that, in general, if ${}^{k}B$ exists, the buckling mode will be local in nature. If ${}^kB$ is zero, the buckling mode is global and there is no critical wavelength.



(a) Global buckling, ${}^kB = 0$ (b) Local buckling, ${}^kB \neq 0$

Fig. 9. Alternative buckling characteristics

Finally, for a local mode, the critical axial force is given by

$$ {}^1W^{\min} = \frac{1}{{}^{1kk}\kappa}\left(E^kC\sqrt{\frac{{}^kB}{E^kC}} + G^kD + {}^kB\sqrt{\frac{E^kC}{{}^kB}} \right) $$

i.e.

$$ {}^1W^{\min} = \frac{1}{{}^{1kk}\kappa}\left(2\sqrt{E^kC^kB} + G^kD \right) \qquad (13) $$

For the channel example shown in Fig. 1, the shapes of the buckling curves in the individual modes are shown in Fig. 10. For short lengths up to about 60 cm, the local and distortional modes are critical. For longer lengths, mode 4, the torsional mode, governs.

If we consider the interaction of more than one buckling mode, the principles remain the same but more terms appear and, in general, the modes become coupled.

Consider the effect of combining several modes in the analysis of buckling for various lengths L on the assumption that all modes have the same shape but different amplitudes:

$$ {}^kV = {}^ka\sin\frac{\pi x}{L} $$

Then, substituting into the basic second-order GBT equation gives a family of equations of the form:

$$\left(\left(E^kC\left(\frac{\pi}{L}\right)^4 + G^kD\left(\frac{\pi}{L}\right)^2 + {}^kB\right){}^ka + {}^1W\left(\frac{\pi}{L}\right)^2\sum_j {}^{1jk}\kappa^j a\right)\sin\frac{\pi x}{L} = 0 \tag{14}$$

Non-trivial solutions give rise to a set of homogeneous equations:

$$\left({}^kP + {}^1W\sum_j {}^{1jk}\kappa\right){}^ka = 0 \tag{15}$$

where $${}^kP = \left(E^kC\left(\frac{\pi}{L}\right)^2 + G^kD + {}^kB\left(\frac{L}{\pi}\right)^2\right) \tag{16}$$

is a constant for a given length L.

This is an eigenvalue problem which can be solved, for any length L, by any of the usual methods to give a vector of eigenvalues, the lowest of which is the buckling load. The associated eigenvector contains the amplitudes of the modes and gives a clear indication of which mode or modes are critical at the length being investigated.

Some results of analyses with different combinations of modes are given in Figs. 11 and 12. In Fig. 11, the symmetrical modes 3, 5 and 7 are shown separately and in combination. It can be seen that the interaction is minimal. Fig. 12 shows the antisymmetrical modes 2, 4 and 6, the individual modes being shown by fine lines and the combined modes by heavy lines. Here there is rather more interaction during the transition from short wavelength distortional buckling (mode 6) to long wavelength torsional buckling (mode 4). There is, of course, no interaction at all between the symmetrical modes shown on Fig. 11 and the antisymmetrical modes shown in Fig. 12. Figs. 11 and 12 also show the results obtained when all modes are combined. It can be seen that at the critical minima, it is sufficient for all practical purposes to consider the individual mode. Thus, for local buckling in mode 7,

Individual mode (equations 12 and 13)	${}^1\sigma^{min}$ = 33.1 kN/cm2;	L_{crit} = 7.8 cm
All modes	${}^1\sigma^{min}$ = 32.6 kN/cm2	

and for local buckling in mode 5,

Individual mode	${}^1\sigma^{min}$ = 27.2 kN/cm2;	L_{crit} = 50.7 cm
All modes	${}^1\sigma^{min}$ = 26.6 kN/cm2	

As the results for individual modes can be calculated from a simple explicit expression, this result has considerable practical significance.

It is also of interest to note that the 'all modes' curves in figs. 11 and 12 appear to be virtually identical to the curve obtained by Lau and Hancock(2) using a spline finite strip analysis. In particular, they quote the local and distortional buckling minima as:

Fig.10. Individual Buckling Modes

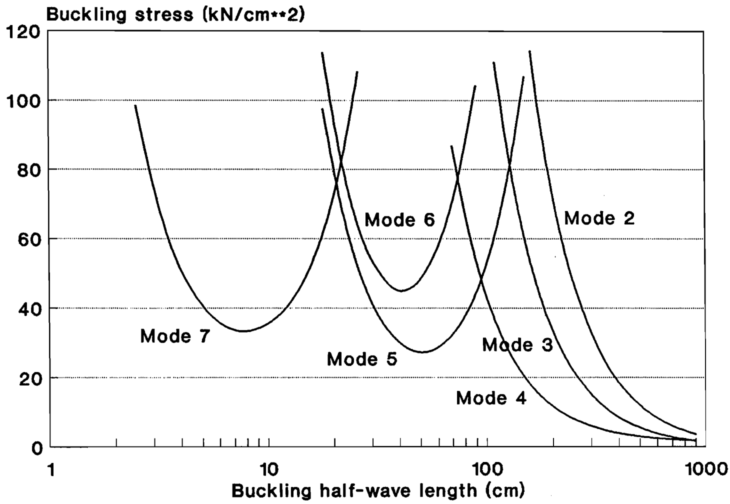


Fig.11. Symmetrical Buckling Modes

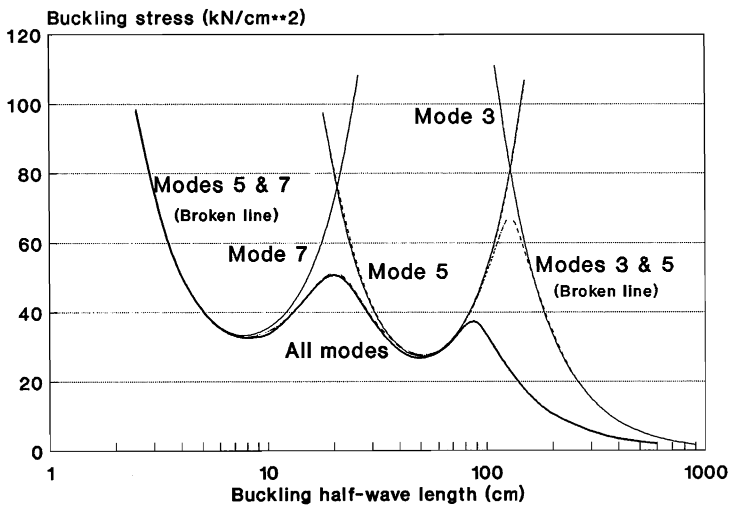


Fig.12. Antisymmetrical Buckling Modes

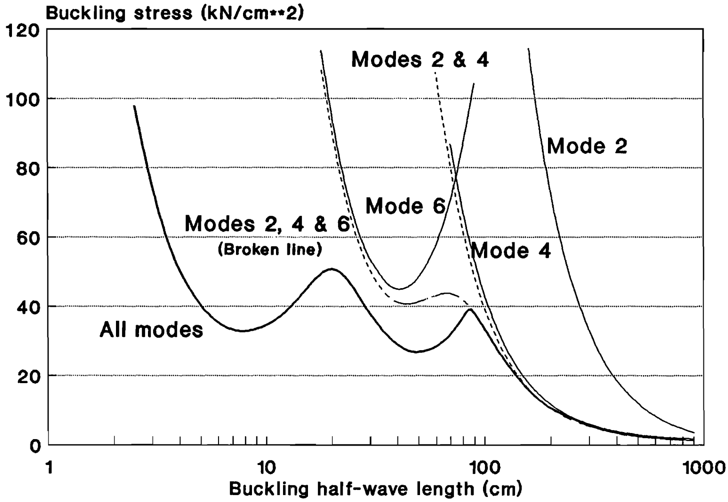
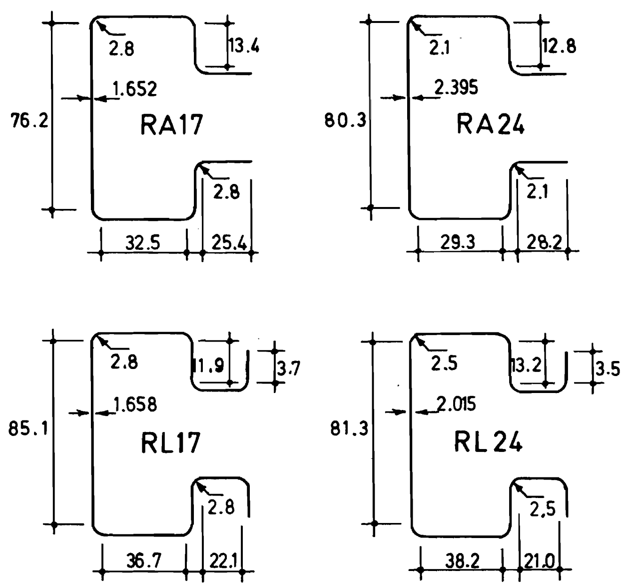


Fig.13 Sections tested by Lau & Hancock



Mode 7 $\quad {}^{1}\sigma^{min} = 315$ MPa $\quad$ at $L_{crit} \approx 80$ mm

Mode 5 $\quad {}^{1}\sigma^{min} = 270$ MPa $\quad$ at $L_{crit} \approx 500$ mm

A further interesting point which will be taken up later is that the distortional buckling mode has a lower critical stress than the local mode and with a half wavelength too long for this mode to be picked up by a standard stub-column test.

Analysis for elastic buckling failure due to pure bending would follow a similar course. The only difference is that the applied load is ${}^{3}W$ and the relevant array of second-order terms is ${}^{3jk}\kappa$, otherwise the calculations are identical.

It is evident from these examples that GBT provides a particularly convenient method of investigating the different buckling modes of cold-formed sections both individually and in combination.

General solution of second-order problems

The above treatment is only applicable to a range of bifurcation problems where the applied load is constant over the length of the member and where the deformed shape of each mode is a sine wave extending over the length of the member. These assumptions are, of course, rather restrictive and for a more general solution it is necessary to resort to numerical methods of analysis. The finite difference method proves to be particularly appropriate and the usual procedures can be readily extended to include the second-order terms. Here, also, the individual modes generally become coupled so that it is not sufficient to consider them independently. However, this coupling is not usually strong so that, provided appropriate steps are taken, it does not dominate the analysis and the benefits obtained by using orthogonal deformation modes are retained.

The simplest general approach to solving the coupled equations (7) is to move the as yet unknown terms in ${}^{i}W\,{}^{j}V$ and their derivatives to the right hand side of the equations where they are treated as load terms together with ${}^{k}q$. These terms are taken as zero in the first iterative solution and successively improved as the iterations proceed. Thus, in each iteration step, only a system of uncoupled linear differential equations of 4th order must be solved. After each step, the right hand sides are updated to include the internal forces and deformations calculated during the previous step and the cycle repeated. In general, the orthogonal nature of the deformation modes ensures that the process converges rapidly.

When using the finite difference method to solve bifurcation problems, a preferable alternative to the iterative procedure described above is to set up the complete coupled differential equations as a single eigenvalue problem. This allows buckling loads to be computed without the need for iteration.

Results of analysis and comparison with tests

Lau and Hancock[2] have described the results of tests and finite strip analyses of 12 different cold-formed sections with fixed ends subject to uniform compression. The shapes were those of typical pallet rack uprights and a selection of these, which are considered in this paper, are shown in Fig. 13. A total of 68 tests were reported and compared with both elastic and inelastic analysis.

The results obtained by Lau and Hancock for the four sections shown in Fig. 13 are compared with those given by GBT in Figs 14 to 17. The legend alongside the test result indicates the mode of failure. Thus 'L' is a local mode, 'D(1)' is a single wave distortional mode and FT is flexural torsional buckling. Six different GBT curves are given, namely:

1. Elastic buckling analysis with fixed end conditions including all modes. This generally gives results which are very similar to the elastic finite strip results. Some slight differences in the curves arise because Lau and Hancock only carried out analyses to correspond to the test points and drawing a smooth curve through these points does not necessarily reveal the undulations due to mode changes. This is a particularly general application of second-order GBT which provides a verification of the method over the full range of mode changes.

2. Elastic buckling analysis with fixed end conditions including only the rigid-body modes. This is, of course, flexural torsional buckling without any interaction with the local and distortional modes and reveals the approximate length at which the mode change from distortional to flexural-torsional may be expected.

3,4 Analysis of the individual local and distortional buckling modes with fixed end conditions.

5,6 Analysis of the local and distortional buckling modes based on a sinusoidal buckling half wave using the explicit expression of equation (11). The boundary conditions implicit in this analysis are simply supported in contrast to the fixed ends used in the tests and all other analyses.

The curves shown are typical of very similar curves obtained for the other sections tested by Lau and Hancock. Setting aside considerations of yield, which could not be considered by GBT, some interesting conclusions can be drawn from these results.

Conclusions from GBT analysis of racking uprights

1. The importance of the distortional buckling mode has been underestimated by the designers of cold-formed sections. For the typical racking posts considered, distortional buckling is more critical than either local buckling or torsional flexural buckling over most of the column lengths of practical interest.

2. The wavelength of distortional buckling is much longer than that of local buckling and is generally so long that conventional stub-column tests will either miss it completely or give an optimistic result. This is particularly so if the specimens used for stub-column tests have fixed

Fig.14. Results for section RA17

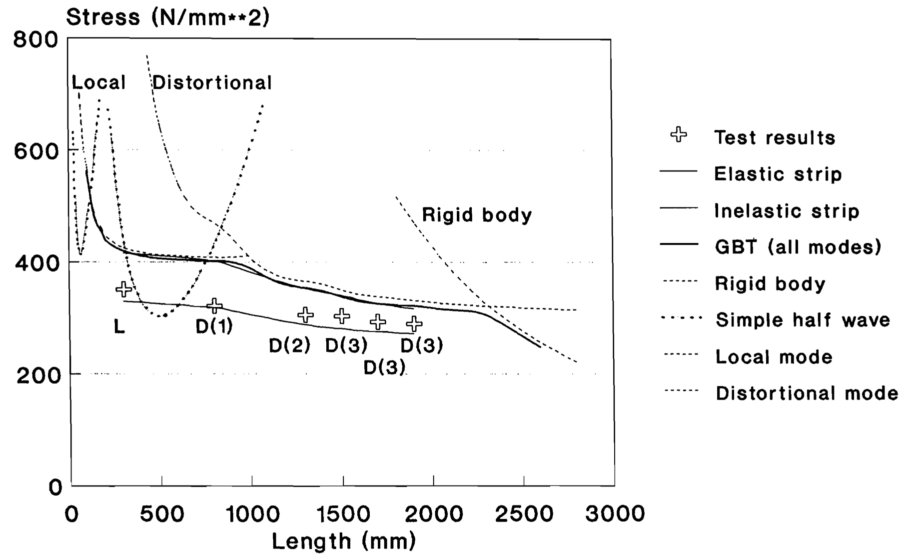


Fig.15. Results for section RA24

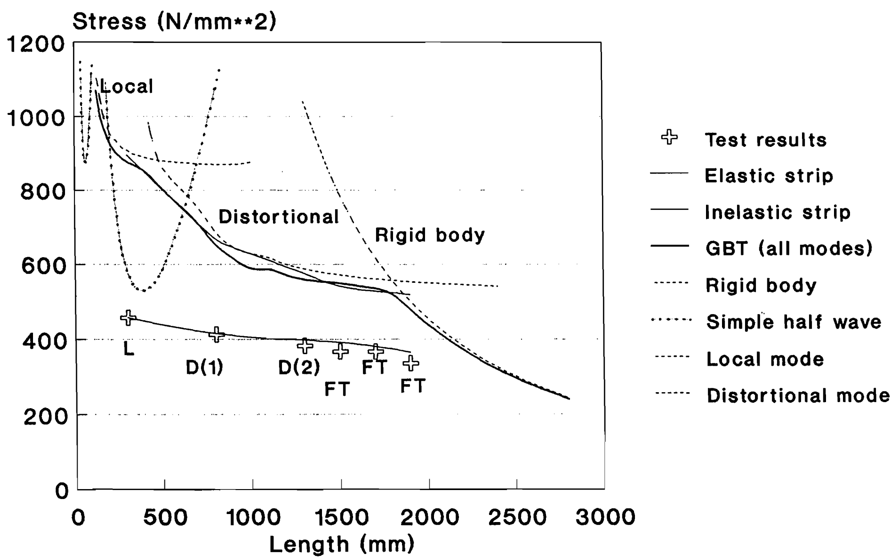


Fig.16. Results for section RL17

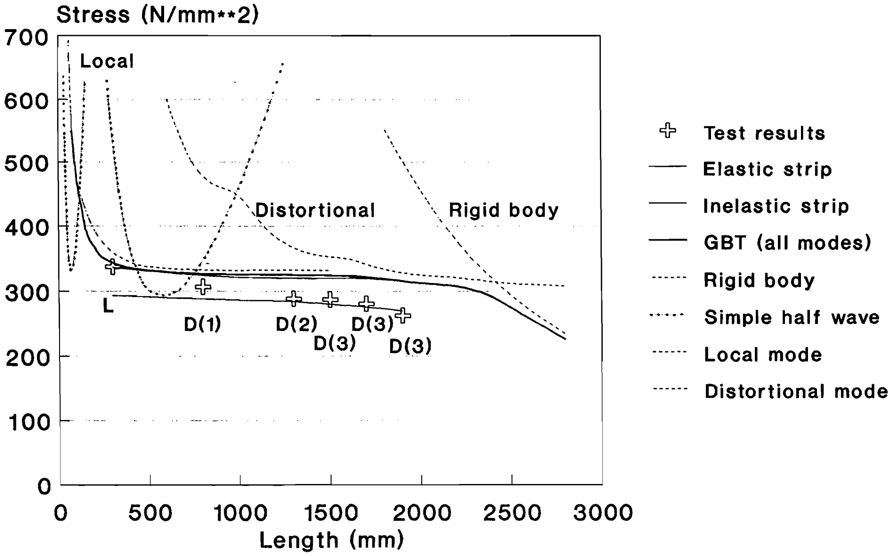
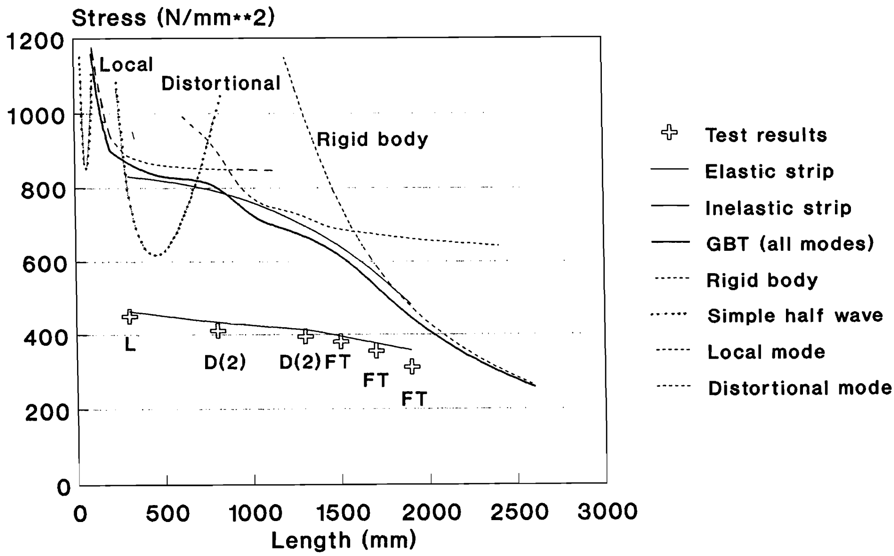


Fig.17. Results for section RL24



ends with respect to the distortional mode.

3. Fixed end conditions have a significant and complex influence on distortional buckling. The half-wave length results with pinned ends are significantly lower than the fixed end results until interaction with torsional-flexural buckling commences. Among other things, this indicates that designers should be particularly cautious when designing uprights with end details that do not inhibit the distortional mode. It would be dangerous to design uprights with loosely fixed base plates on the basis of tests with welded end plates.

   Conversely, theoretical results based on a buckled half-wave length may prove to be pessimistic when applied to longer sections with fixed ends.

4. Provided that the critical buckling mode and its boundary conditions can be identified, it is generally sufficient to consider the critical mode on its own. There is very little interaction between local, distortional and flexural torsional buckling.

5. Clearly, the distortional buckling of cold-formed sections is a subject that requires further research. By allowing consideration of individual buckling modes with different boundary conditions, GBT offers a particularly appropriate methodology for this.

General conclusions

Generalised beam theory is worthy of wider recognition for two reasons:

1. It represents a major advance in structural mechanics and, therefore, has theoretical importance.

2. It provides a powerful and elegant method of analysis which is particularly applicable to practical problems of cold-formed section design.

In particular, GBT is particularly successful in dealing with problems of cross-sectional distortion which have often been neglected by the designers of cold-formed sections. Both first-order and second-order problems can be tackled. In the latter case, the separation of the alternative buckling modes means that mode interaction can be studied in a way that is not possible with other methods.

The authors hope that this paper will provide a useful introduction of GBT to the English-speaking world, particularly its second-order form. Interested readers may then pursue the subject further with the aid of Reference 1 and the other (German) references to be found in its Bibliography.

Acknowledgement

The authors wish to acknowledge Professor R Schardt as the pioneer of Generalised Beam Theory and to thank him for the many helpful discussions that have contributed to their present understanding of the subject.

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