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Introduction to the Theory and Finite Element Implementation of (Steel) Plasticity

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Introduction to the Theory and Finite Element Implementation of (Steel) Plasticity

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Abstract

This paper tries to enlighten the subject of (numerical) plasticity by presenting fundamental theory and some very practical examples. For a two-dimensional plain stress state, a Hubert-Henky yield criterion is derived. The flow rule is discussed, including an explanatory numerical example. The yield criteria and flow rule are theoretically applied in a four node finite element. This element is used to show some of the conditions and limitations of a plastic calculation in the finite element method. Finally, a real finite element calculation is made for illustrating the theory derived.

Introduction

An overwhelming amount of books is available on the subject of (steel) plasticity. However, structural engineers -and even scientists in the field of structural steel design- do often know not more than that a yield stress exists, and that it is influenced by hardening and residual stresses. Starting than with finite element simulations including plastic steel behavior, it is not unlikely that the simulations do not predict structural behavior well. Using the information in this article, plastic finite element calculations can be carried out on (thin-walled) steel structures, see figure 1 [Hofm00a], having basic understanding of theories used.

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1 Elastic stresses

For a plane stress state, normal stresses and shear stresses can be defined for a coordinate system x-y.
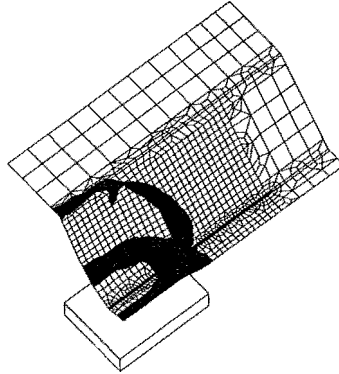

Figure 1, plastic analysis of a thin-walled structure [Hofm00a]. *Are these results trustworthy?*

For a positive angle α rotated coordinate system $\xi-\eta$, stresses can be determined as follows [Verr99a]:

$$
\begin{bmatrix} \sigma_\xi \\ \sigma_\eta \\ \sigma_{\xi\eta} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha & \sin^2\alpha & 2\sin\alpha\cos\alpha \\ \sin^2\alpha & \cos^2\alpha & -2\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin\alpha\cos\alpha & \cos^2\alpha-\sin^2\alpha \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} \quad (1)
$$

If the stresses for a given location are known, elastic strains can be calculated if Young's modules E and Poison's constant v are known:

$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -v & -v & 0 & 0 & 0 \\ -v & 1 & -v & 0 & 0 & 0 \\ -v & -v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+v & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+v & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+v \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix} \quad (2)
$$

We define an average stress σ_{avg} and deviator stresses s_x, s_y, and s_z:

$$\sigma_{avg} = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right) \quad (3)$$

$$s_i = \sigma_i - \sigma_{avg}, i = x,y,z \quad (4)$$

The average stress indicates whether a volume change of the material occurs. The deviator stresses indicate a shape change of the material. Using formula 2 and the knowledge that $s_x + s_y + s_z = 0$, deviator strains e_i can be determined:

$$\begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1+\nu & 0 & 0 \\ 0 & 1+\nu & 0 \\ 0 & 0 & 1+\nu \end{bmatrix} \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \quad (5)$$

Although this paper presents two-dimensional plane stress problems, in the above presented formulae 2 to 5 the third dimension is taken into account. This because even if stress σ_z is zero for formula 4, deviator stress s_z will still have a value.

2 Yield condition (Henky)

In this paper, we assume that we know yield strength f_y [N/mm^2] for the steel tensile test, a (modeled) one-dimensional stress state. Now we are puzzled with defining yielding for a two-dimensional, plane-stress situation. Reference [Prag51a] beautifully explains how mankind has find solutions for the above mentioned problem, by comparing simple developed material models with experiments, and giving explanations afterwards for the corrections made. We will use the model of Hencky here [Henc24a]. For the elastic state of a material, the mechanical work during deformation can be regarded as consisting out of two parts: one part of work that is needed to generate a change of volume (1) and one part of work needed to change the shape of the material (2). Hencky showed that if steel yields, for this yielding always the same amount of mechanical work of type 2 is needed, regardless of the observation of a one-dimensional or a two-dimensional state of stress. If for a work equation the deviator stresses and strains are used, only the work associated with a shape change of the material is predicted (mechanical work of type 2). Index "1" stands for the simple tension test:

$$w_{e;1} = \frac{1}{2}(e_x s_x + e_y s_y + e_z s_z) =$$

$$\frac{1}{2}\left(\frac{1}{E}(1+\nu)\left(\frac{2}{3}f_y\right)^2 + 2\frac{1}{E}(1+\nu)*\left(-\frac{1}{3}f_y\right)^2\right) = \frac{1+\nu}{3E}f_y^2 \quad (6)$$

The elastic work (only for shape changes for a random two-dimensional plane stress state (index "2") equals:

$$w_{e;2} = \frac{1+\nu}{3E}(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\sigma_{xy}^2) \quad (7)$$

Setting the elastic mechanical work for the one-dimensional case (formula 6) equal to the work for a random two-dimensional case (formula 7), yields the following:

$$w_{e;1} = w_{e;2} \Leftrightarrow \frac{(1+\nu)}{3E}f_y^2 = \frac{(1+\nu)}{3E}\left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\sigma_{xy}^2\right) \Leftrightarrow$$

$$\left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\sigma_{xy}^2\right) - f_y^2 = 0 \quad (8)$$

For the condition presented above, the material in a random two-dimensional plane-stress state is yielding. The above function can simply be defined as yield function f, and then yielding is defined when f equals zero:

$$f\left(\sigma_x, \sigma_y, \sigma_{xy}, f_y\right) = \sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\sigma_{xy}^2 - f_y^2 \quad (9)$$

For plastic deformations, df (the change of yield function f for changing variables) should be zero, because if df is smaller than zero, this indicates that for further deformations function f will be smaller than zero. If f is smaller than zero, no plastic deformations will occur. Furthermore, df cannot be larger than zero, because this indicates that the function f will be larger than zero, which is physically impossible.

3 Plastic flow

If material yields, the constitutive equations should be determined, in other words, the relation between (plastic) strains and stresses should be derived. For

the elastic state we have this information given by equation 2. Reference [Kali89a] gives a good overview on how to find the constitutive equations for the plastic state, and this overview will be used here rewritten for a plane-stress case and avoiding the rather complex tensor-notations. For plastic states, more strain values can be linked to one stress value: there is no direct connection between strains and stresses. To work around this, increments instead of total values will be used. This means a function will be derived between strain increments and stress increments, for a specific stress state. We assume that strains can be decomposed into elastic and plastic strains, and this implies that also strain increments can be decomposed:

$$d\varepsilon_i = d\varepsilon_i^e + d\varepsilon_i^p, \quad i = x,y,xy \quad (10)$$

See figure 2. A three-dimensional stress space (σ_x, σ_y, σ_{xy}), for a plane-stress situation, is presented schematically in a two-dimensional figure. Yield function $f = 0$ is given schematically by the oval: this is the yield surface. From the origin, a stress state is given by vector σ^A; point A represents the stress values (σ_x, σ_y, σ_{xy}). As mentioned function f cannot be larger than zero physically, this means that stress vectors like σ^A cannot reach the outside of the yield surface. Assume that due to a external load, the stress at one location in the material will change from point A to point B, then to point C, and finally back to point A. The additional work (the work due to σ^A is not regarded) during this load can be written as:

$$w = \oint \left(\sigma_x - \sigma_x^A \right) d\varepsilon_x + \oint \left(\sigma_y - \sigma_y^A \right) d\varepsilon_y + \oint \left(\sigma_{xy} - \sigma_{xy}^A \right) d\varepsilon_{xy} \quad (11)$$
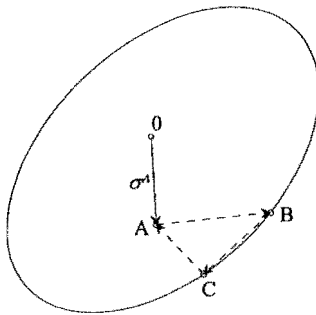

Figure 2, plastic yield criterion in stress space, stress space is for a plane stress situation three-dimensional. Here, only two dimensions are drawn.

Equation 11 can be rewritten because of the following items. The stability postulate of materials [Druc52a] defines that the additional work cannot be negative for perfectly plastic materials (1). Total strains can be decomposed in elastic and plastic strains (2). Elastic energy is restored for a complete load cycle (starting from A and return to A). Thus, the sum of these terms equals zero and can be deleted (3). Assume that point B and C are closely together, making the plastic work between B and C infinite small (4). No plastic work will be generated once traveling back from C to A (5). Assume that the path from A to B is linear (6):

$$\left(\sigma_x{}^B - \sigma_x{}^A\right)d\varepsilon_x^p + \left(\sigma_y{}^B - \sigma_y{}^A\right)d\varepsilon_y^p + \left(\sigma_{xy}{}^B - \sigma_{xy}{}^A\right)d\varepsilon_{xy}^p \geq 0 \Leftrightarrow$$

$$\left(d\sigma_x\right)d\varepsilon_x^p + \left(d\sigma_y\right)d\varepsilon_y^p + \left(d\sigma_{xy}\right)d\varepsilon_{xy}^p \geq 0 \quad (12)$$

The stress increments in equation 12 can be written as a vector $d\sigma(d\sigma_x, d\sigma_y, d\sigma_{xy})$ and the strain increments also as $d\varepsilon(d\varepsilon_x, d\varepsilon_y, d\varepsilon_{xy})$. If the strain vector is drawn in a three-dimensional strain space, and this space is coaxial with the stress space, the angle between the two vectors should not exceed 90 degrees. This because the product of the two vectors should be larger than zero (equation 12). It can be proven that this rule implies that the oval shape of function $f = 0$ should be convex, and that the incremental plastic strain vector should be perpendicular to the yield surface, but this will not be proven here. Figure 3 shows that the strain vector, which starting point can be positioned arbitrarily in space because it is incremental, is perpendicular to the yield surface.
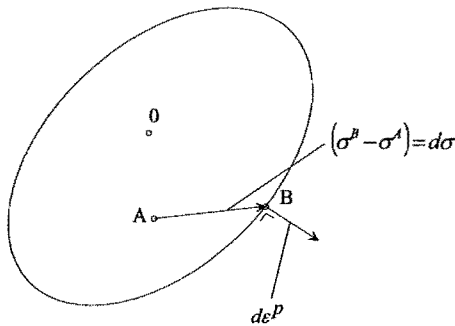

Figure 3, the incremental strain vector is perpendicular to the yield surface.

Another way (instead of equation 12) to define the incremental strain vector is now to derive the normal vector (at location B) of the yield function. This is straightforward:

$$n_i = \frac{\partial f(B)}{\partial \sigma_i}, i = x, y, xy \quad (13)$$

We know that the incremental strain vector should coincide this normal vector, but the size of it remains unknown for which we thus will use a defined factor $d\lambda$:

$$d\varepsilon_i^p = d\lambda \frac{\partial f(\sigma^B)}{\partial \sigma_i}, i = x, y, xy \quad (14)$$

If we can solve this defined factor $d\lambda$, the incremental constitutive equation is solved. It is possible to show how factor $d\lambda$, is solved theoretically. However, in this paper this will only be presented for an practical example, see next section.

4 Example plastic flow

Formula 2 can be rewritten for stresses as a function of strains. For a plane stress situation, σ_z, σ_{xz}, and σ_{yz} are zero and the following is valid:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{E}{1-v^2} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & 1-v \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix} \quad (15)$$

The strains presented in equation 15 elastic strains. Elastic strains can be regarded as total strains minus plastic strains. Rewritten for increments and substituting the flow rule (equation 14):

$$d\sigma_x = E_1 \cdot \left(d\varepsilon_x - d\lambda \frac{\partial f}{\sigma_x} \right) + E_2 \cdot \left(d\varepsilon_y - d\lambda \frac{\partial f}{\sigma_y} \right) \quad (16)$$

$$d\sigma_y = E_2 \cdot \left(d\varepsilon_x - d\lambda \frac{\partial f}{\sigma_x} \right) + E_1 \cdot \left(d\varepsilon_y - d\lambda \frac{\partial f}{\sigma_y} \right) \quad (17)$$

$$d\sigma_{xy} = E_3 \cdot \left(d\varepsilon_{xy} - d\lambda \frac{\partial f}{\sigma_{xy}} \right) \quad (18)$$

The left and right side of equation 16 to 18 can be multiplied with an arbitrarily chosen factor:

$$\frac{\partial f}{\partial \sigma_x} d\sigma_x = \frac{\partial f}{\partial \sigma_x} E_1 \cdot \left(d\varepsilon_x - d\lambda \frac{\partial f}{\sigma_x} \right) + \frac{\partial f}{\partial \sigma_x} E_2 \cdot \left(d\varepsilon_y - d\lambda \frac{\partial f}{\sigma_y} \right) \quad (19)$$

$$\frac{\partial f}{\partial \sigma_y} d\sigma_y = \frac{\partial f}{\partial \sigma_y} E_2 \cdot \left(d\varepsilon_x - d\lambda \frac{\partial f}{\sigma_x} \right) + \frac{\partial f}{\partial \sigma_y} E_1 \cdot \left(d\varepsilon_y - d\lambda \frac{\partial f}{\sigma_y} \right) \quad (20)$$

$$\frac{\partial f}{\partial \sigma_{xy}} d\sigma_{xy} = \frac{\partial f}{\partial \sigma_{xy}} E_3 \cdot \left(d\varepsilon_{xy} - d\lambda \frac{\partial f}{\sigma_{xy}} \right) \quad (21)$$

It was already mentioned that df should be zero (see section 2). The sum of the left terms of equation 19 to 21 equals df and thus should be zero. Now $d\lambda$ can be solved using the sum of the right terms equal to zero. The term $d\lambda$ can be substituted into equations 19 to 21 and then the relationship between stress increments and total strain increments is known. An explanatory example will be presented, figure 4.
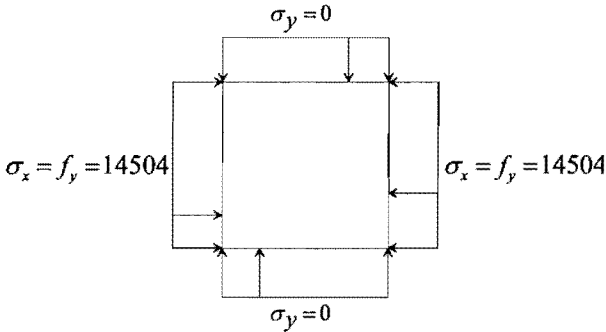
Figure 4, steel plate, 3.93*3.93 inch, thickness 0.39 inch, loaded horizontally with the yield stress, not loaded vertically.

We will take a thin plate of steel, and although the dimensions are not relevant for the example, assume that the size is 3.93*3.93 inch (100*100 mm) and the thickness equals 0.39 inch (10 mm), see figure 4. Young's modulus is taken 3.046E7 psi (210,000 N/mm^2) and Poison's ratio equals 0.3 [1]. This makes it

possible to calculate E_1, E_2, and E_3 (equation 15). We use the yield function as derived in section 2, formula 14. For the yield strength 14504 psi (100 N/mm^2) is taken. The steel plate is compressed in horizontal direction with the yield stress and in horizontal direction no stresses are applied (the plate is able to deform freely), see figure 4.

Factor $d\lambda$ can be calculated using formulae 19 to 21:

$$d\lambda = (3.084E - 5)d\varepsilon_x + (-7.258E - 6)d\varepsilon_y + 0d\varepsilon_{xy} \quad (22)$$

These values can be substituted into equation 16 to 18, and the result can be written conveniently into matrix notation:

$$\begin{bmatrix} d\sigma_x \\ d\sigma_y \\ d\sigma_{xy} \end{bmatrix} = \begin{bmatrix} 8015789 & 16031579 & 0 \\ 16031579 & 32063158 & 0 \\ 0 & 0 & 23430769 \end{bmatrix} \begin{bmatrix} d\varepsilon_x \\ d\varepsilon_y \\ d\varepsilon_{xy} \end{bmatrix} \quad (23)$$

To compare these values with elastic values, formula 16 to 18 are used with elastic strains only. Young's modulus is taken 3.046E7 psi (210,000 N/mm^2) and Poison's ratio equals 0.3 [1]:

$$\begin{bmatrix} d\sigma_x \\ d\sigma_y \\ d\sigma_{xy} \end{bmatrix} = \begin{bmatrix} 33472527 & 10041758 & 0 \\ 10041758 & 33472527 & 0 \\ 0 & 0 & 23430769 \end{bmatrix} \begin{bmatrix} d\varepsilon^e_x \\ d\varepsilon^e_y \\ d\varepsilon^e_{xy} \end{bmatrix} \quad (24)$$

If plastic (equation 23) and elastic (equation 24) behavior are compared, a few notes can be made. Firstly, both matrices are symmetric. Secondly, in the compression direction, the plastic system gets less stiff, which is expected, and in the other direction the system gets stiffer. For the shear stiffness, no differences occur.

The practical example in figure 4 will be continued a little bit further by applying a prescribed strain in x-direction, and by letting the material free in y-direction, thus σ_y equals zero. Furthermore, σ_{xy} equals zero in this situation. See table 1.

Table 1, loading the example of figure 4.

$d\varepsilon_x$	$d\varepsilon_y$	$d\sigma_x$	σ_x	ε_x	ε_y	$d\varepsilon_x / d\varepsilon_y$
0	0	0	0	0.00000	0.00000	
0.0001	-3.000E-05	3046	3046	0.00010	-0.00003	-3.300
0.0001	-3.000E-05	3046	6092	0.00020	-0.00006	-3.300
0.0001	-3.000E-05	3046	9138	0.00030	-0.00009	-3.300
0.0001	-3.000E-05	3046	12184	0.00040	-0.00012	-3.300
0.00007615	-2.285E-05	2320	14504	0.00048	-0.00014	-3.300
0.0001	-2.000E-04	0	14504	0.00058	-0.00034	-0.5000

Starting with a strain increment $d\varepsilon_x$, strain increment $d\varepsilon_y$ and stress increment $d\sigma_x$ can be calculated using matrix-formula 24 (this because $\sigma_y = \sigma_{xy} = 0$). The total stress σ_x is found by summating all stress increments. Total strains ε_x and ε_y are found likewise. At a certain moment, the stress state will be such that the plastic state is reached. In this case, this is the moment where σ_x equals 14504 psi (100 N/mm^2) (because $\sigma_y = \sigma_{xy} = 0$). From this moment on, not equation 24 but equation 23 should be used to calculate strain increment $d\varepsilon_y$ and stress increment $d\sigma_x$. In the plastic state stress increment $d\sigma_x$ equals zero, and thus the total stress remains 14504 psi (100 N/mm^2), see figure 5. Reading through sections 1 to 3, the reader can be surprised on the amount of formulae and effort necessary to correctly describe the simple example of figure 4. Hopefully, this makes clear that plasticity is not easy to model and to understand and that especially finite element calculations using principles of plasticity should be used with great care.

4 Finite element theory

The most fundamental step in finite element theory is to derive the element stiffness: the relationship between nodal displacements and forces. Without explaining every detail, the derivation of element stiffness is presented here (see for more information [Cook95a]). See figure 5. We interpolate the horizontal $u(x,y)$ and vertical displacements $v(x,y)$ in the element if given the eight nodal displacements d_i. Both the displacements in the element and the nodal displacements are written in matrix notation, the interpolation itself is given by matrix N:

$$[u] = [N][d] \quad (25)$$

If the displacements in the element are known (relative to the nodal displacements), the strains in the element can be calculated using the kinematical relations. Matrix B occurs to relate element strains and nodal displacements:

$$[\varepsilon] = [B][d] \quad (26)$$

If strains in the element are known, we can calculate stresses easily using the constitutive relations. Now, the element stiffness matrix can be derived by setting the variation of potential energy of the system to zero. As a result [Cook95a] the stiffness matrix can be found by:

$$[K] = \int_{x=0}^{x=b} \int_{y=0}^{y=h} \left([B]^T [E][B] \right) dx dy \quad (27)$$

Formula 27 can be understood as summating (the integral signs) some components of stress and strain (this can be more or less understood seeing the term $[B]^T[E][B]$ and formula 26) at every location in de element (integral boundaries 0 to b and 0 to h). We derived the stiffness matrix using formula 27 for an element with dimensions as shown in figure 4. The element was loaded with an equally distributed strain by prescribed displacements as shown in figure 5. This results in an equal stress distribution for all locations in the element. At the moment the yield stress is reached, not matrix $[E]$ and formula 24, but plastic constitutive equations, formula 23, are used to calculate the element stiffness matrix with formula 27. As a result, using manual calculations (no finite element program), the same stress strain curve was found for the finite element and the example of chapter 4 [Doet04a]. A problem occurs when the element is loaded by only one prescribed displacement. The stress field in the element is not equally distributed now, and only some specific locations in the element will yield. If the stiffness of the element has to be updated, formula 27 cannot be used because if we are at a yielding location in the integration domain, we have to use the plastic constitutive equations, and if we are at a non-yielding location, the elastic constitutive equations have to be used. But how can these elastic and plastic domains be defined mathematically? Therefore, the integral of formula 27 is solved numerically. This means that the equations are solved for specific values and these values are weighted, formula 28 [Cook85a, page 172]. In practice this means that stress and strain are only calculated at the points seen in

figure 5, the so-called integration points. If such an integration point yields, the plastic constitutive relationship is used, otherwise the elastic relationship:

$$[K] = \int_{x=0}^{x=b} \int_{y=0}^{y=h} \left([B]^T [E][B] \right) dx\, dy \approx$$

$$(1)[B(x_1,y_1)]^T [E(x_1,y_1)][B(x_1,y_1)] + (1)[B(x_2,y_2)]^T [E(x_2,y_2)][B(x_2,y_2)] +$$

$$(1)[B(x_3,y_3)]^T [E(x_3,y_3)][B(x_3,y_3)] + (1)[B(x_4,y_4)]^T [E(x_4,y_4)][B(x_4,y_4)] \quad (28)$$
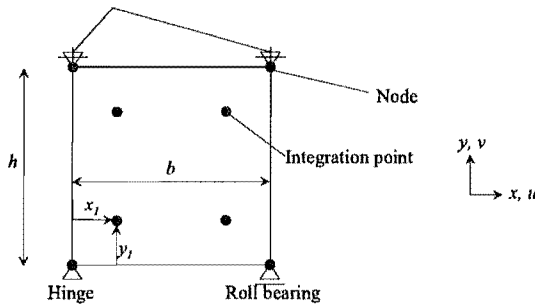

Figure 5, a finite element loaded with two equal prescribed displacements.

This means that the element only "feels" plasticity at the integration points, which is a very important conclusion for practice. For instance, if a shell element is bended only two integration points along the shell height means that the cross-section is only able to simulate full elastic behavior or full plastic behavior. If a shell element is used with four integration points along the surface, and a yield line is parallel moving through the surface, the line will move discontinue, because of the fact that it can only exist at the integration points.

5 Real examples (Ansys)

Using the finite element program Ansys, the two above presented load cases (equal prescribed displacements, only one prescribed displacement) have been simulated, see figure 6. The program output is as expected, especially for the case with only one prescribed displacement. This means that explanations and numerical calculations as presented in this paper are likely correct.
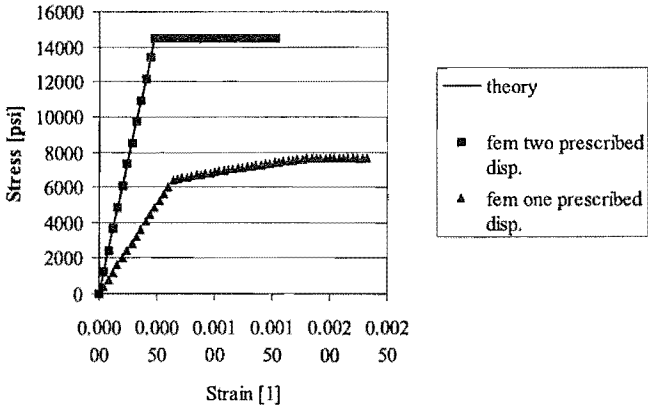
Figure 6, load-deformation curves for one and two prescribed displacements.

6 Conclusions

It is interesting to note the amount of formulae and effort necessary to correctly describe a very simple example of plasticity (figure 4). This makes clear that plasticity is not easy to model and to understand and that especially finite element calculations using principles of plasticity should be used with great care.

A finite element only "feels" plasticity at the integration points. For instance, if a shell element is bended only two integration points along the shell height means that the cross-section is only able to simulate full elastic behavior or full plastic behavior. If a shell element is used with four integration points along the surface, and a yield line is parallel moving through the surface, the line will move discontinue, because of the fact that it can only exist at the integration points.

Using the finite element program Ansys, two load cases (equal prescribed displacements, only one prescribed displacement) were simulated. The program output is as expected, especially for the case with only one prescribed deformation. This means that explanations and numerical calculations as presented in this paper are likely to be correct. Solution procedures to solve the

problem presented are not discussed in this paper. Note that they can produce as many problems and questions as for the application of the flow rule here.

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8 Notations

σ_{ij}	Stress in ij-direction	[psi, N/mm^2]
σ_{avg}	Average stress	[psi, N/mm^2]
s_i	Deviator stress in i-direction	[psi, N/mm^2]

f_y	Yield stress	[psi, N/mm^2]
ε_{ij}	Total (or elastic) strain in ij-direction	
$\varepsilon_{ij}{}^{p,e}$	Plastic, elastic strain in ij-direction	
e_i	Deviator strain in i-direction	
α	Angle of rotation	[rad.]
E	Young's modulus	[psi, N/mm^2]
ν	Poison's constant	
x, y, z, ξ, η	Variables for axis definition	
$w_{e;1}$	Elastic shape change work, 1-D	
$w_{e;2}$	Elastic shape change work, 2-D	
w	Plastic work	
f	Flow function	
$d\lambda$	Scale factor	
$[u]$	Vector containing displacements as function of x,y	
$[\varepsilon]$	Vector containing strains as function of x,y	
$[d]$	Vector containing displacements of nodes in x,y-direction	
$[N]$	Matrix relates $[u]$ and $[d]$.	
$[B]$	Matrix relates $[\varepsilon]$ and $[d]$.	
$[K]$	Matrix relates nodal forces and nodal displacements.	