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THE ANALYSIS OF SANDWICH PANELS WITH PROFILED FACES

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1. Introduction

Classical methods of analysis for sandwich panels consisting of a relatively flexible core and relatively stiff metal faces have been available for a number of years. However, the governing differential equations are somewhat cumbersome, and explicit solutions have only been obtained for a small number of simple cases. There has, therefore, been a search for more general solutions and the current state of the art is reviewed in this paper. A particular aspect of this search has been concerned with finite element solutions and some new possibilities are introduced, including an exact, explicit, general finite element for panels with profiled metal faces. Attention is confined to the general case of panels with profiled faces as panels with plane faces can always be treated as a special case. Some of the considerations are illustrated by a comprehensive example.

2. Classical methods of analysis

The most readily accessible explicit solutions of the governing differential equations are those presented by Hartsock and his colleagues. Reference 1 gives solutions for simply supported panels subject to central point load, uniformly distributed load and uniform temperature difference between the faces. It also gives FORTRAN listings of computer programs for the various calculations involved. The theory is repeated in references 2 and 3 which also include comparison with some test results. Finally, it is shown in reference 4 how the previous results may be combined to give a solution to the important case of a continuous panel of two equal spans subject to either a uniformly distributed load or a temperature difference between the faces.

Allan [5,6] gives some similar equations for simply supported panels subject to uniformly distributed load and central point load.

An alternative formulation, presented within the framework of a complete treatise on sandwich panels for building construction is given by Stamm and Witte[7]. As this work is not readily available to English-speaking readers and as it is the neatest and most comprehensive interpretation of the classical approach, the basic equations and the most important solutions will be repeated here. The same notation and basic equations will also be used later in the derivation of a completely general finite element solution.

The author has programmed both the Hartsock and Stamm and Witte solutions. They give identical solutions though the Stamm and Witte equations may be somewhat better conditioned.

Figure 1 shows a typical sandwich panel with profiled metal faces and Figure 2 shows the relevant stress resultants and deformations. The

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relationships between the stress resultants and deformations are:

$$\left.\begin{aligned} M_S &= B_S(\gamma_2^1 + \Theta) = B_S(\gamma^1 - w^{11} + \Theta) \\ M_1 &= -B_1 w^{11} \;;\quad M_2 = -B_2 w^{11} \\ Q_S &= A\, G_{eff}\, \gamma \\ Q_1 &= -B_1 w^{111} \;;\quad Q_2 = -B_2\, w^{111} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where, in addition to the quantities defined on Figures 1 and 2, a prime denotes differentiation with respect to x which is measured along the axis of the panel and

B_S = bending stiffness of sandwich part of cross-section

= $E_1 A_1 E_2 A_2 D^2 /(E_1 A_1 + E_2 A_2)$

B_1 = $E_1 I_1$ = bending stiffness of upper face

B_2 = $E_2 I_2$ = bending stiffness of lower face

A_1, A_2 = areas of faces

E_1, E_2 = Young's moduli of faces

A = $B D_c$ = area of foam core

G_{eff} = effective shear modulus of core = $G_{nom} D/D_c$

G_{nom} = nominal shear modulus of core

q = uniformly distributed load on panel

w = total deflection

γ = shear strain in foam core (divergence from the normal to the axis of the cross-section)

Θ = $\alpha_2 T_2 - \alpha_1 T_1)/D$ = curvature resulting from a temperature difference between the faces.

Because the stress resultants in the two faces are proportional to the same deformations, it is convenient to treat them together. Thus

$$\left.\begin{aligned} M_D &= M_1 + M_2 \;;\quad Q_D = Q_1 + Q_2 \\ M &= M_D + M_S \;;\quad Q = Q_D + Q_S \\ B_D &= B_1 + B_2 \\ B &= B_D + B_S \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2), the following differential equations are obtained

$$AG_{eff}\,\gamma - B_D w^{111} = Q \qquad \text{..................(3)}$$

$$B_S\,(\gamma^{1} + \Theta) - Bw^{11} = M$$

Eliminating γ and noting that $Q^1 = -q$, a fourth order differential equation in w is obtained.

$$w^{1V} - (\frac{\lambda}{L})^2 w^{11} = (\frac{\lambda}{L})^2 \frac{M}{B} + \frac{1+\alpha}{\alpha}\frac{q}{B} - (\frac{\lambda}{L})^2 \frac{\Theta}{1+\alpha} \qquad \text{.........(4)}$$

where L is the total length of the panel and

$$\alpha = \frac{B_D}{B_S}\ ; \qquad \beta = \frac{B_S}{AG_{eff}L^2}\ ; \qquad \lambda^2 = \frac{1+\alpha}{\alpha\beta} \qquad \text{...............(5)}$$

Similarly, eliminating w from (3)

$$\gamma^{11} - (\frac{\lambda}{L})^2 \gamma = -\frac{1}{B}\,\beta\lambda^2 Q \qquad \text{..................(6)}$$

The equations in the above form are particularly useful when the distributions of bending moment M and shear force Q are known, i.e. for statically determinate systems. For such cases, the general solutions of (4) and (6) are

$$w = C_1 \cosh\frac{\lambda x}{L} + C_2 \sinh\frac{\lambda x}{L} + C_3 + C_4 x + w_p$$

$$\gamma = D_1 \cosh\frac{\lambda x}{L} + D_2 \sinh\frac{\lambda x}{L} + \gamma_p \qquad \text{.......(7)}$$

where w_p and γ_p are particular integrals which depend on the loading etc. As these solutions must also satisfy (3) it is easy to show that

$$D_1 = (1+\alpha)\frac{\lambda}{L} C_2\ ; \quad D_2 = (1+\alpha)\frac{\lambda}{L} C_1 \qquad \text{.............(8)}$$

Thus, the number of constants of integration reduce to four and these can be determined from the boundary conditions, e.g. for a simply supported panel,

$$w(o) = 0;\ w^{11}(o) = 0;\ w(L) = 0;\ w^{11}(L) = 0 \qquad \text{..........(9)}$$

Stamm and Witte give three solutions of the above equations, namely for simply supported panels subject to (a) uniformly distributed load

(b) point load anywhere in span

and (c) uniform temperature difference between faces

and from combinations of these, a number of other important cases can also be derived. Indeed, one of the significant features of this work is the explicit solution for case (b) with the point load applied anywhere in the span. As far as the author is aware, an explicit solution for this case has not been given elsewhere and it allows, for instance, a solution for the important case of a three-span panel under uniformly distributed and temperature loading to be derived. The explicit solutions are as follows where $\xi = x/L$:

(a) Simply supported panel with uniformly distributed load q

With

$$M = \frac{q}{L}(Lx - x^2); \quad Q = \frac{q}{2}(L - 2x) \qquad \ldots\ldots(10)$$

the particular integrals in equation (7) are

$$\left.\begin{aligned} w_p &= \frac{q}{24B}\left(x^4 - 2L\,x^3 - \frac{12}{\alpha\lambda^2}L^2\,x^2\right) \\ \gamma_p &= \frac{qL^2\,\beta}{2B}(L - 2x) \end{aligned}\right\} \qquad \ldots\ldots(11)$$

giving the solutions

$$\left.\begin{aligned} w &= \frac{qL^4}{B}\left[\frac{1}{24}\xi(1 - 2\xi^2 + \xi^3) + \frac{1}{2\alpha\lambda^2}\,\xi(1 - \xi) - \frac{1}{\alpha\lambda^4}\,\frac{\cosh\lambda/2 - \cosh\lambda(1-2\xi)/2}{\cosh\lambda/2}\right] \\ \gamma &= \frac{qL^3}{B}\beta\left[\frac{1}{2}(1 - 2\xi) - \frac{1}{\lambda}\,\frac{\sinh\lambda(1-2\xi)/2}{\cosh\lambda/2}\right] \end{aligned}\right\} \qquad \ldots\ldots(12)$$

$$\left.\begin{aligned} M_S &= q\,L^2\,\frac{1}{1+\alpha}\left[\frac{1}{2}\xi(1 - \xi) - \frac{1}{\lambda^2}\,\frac{\cosh\lambda/2 - \cosh\lambda(1-2\xi)/2}{\cosh\lambda/2}\right] \\ M_D &= q\,L^2\,\frac{\alpha}{1+\alpha}\left[\frac{1}{2}\xi(1-\xi) + \frac{1}{\alpha\lambda^2}\,\frac{\cosh\lambda/2 - \cosh\lambda(1-2\xi)/2}{\cosh\lambda/2}\right] \\ Q_S &= q\,L\,\frac{1}{1+\alpha}\left[\frac{1}{2}(1-2\xi) - \frac{1}{\lambda}\,\frac{\sinh\lambda(1-2\xi)/2}{\cosh\lambda/2}\right] \\ Q_D &= q\,L\,\frac{\alpha}{1+\alpha}\left[\frac{1}{2}(1-2\xi) + \frac{1}{\alpha\lambda}\,\frac{\sinh\lambda(1-2\xi)/2}{\cosh\lambda/2}\right] \end{aligned}\right\} \qquad \ldots(13)$$

(b) Simply supported panel with point load P

If the point load P is applied at a position given by x = e, i.e. $\xi = e/L = \varepsilon$, the bending moment and shearing force are given by

$$M = \frac{P}{L}(L-e)x - P\{x-e\}; \quad Q = \frac{P}{L}(L-e) - P\{x-e\}^0 \qquad \ldots\ldots(14)$$

where, according to Macauley's notation, the quantities in the curly brackets are set equal to zero when negative.

The particular integrals in equation (7) are then

$$\left.\begin{aligned} w_p &= \frac{P}{6BL}\left[-(L-e)x^3 + L\ \{x-e\}^3\right] - \frac{pL}{B\lambda^2}\left[(L-e)x + \frac{L}{\alpha}\left(x-e - \frac{\sinh\lambda(x-e)/L}{\lambda/L}\right)\{x-e\}^o\right] \\ \gamma_p &= \frac{\beta PL}{B}\left[L-e-L\left(1-\cosh\frac{\lambda(x-e)}{L}\right)\{x-e\}^o\right] \end{aligned}\right\}\quad \ldots\ldots(15)$$

giving, with index 1 valid for $0 \leq \xi \leq \varepsilon$ and index 2 for $\varepsilon \leq \xi \leq 1$

$$\left.\begin{aligned} w_1 &= \frac{PL^3}{B}\left[\frac{1}{6}(1-\varepsilon)\xi(2\varepsilon-\varepsilon^2-\xi^2) + \frac{1}{\alpha\lambda^2}(1-\varepsilon)\xi - \frac{1}{\alpha\lambda^3}\frac{\sinh\lambda(1-\varepsilon)}{\sinh\lambda}\sinh\lambda\xi\right] \\ w_2 &= \frac{PL^3}{B}\left[\frac{1}{6}\varepsilon(1-\xi)(-\varepsilon^2+2\xi-\xi^2) + \frac{1}{\alpha\lambda^2}\varepsilon(1-\xi) - \frac{1}{\alpha\lambda^3}\frac{\sinh\lambda\varepsilon}{\sinh\lambda}\sinh\lambda(1-\xi)\right] \\ \gamma_1 &= \frac{PL^2}{B}\beta\left[1-\varepsilon+\frac{\sinh\lambda(1-\varepsilon)}{\sinh\lambda}\cosh\lambda\xi\right] \\ \gamma_2 &= \frac{PL^2}{B}\beta\left[-\varepsilon+\frac{\sinh\lambda\varepsilon}{\sinh\lambda}\cosh\lambda(1-\xi)\right] \end{aligned}\right\}\quad \ldots\ldots(16)$$

$$\left.\begin{aligned} M_{S_1} &= PL\ \frac{1}{1+\alpha}\left[(1-\varepsilon)\xi - \frac{\sinh\lambda(1-\varepsilon)}{\lambda\sinh\lambda}\sinh\lambda\xi\right] \\ M_{S_2} &= PL\ \frac{1}{1+\alpha}\left[\varepsilon(1-\xi) - \frac{\sinh\lambda\varepsilon}{\lambda\sinh\lambda}\sinh\lambda(1-\xi)\right] \\ M_{D_1} &= PL\ \frac{\alpha}{1+\alpha}\left[(1-\varepsilon)\xi + \frac{\sinh\lambda(1-\varepsilon)}{\alpha\lambda\sinh\lambda}\sinh\lambda\xi\right] \\ M_{D_2} &= PL\ \frac{\alpha}{1+\alpha}\left[\varepsilon(1-\xi) + \frac{\sinh\lambda\varepsilon}{\alpha\lambda\sinh\lambda}\sinh\lambda(1-\xi)\right] \\ Q_{S_1} &= P\ \frac{1}{1+\alpha}\left[1-\varepsilon-\frac{\sinh\lambda(1-\varepsilon)}{\sinh\lambda}\cosh\lambda\xi\right] \\ Q_{S_2} &= P\ \frac{1}{1+\alpha}\left[-\varepsilon+\frac{\sinh\lambda\varepsilon}{\sinh\lambda}\cosh\lambda(1-\xi)\right] \\ Q_{D_1} &= P\ \frac{\alpha}{1+\alpha}\left[1-\varepsilon+\frac{\sinh\lambda(1-\varepsilon)}{\alpha\sinh\lambda}\cosh\lambda\xi\right] \\ Q_{D_2} &= P\ \frac{\alpha}{1+\alpha}\left[-\varepsilon-\frac{\sinh\lambda\varepsilon}{\alpha\sinh\lambda}\cosh\lambda(1-\xi)\right] \end{aligned}\right\}\quad \ldots\ldots(17)$$

(c) Simply supported panel with temperature difference between the faces

If the temperatures of the two faces are T_1 and T_2 with coefficients of linear extension α_1 and α_2, then in the absence of bending stiffness, the panel would bend into a curvature of

$$\Theta = \frac{\alpha_2 T_2 - \alpha_1 T_1}{D} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

It is convenient to use θ as a parameter in the equations for this case and θ has been included in the derivations of the governing differential equations (4) and (6). The particular integrals in equations (7) for this case are

$$w_p = -\frac{\theta}{2(1+\alpha)} x^2 \; ; \qquad \gamma_p = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Thus the complete solutions which follow may be obtained:

$$\left.\begin{aligned} w &= \frac{\theta L^2}{1+\alpha}\left[\frac{1}{2}\xi(1-\xi) - \frac{1}{\lambda^2}\frac{\cosh \lambda/2 - \cosh \lambda(1-2\xi)/2}{\cosh \lambda/2}\right] \\ \gamma &= -\frac{\theta L}{\lambda}\frac{\sinh\lambda(1-2\xi)/2}{\cosh \lambda/2} \end{aligned}\right\} \ldots (20)$$

$$\left.\begin{aligned} M_S &= -\frac{\alpha\theta Bs}{1+\alpha}\frac{\cosh \lambda/2 - \cosh \lambda(1-2\xi)/2}{\cosh\lambda/2} \\ M_D &= +\frac{\alpha\theta Bs}{1+\alpha}\frac{\cosh \lambda/2 - \cosh \lambda(1-2\xi)/2}{\cosh\lambda/2} \\ Q_S &= -\frac{\theta Bs}{\beta\lambda L}\frac{\sinh\lambda(1-2\xi)/2}{\cosh\lambda/2} \\ Q_D &= +\frac{\alpha\theta Bs}{\alpha\beta\lambda L}\frac{\sinh\lambda(1-2\xi)/2}{\cosh\lambda/2} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (21)$$

3. Solutions for two and three span panels

Solutions for panels with two equal spans L subject to uniformly distributed load follow as a combination of cases (a) and (b) above as shown in Figure 3. It is merely necessary to add together the solutions for a single span of 2L subject to (Figure 3b) the uniformly distributed load and (Figure 3c) an upward point load P at mid-span where P is chosen so that the two deflections Δ are equal.

Solutions for two span panels subject to a temperature difference between the faces may be obtained similarly as a combination of cases (b) and (c).

Panels with three equal spans L may also be solved as a combination of the solutions for a simply supported span of 3L for (Figure 4b) the applied uniformly distributed or temperature loading and (Figures 4c and 4d) a point load P at the third point. Figures 3c and 4d are, of course, mirror images and P is chosen so that

$$\Delta_1 + \Delta_2 = \Delta.$$

The author has programmed the complete set of equations given above, together with the extension to panels with two and three equal spans. The solutions are relatively stable numerically and it is considered that these equations represent by far the best approach to regular situations. The finite element methods which are discussed later are essentially applicable to irregular situations such as unequal spans or non-uniform loading.

4. Approximate solutions for simply supported panels

A significant landmark in the development of sandwich panel technology was achieved when the German firm of Hoesch-Siegerlandwerke AG was awarded a "Zulassung" for both its wall panel system [8] and its roof deck system [9]. A Zulassung is an official approval document which implies a high level of technical assessment and these first formal approvals for the use of sandwich panels in Germany were only obtained after several years of intensive experimental and theoretical research. These approval documents are worthy of careful study by engineers responsible for the design of sandwich panels for use as the walls and roofs of buildings.

Incorporated in these Zulassung documents is an approximate solution for the stresses and deflections in sandwich panels which gives results of acceptable accuracy for simply supported panels. The background to this solution has been given by Wölfel [10].

A factor k is first calculated which describes the influence of core shear on the deformation of the panel. Some expressions for k are given in Figure 5. It is calculated (for example using the principle of virtual work) on the assumption that the applied load is shared between two independent systems, namely by the sandwich part, which includes the influence of core shear, and by bending of the flanges. The deflections of these two independent systems coincide at mid-span.

On this basis, an applied distributed load q is divided into a sandwich part q_S and a flange part q_D where

$$q_S = \frac{B_S q/(1+k)}{B_D + B_S/(1+k)}$$

$$q_D = \frac{B_D\, q}{B_D + B_S/(1+k)} \qquad \text{......................(22)}$$

and where B_S and B_D, the respective bending stiffnesses, have been defined previously. The corresponding stresses then follow from simple beam theory and the mid-span deflection is given by

$$w_{max} = \frac{5(1 + k)}{384} \frac{q_s L^4}{B_S} \qquad \ldots\ldots\ldots\ldots\ldots(23)$$

Similarly, a temperature difference giving rise to a curvature coefficient $\theta = (\alpha_2 T_2 - \alpha_1 T_1)/D$ can be shown to give rise to component bending moments M_S and M_D in the sandwich and flange parts given by

$$-M_S = M_D = B_D \Theta \frac{B_S/(1 + k)}{B_D + B_S/(1 + k)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

and a central deflection of

$$w_{max} = \frac{L^2\theta}{8} \frac{B_S/(1 + k)}{B_D + B_S/(1 + k)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(25)$$

These approximate procedures can be extended, with similar simplicity, to two-span beams but the accuracy is then diminished to the point where their value is questionable. It may be noted that the division of the load-carrying behaviour of a sandwich panel into the sandwich part and the flange part is of fundamental importance and will be used in the development of the finite element methods which follow.

5. Numerical methods of analysis

Evidently, if general solutions for sandwich panels with arbitrary loading and boundary conditions are to be obtained, recourse must be made to numerical methods of analysis. The first general method appears to have been derived by Berner[11] who developed a finite difference solution of the governing differential equations. This solution has subsequently been reported by Berner and Jungbluth[12] in a review of recent German research.

More recently, Schwarze[13] has developed an exact general solution which may be described as a quasi-finite element method in that partial solutions of the governing differential equations allow solutions of complete general problems to be set up as a set of linear simultaneous equations. However, the procedures are rather specialised and cumbersome and in the author's opinion, the finite element solutions which follow are to be preferred.

6. Finite element solutions using available programs

It is clear from the preceding sections that in a sandwich panel there are three essentially separate load-carrying systems, namely
(a) bending in the upper profiled face;
(b) bending in the lower profiled face;
(c) sandwich action involving axial forces in the faces together with shear in the core;
and that, for most purposes, the first two can be combined.

It follows that the most direct way to model a sandwich panel for finite element analysis is to represent the upper and lower faces by beam elements with the relevant cross-sectional area and second moment of area and with three degrees of freedom per node and to connect these together with a suitable plane stress element to represent the properties of the core. Such a representation is shown in Figure 6.

As the fundamental behaviour is modelled precisely, this finite element representation gives accurate answers and it can, of course, be used to analyse any situation however complex or irregular. However, as connection between the flanges and the core is only provided at discrete points and as the shear force in the core can change quite rapidly, it is necessary to use a fairly large number of elements per span (of the order 10 to 20) in order to obtain adequate accuracy in many situations.

It should be noted that in the classical solutions, axial strain in the core is neglected and there is no attempt to account for the distribution of shear stress within the depth of the core which is assumed to be constant. These assumptions are found to give more than adequate accuracy. It follows that there is no virtue in sophisticated modelling of the core and indeed all that is required is an element capable of modelling the deformation of the core due to a state of pure shear between the nodes. Thus the rectangular finite element shown in Figure 6 may be the simplest possible plane stress element with two degrees of freedom per node and there are a total of six degrees of freedom on each nodal line.

The advantage of this method is that it does not require special programming provided that a program package is available which combines beam or frame elements with rectangular plane stress elements. A further advantage is that the important load case of temperature difference between the faces can be included very easily in this formulation as it merely involves the introduction of appropriate axial strain in the flanges.

A useful variation on the above finite element model is obtained when it is realised that the state of pure shear between the nodal lines can be modelled with negligible loss of accuracy by replacing the rectangular plane stress elements by the internal members of a truss as shown in Figure 7. The internal members have axial stiffness only. The verticals have sufficient area to make the relative vertical displacement between the faces negligible. The area A_d of the diagonals is given by

$$A_d = \frac{B.G_{eff}\,\ell^3}{2pE_dD} \qquad \text{.................(26)}$$

where ℓ = length of diagonal = $\sqrt{D^2+p^2}$

E_d = Young's modulus of diagonal member

The particular advantage of the truss analogy is that the problem may be solved using any available program for plane frame analysis. As nowadays, suitable programs are available on almost all computers used by structural engineers, this means that the analysis of arbitrary sandwich panels is readily available without special programming. The disadvantage is that the data preparation is onerous although it is not difficult to write a simple data generator and post-processor in order to remove this problem.

As a number of elements are necessary to model a given span, there is little loss of accuracy if distributed loads are applied as point loads at the nodes.

7. Exact finite element formulation

If a suitable finite element is subject to nodal forces only (including nodal equivalent forces for distributed loads and temperature gradients etc.) the total shear force Q in the element will be constant and the bending moment M linear. Differentiating equations (4) and (6) twice then gives the following governing differential equations in which the bending moment and shear terms are eliminated.

$$\left.\begin{aligned} w^{vi} - \left(\frac{\lambda}{L}\right)^2 w^{iv} &= 0 \\ \gamma^{iv} - \left(\frac{\lambda}{L}\right)^2 \gamma^{11} &= 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(27)$$

The general solutions of these equations, in a form which reduces possible numerical ill-conditioning, are

$$\left.\begin{aligned} w &= C_1 e^{-\lambda x/L} + C_2 e^{-\lambda \bar{x}/L} + C_3 x^3 + C_4 x^2 + C_5 x + C_6 \\ \gamma &= D_1 e^{-\lambda x/L} + D_2 e^{-\lambda \bar{x}/L} + D_3 x + D_4 \end{aligned}\right\} \ldots\ldots(28)$$

where $\bar{x} = L-x$. Equations(28) must also satisfy (3) and it follows that:

$$\left.\begin{aligned} D_1 &= -(1+\alpha)\left(\frac{\lambda}{L}\right) C_1 \; ; \quad D_2 = (1+\alpha)\left(\frac{\lambda}{L}\right) C_2 \\ D_3 &= 0 \; ; \quad D_4 = -\frac{6Bs}{AG_{eff}} C_3 \end{aligned}\right\} \ldots\ldots\ldots(29)$$

so that the number of arbitrary constants in the complete solution reduces to the six in the first equation (28). An exact finite element formulation therefore requires 3 degrees of freedom per node together with the corresponding nodal forces. The best choice appears to be the total shear force Q together with the components of bending moment M_D and M_S in the faces and sandwich part respectively. Thus, the displacement vector at each node takes the form $\{w, w^1, w^1 - \gamma\}$ and the relationship between the nodal displacements and arbitrary constants is

$$\begin{bmatrix} w_1 \\ w^1_1 \\ w^1_1 - \gamma_1 \\ w_2 \\ w_2^1 \\ w_2^1 - \gamma_2 \end{bmatrix} = \begin{bmatrix} 1 & e^{-\lambda} & 0 & 0 & 0 & 1 \\ -\left(\frac{\lambda}{L}\right) & \frac{\lambda}{L}e^{-\lambda} & 0 & 0 & 1 & 0 \\ \alpha\frac{\lambda}{L} & -\alpha\frac{\lambda}{L}e^{-\lambda} & \frac{6Bs}{AG_{eff}} & 0 & 1 & 0 \\ e^{-\lambda} & 1 & L^3 & L^2 & L & 1 \\ -\frac{\lambda}{L}e^{-\lambda} & \frac{\lambda}{L} & 3L^2 & 2L & 1 & 0 \\ \alpha\frac{\lambda}{L}e^{-\lambda} & -\alpha\frac{\lambda}{L} & 6\frac{Bs}{AG_{eff}} + 3L^2 & 2L & 1 & 0 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix} \ldots\ldots\ldots\ldots(30)$$

i.e. $w = AC$ ....(31)

and $C = A^{-1} w$ ....(32)

The nodal force vector at each node takes the form $\{Q, M_D, M_S\}$ and the individual terms can be evaluated in terms of the arbitrary constants C_i using the equations

$$
\begin{aligned}
Q &= AG_{eff}\gamma - B_D w^{111} \\
M_D &= -B_D w^{11} \\
M_S &= B_S(\gamma^{1} - w^{11})
\end{aligned}
\qquad \text{.....................(33)}
$$

After a little rearrangement and changing the signs of the terms in Q_1, M_{D2}, M_{S2} to accord with the usual sign convention for finite element analysis (see Figure 8) this gives

$$
\begin{bmatrix} Q_1 \\ M_{D1} \\ M_{S1} \\ Q_2 \\ M_{D2} \\ M_{S2} \end{bmatrix}
\begin{bmatrix}
0 & 0 & 6B & 0 & 0 & 0 \\
-B_D(\frac{\lambda}{L})^2 & -B_D(\frac{\lambda}{L})^2 e^{-\lambda} & 0 & -2B_D & 0 & 0 \\
B_D(\frac{\lambda}{L})^2 & B_D(\frac{\lambda}{L})^2 e^{-\lambda} & 0 & -2B_S & 0 & 0 \\
0 & 0 & -6B & 0 & 0 & 0 \\
B_D(\frac{\lambda}{L})^2 e^{-\lambda} & B_D(\frac{\lambda}{L})^2 & 6LB_D & 2B_D & 0 & 0 \\
-B_D(\frac{\lambda}{L})^2 e^{-\lambda} & -B_D(\frac{\lambda}{L})^2 & 6LB_D & 2b_S & 0 & 0
\end{bmatrix}
\begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{bmatrix}
\quad \text{..(34)}
$$

i.e. $F = BC = BA^{-1} w$ ..... (35)

The stiffness matrix is thus obtained as the product of two 6 X 6 matrices and for sandwich panels subject to point loads and moments applied at the nodes, this is all that is required for exact solutions. Element stiffness matrices derived in this way can be assembled to form the global stiffness matrix of the complete sandwich panel according to the usual rules which will be found in any text book dealing with the matrix analysis of structures. The global stiffness equations may then be solved to give the displacement vectors $\{w, w^{1}, w^{1} - \gamma\}$ at the nodes. The nodal stress resultants then follow from the element stiffness equations (35).

If values of the displacements or nodal stress resultants are required at points other than the natural nodes, the simplest solution is to insert additional nodes as necessary. Alternatively, equations (30) and (32) can be used to evaluate the arbitrary constants in equations (28) and the complete pattern of deflections and forces obtained with the aid of (33).

The applied load vector for a temperature difference between the faces is simple to derive. In the absence of bending stiffness in the faces, temperatures T_1 and T_2 in the faces give rise to a curvature given by $\Theta = (\alpha_2 T_2 - \alpha_1 T_1)/D$. This curvature could also be obtained by applying a uniform moment M_{ST} to the sandwich part of the profile where

$$M_{ST} = B_S \theta$$

It is therefore merely necessary to apply a load vector

$$\{Q_1, M_{S_1}, M_{D_1}, Q_2, M_{S_2}, M_{D_2}\} = \{0, 0, B_S \Theta, 0, 0, -B_S \Theta\} \quad \text{.........(37)}$$

and then to subtract these artificially applied moments at the conclusion of the analysis.

When an element is subject to a uniformly distributed load it is necessary to apply nodal equivalent forces and these require a separate calculation. As shown in Figure 8, the nodal equivalent forces are "fixed end moments" and shears reversed and the simplest way to calculate these is to return to the original fourth order differential equation (4) and its companion equation (6) and to insert the appropriate boundary conditions into the solution.

If $M_F = M_{FS} + M_{FD}$ is the total fixed end moment, the complete solution for the situation shown in Figure 8(b) is

$$w = C_1 e^{-\lambda x/L} + C_2 e^{-\lambda \bar{x}/L} + C_3 + C_4 x - \frac{1}{B}\left[M_F \frac{x^2}{2} + \frac{qL^4}{24}\left(-\frac{x^4}{L^4} + \frac{2x^3}{L^3} + \frac{12x^2}{\lambda^2 \alpha L^2}\right)\right] \quad \text{...(38)}$$

$$\gamma = (1+\alpha)\left[-\frac{\lambda}{L} C_1 e^{-\lambda x/L} + \frac{\lambda}{L} C_2 e^{-\lambda \bar{x}/L}\right] + \frac{(1+\alpha)}{B\lambda^2 \alpha} \frac{qL^3}{2}\left(1 - \frac{2x}{L}\right) \quad \text{.......}$$

The boundary conditions for fixed ends are

$$\text{at } x = 0, L; \qquad w = 0, \quad w^{1} = 0, \quad \gamma = 0 \quad \text{..........(39)}$$

Inserting these conditions and after some manipulation,

$$Q_F = A\gamma - B_D w^{111} = \frac{qL}{2}$$

$$M_{FD} = B_D w^{11} = \frac{B_D}{B} \frac{qL^2}{12}\left[-1 - \left(\frac{1+e^{-\lambda}}{1-e^{-\lambda}}\right)\frac{6}{\lambda\alpha} + \frac{12}{\lambda^2 \alpha}\right] \quad \text{..............(40)}$$

$$M_{FS} = B_S(\gamma^1 - w^{11}) = \frac{B_S}{B} \frac{qL^2}{12}\left[-1 + \left(\frac{1+e^{-\lambda}}{1-e^{-\lambda}}\right)\frac{6}{\lambda} - \frac{12}{\lambda^2}\right]$$

It may be noticed that, as a direct consequence of the boundary condition $\gamma = 0$, the core shear $Q_S = A\gamma$ at a fixed end or at the internal support of a symmetrical two span beam must be zero and all of the shear force is carried in the flanges.

Equations (40) complete the information required for the finite element solution of any arbitrary sandwich panel loaded by a combination of point loads, distributed loads and temperature differences.

An example will now be considered.

8. Example

The example given in this section was first presented by Schwartze[13]. The cross-section is shown in Figure 9 and the general arrangement and loading in Figure 10. The following properties for a panel of one metre width were used in the analysis in addition to the dimensions shown in Figure 9.

$E_1 = E_2 = 210$ kNm/mm^2

$G_{eff} = 4$ N/mm^2

$A_1 = 845$ mm^2

$A_2 = 510$ mm^2

$I_1 = 172700$ mm^4

$I_2 = 11.1$ mm^4

The distributions of bending moment, shearing force and deflection given by the exact finite element analysis, together with the more important numerical values, are in Figure 11. The corresponding distributions using the truss analogy (16 elements per span) were also obtained and in general these are so similar to those shown that separate representation is not possible. Numerical values obtained using the truss analogy are given in brackets. The only aspects of the behaviour that are not accurately reproduced by the truss analogy are, not surprisingly, the very sharp peaks in the flange member shear force adjacent to the point load and central support and the corresponding distributions of shear force in the core.

Continuous sandwich panels, and panels subject to point loads, invariably exhibit rapid fluctuations of shear force and approximate methods of analysis generally find these difficult to follow. This is true of approximate solutions of the type discussed in section 4 and finite difference methods as well as non-exact finite element procedures and is why the truss analogy requires a relatively large number of nodes per span.

It is also of interest to note the typical result that the maximum shear force in the core, which is a factor which may influence the design, is within the span and the core shear forces at the support and at the load point are both relatively small.

The distributions of bending moment and shear force appear to agree well with those obtained by Schwartze[13] although he only gives numerical values for the moments at the internal support which may be compared as follows:-

	Exact FE	Schwartze	Truss Analogy
M_S (kNm/m)	0.813	0.808	0.775
M_D (kNm/m)	0.801	0.804	0.791

9. Conclusions

The state of the art regarding the analysis of structural sandwich panels has been reviewed and a number of new approaches presented. For the vast majority of practical cases, including panels spanning over one, two or three equal spans subject to uniformly distributed load or temperature difference between the faces, explicit solutions of the governing differential equations is possible and solutions have been given.

For irregular cases, which will generally involve unequal spans or non-uniform loading, recourse must be made to numerical methods.

An exact finite element solution has been given and, of course, the solutions given by this method are precise. At the less sophisticated end of the scale, acceptable solutions can be obtained using existing plane frame analysis programs. These require no programming effort but a penalty is paid in terms of the data preparation required.

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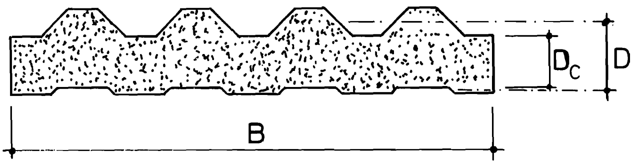
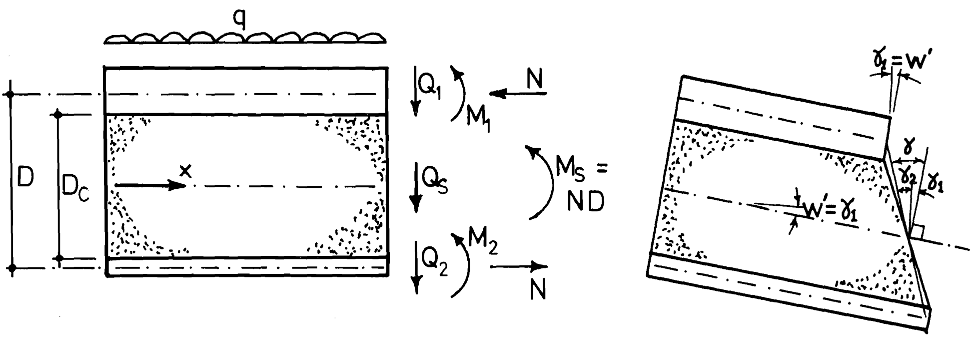


Fig. 1 Typical sandwich panel with profiled faces



(a) dimensions and stress resultants

(b) deformed element

Fig. 2 Forces and deformations in typical sandwich element

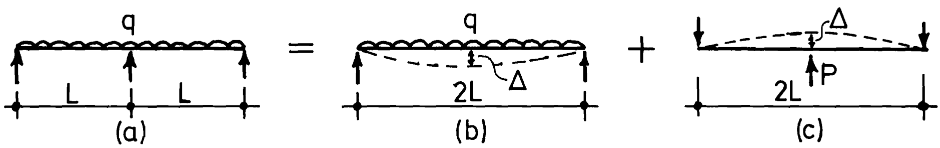


Fig. 3 Solution for two-span panel

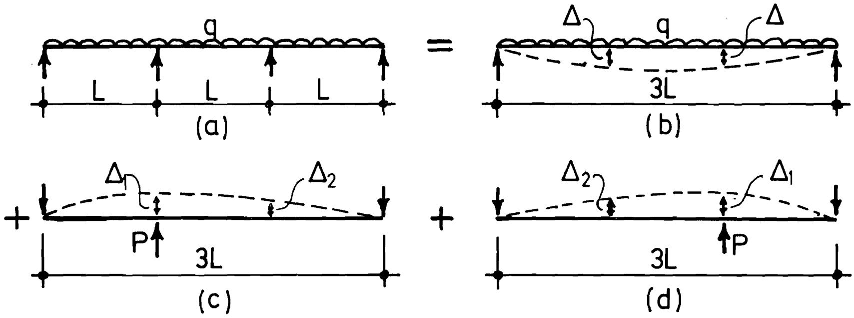


Fig. 4 Solution for three-span panel

General arrangement	Expression for k
uniformly distributed q; L	$9.6 \dfrac{B_S}{G_{eff} A L^2}$
$q_0 \sin \frac{\pi x}{L}$; L	$\pi^2 \dfrac{B_S}{G_{eff} A L^2}$
temperature difference ΔT; L	$8 \dfrac{B_S}{G_{eff} A L^2}$

Fig. 5 Expressions for shear factor k

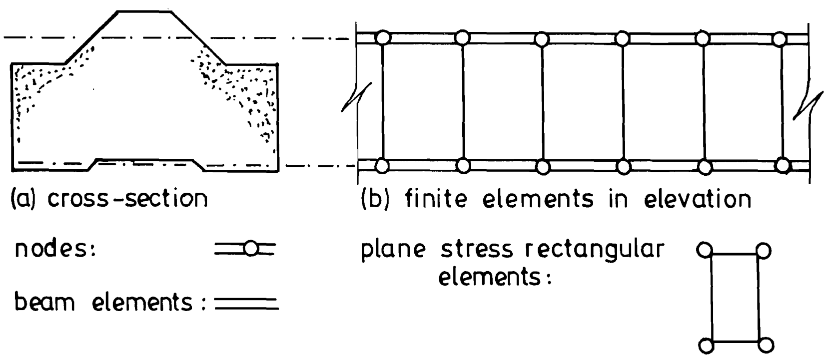


Fig. 6 Finite element representation of sandwich element

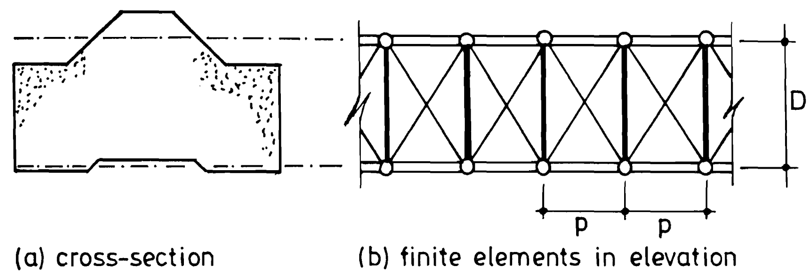
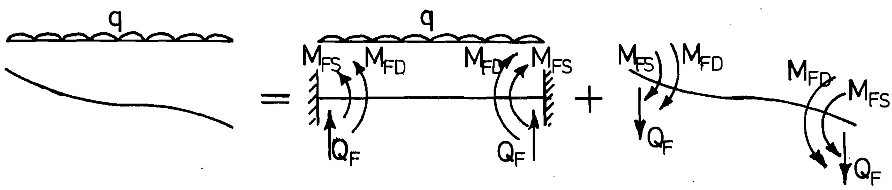


Fig. 7 Plane frame simulation of sandwich element (truss analogy)



(a) element with U.D.L **(b)** fixed-end element with U.D.L **(c)** fixed-end forces reversed

Fig. 8 Nodal equivalent forces for uniformly distributed load

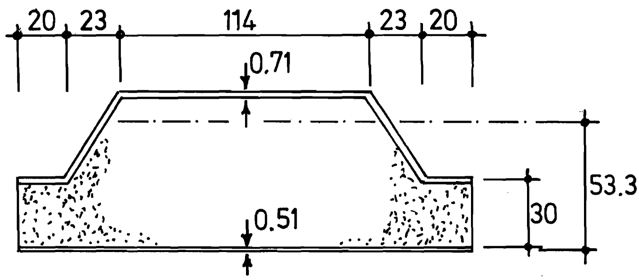


Fig. 9 Cross-section for example (Hoesch Isowand TL66)

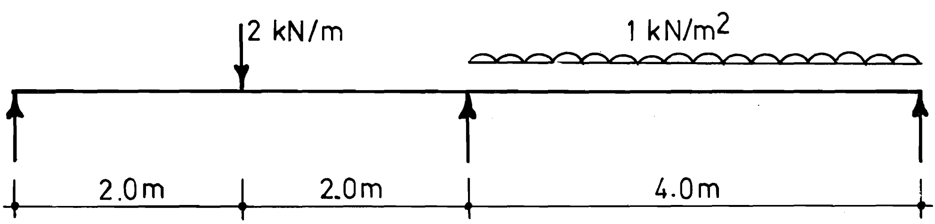


Fig. 10 General arrangement for example

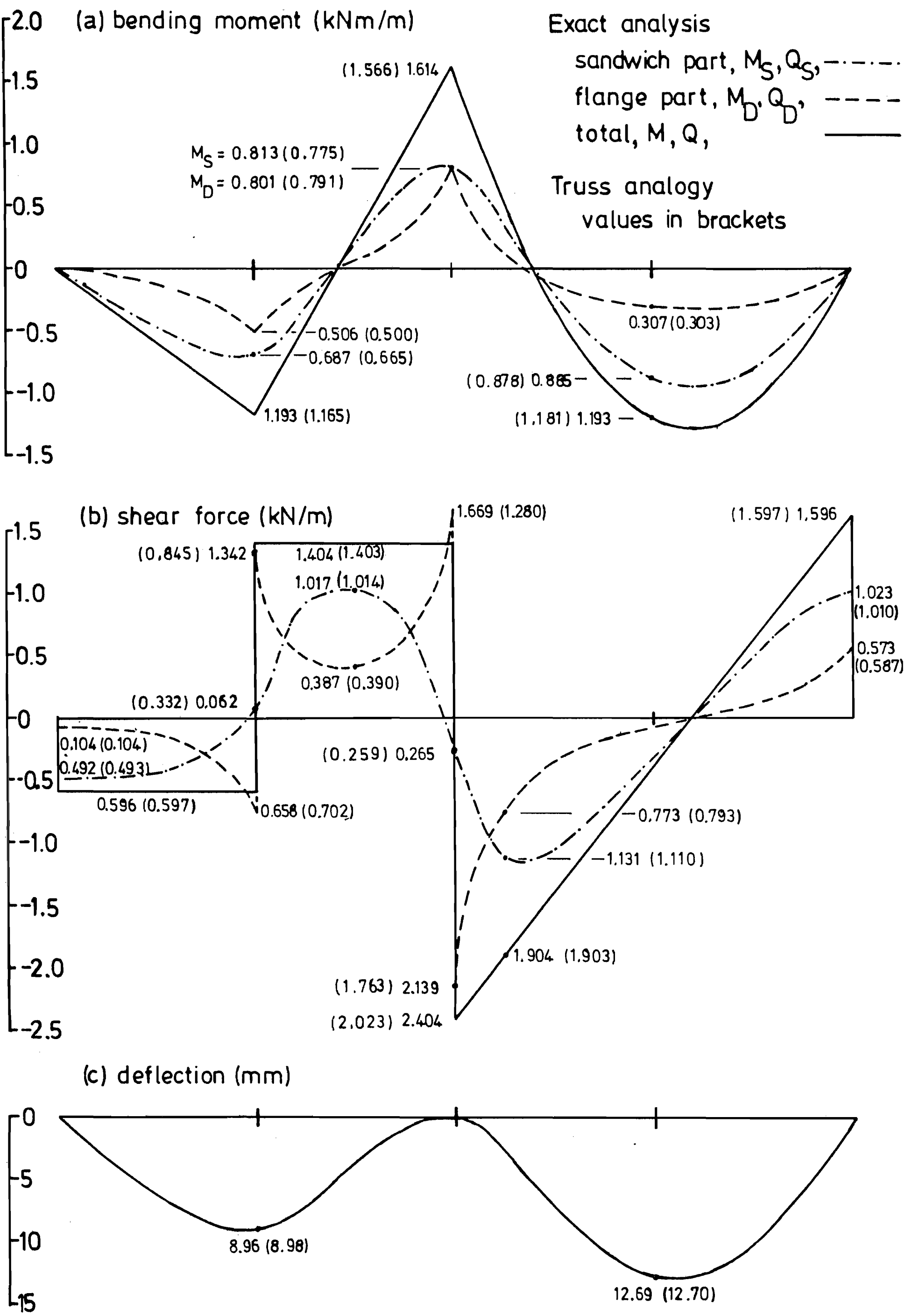


Fig. 11 Stress resultants and deflections for two-span panel