A study of viscous frictional effects and relay characteristics on the operation of a positional relay servomechanism

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A STUDY OF VISCOUS FRICTIONAL EFFECTS AND RELAY CHARACTERISTICS ON THE OPERATION OF A POSITIONAL RELAY SERVOMECHANISM

BY

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A

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ABSTRACT

The thesis describes some experimentally observed data and analytical approaches to the study of a relay servomechanism. The experimental data were secured from an analog simulation of a relay servo system. The results of this data were compared to the transient approach by use of the phase-plane method. The stability of the system was determined using a describing-function. The analytical approaches are described in the presentation.

A study of the effects of viscous friction is described, and the characteristic of the relay used in the system is given special attention. The response to input driving function was considered, and rise time, overshoot, settling time and steady state errors were considered as important criteria for determining the over-all performance.

Finally, a recommendation is given as to the approach to the synthesis and analysis of a relay servomechanism.

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For the many hours of typing by my wife, Marjorie, I give my thanks. To the patience of two youngsters, Dennis and Laura, who had other ideas, my appreciation.

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LIST OF SYMBOLS

Symbol	Analytical or Physical Meaning of Symbol
e	Instantaneous value of error signal in the relay servo.
e_m	Maximum value of error signal in relay servo.
e_a	Armature voltage to servo motor.
R_a	Armature resistance of servo motor.
K_t	Motor torque constant.
K_e	Counter emf constant of motor.
N	Gear reduction ratio.
θ_i	Input shaft position.
θ_L	Load shaft position.
J_{eL}	Effective polar moment of inertia of servo motor and load referred to load shaft.
f_{eL}	Effective viscous friction of servomotor and load referred to load shaft.
f_L	Viscous friction of load.
f_m	Viscous friction of servo motor.
J_L	Polar moment of inertia of load.
J_M	Polar moment of inertia of motor.
i_a	Servomotor armature current.
τ	Time constant of motor and load.
K	Gain constant of motor and load.
G_d	Describing-function of relay.
G	Transfer function of linear components of system.

M_p	Peak value of output to input ratio.
α	Attenuation constant of lead network.
C_1, C_2	Phase lead network capacitors.
R, R_1	Phase lead network resistors.
P	"Pull in" voltage of relay.
D	"Drop out" voltage of relay.
$\bar{\theta}_o$ (S)	Laplace Transform of θ_o (t).
Δ	Dead zone width of relay.
h	Hysteresis width of relay.
E_o	Output of phase lead network which becomes the input to relay in the circuitry used.
E_i	Input to phase lead network which is the error signal in circuitry used.

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I. INTRODUCTION

Definition of a feedback control system could be expressed as follows:

A feedback control system is one in which a command signal is compared to a signal from a controlled mechanism. The difference between the two signals, called the error signal, is used to operate a power element which changes the controlled variable in such a manner as to reduce the error signal to zero, or nearly zero.

Feedback control systems, as known today, have been divided into two distinct groups based on their inherent operating characteristics. These two groups have become known as the linear and the nonlinear feedback control systems. The former systems are given a great deal of attention and investigation in most undergraduate and also in the first graduate level electrical engineering curricula. A brief introduction to nonlinear systems is generally included in this sequence. It is unfortunate, however, that most programs end at this level, because it has been estimated that seventy per cent of the control systems in operation today are nonlinear systems.

Linear systems are few because of the introduction of nonlinear components into physical systems. Some of these components include backlash in gearing, amplifier saturation, torque saturation in servo motors and other such nonlinearities. It is also known that direct introduction of nonlinear elements has been advantageous in compensation, hence, some systems are deliberately made nonlinear.

If these nonlinearities are small, then the system can be analyzed using linear methods. If not, nonlinear methods must be used. Nonlinear analysis is not as well defined as linear methods and various techniques are available to analyze and synthesize nonlinear systems.

Feedback control systems can also be classified according to the type of correction used in reducing the error between command signal and the controlled variable. The continuous type system is one in which there is always a correction, when an error exists between command signal and the controlled variable. This would indicate, that as the error reduces, the amount of correction also reduces. This type of system can be made to have a fast response, good stability and operating characteristics, if enough gain is supplied and enough compensation is introduced. This, then, leads to a large system, expressed in weight, cost and size.

The discontinuous system can be made to have the desirable characteristics of this foregoing linear system with less components. The system is characterized by supplying a large correction signal if the error is above a certain value, as specified by the system, otherwise there is no correction at all. The relay type servo system operates with this type of control. This means that a relay servo, utilizing a servo motor, produces rated torque output in the proper direction to reduce the error. Systems of this sort have a tendency to oscillate at small amplitudes but can be stabilized by use of damping and compensation networks.

A simple relay regulator servo, for example, has been in use for years in home heating control, (the thermostat in combination with a heat source). The thermostat, acting as the relay in this case, controls a large source of heat. Being an on-off system, the thermostat demands full heat or nothing at all. Depending on the temperature differential of the thermostat, the room temperature oscillates about a pre-selected temperature. Much has been done to re-design the system to compensate for such conditions as; sudden changes in outside temperature or wind velocities. In this manner the heat control system can anticipate these changes and take corrective action so that the temperature variation can be kept to a minimum.

Other examples of a simple nature could be remote steering devices for ships, liquid level regulation, voltage regulation systems and speed regulation. The major difficulties with these systems which use on-off control arises when high accuracy must be combined with stability.

This thesis considers the operation of a second-order type 1, position control relay servo system when such variables as relay contact spacing are considered and methods by which system performance can be improved. The aspect of using viscous friction as a stabilizing element is considered; and the relay characteristic is studied from the standpoint of stability, rise time and overshoot. The advantages and disadvantages of the relay servo can then be determined in comparison to an equivalent linear servo.

The Royal McBee LPG 30 Digital Computer, located at the Missouri School of Mines Computing Center was used for the following purposes:

1. Obtaining data for isoclines on the phase-plane plots.
2. Securing data for the Bode plots of the servo motor and load.
3. Data for plotting the describing-function loci.

The entire relay system was simulated on the Electrical Engineering Department's analog computer, from which experimental data was secured to enable a comparison between the theoretical results and the actual test results.

II. REVIEW OF LITERATURE

The relay type servomechanism is not a recent development, for these systems have been in existence in the form of regulator controls for some time. Nearly all the literature, however, is restricted to periodicals within the last ten years. One of the earliest contributions was a paper by Weiss (1)* in 1946 on a simple servo relay system. Weiss applied the methods, which Minorsky (2) previously used for the graphical representation of displacement and velocity in the field of nonlinear mechanics, to the relay servo. The representation of position and velocity of the servo in two dimensions is called the position-velocity plane or more commonly called the phase plane. In this plane the discontinuous nature of the driving torques divide the phase plane into regions where, within each region, motion can be represented by a simple differential equation. The two response relationships of position and its first derivative are expressed. Time is eliminated as a variable and made a parameter. These quantities, when plotted, form a graphical representation in two dimensions, called the phase-plane. By forming the trajectory paths for step inputs in velocity or position, or both, the output can be graphically expressed.

*All references appear in the Bibliography.

In 1949 Kahn (3) applied Laplace Transform methods to a nonlinear system by subdividing the discontinuous regions into as many piece-wise linear regions as necessary. Each region was analyzed individually and when appropriate boundary conditions were applied, a solution was obtained in the time domain. The difficulty which arises is that synthesis of a system by this method becomes extremely difficult.

In 1950, Kochenberger (4), seeking to solidify all previous contributions to relay servos, presented a method to synthesize a system using a describing-function approach. This is a so-called frequency response method whereby the nonlinearities which are periodic can be represented as terms of a Fourier series. By considering only the first term of such a series, a linear approach can be approximated. Synthesis of a system by the means of frequency response gives insight as to the selection of compensating networks more readily than the cut and try methods of the phase-plane, or the differential equation approach. Nyquist's stability criterion, therefore, applies so that stability can be studied in the polar plane.

The aircraft industry, interested in rugged servo controls which were reliable and simple, promoted the use of the on-off type of systems. Parziale and Tilton (5), in 1950;

and Stuelpnagle and Dallas (6), in 1953, introduced on-off magnetic clutch type servos. The clutch servo incorporates a driver which operates at a constant velocity and through the use of clutches, either positive or negative torques are applied to the controlled variable. These torques are maximum restoring torques and produce characteristics similar to relay servomechanisms.

Johnson (7) presented more comprehensive methods when analyzing nonlinearities in feedback control systems. He also introduced an insight as to the magnitude of the errors involved by using describing-function methods. Kazda (8) elaborated on the errors involved in relay servos under certain conditions of operation.

It is interesting to note that the authors of papers mentioned, prescribe analysis and synthesis for specified input functions. The driving function has a very decided effect on the response of the system and in general, for this reason, a system which produces a desirable response to one input may produce an unsuitable response for another form of input.

The phase-plane method, as a graphical method, is the two dimensional analysis of the more general phase-space method. Second order systems can be completely represented in two dimensions. Bogner and Kazda (9); Kuba and Kazda (10) have investigated switching and synthesis of nonlinear systems by phase-space methods for higher order systems with inputs limited to step functions. The systems involved must be describable over every interval of its operation by a linear differential equation. A considerable amount of the data these authors obtained was from analog computer studies of the systems considered.

Hart (11) contributed a paper on an analytical design of a relay servomechanism when investigating "dead beat" systems. "Dead beat" systems are systems where one application of restoring torque is sufficient to cause the error and all derivatives of the error to become zero simultaneously. Conclusions reached were that for the higher order systems, solutions of simultaneous transcendental equations were required. As a practical design approach, it was further suggested that approximate or empirical methods be used.

Chao (12) elaborated on the describing-function methods of Kochenburger and supported his conclusions by means of analog computer studies. It was found that in the systems studied, an accuracy of about 7 per cent can be expected when analysis is made by describing-function methods on simple systems.

In the last three years more attention has been given to compensation methods by linear and nonlinear circuits. It is a known fact that if fair static accuracy of relay systems is to be accomplished, compensation must be included. The nonlinear circuits, as proposed by McDonald (13) are being expounded to provide good static accuracy, excellent rise time and small overshoot. Embler and Weaver (14); Harris, McDonald, Thaler (15); McDonald, Thaler (16) are chiefly responsible for the literature in this field.

In the last two years, Buland and Furumoto (17) have promoted use of dual mode systems. By the addition of high speed switching relays to relay servos, optimum response can be obtained by switching from nonlinear modes of operation to linear modes of operation at the proper time.

III. THE BASIC SYSTEM

The basic system which was chosen for study was to lend itself to answering the following questions:

1. The effects of viscous friction were to be studied on the relay servo system response. Viscous friction is known to have a stabilizing affect on systems, but how much is desirable? If a desirable value is inserted into the relay system, how then does it compare to an equivalent linear servo without a relay?

2. In addition to viscous friction, what effect does the relay characteristic itself play in a successful system? Is an ideal relay desirable? What effect does the relay characteristics of a typical manufactured relay have on a relay system? Does the fact that this relay has a different "pull in" and "drop out" voltage affect the relay servo?

3. What effect do compensation circuits have on a relay servo?

To study these effects, a simple relay system was chosen which consisted of a controller, relay, motor and load. The motor-load combination selected was a type 1 second order system. The system is schematically represented in Fig. 1

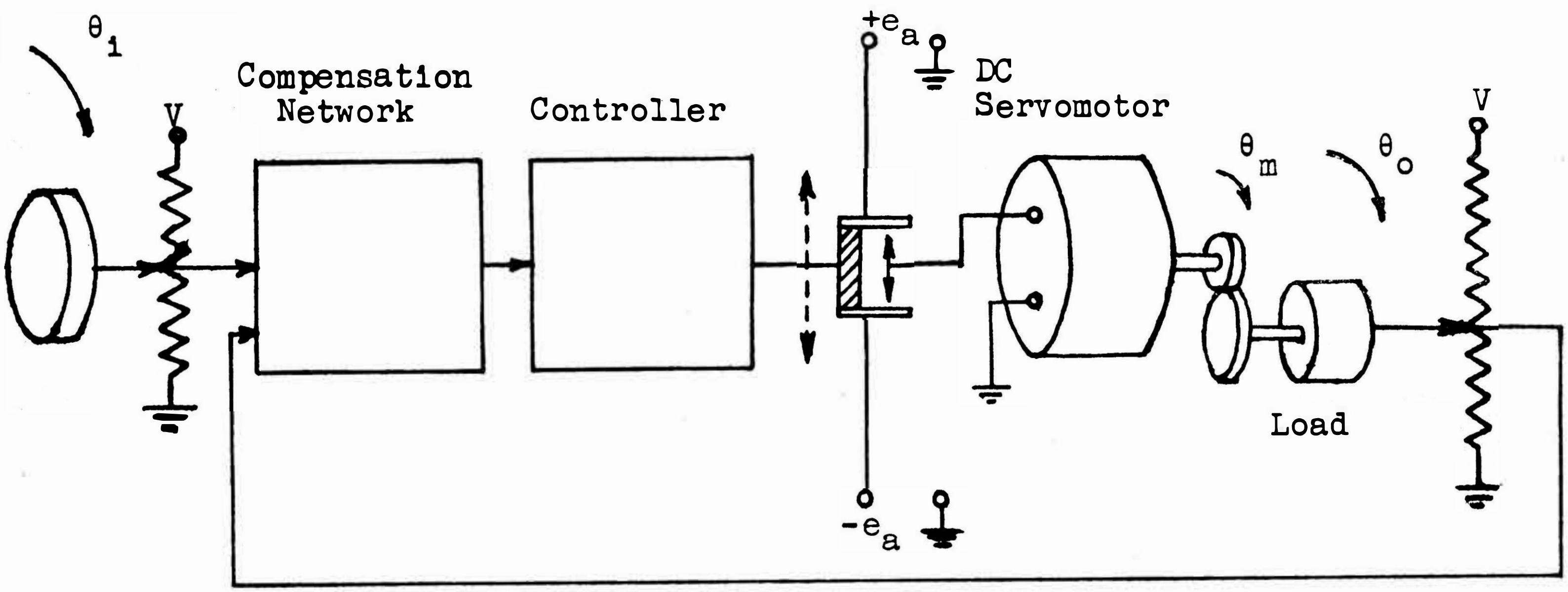


FIGURE 1 SIMPLE RELAY SYSTEM

The error signal, which is a function of the positional error between θ_i and θ_o is applied to a relay. Here the error signal is $e = \theta_i - \theta_o$. The relay is polarized so that it is both sensitive to the magnitude and polarity of the error signal. The relay, therefore, has control of the direction of rotation of the reversible motor. The relay operating characteristic can take three forms. These are represented in figure 2 and the relay analog computer simulation for each is also shown. The analog computer circuit for simulation of the relay appears in Appendix II.

The motor-load differential equation for the system under consideration is derived in Appendix II. Rewritten here, itis:

$$\frac{e_a}{R_a} K_t N = J_{eL} \frac{d^2 \theta_o}{dt^2} + \left(f_{eL} + \frac{K_e K_t N^2}{R_a}\right) \frac{d \theta_o}{dt} \qquad (1)$$

The viscous friction term appears as the coefficient of the first derivative of θ_o. The damping effect was observed with regard to system, stability and response. However, other than the stability produced by viscous friction, its occurance is objectionable as it increases the response time of a system by reducing the runaway velocity. By proper use of linear compensation and proper addition of viscous friction, desirable relay servo systems can be obtained.

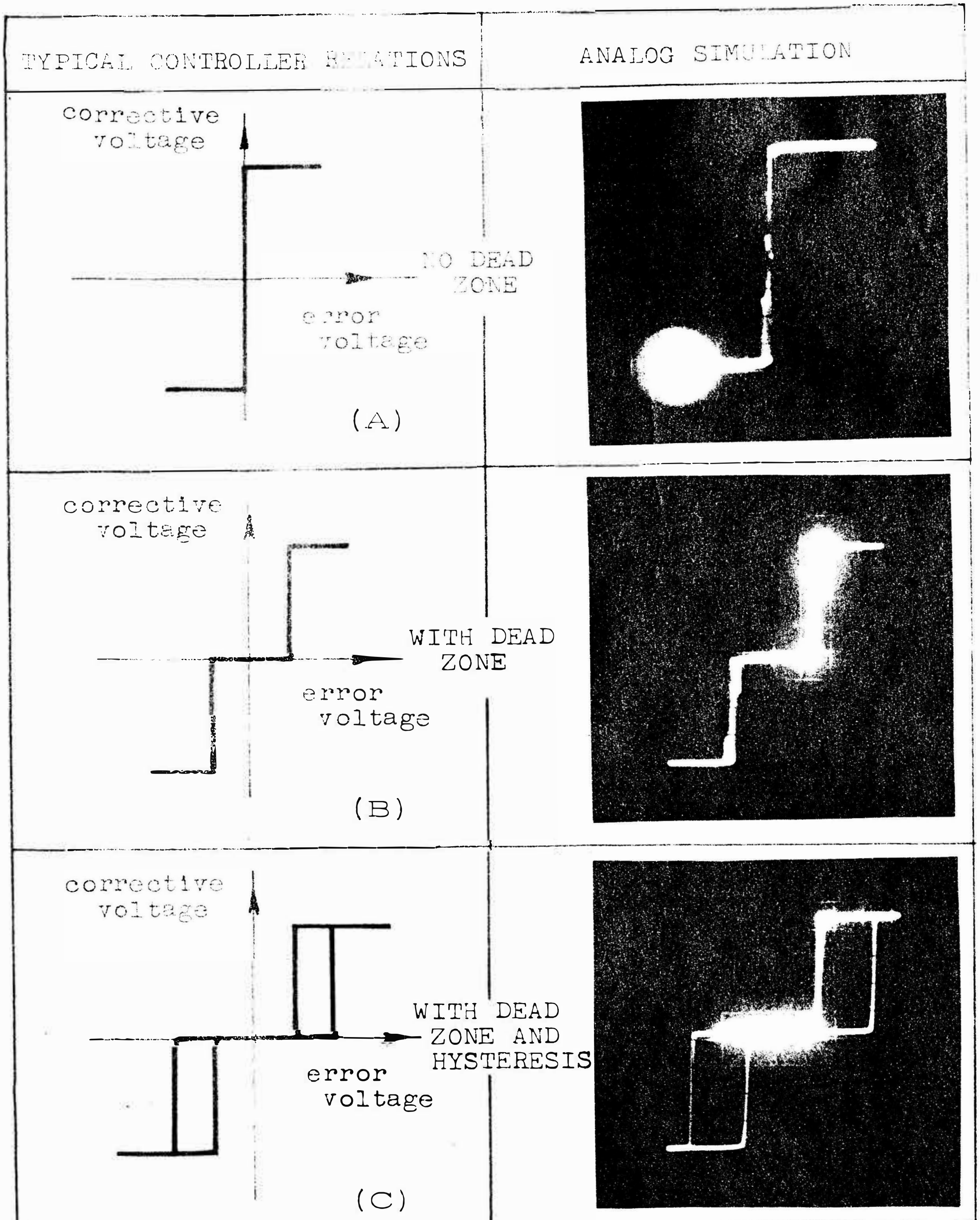


FIGURE 2 RELAY CHARACTERISTICS

IV. ANALYTICAL METHODS

Before proceeding, it will be necessary to briefly mention the techniques used in the course of this investigation. For more detailed analysis references, (4), (9), (10), (18) and (19) are suggested.

(A) THE PHASE-PLANE

The phase-space method, as applied to the study of ordinary nonlinear differential equations, is a graphical method in which a variable and its time derivatives are plotted. The space dimension required is determined by the number of degrees of freedom of the particular equation. When a dynamic system, having only one degree of freedom is studied, a two dimensional phase-space results, commonly called the phase-plane.

For the second-order positional relay servo to be studied here; the dependent variable and the time rate of change of this variable was plotted in the phase-plane. A trajectory in the phase-plane is a locus of the path that a dynamic system follows when given some initial starting point. Moreover, there will be one and only one trajectory through each point of the phase-plane.

Phase-plane analysis in general is not applicable to systems subjected to external driving functions. The phase-plane method can, therefore, be used to predict transient response limited to systems when subjected to some initial condition of step driving functions.

The trajectories can be plotted for a system by any of the following methods:

Consider a second order differential equation of the form:

$$\frac{Ad^2 x}{dt^2} + B \frac{dx}{dt} + Cx = D \qquad (2)$$

(a) By integration, a time solution of $\dot{x} = f(x,t)$ and $x = g(t)$ can be obtained. By elimination of time between these equations, a relation of $\dot{x} = h(x)$ could be obtained and x versus $\dot{x}$ could be plotted. In general, however, if x can be found as a function of time, the transient performance is already defined and there is no reason to resort to phase-plane methods.

(b) An alternative to the solution above is to change the equation (2) into another form by substitution:

By letting

$$\frac{dx}{dt} = \dot{x} = p \qquad Q = x$$

We obtain:

$$A\dot{p} + Bp + CQ = D \qquad (3)$$

By forming:

$$\frac{\dot{p}}{p} = \frac{dp/dt}{dx/dt} = \frac{dp}{dx} \qquad (4)$$

Then by solving (3) for $\dot{p}$ and substituting into (4) yields:

$$\frac{D - (Bp + CQ)}{A} dx - pdp = 0 \qquad (5)$$

Integration of (5) yields a solution for the trajectories. It may be that this solution is difficult, if not impossible. Therefore, the more general procedure is to graphically find the solution of (5).

By rewriting (5) in the form:

$$\frac{dp}{dx} = \frac{D - (Bp + CQ)}{p} = \frac{d\dot{x}}{dx} = k \qquad (6)$$

where k = the slope in the phase plane.

By assigning values for k an equation of a family of isoclines is obtained. An isocline is a locus of points where the phase trajectories in the phase have a constant slope k. That is, if the trajectories cross the isocline, they must cross at a slope equal to k. The latter graphical method is the method used in the course of this study.

(B) DESCRIBING-FUNCTION

The describing-function method is based on the premise that if the input to a nonlinear element is sinusoidal, then the output, if periodic, can be represented as the fundamental component of a Fourier series. The assumption made in the analysis is that the higher frequency components are negligible. In most control systems this assumption is justified because most of the components used in systems have time constants which appear in the denominator of their transfer functions. For this reason the system transfer functions are fundamentally low pass filters and, therefore, produce larger attenuation for the higher frequencies than for the fundamental frequency. Most servo components are low pass devices. For this reason, the more complex the system, the greater the high frequency attenuation. Describing-functions are, therefore, very applicable to higher order systems.

Figure (3) represents a block diagram of a typical nonlinear system. Represented are a describing-function, G_d, which is a function of the magnitude of the error signal and can be a function of the frequency of the error signal. The transfer function of the linear components of the system G, is a function of the frequency and independent of signal amplitude.

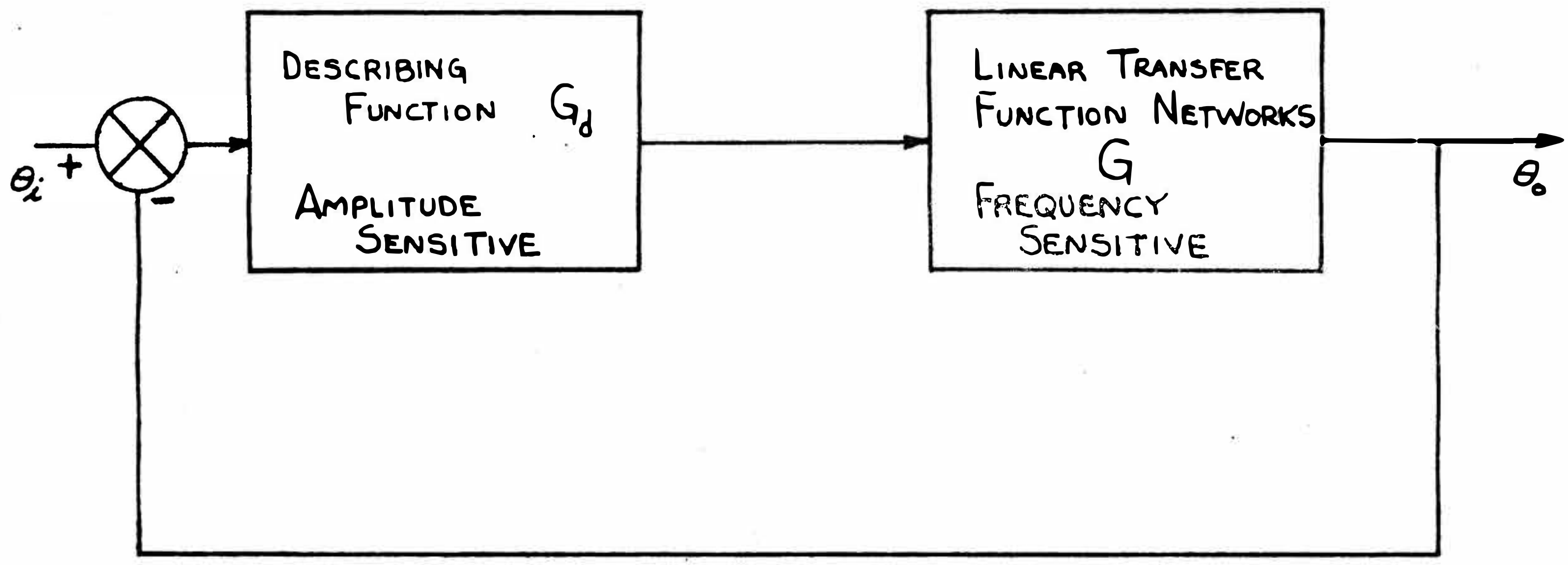


FIGURE 3. BLOCK DIAGRAM OF BASIC RELAY SYSTEM SHOWING AMPLITUDE AND FREQUENCY SENSITIVE SECTIONS

The describing-function is analogous to the transfer function of a linear system. By assuming various amplitudes of error signal, the coefficient of the fundamental component of the output becomes G_d.

Stated mathematically this is:

$$G_d = \frac{D_1}{E} \angle \gamma \qquad (7)$$

where γ is the phase shift between input and output and $\frac{D_1}{E}$ is the magnitude ratio of output to input.

The system under analysis is thus linearized to the extent that the open loop frequency response methods can be applied. Nyquist's stability criterion gives insight to the absolute stability and an indication of the relative stability of the system.

Consider the system given in Figure (3). The closed loop transfer function of which is:

$$KG_1 = \frac{\theta_o}{\theta_i} = \frac{GG_d}{1 + GG_d} \qquad (8)$$

The characteristic equation of this transfer function is:

$$1 + G\ G_d = 0$$

The critical point will occur when:

$$G\ G_d = -1 + jo \qquad (9)$$

For a large number of nonlinear systems, G_d will be only a function of signal amplitude and not of the frequency. It is, therefore, possible to investigate on a polar plot, critical frequencies and signal amplitudes where instability might result.

Ordinarily, it is easier to plot $-G_d$ and G^{-1} separately on the polar plot and investigate possible intersections of the loci.

Then, on rearranging, equation (9) becomes:

$$-G_d = \frac{1}{G} = G^{-1} \tag{10}$$

The rules for determination of Nyquist's stability criterion are as follows:

(a) That portion of the describing-function locus, $-G_d$, that lies to the right of the G^{-1} locus when the locus is traversed in a direction of increasing frequency represents a region of instability.

(b) The point where the two loci intersect (not a necessary condition) represents an equilibrium point.

In addition to polar plots, the Bode diagrams and root locus plots can be formed. The root locus, Bode, and polar plots, however, are restricted to describing-functions which are independent of frequency.

When this is true, a gain factor is involved in the root locus diagram and a change in location of the roots on the root locus plot occurs. If the describing-function is a function of frequency, a root locus plot would be required for every magnitude of gain and, hence would not be a productive method.

In Figure 4, three examples are shown using the rules stated. Sketch (a) represents a system absolutely stable for any amplitude of input signal, the final operation ending in the dead zone of the relay. Sketch (b) indicates a system which is unstable for any amplitude of signal input. Sketch (c) represents an equilibrium point with a steady state oscillation resulting. The magnitude of the oscillation is $|G_{d_1}|$ and the frequency of oscillation is ω_1 . For error signal amplitudes greater than $|G_{d_1}|$, operation is confined to the stable region and hence error amplitude must decrease, forcing operation to the intersection of the loci. For error signals smaller than $|G_{d_1}|$, operation is confined to an unstable region and, hence error signal amplitudes increase again to the intersection of the loci.

C. THE DESCRIBING-FUNCTION FOR A RELAY

The describing-function for the relay under consideration

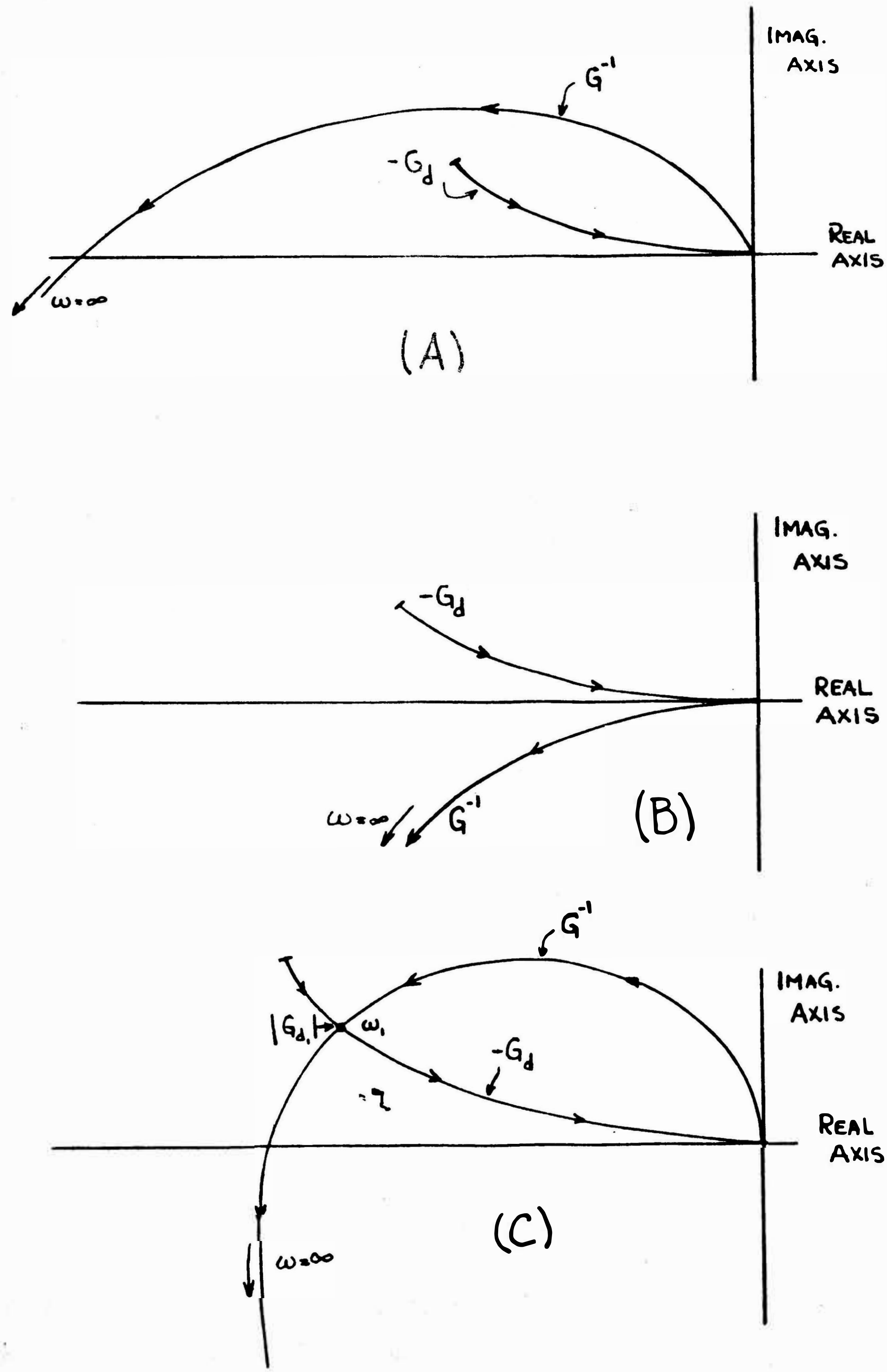


FIGURE 4 FIGURES ILLUSTRATING DESCRIBING-FUNCTION TECHNIQUES FOR DETERMINING STABILITY.

is obtained as follows, with the aid of Figure (5). The correction signal can be summarized over a cycle as follows:

Correction signal	(e_a)	=	0	for	$0 \le \omega t \le \alpha$
"	"	=	+200	"	$\alpha \le \omega t \le \beta$
"	"	=	0	"	$\beta \le \omega t \le \alpha + \pi$
"	"	=	-200	"	$\alpha + \pi \le \omega t \le \beta + \pi$
"	"	=	0	"	$\beta + \pi \le \omega t \le 2\pi$

To determine the describing-function, the fundamental components of the Fourier series must be found. This is:

$$A_1 = 2 \cdot \frac{1}{\pi} \int_{\alpha}^{\beta} e_a \cos \omega t \; d\omega t = \frac{2e_a}{\pi} \int_{\alpha}^{\beta} \cos \omega t \; d\omega t = \frac{2e_a}{\pi} \left[\sin \beta - \sin \alpha \right]$$

Where: $\sin \alpha = \frac{P}{e_m}$; $\sin \beta = \frac{D}{e_m}$

$$A_1 = \frac{2e_a}{\pi} \left[\frac{D - P}{e_m} \right]$$

$$B_1 = 2 \cdot \frac{1}{\pi} \int_{\alpha}^{\beta} e_a \sin \omega t \; d\omega t = \frac{2e_a}{\pi} \left[-\cos \beta + \cos \alpha \right]$$

Where: $\cos \alpha = \frac{\sqrt{e_m^2 - P^2}}{e_m}$; $\cos \beta = - \frac{\sqrt{e_m^2 - D^2}}{e_m}$

$$B_1 = \frac{2e_a}{\pi} \left[\sqrt{1 - \left(\frac{D}{e_m}\right)^2} + \sqrt{1 - \left(\frac{P}{e_m}\right)^2} \right]$$

By letting: $R = \frac{D}{e_m}$ and $T = \frac{P}{e_m}$

$$A_1 = \frac{2\, e_a}{\pi} \left[R - T \right] \qquad B_1 = \frac{2\, e_a}{\pi} \left[\sqrt{1 - R^2} + \sqrt{1 - T^2} \right]$$

The describing-function is then:

$$G_d = \frac{1}{e_m}\sqrt{A_1^2 + B_1^2} \;\angle \tan^{-1} A_1/B_1$$

$$= \frac{2e_a}{e_m \pi}\sqrt{2 \quad (1 - RT) + 2\sqrt{(1 - R^2)(1 - T^2)}}$$

$$\angle \tan^{-1} \frac{R - T}{\sqrt{1 - R^2} + \sqrt{1 - T^2}} \qquad (1)$$

In addition to the describing-function of the general relay which has been derived, two special cases were studied.

1. Ideal relay (i.e. a relay with no dead zone (Δ) and no hysteresis zone (h). Physically this describes a relay which operates for any amount of applied signal and hence has a "pull in" and "drop out" voltage of zero. The describing-function for this case is:

$$G_d = \frac{4e_a}{\pi e_m} \;\angle 0^\circ \qquad \text{When } P = D = 0 \qquad (2)$$

2. The relay with dead zone (Δ). The hysteresis zone (h) is zero and the relay operation represents a relay having equal "pull in" and "drop out" voltages. The describing-function for this case is:

$$G_d = \frac{4e_a}{\pi e_m}\sqrt{1 - \left(\frac{\Delta}{2\, e_m}\right)^2} \;\angle 0^\circ \qquad \text{When } P = D \qquad (3)$$

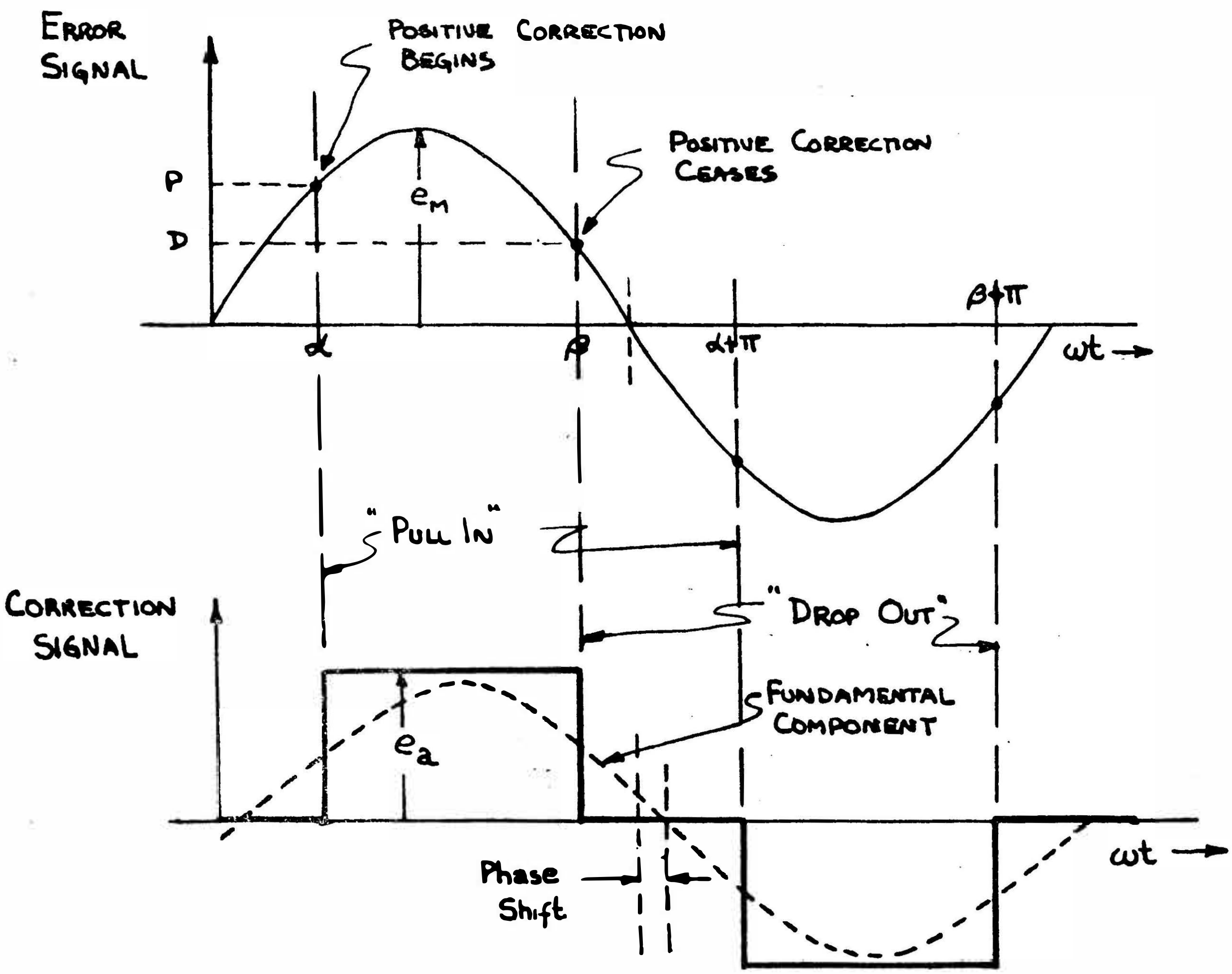


FIGURE 5. DIAGRAM FOR THE DETERMINATION OF THE DESCRIBING FUNCTION FOR A RELAY.

V. STUDY OF THE RELAY CHARACTERISTIC WITH VARIABLE VISCOUS FRICTION

The study of the relay has been divided into three sections. These sections are concerned with the ideal relay, the relay with dead zone and the relay with hysteresis and dead zone.

In the course of the study of each of these foregoing sections, the viscous friction was varied on the motor load combination of the servo system. Three particular values were chosen upon which to make recordings and for making basic analytical calculations. In this manner it was possible to make comparisons and better exhibit the conclusions of the study. The three particular values of viscous friction chosen appear in Appendix II and are represented by motor-load time constants of .107, .1552 and .222.

The compensation network used was a simple phase lead circuit which appears in figure 14. Capacitances C_1 and C_2 were varied with R_1 and R_2 fixed in magnitude. All Brush oscillograph recordings, using compensation, show the values of these capacitances.

A. THE BASIC SYSTEM WITH STEP INPUT OF POSITION USING AN IDEAL RELAY

The block diagram of the system is shown in Figure 6. The describing-function for an ideal relay is independent of the frequency of the error signal and rewritten from Chapter IV is:

$$G_d = \frac{4\ e_a}{\pi\ e_m} \angle 0^\circ \qquad (1)$$

where:

e_m is the maximum value of the sinusoidal error signal.

e_a is the magnitude of the relay output voltage. The value of e_a chosen was the rated voltage of the motor under consideration which was ± 200 volts.

Equation (1) then becomes:

$$G_d = |254.65/e_m| \qquad (2)$$

The polar plot of the function $(-G_d)$ is shown in Figure 7 for values of e_m. The transfer function of the motor-load (G^{-1}) is also plotted for three cases of variable amounts of viscous damping. The Bode plots of the open loop response of the motor and load; derivation of the motor load transfer functions for the three cases of viscous friction, and the

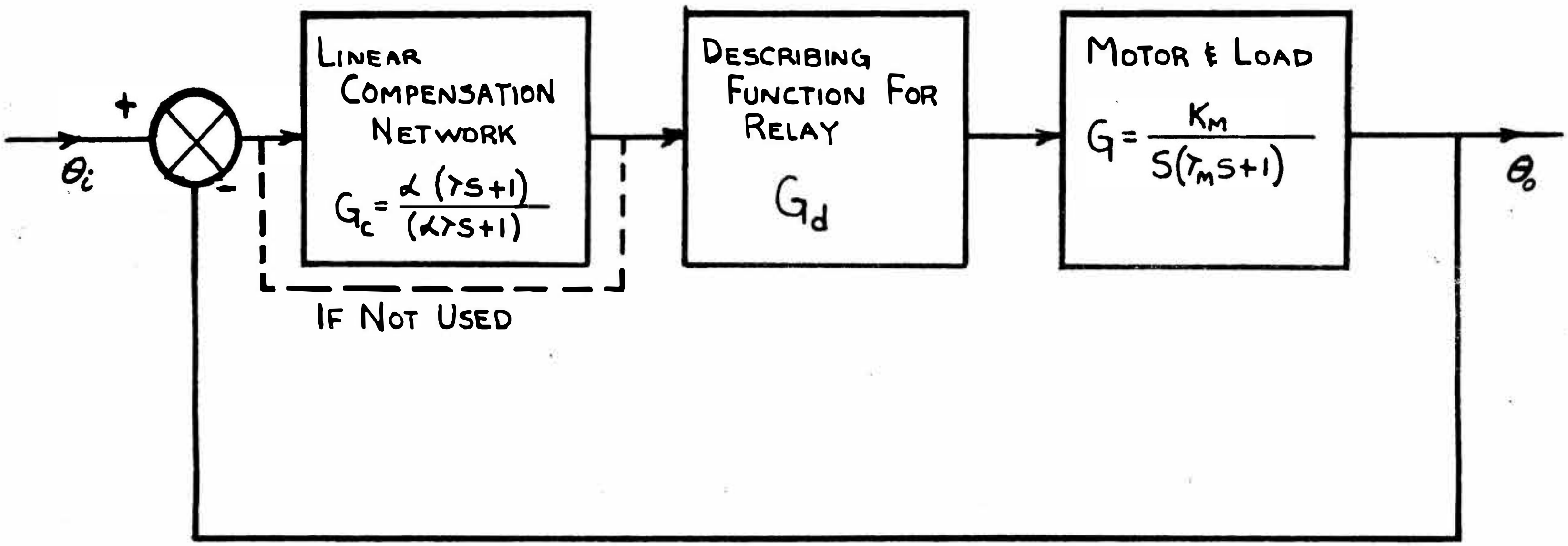


FIGURE 6. BLOCK DIAGRAM OF SYSTEM SHOWING TRANSFER FUNCTIONS OF SYSTEM COMPONENTS.

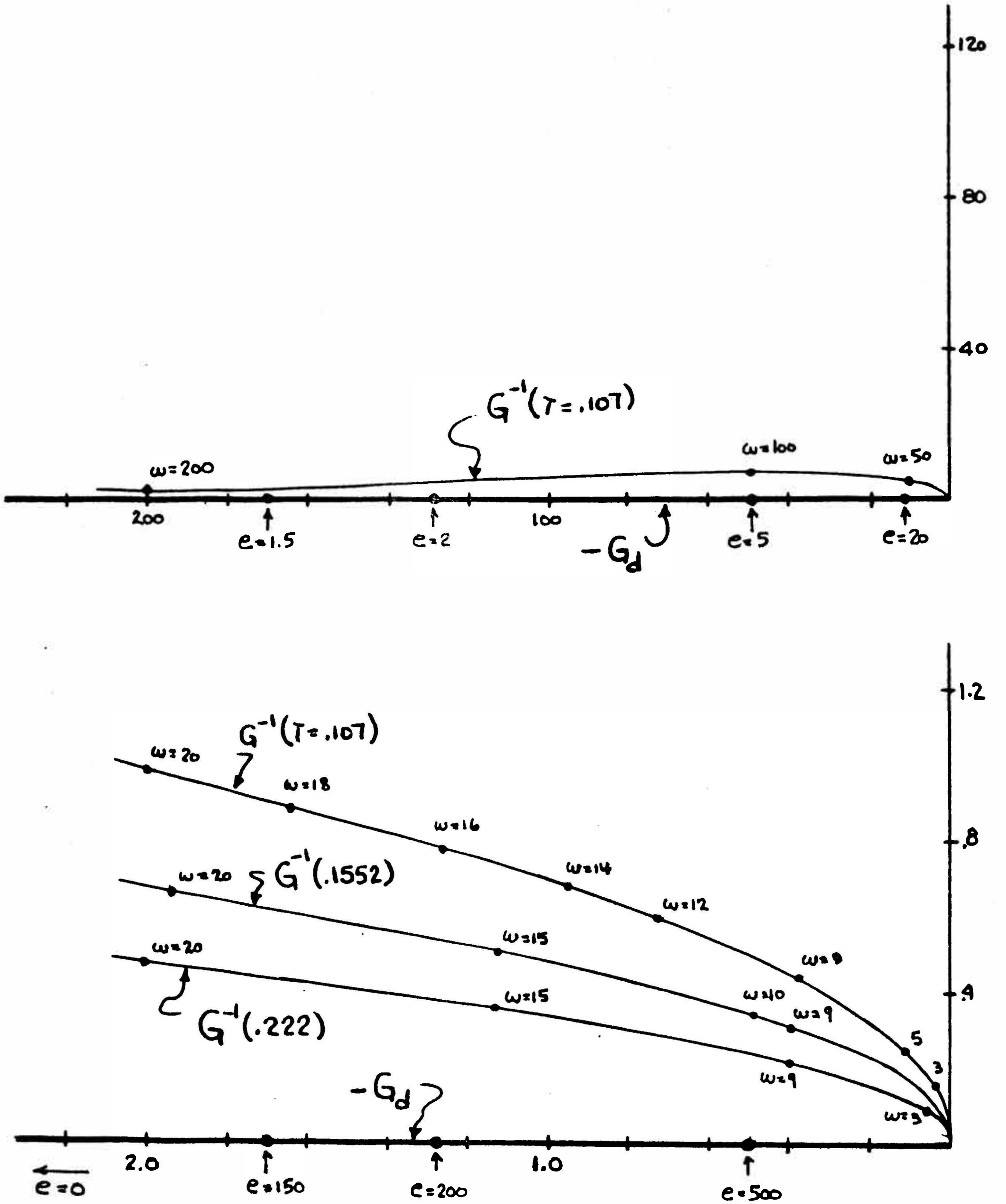


FIGURE 7. POLAR PLOT OF DESCRIBING-FUNCTION RELAY WITH THREE VALUES OF VISCOUS DAMPING.

selection of the parameters of the system appear in Appendix I and II.

Interpretation of the polar locus of Figure 7 indicates that the system has absolute stability. The plot also shows the system should oscillate at infinite frequency with zero amplitude for any magnitude of error signal input.

The phase-plane trajectories for the system under consideration were formed by the phase-plane methods described in Chapter III. Rewriting equation (7) of Appendix I for the motor with load:

$$\frac{e_a K_t N}{R_a} = J_{eL} \frac{d^2\theta_L}{d t^2} + f_{eL} + \frac{K_e K_t N^2}{R_a} \frac{d \theta_L}{d t}$$

where $f_{eL} = f_m N^2 + f_L$ and $J_{eL} = J_m N^2 + J_L$

(7) Appendix I.

$$\text{or:} \quad Ae_a = \frac{d^2\theta_o}{dt^2} + B\frac{d\theta_o}{dt} \qquad (1)$$

Where the constants A and B are:

$$A = \frac{K_t N}{R_a J_{eL}} \qquad B = \frac{f_{eL} R_a + K_e K_t N^2}{R_a J_{eL}}$$

The error signal when referring to Figure I is:

$$e = \theta_L - \theta_o$$

for a step input at $t = 0^{+}$

$$\frac{de}{dt} = -\frac{d\theta_L}{dt} \text{ and } \frac{d^2e}{dt^2} = -\frac{d^2\theta_L}{dt^2} \tag{2}$$

Substitution of equation (2) into equation (1) introduces the error signal. The result is:

$$Ae_a = -\left[\frac{d^2e}{dt^2} + B\frac{de}{dt}\right] \tag{3}$$

by letting:

$$\frac{de}{dt} = \dot{e} \tag{4}$$

then:

$$\frac{d^2e}{dt^2} = \frac{d\dot{e}}{dt} \tag{5}$$

Introducing these quantities into equation (3) and rearranging yields:

$$\frac{d\dot{e}}{dt} = Ae_a - B\dot{e} \tag{6}$$

Dividing both sides of the equation by equation (4) gives the final form:

$$\frac{d\dot{e}}{de} = \frac{Ae_a - B\dot{e}}{\dot{e}} \tag{7}$$

The constant A is independent of the coefficient of viscous friction while B is dependent on amount of friction present. Introducing this factor into (7) yields:

$$\frac{d\dot{e}}{de} = \frac{e_a - (4.5 + .968\ f_{eL})\ \dot{e}}{\dot{e}} \qquad (8)$$

For the ideal relay with no dead zone:

$e_a = 200$ for $e > 0$

$e_a = -200$ for $e < 0$

Equation (8) becomes the equation of a system of straight lines having zero slope. These lines are the isoclines for the phase-plane plot when $\frac{d\dot{e}}{de}$ is set equal to a constant. The curves have the property that all trajectories, if they cross the curve, must cross at the slope assigned.

Phase-plane trajectories for the three variable amounts of viscous damping appear in Figures 8, 9 and 10. The initial error corresponds to 5 radians. These trajectories correspond to the describing-function loci that appear in Figure 7. In all cases, a stable condition results, in that the oscillatory motion of the servo comes to rest in approximately five relay cycles. The first overshoot is roughly 8 per cent, indicating a damping ratio of about .5 by linear standards.

The relative stability of the system can be determined by the relative proximity of the describing-function locus ($-G_d$) and the transfer function locus (G^{-1}). The further

K	ė	
	$e<0$	$e>0$
0	21.46	-21.46
1	19.38	-19.38
3	16.23	-16.23
10	10.35	-10.35
20	6.8	-6.8
∞	0	0
-40	-6.51	6.51
-22.6	-15	15
-20	-18.73	18.73

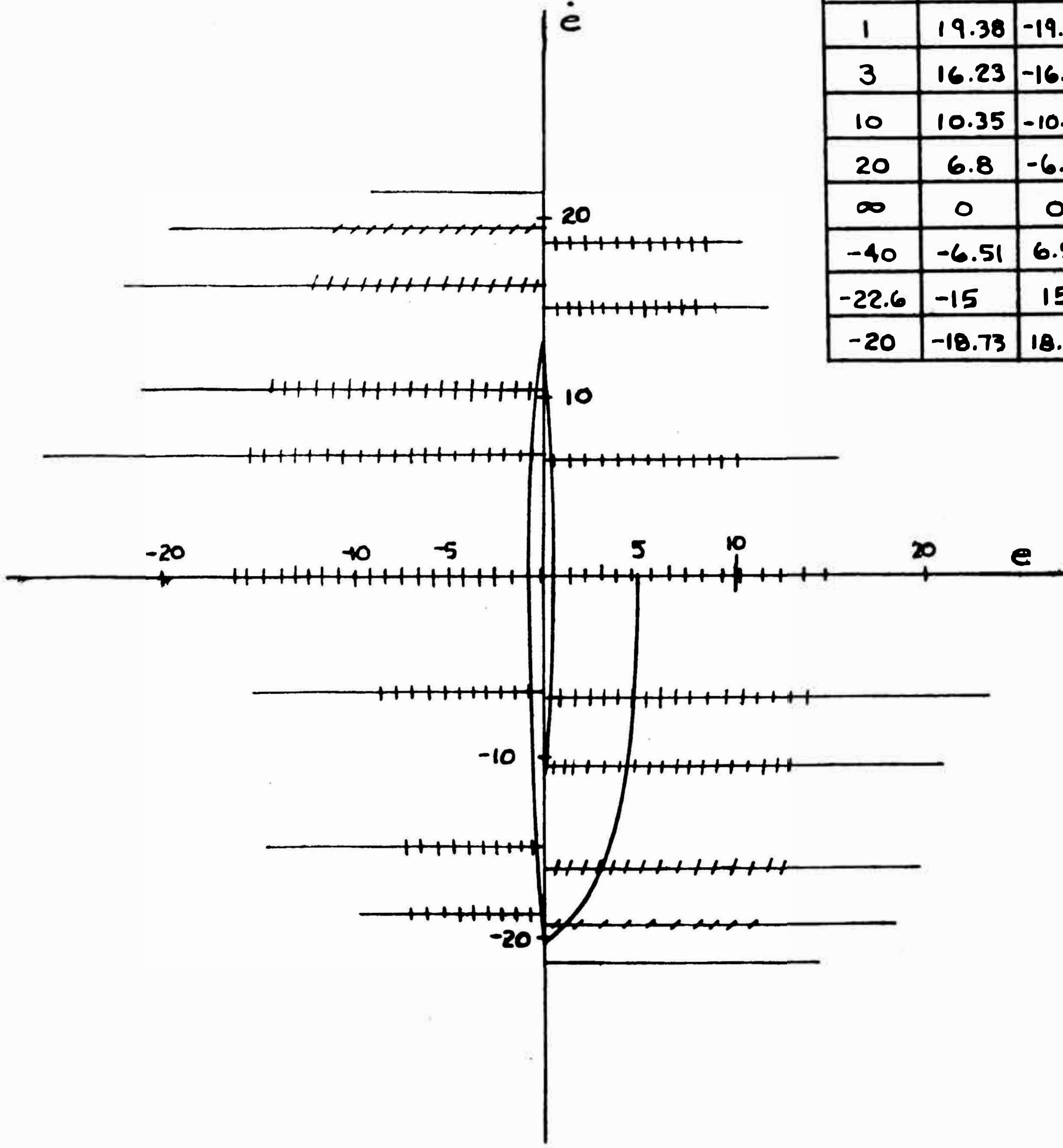


FIGURE 8. PHASE-PLANE PLOT FOR IDEAL RELAY AND A MOTOR LOAD TIME CONSTANT = .107.

K	ė	
	e<0	e>0
0	31.1	-31.1
1	26.9	-26.9
3	21.2	-21.2
10	12.16	-12.16
∞	0	0
-20	-14.78	14.78
-15	-23.4	23.4

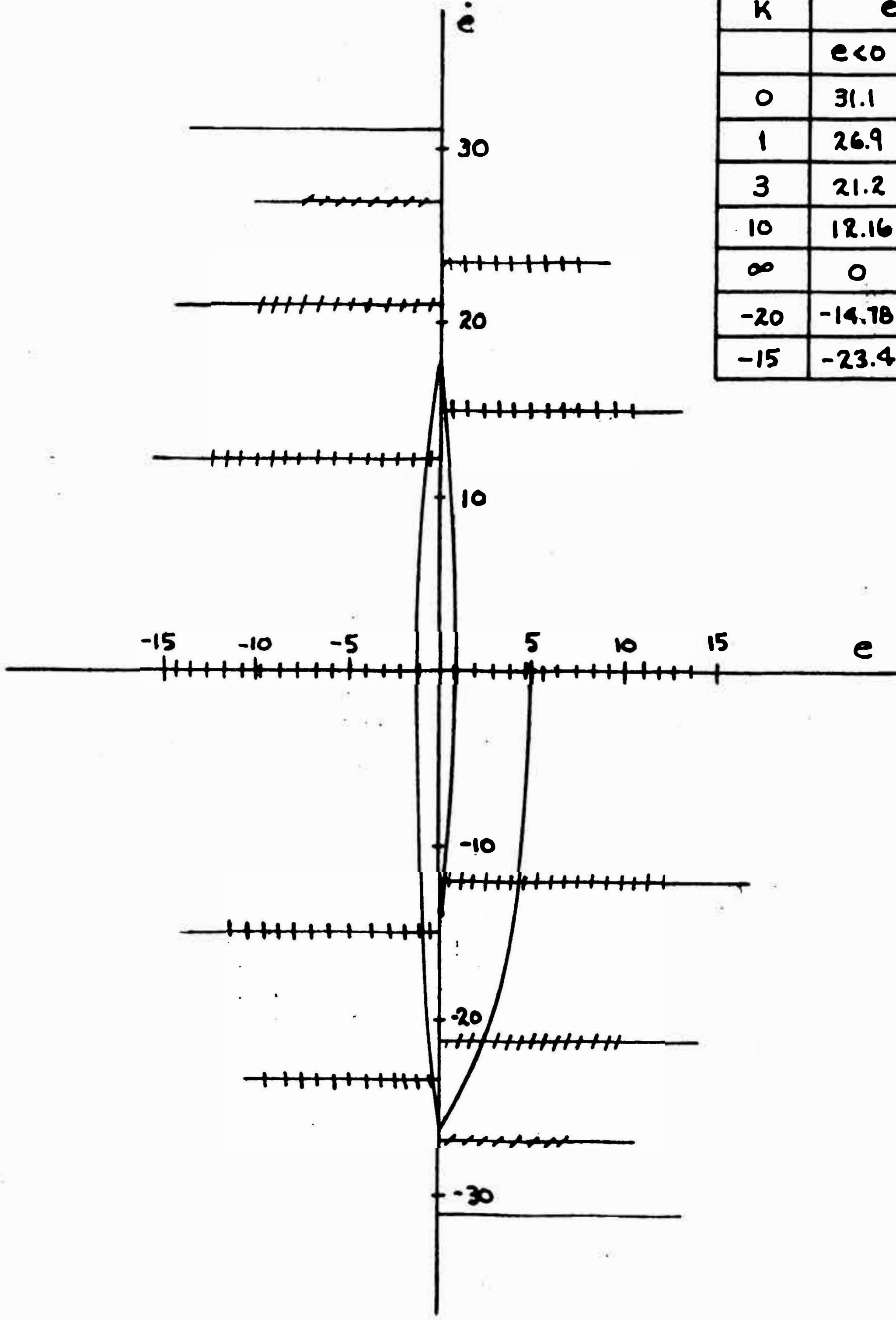


FIGURE 9. PHASE-PLANE PLOT FOR IDEAL RELAY AND A MOTOR LOAD TIME CONSTANT = .1552.

K	ė	
	e<0	e>0
1	37.4	-37.4
2.31	30	-30
5	21.4	-21.4
10	13.99	-13.93
∞	0	0
-20	-12.78	12.78
-12.35	-25	25
-10	-35.4	35.4

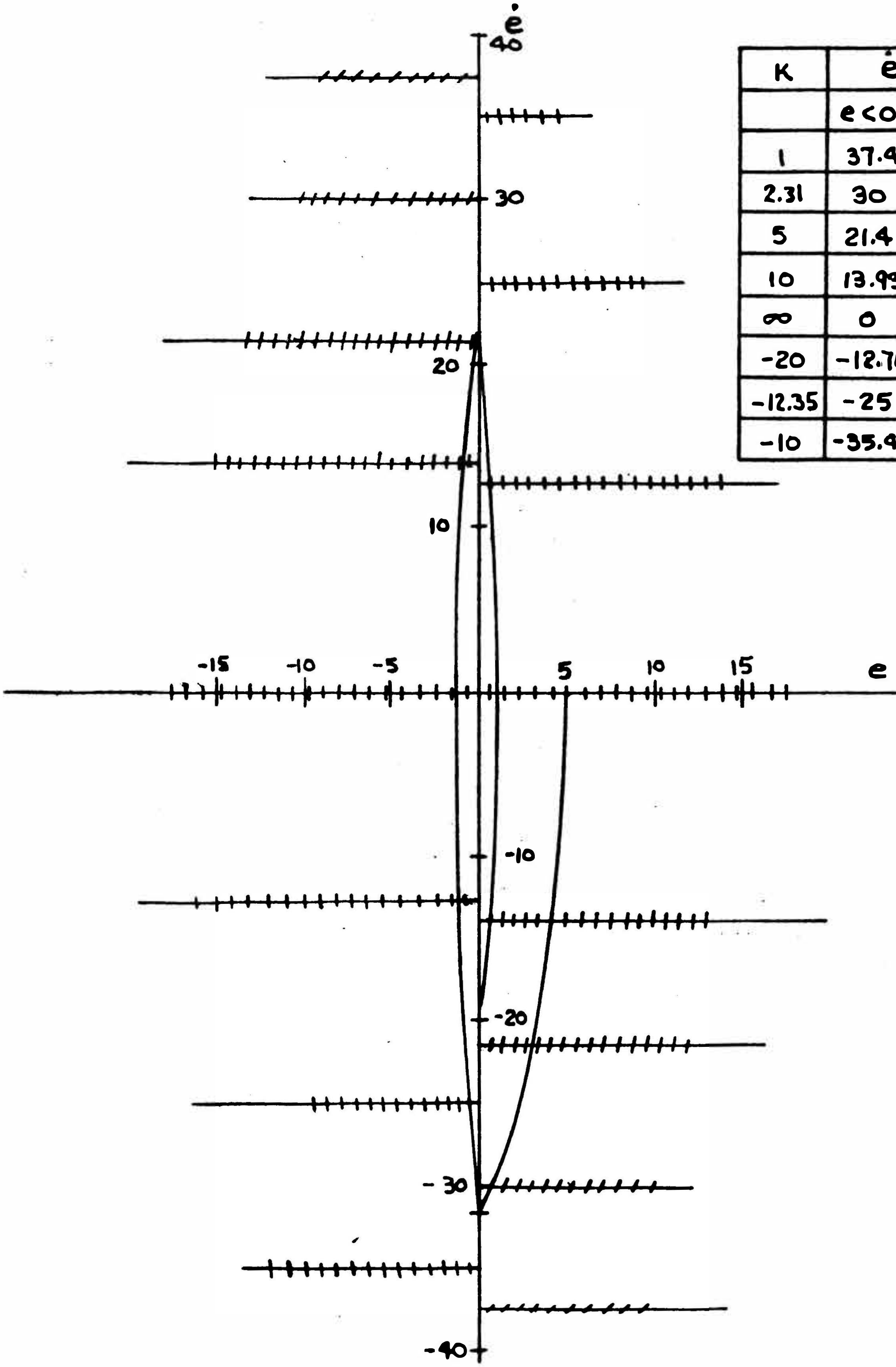


FIGURE 10. PHASE-PLANE PLOT FOR IDEAL RELAY AND A MOTOR LOAD TIME CONSTANT = .222.

the two loci are displaced, the greater will be the stability. The peak value of the output to input ratio of the servo is called M_p. Expressed thus:

$$M_p = \frac{\theta_o}{\theta_i} \qquad \text{(maximum)}$$

The value of M_p is determined by the shortest distance on the polar plot between any given input on the describing-function locus to the transfer function locus, G^{-1}. If greater stability with less oscillatory output is to be obtained, it is apparent that M_p will have to be increased. One apparent way to increase M_p is to reduce the runaway velocity. However, this decreases the speed of response and hence is not as desirable as using compensation. Linear theory suggests the use of lead networks for compensation. This matter is next considered.

The phase-plane of the relay servo indicates that performance could be improved if the output torque of the motor could be reversed before the error becomes zero. That is, for some given input driving function, some combination of error and error rate could be chosen as the point where the motor excitation reversal could take place.

Expressed thus: $Ke + B\dot{e} \gtrless 0 \qquad (9)$

$$B\dot{e} = C - Ke$$

or $$\dot{e} = \frac{C}{B} - \frac{K}{B}e \qquad (10)$$

A simple lead network ahead of the relay performs such a task by causing the output to lead the input voltage to the network. Equations (9) and (10) describe the new boundaries in the phase-plane as straight lines having a negative slope which pass through the origin. A simple linear lead network is shown in Figure 14. The transfer function of which is:

$$E_o/E_i = \alpha \cdot \frac{\lambda S+1}{\alpha\lambda S+1} \quad \text{where} \quad \alpha = \frac{R_1}{R+R_1}$$

$$\lambda = RC \qquad (11)$$

The effect of the addition of compensation is to relocate the G^{-1} locus on the polar plot. The additional lead angle gain increases the distance between the loci and hence increases the relative stability. One such plot is made of $\lambda = .1552$ and is shown in Figure 11.

The phase-plane plot for the same value of $\lambda = .1552$ is shown in Figure 12 and shows how the torque reversal region has been shifted.

In general, a time constant in the numerator of a transfer function shifts the torque reversal region clockwise (i.e. a phase lag condition). A time constant in the

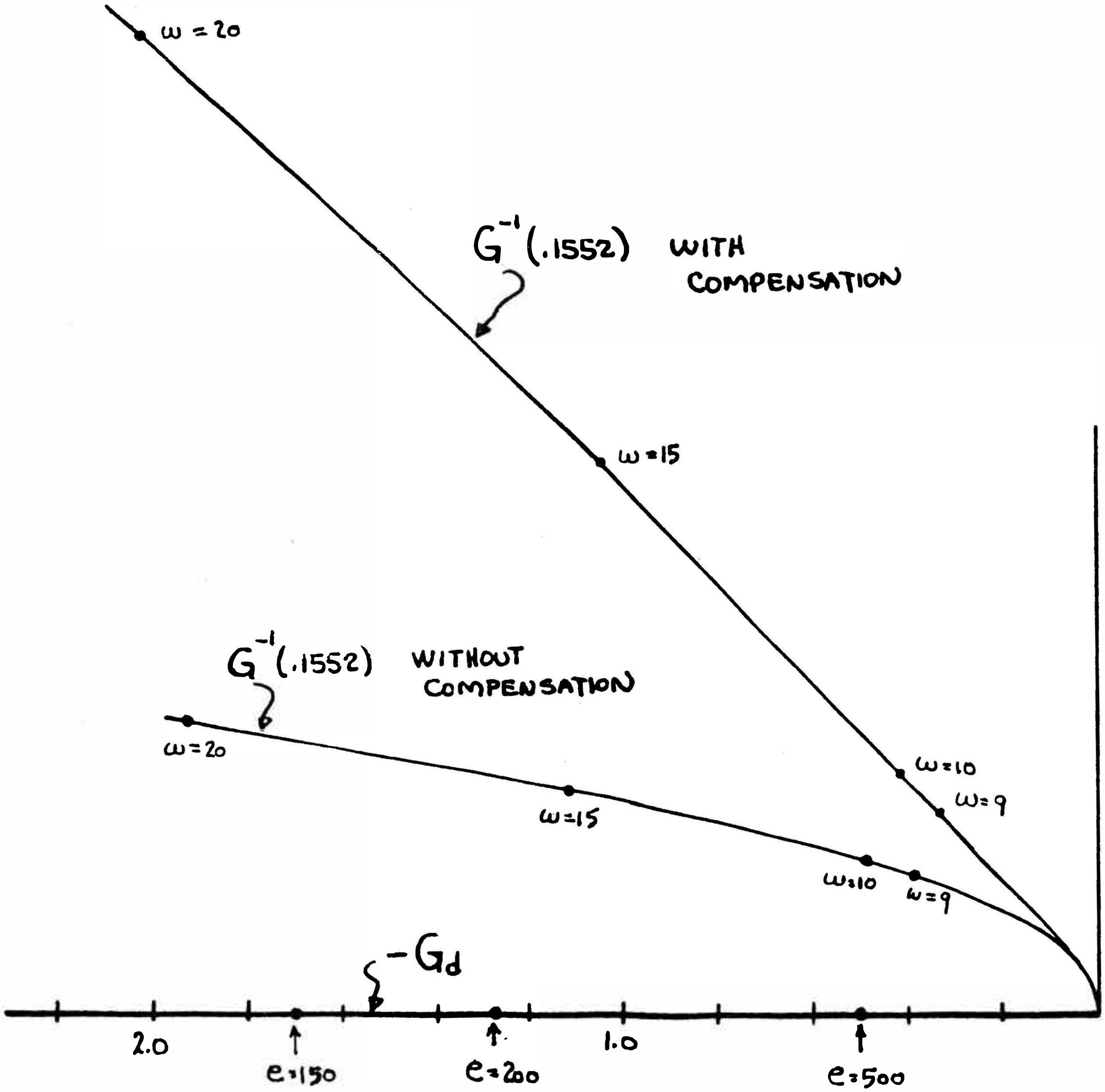


FIGURE 11. POLAR PLOT FOR THE DESCRIBING FUNCTION OF AN IDEAL RELAY (λ=.1552) WITH AND WITHOUT PHASE LEAD COMPENSATION.

K	ė	
	e<0	e>0
0	31.1	-31.1
1	26.9	-26.9
3	21.2	-21.2
10	12.16	-12.16
∞	0	0
-20	-14.78	14.78
-15	-23.4	23.4

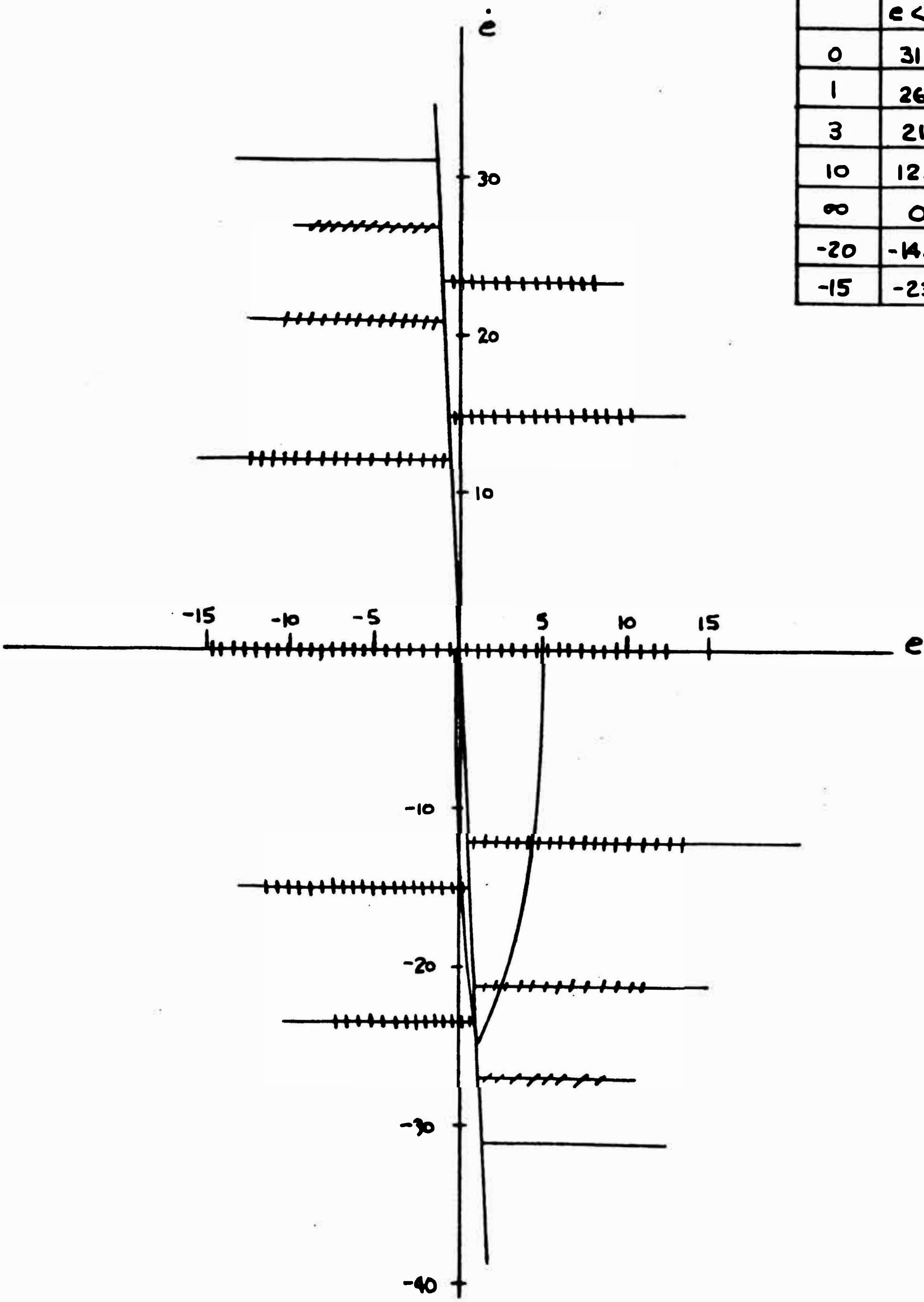


FIGURE 12. PHASE-PLANE PLOT OF RELAY SERVO WITH IDEAL RELAY (λ=.1552) AND PHASE LEAD COMPENSATION.

denominator of the transfer function produces a counter clockwise rotation to the torque reversal region (i.e. a phase lead condition).. The location of the torque reversal region for the phase lead network is approximated by considering a power series expansion of the phase lead transfer function. The transfer function of the phase lead network is:

$$E_o/E_i = \alpha\frac{\lambda S+1}{\alpha\lambda S+1} = \alpha\left[(\lambda S+1)(1-\alpha\lambda S+\alpha^2\lambda^2 S^2-\alpha^3\lambda^3 S^3+\ldots)\right]$$

by neglecting higher power terms in S

$$E_o/E_i \cong \alpha(\lambda S+1)(1-\alpha\lambda S) \cong \alpha\left[1+(\lambda-\alpha\lambda)\,S-\alpha\lambda^2 S^2\right]$$

Again neglecting higher power terms in S this becomes:

$$E_o/E_i \cong \alpha\left[1+(\lambda-\alpha\lambda)\,S\right]$$

$$E_o\,(t) \cong \alpha E_i\,(t) + (\lambda-\alpha\lambda)\,\dot{E}_i\,(t)$$

where E_o = input to relay or output of phase lead network.

E_i = error signal or input to phase lead network.

The condition for torque reversal occurs when $E_o = 0$. The equation of the torque division line results which is a straight line passing through the origin.

Figure 13 indicates the analog used to simulate the contactor servo with variable damping. The compensating network is a linear lead network which is depicted in Figure 14. By experimenting with values of C_1 and C_2, various lead angles are introduced into the circuit. C_1 and C_2 have no effect on the attentuation factor so that through the course of the experimentation the gain remained constant.

A Brush recorder was used in recording the response of the system. Figure 15 illustrates several responses with the relay circuit, having no dead zone, no hysteresis or compensating network and also shows the results of compensation with a lead network. To compare the relay servo operation with a linear servo (no relay), Figure 15 also shows the same motor-load and viscous frictional constant. The gain of the system is 200 to correspond with the relay servo.

The experimental results for the relay servo with ideal relay do not verify the analytical results completely. Analytically we expect oscillation at infinite frequency and zero amplitude without compensation but the simulation of the system results in a limit cycle or continuous oscillation.

The diode relay simulation is not a perfect simulation. The largest source of error originating in generating a hysteresis loop originates in the loop itself. This loop is used

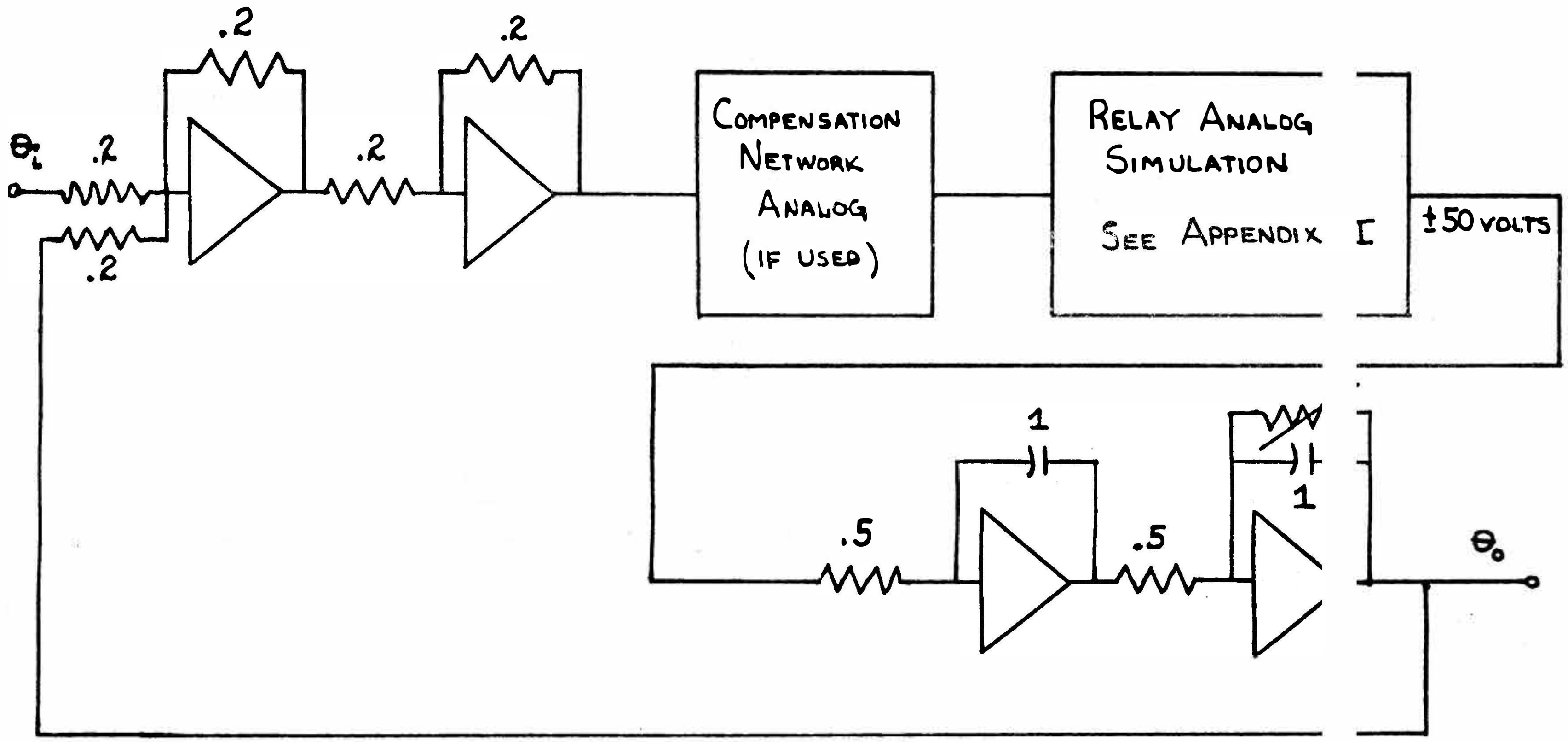
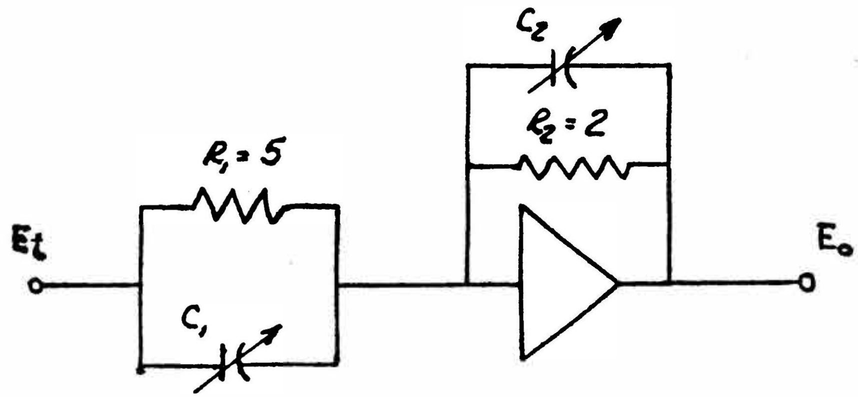
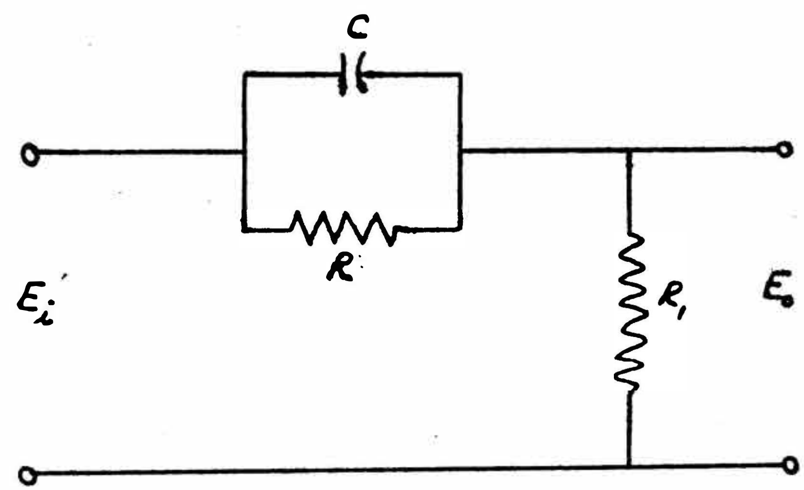


FIGURE 13. ANALOG COMPUTER SIMULATION FOR RELAY SERVO



$$\frac{E_o}{E_i} = -\frac{R_2}{R_1} \cdot \frac{(R_1 C_1 S + 1)}{(R_2 C_2 S + 1)}$$



$$\frac{E_o}{E_i} = \alpha \frac{(\tau S + 1)}{(\alpha \tau S + 1)}$$

$$\alpha = \frac{R_1}{R + R_1} \qquad \tau = RC$$

FIGURE 14. PHASE LEAD CIRCUIT AND THE ANALOG COMPUTER SIMULATION OF PHASE LEADCIRCUIT.

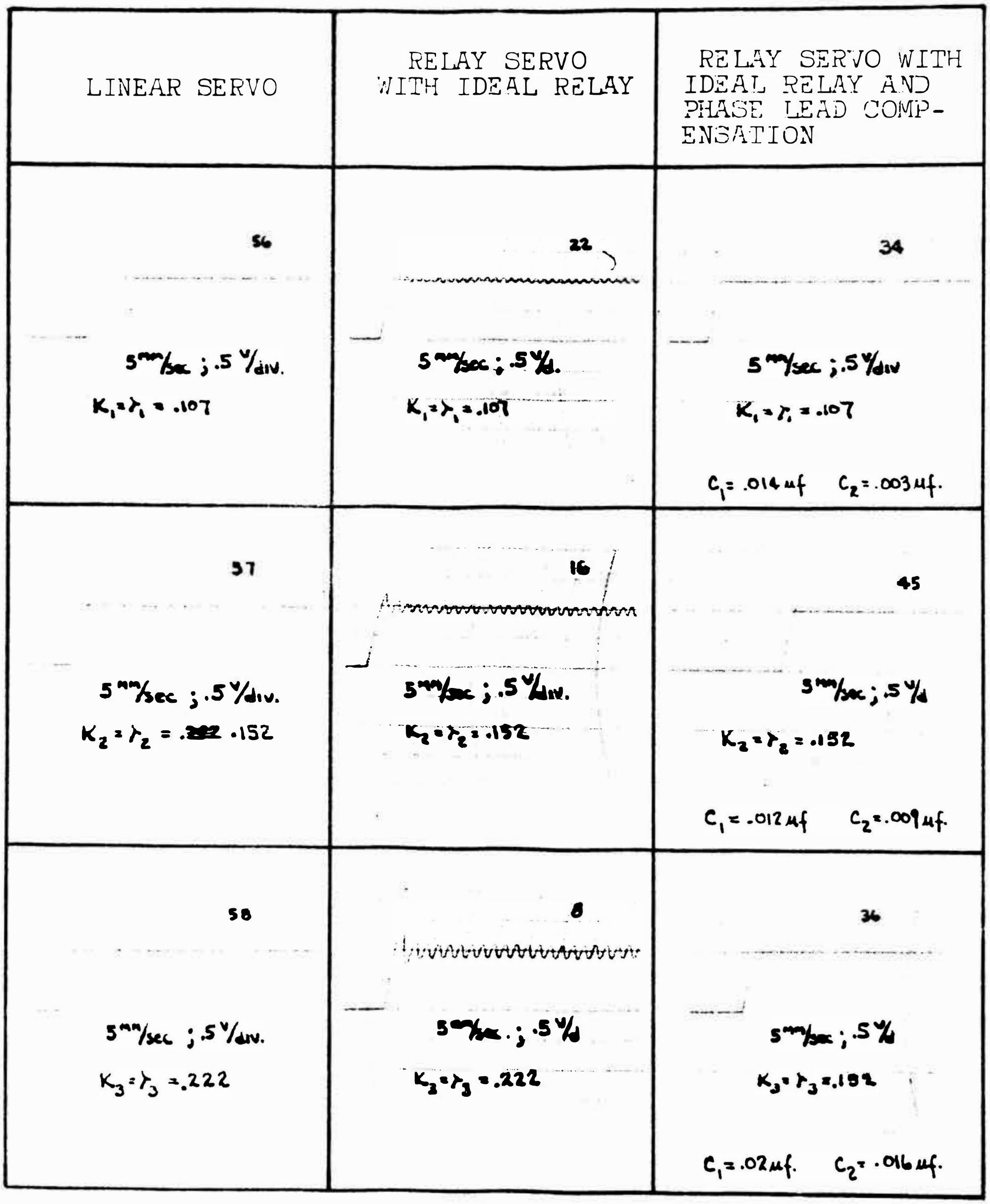


FIGURE 15 TRANSIENT RESPONSE OF A VARIABLE DAMPED RELAY SERVO WITH AND WITHOUT COMPENSATION AND A COMPARABLE LINEAR SERVO SYSTEM WHEN SUBJECTED TO A STEP DISPLACEMENT

as a triggering circuit for the diodes. The loop is somewhat frequency sensitive which affects the initial transient. The steady state oscillation (limit cycle) is the result of ragged diode cut-off which appears to the relay servo as a time lag (i.e. different "pull in and "drop out" voltages). When the other relay characteristics are simulated, those having dead zone and contact hysteresis, this problem of time lag can be accounted for by adjustment of circuit parameters and is not detrimental to the simulation.

Figure 16 shows the relationship between the error signal and the output voltage of the relay. The phase-plane photographs indicate the limit cycles of the relay servo with no external viscous friction. Note the reduced amplitude of the error and error rate with phase lead compensation. This figure is interesting in that it verifies that the error signal is a sinusoidal voltage. The describing-function for the relay was derived on the assumption that the input was sinusoidal in nature and the figure substantiates this fact.

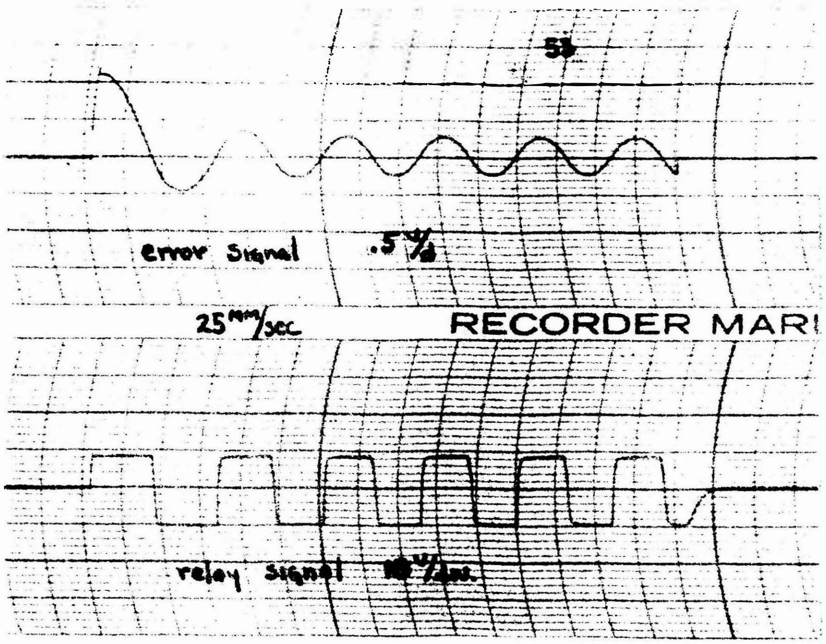
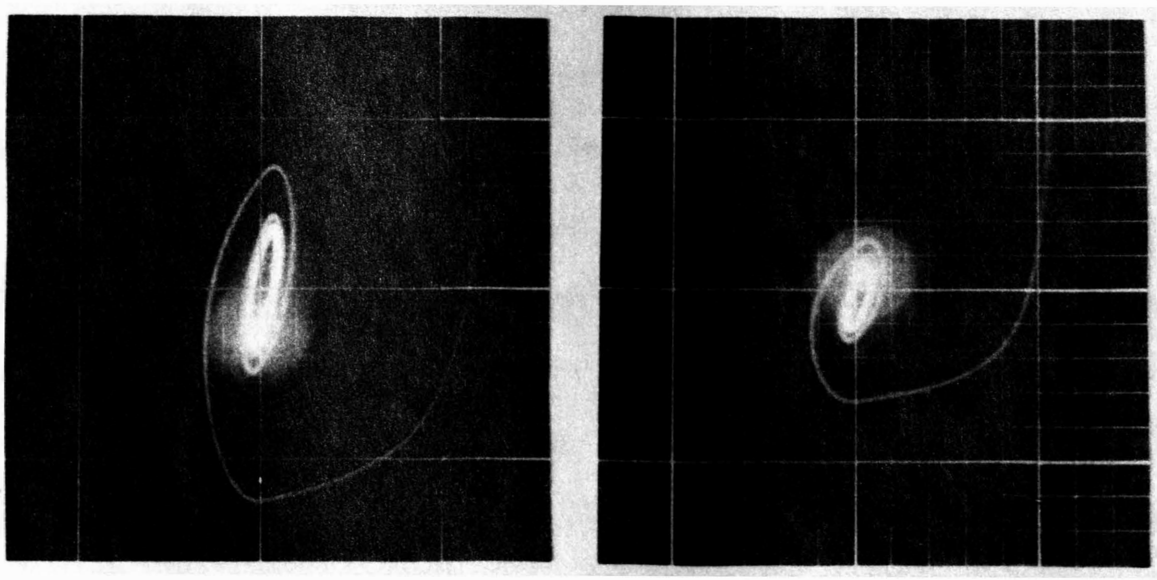


FIGURE 16 ERROR SIGNAL & RELAY OUTPUT



NO PHASE LEAD COMPENSATION

WITH PHASE LEAD COMPENSATION
$C_1 = .02\mu fd.$ $C_2 = .016\mu fd.$

$K_3 = T_3 = .222$

FIGURE 16 PHASE PLANE PHOTOS WITH NO VISCOUS FRICTION

B. THE BASIC SYSTEM WITH STEP INPUT OF POSITION USING A RELAY WITH DEAD ZONE

When a relay with dead zone is introduced into the system, the relationship defining the error signal to relay output is:

$$e_a = 200 \text{ for } e \geq a$$

$$e_a = -200 \text{ for } e \leq b$$

$$e_a = 0 \text{ for } a > e > b$$

When a = b a symmetrically operating relay is produced and is the type considered here.

In general, a dead zone has a stabilizing affect on the transient response. The disadvantage, however, is the loss in static accuracy. The larger the dead zone, the greater will be the chances of positional static error. One consideration which must be taken into account when choosing a relay for a system, is the magnitude of the dead zone, as this will determine the resulting steady state error. Figure 2b pictures the relay operation depicted in this section.

The describing-function for the relay with dead zone is determined with a sinusoidal input. For symmetrical opertion the describing-function derived in Chapter IV is:

$$G_d = \frac{4\ e_a}{\pi e_m}\sqrt{1 - \left(\frac{\Delta}{2\ e_m}\right)^2}\ \angle 0^\circ \qquad (1)$$

where: Δ is the dead zone width.

e_m is the maximum value of the sinusoidal error signal.

As before, the function G_d is independent of the frequency but is a function of the maximum value of the error signal. The describing-function locus for G_d is shown in Table 1 for parameters of dead zone width (Δ). Included also are values of the describing-function for an ideal relay. This data appears graphically in Figure 17. It is noticed that the only perceptible change in values of G_d, when compared to the ideal relay, occurs for small input signal, which are of about the same amplitude as the dead zone width. This is reasonable, for if large sinusoidal errors are involved and the dead zone width is small, the relay is operating nearly as an ideal relay.

The dead zone affect then tends to crowd the describing-function locus into the origin for the small magnitude of error signal. This would indicate an increase in the relative stability of the system. In general, it will suffice to say

e_M	IDEAL RELAY	RELAY WITH DEAD ZONE VALUES OF Δ (dead zone width)			
		.5	1	2	5
.5	509.3	441	0	NO RELAY ACTION	NO RELAY ACTION
1	254.65	247.5	220.5	0	NO RELAY ACTION
2	127.32	126.2	123.1	111.1	0
5	50.93	50.82	50.7	49.9	44.1
7	36.38	36.3	36.26	36	34.2
10	25.46	25.4	25.38	25.35	24.7
20	12.73	12.73	12.70	12.67	12.62

G_d (I AL RELAY) =

$$\frac{4\, e_a}{\pi\, e_M} \angle 0^\circ$$

G_d (R Y WITH DEAD ZONE)

$$= \frac{4 e_a}{\pi e_M}\sqrt{1-\left(\frac{\Delta}{2e_M}\right)^2}$$

TABLE 1. DESCRIBING-FUNCTION VALUES FOR IDEAL REL Y AND A RELAY WITH DEAD ZONE.

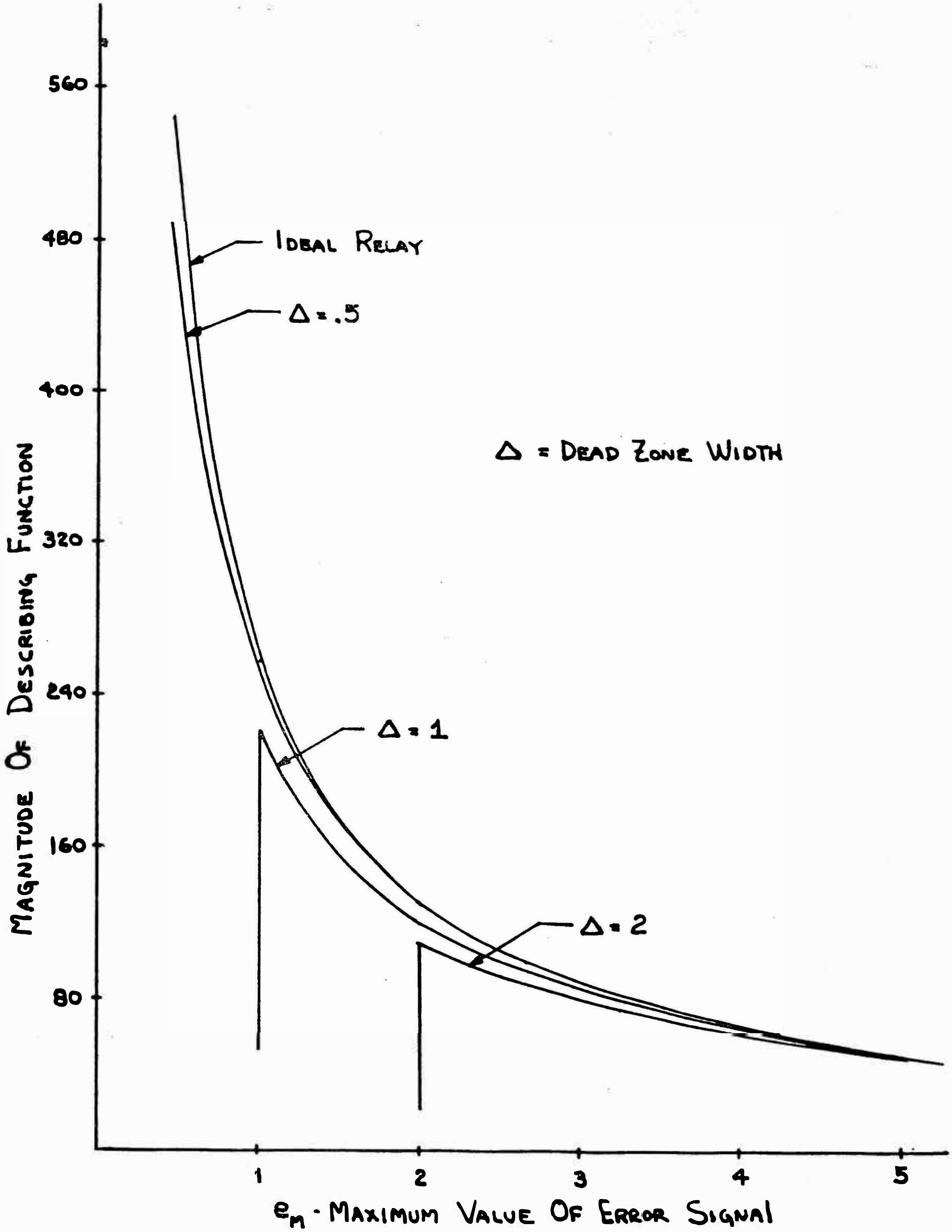


FIGURE 17. COMPARISON OF DESCRIBING-FUNCTION MAGNITUDES FOR RELAY WITH DEAD ZONE AND CHANGES IN ERROR SIGNAL.

that increasing the dead zone will make a relay system stable, provided some damping is present. The term, stable, meaning that no limit cycle in the output will result.

From a dynamic point of view, the phase portrait of the system is more enlightening than the polar locus plot. A phase portrait of the system is shown in Figure 18. The drive motor, when operating in the dead zone, has no applied voltage to the armature and, therefore, loses its counter emf. The counter emf is a viscous damping term, so unless some additional damping is present, a continuous oscillation in the output will result. The trajectories would then be horizontal lines in the dead zone. The greater the negative slope of the trajectories in the dead zone, the faster the oscillations in the output disappear. Some damping is then desirable to produce stability.

Several degrees of output damping and dead zone widths were studied on the analog computer. These studies appear in Figures 19 through 25. Figure 19 represents the response of a system with substantial output damping and a 5 volt step input signal with a dead zone width of 1 volt. Phase lead compensation can be noted to have desirable effects when various lead networks are introduced. Figures 20 and 21 represent the same dead zone width and same step input signal as in

K	$\dot{e}$	
	$e<-2$	$e>2$
0	21.46	-21.46
1	19.38	-19.38
3	16.23	-16.23
10	10.35	-10.35
20	6.8	-6.8
∞	0	0
-40	-6.51	6.51
-22.6	-15	15
-20	-18.73	18.73

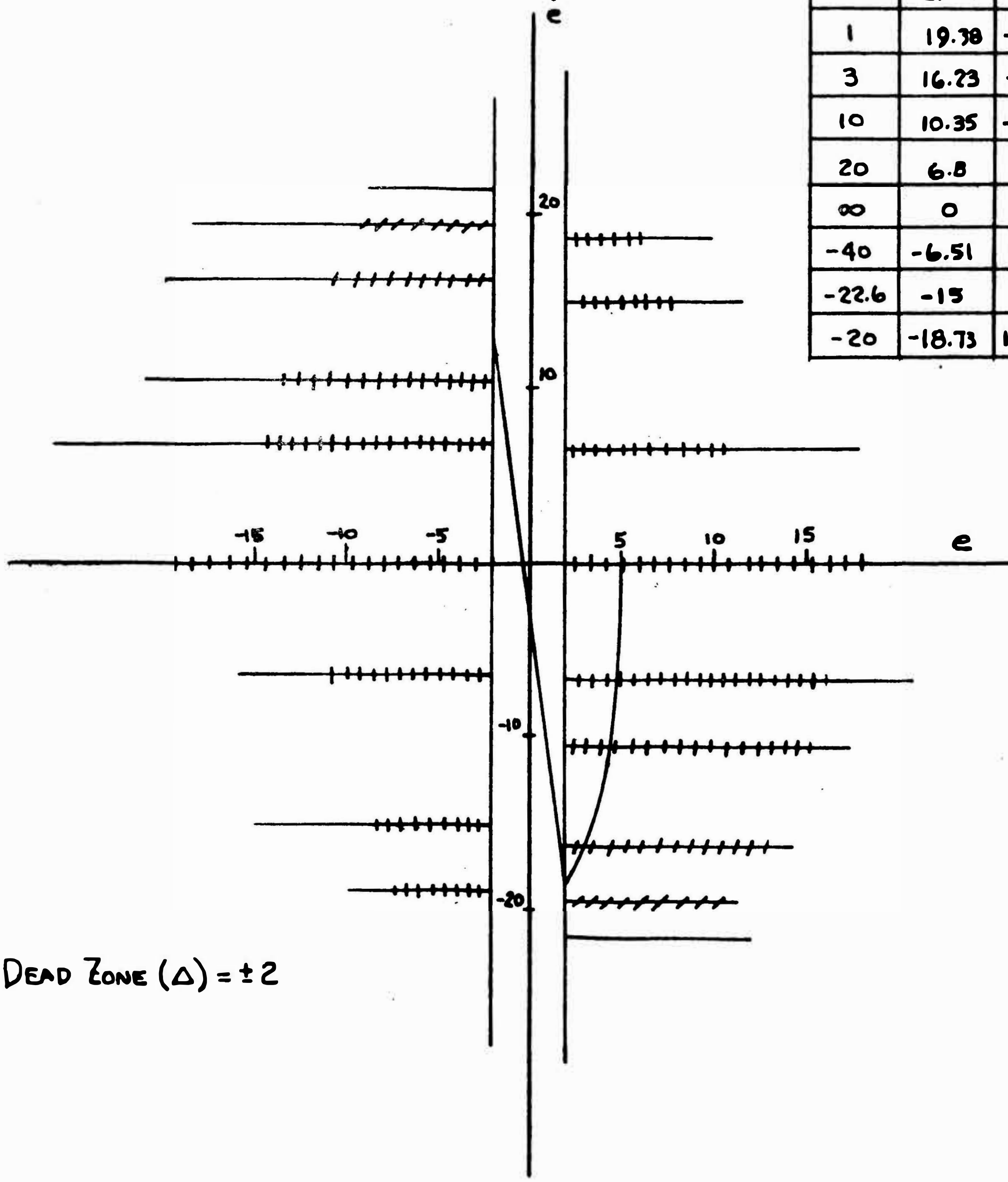


FIGURE 18. PHASE-PLANE PLOT OF RELAY SERVO WITH DEAD ZONE (λ=.107)

Figure 19 but with two lesser degrees of viscous damping. Note that the resulting steady state response output is within ± .5 volts of the actual desired output, which is consistent with the limits established on the dead zone of the relay. As the viscous damping decreases, the phase lead compensation becomes less effective in improving system performance.

Figures 22, 23 and 24 compare an uncompensated and a compensated relay servo when the dead zone width is varied. Each figure also represents a different degree of damping. The tendency toward oscillation is apparent and is directly related to decreasing dead zone width and decreased output damping. Figure 25 is the phase-plane portraits for three values of the output damping of a relay servo with the same dead zone width. The increasing number of oscillations is apparent as the damping is decreased.

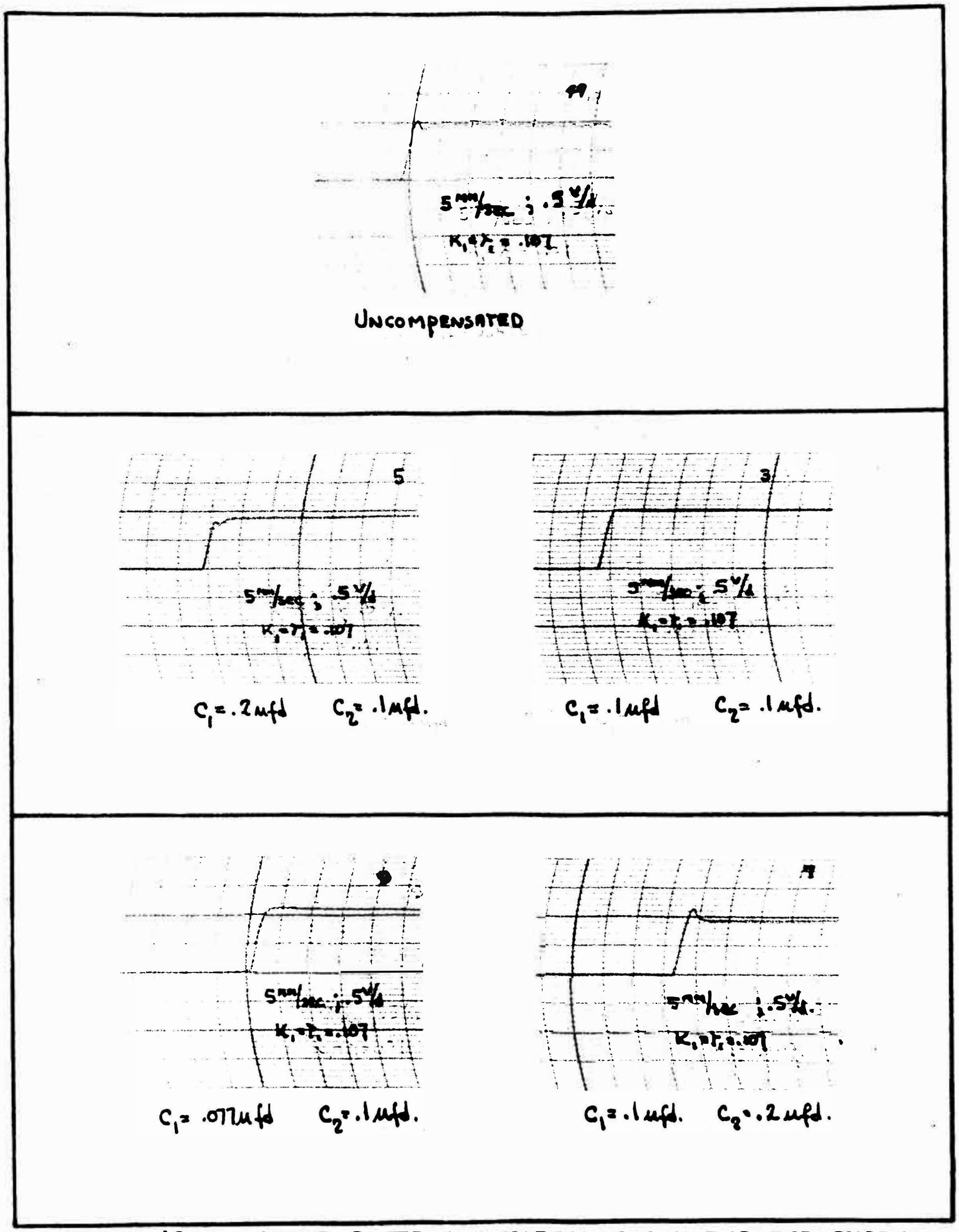


FIGURE 19 UNCOMPENSATED AND VARIABLE COMPENSATED SYSTEM WITH RELAY DEAD ZONE. THE DEAD ZONE WIDTH CORRESPONDING TO 20% OF THE STEP INPUT SIGNAL.

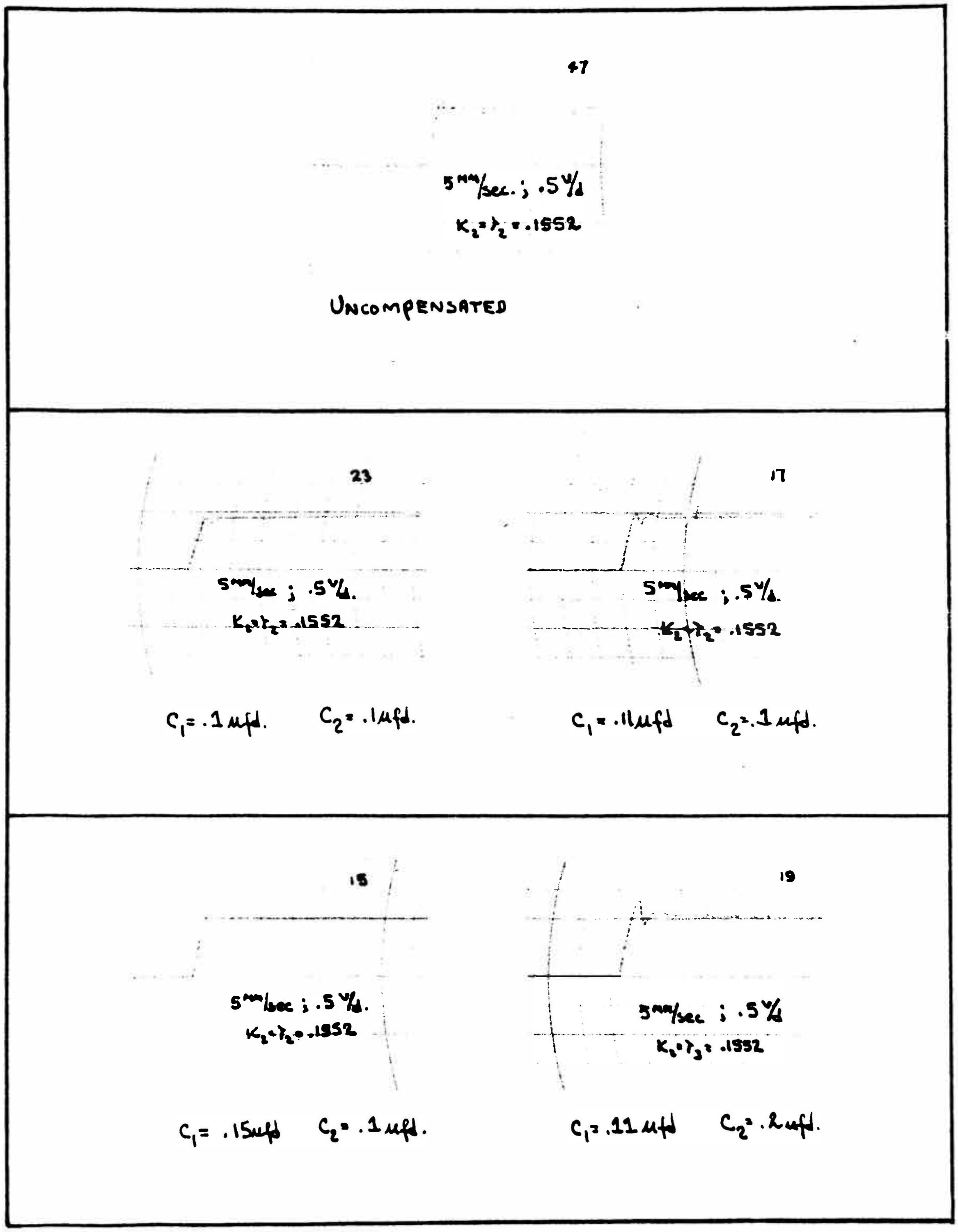


FIGURE 20 UNCOMPENSATED AND VARIABLE COMPENSATED SYSTEM WITH RELAY DEAD ZONE. THE DEAD ZONE WIDTH CORRESPONDING TO 20% OF THE STEP INPUT SIGNAL.

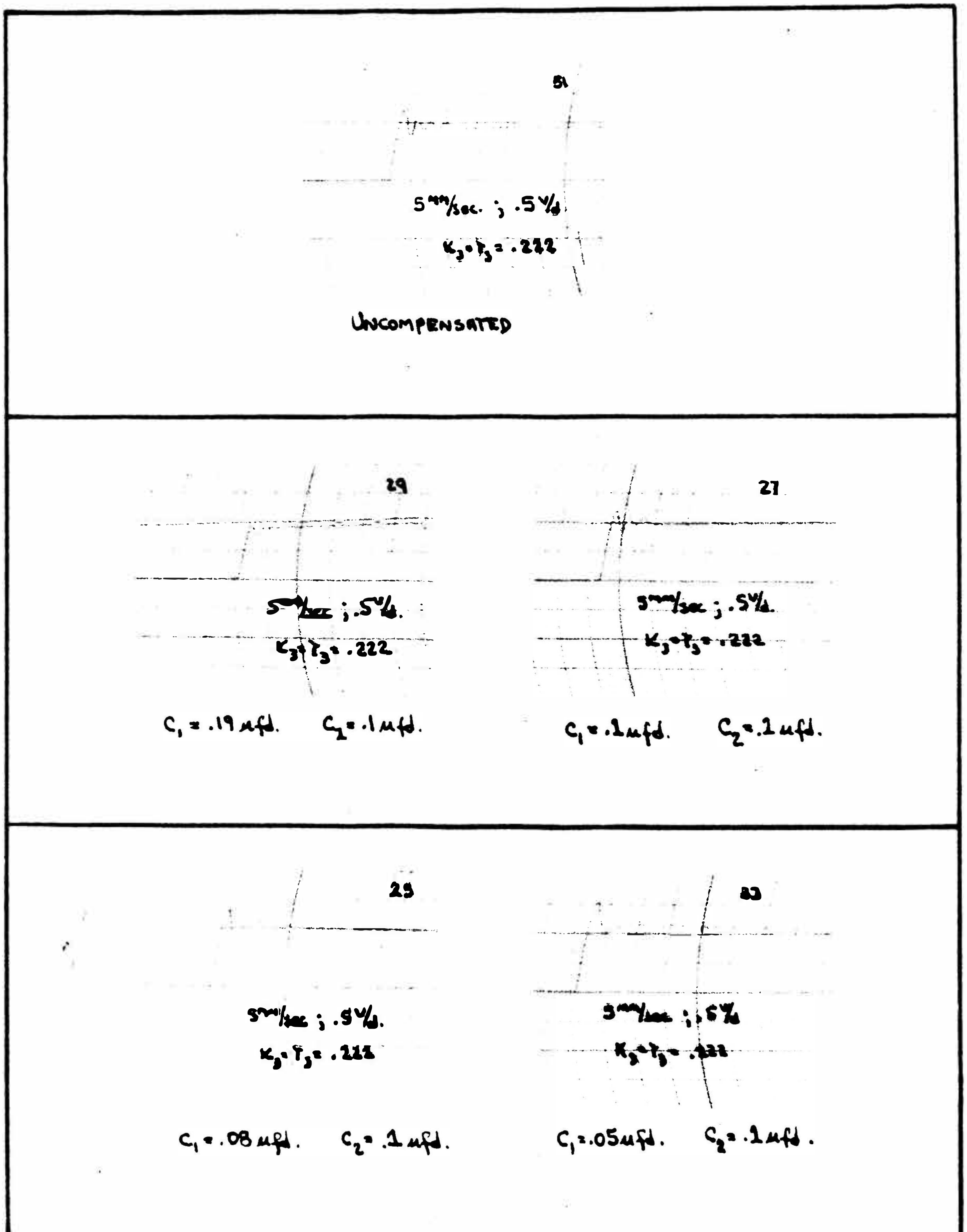


FIGURE 21 UNCOMPENSATED AND VARIABLE COMPENSATED SYSTEM WITH RELAY DEAD ZONE. THE DEAD ZONE WIDTH CORRESPONDING TO 20% OF THE STEP INPUT SIGNAL.

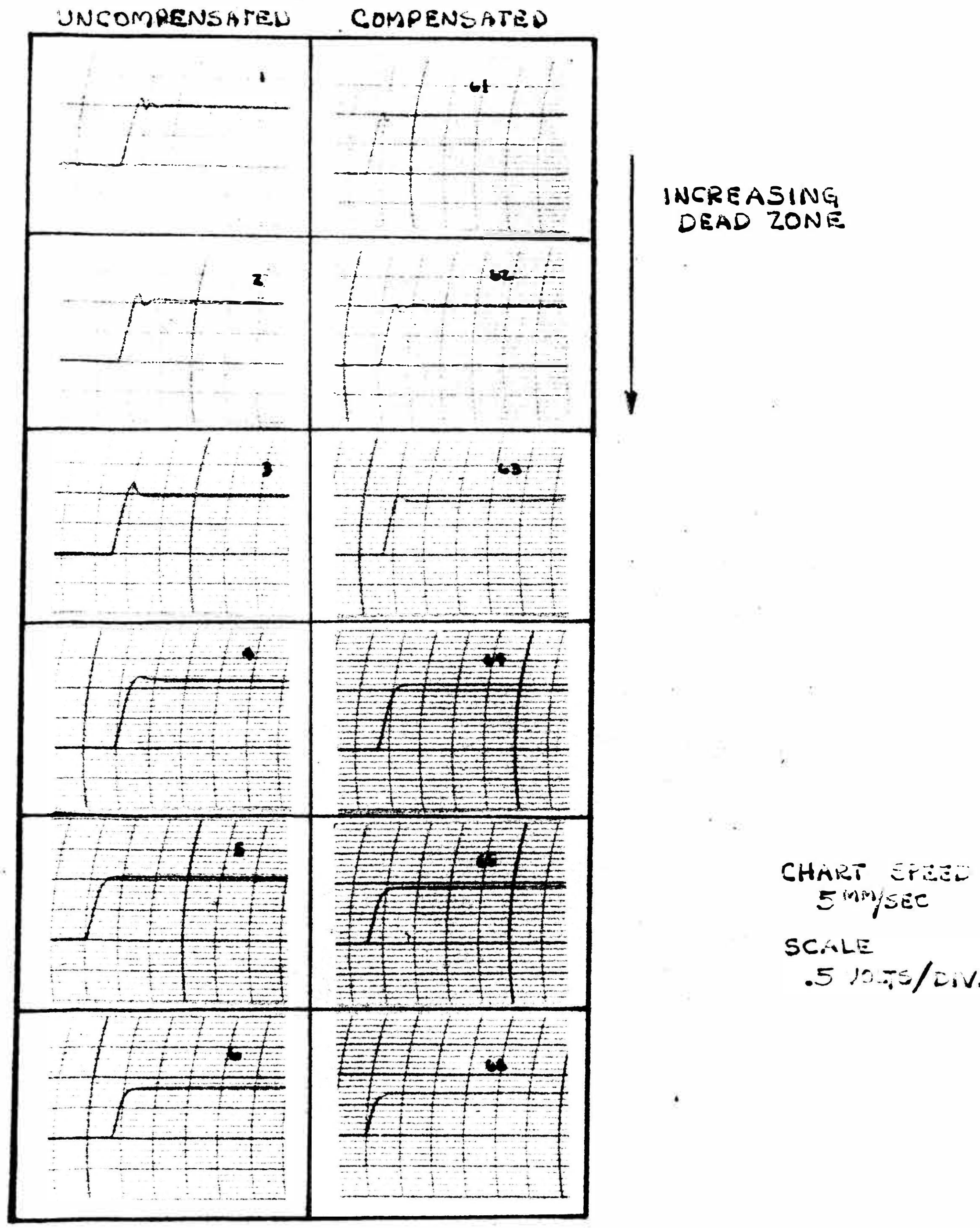


FIGURE 22. AN UNCOMPENSATED AND A COMPENSATED RELAY SERVO WITH VARIABLE RELAY DEAD ZONE AND A MOTOR LOAD TIME CONSTANT OF τ =.1552.

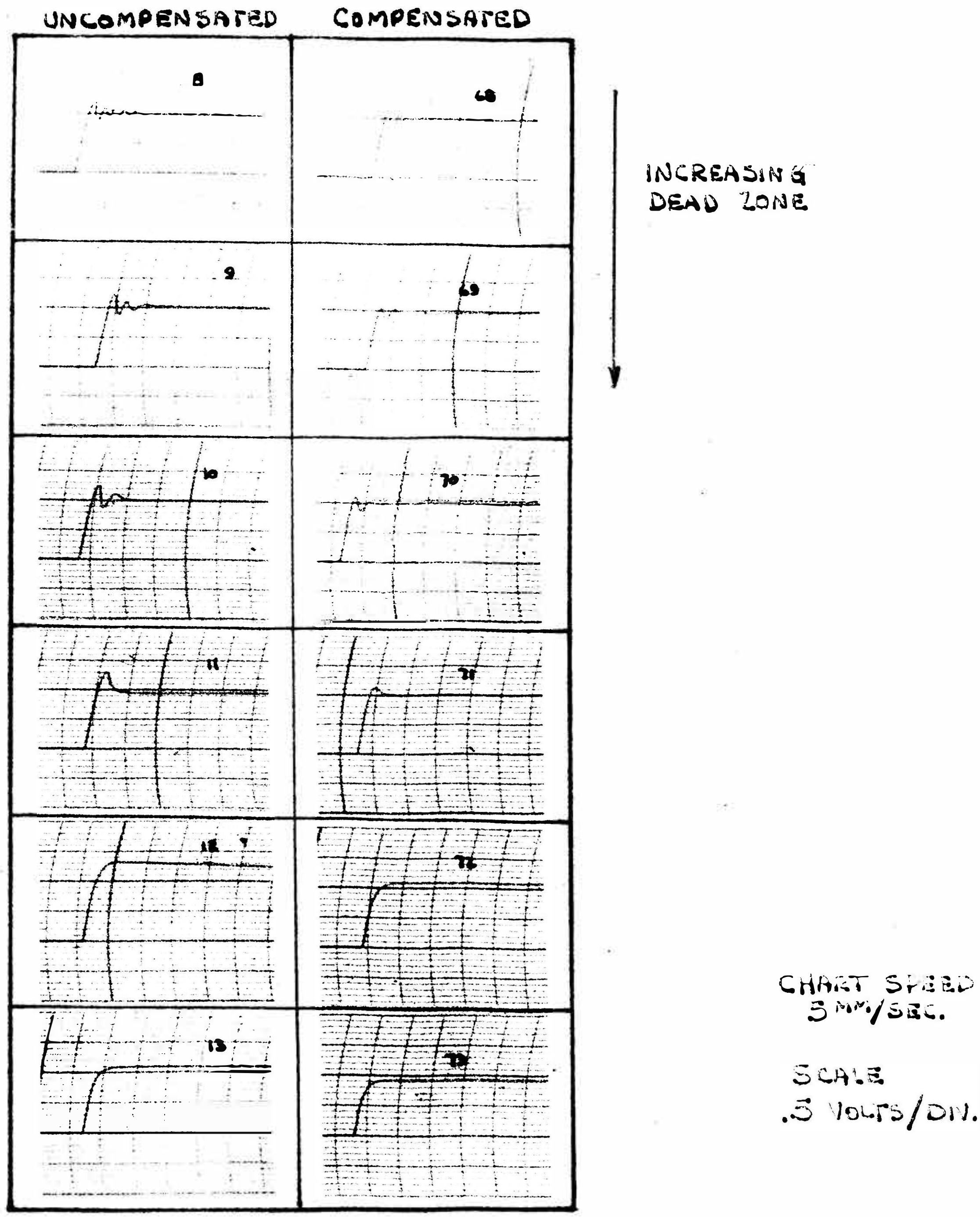


FIGURE 23. AN UNCOMPENSATED AND A COMPENSATED RELAY SERVO WITH VARIABLE RELAY DEAD ZONE AND A MOTOR LOAD TIME CONSTANT OF τ =.1552.

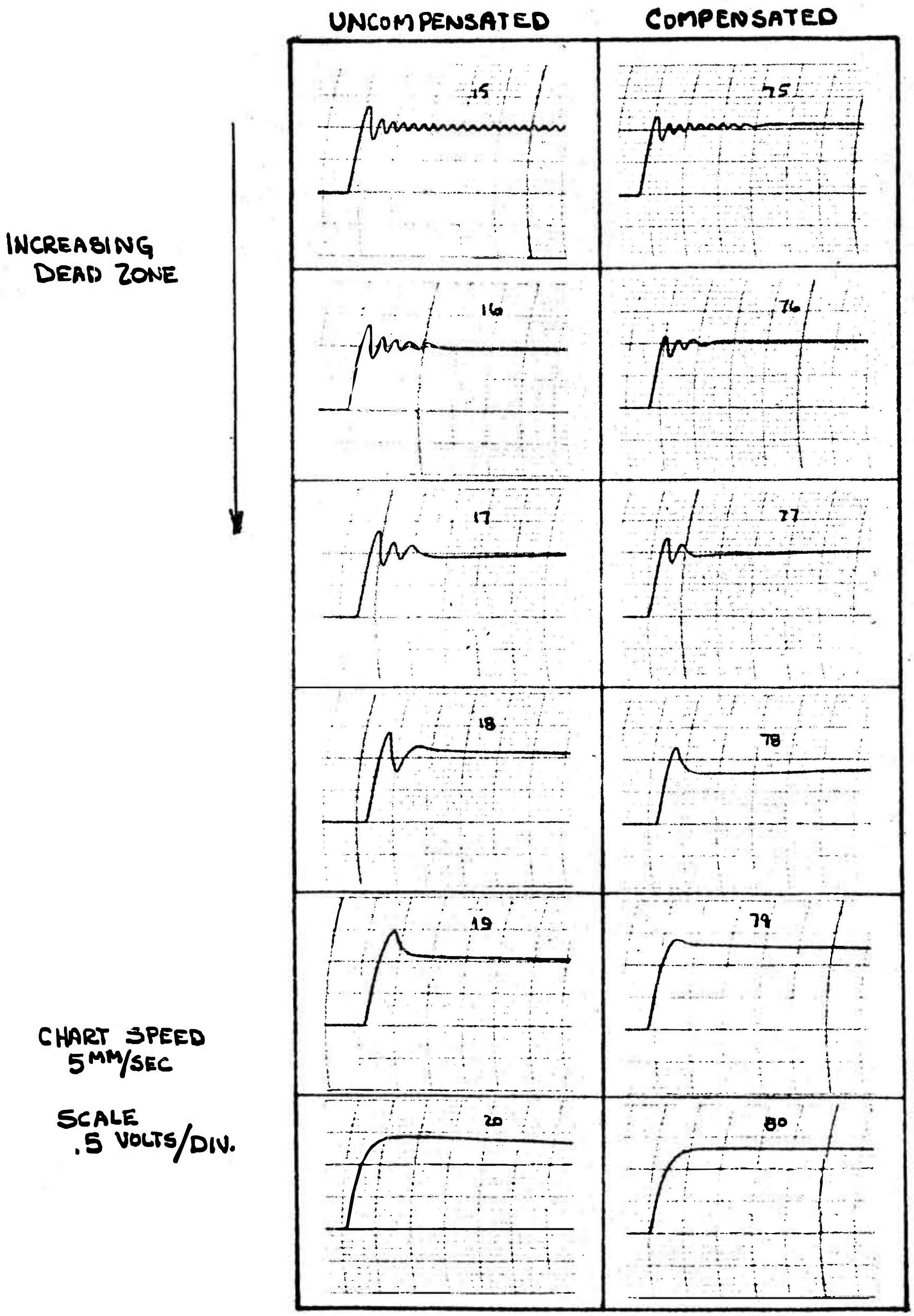
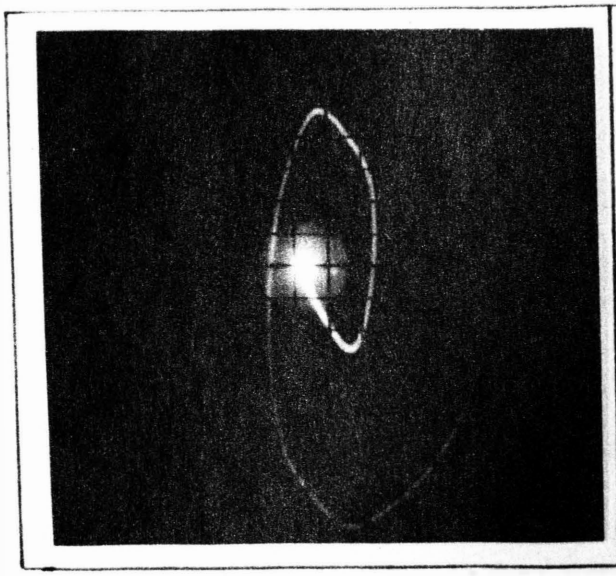
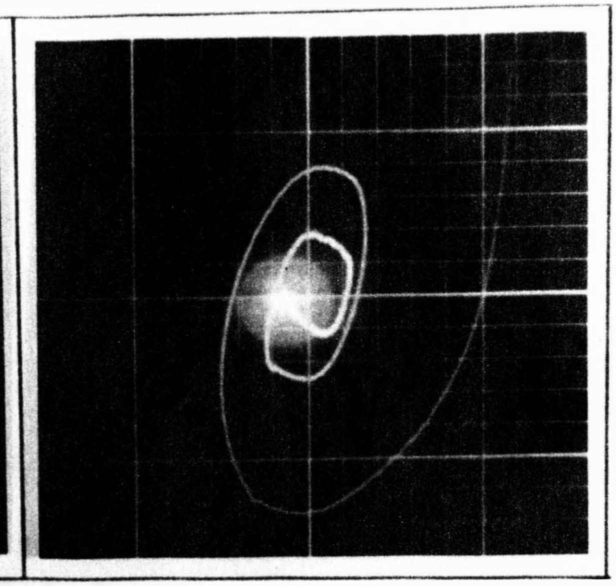


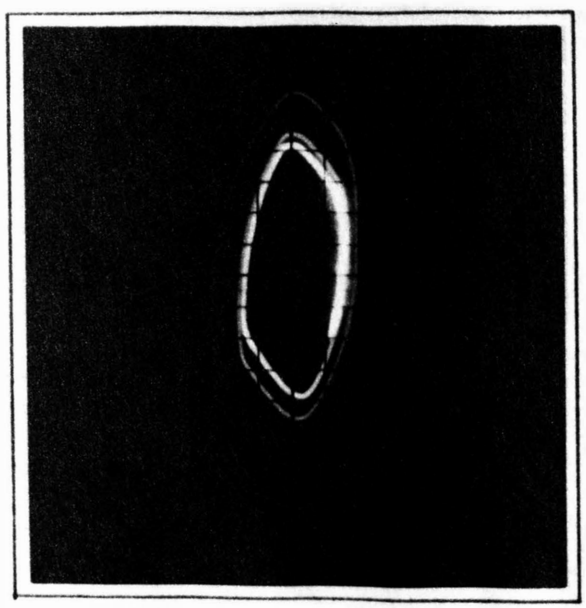
FIGURE 24. AN UNCOMPENSATED AND A COMPENSATED RELAY SERVO WITH VARIABLE RELAY AND A MOTOR LOAD TIME CONSTANT OF $\lambda = .222$.



λ = .107



λ = .1552



λ = .222

FIGURE 25. PHASE-PLANE PHOTOGRAPHS SHOWING THE EFFECT OF VARIATION OF THE DEGREE OF VISCOUS DAMPING ON A RELAY SERVO.

C. THE BASIC SYSTEM WITH STEP INPUT OF POSITION USING A RELAY WITH DEAD ZONE AND HYSTERESIS

In figure 2C a practical relay is illustrated. The relay will initiate either a positive or negative torque correction depending on the sign of the error signal. The hysteresis zone indicates a range where a greater error signal is necessary to produce a correction torque than to cease restoring action.

In practice, a practical limit must be specified for the relay, as the over-all system performance is directly concerned with the relay characteristic. Performance requirements must be met by specification of the dead zone and the speed of response of the system which is determined by specifying the run-away velocity.

The describing-function for the relay under consideration here is from Chapter IV.

$$G_d = \frac{2\, e_a}{\pi e_m} \sqrt{2(1 - ST) + 2\sqrt{(1 - S^2)(1 - T^2)}} \;\angle \tan^{-1} \frac{S - T}{\sqrt{1 - S^2} + \sqrt{1 - T^2}}$$

The phase shift is a function of the error signal amplitude, the dead zone width and the hysteresis zone. The phase relation and describing-function magnitude have been plotted in

Figure 26, with dead zone width of 1 volt and hysteresis zone width ± .2 volts. The describing function, as in the two previously considered cases, is independent of the frequency.

When the hysteresis zone is zero, the phase angle becomes zero, which is the condition described in Chapter V, Section B. When the input becomes less than P there is no relay operation, hence $G_d = 0$. In general, the hysteresis action is detrimental to the operation of the relay servo.

It is evident from Figure 5 that the phase shift is dependent on the "pull in" and "drop out" voltages of the relay. That is, the fundamental component of this Fourier series will undergo a phase shift in relation to the error signal as the correction pulse shifts.

The phase-plane portrait of a damped relay servo, with a motor-load time constant equal to .107, appears in Figure 27. It is apparent that oscillations in the output result and a limit cycle is reached. Figures 28 through 30 show the effect of variable hysteresis zone width (h) and variable dead zone (Δ). The static positional error is, however, noticeable as the dead zone width is increased. The three figures represent the three conditions of damping presented throughout

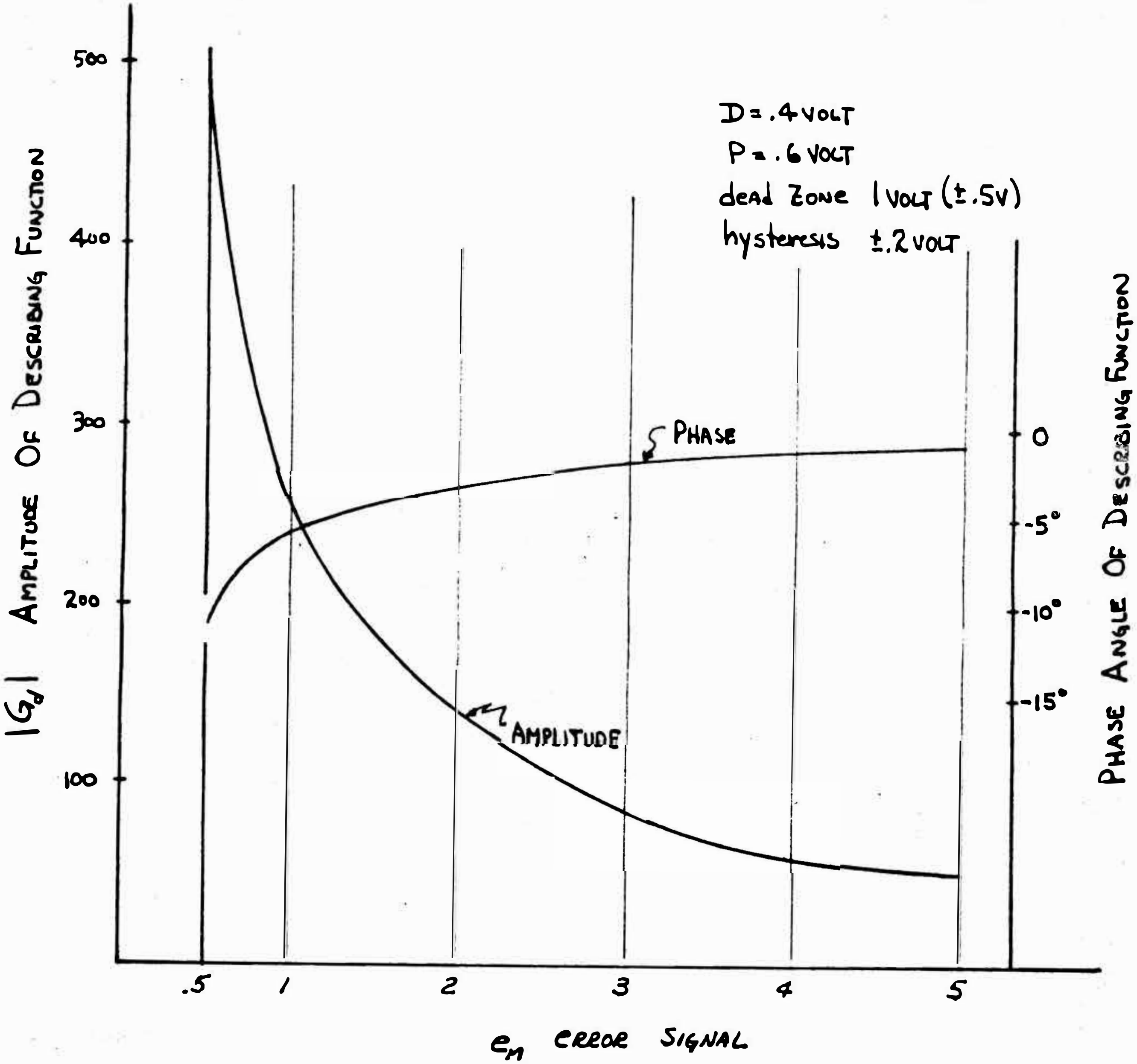


FIGURE 26. PHASE RELATIONS AND DESCRIBING-FUNCTION VALUES FOR THE RELAY WITH DEAD ZONE AND HYSTERESIS.

K	ė	
	e < 2	e > 2
0	21.46	-21.46
1	19.38	-19.38
3	16.23	-16.23
10	10.35	-10.35
20	6.8	-6.8
∞	0	0
	e < 3	e > 3
-40	-6.51	6.51
-22.6	-15	15
-20	-18.73	18.73

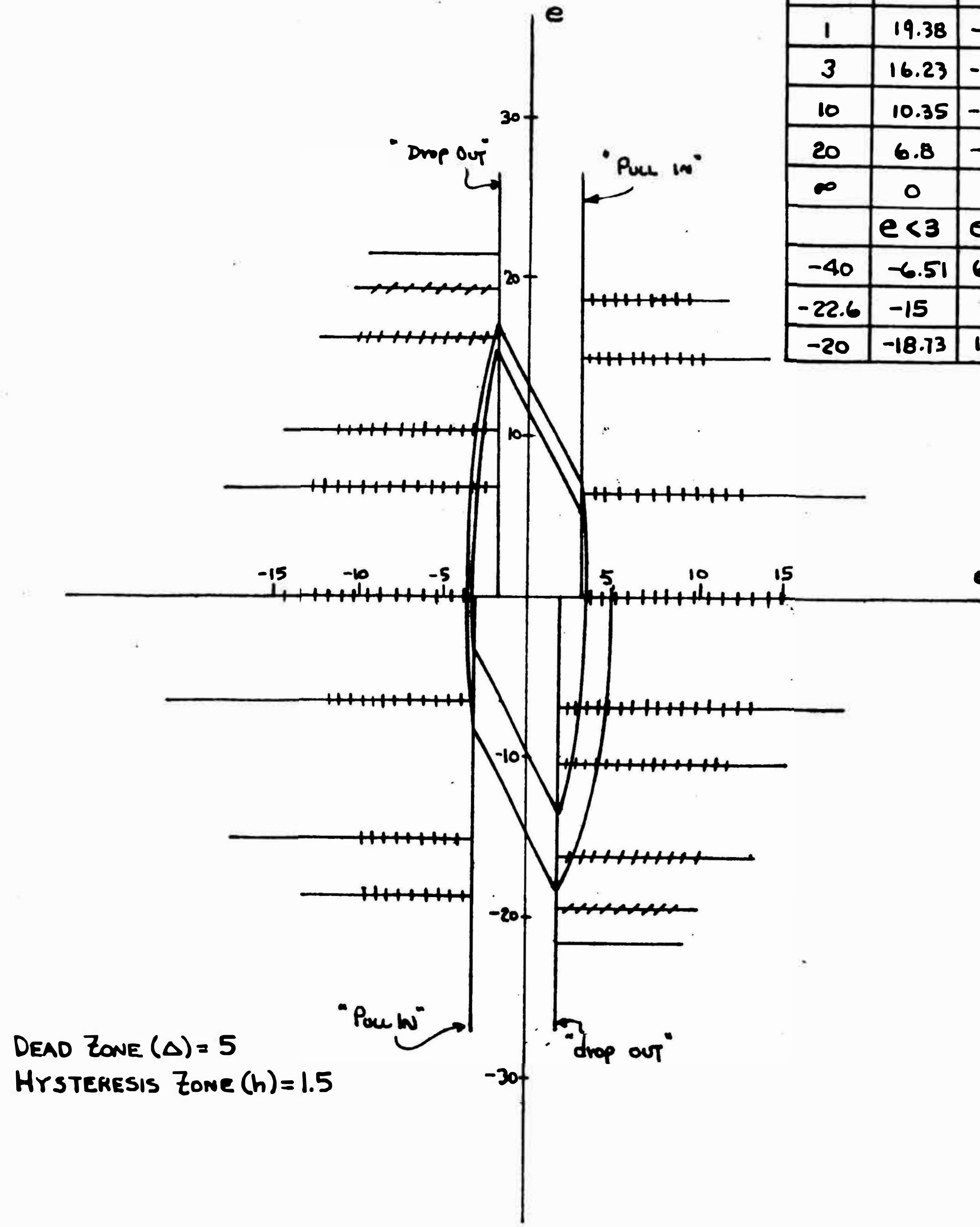


FIGURE 27. PHASE-PLANE PLOT OF RELAY WITH DEAD ZONE AND HYSTERESIS

the thesis and it is apparent that damping is necessary in relay servos. Figure 31 shows the results of linear lead compensation of the systems in Figures 28 through 30. For comparison of data, the following chart is included.

Figure 31	Corresponding for identical damping, dead zone and hysteresis zone.
Column 1	Figure 28 Column A
Column 2	Figure 29 Column F
Column 3	Figure 30 Column D

To make the results more evident, the responses of Figure 31 Column (2), for a motor-load time constant .1552, have been reproduced with the results of Figure 29, Column F. These results appear in Figure 32.

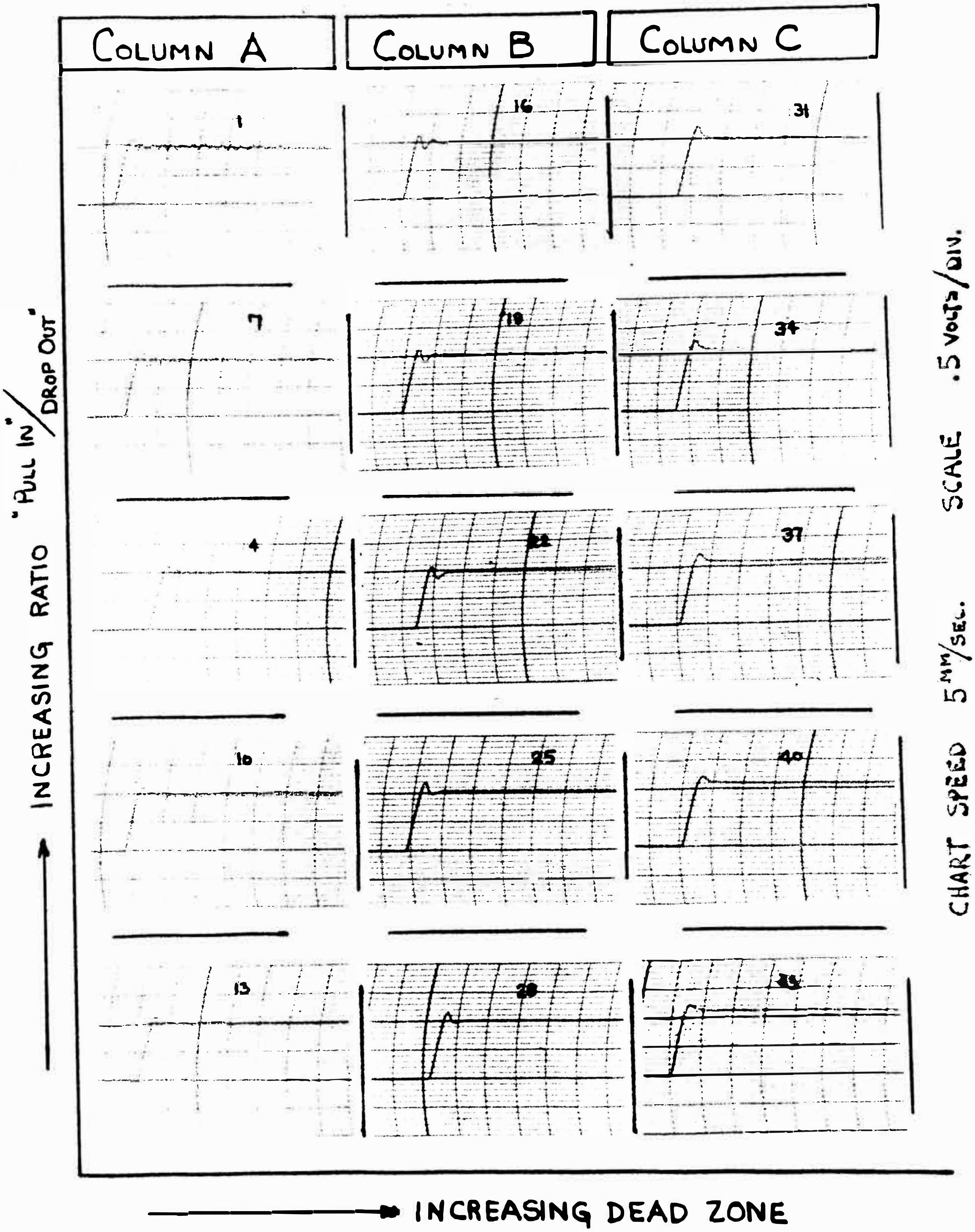


FIGURE 28. ANALOG COMPUTER RESULTS FOR A RELAY SERVO INCLUDING VARIABLE DEAD ZONE AND HYSTERESIS. THE MOTOR LOAD TIME CONSTANT λ =.107.

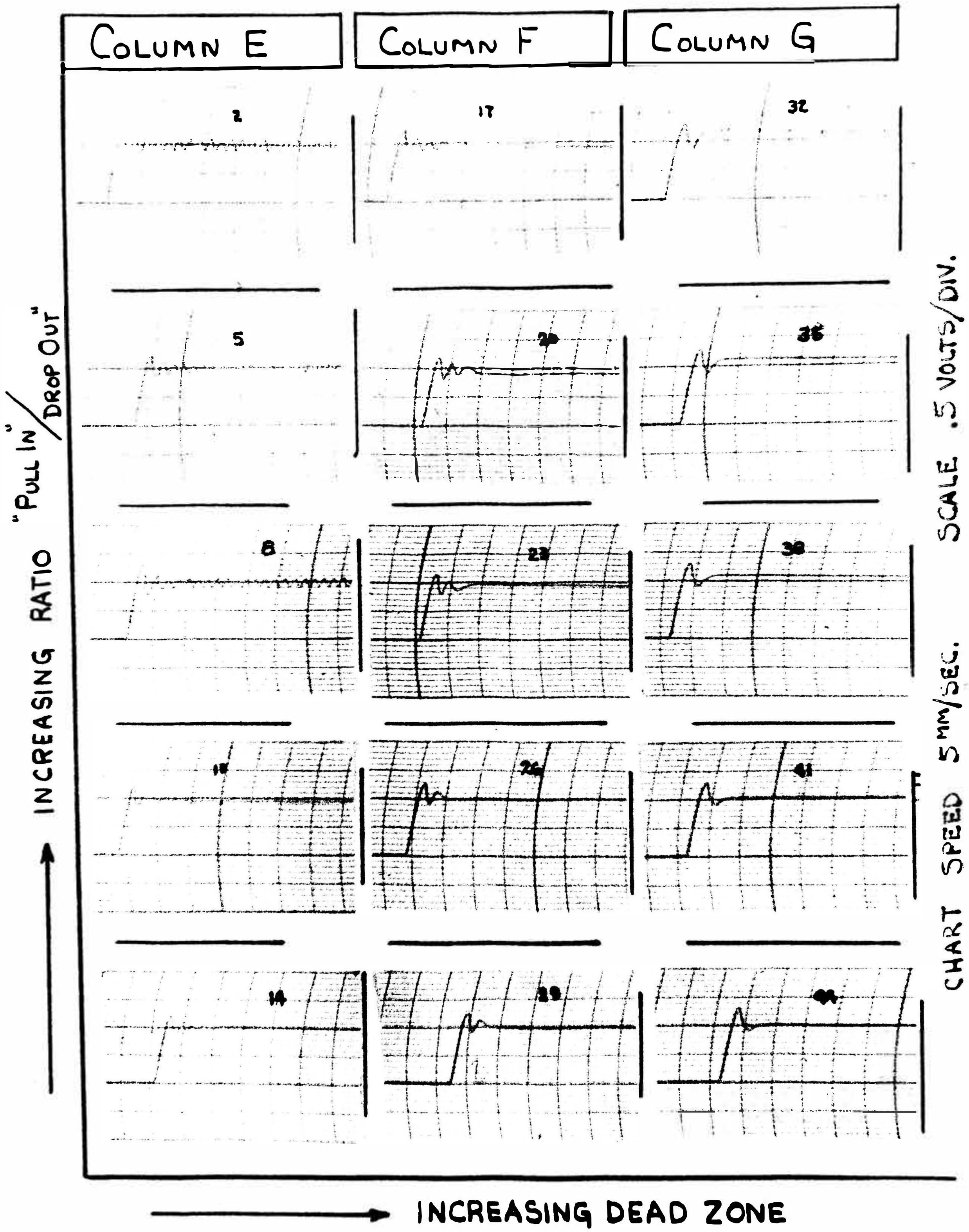


FIGURE 29. ANALOG COMPUTER RESULTS FOR A RELAY SERVO INCLUDING VARIABLE DEAD ZONE AND HYSTERESIS. THE MOTOR LOAD TIME CONSTANT τ = .1552.

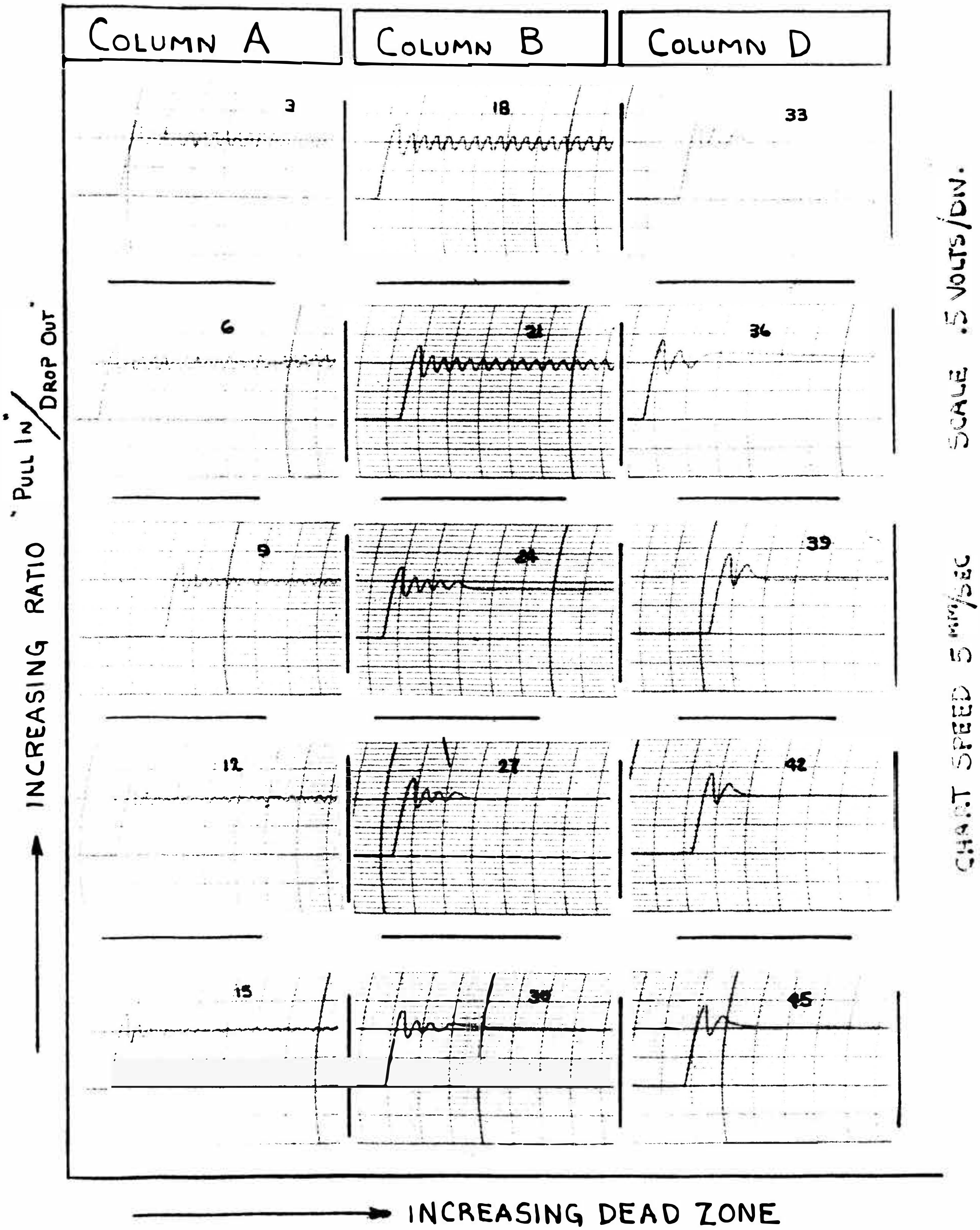


FIGURE 30. ANALOG COMPUTER RESULTS FOR A RELAY SERVO INCLUDING VARIABLE DEAD ZONE AND HYSTERESIS. THE MOTOR LOAD TIME CONSTANT λ =.222.

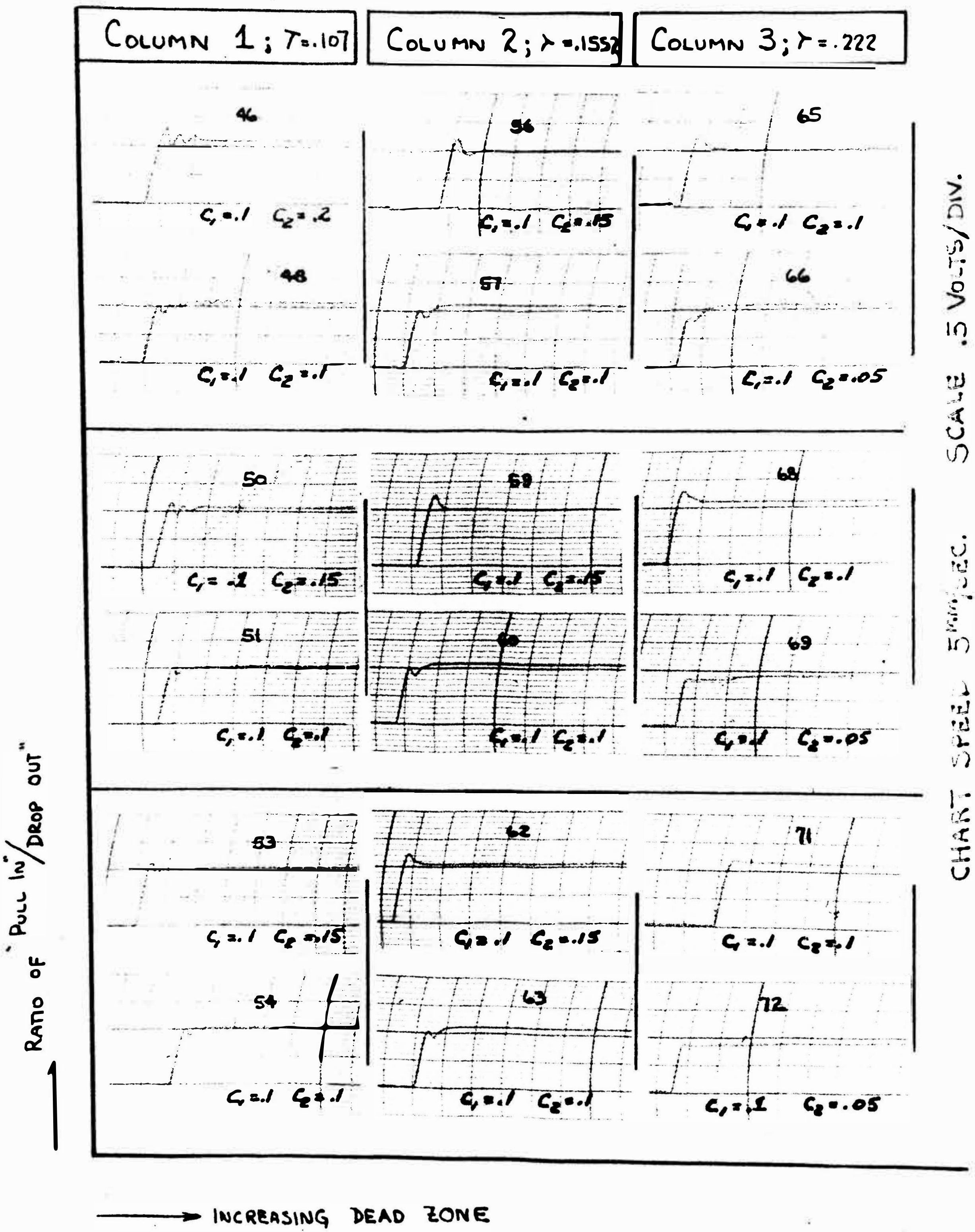


FIGURE 31. ANALOG COMPUTER RESULTS FOR A RELAY SERVO WITH COMPENSATION FOR VARIABLE DEGREES OF RELAY DEAD ZONE AND HYSTERESIS.

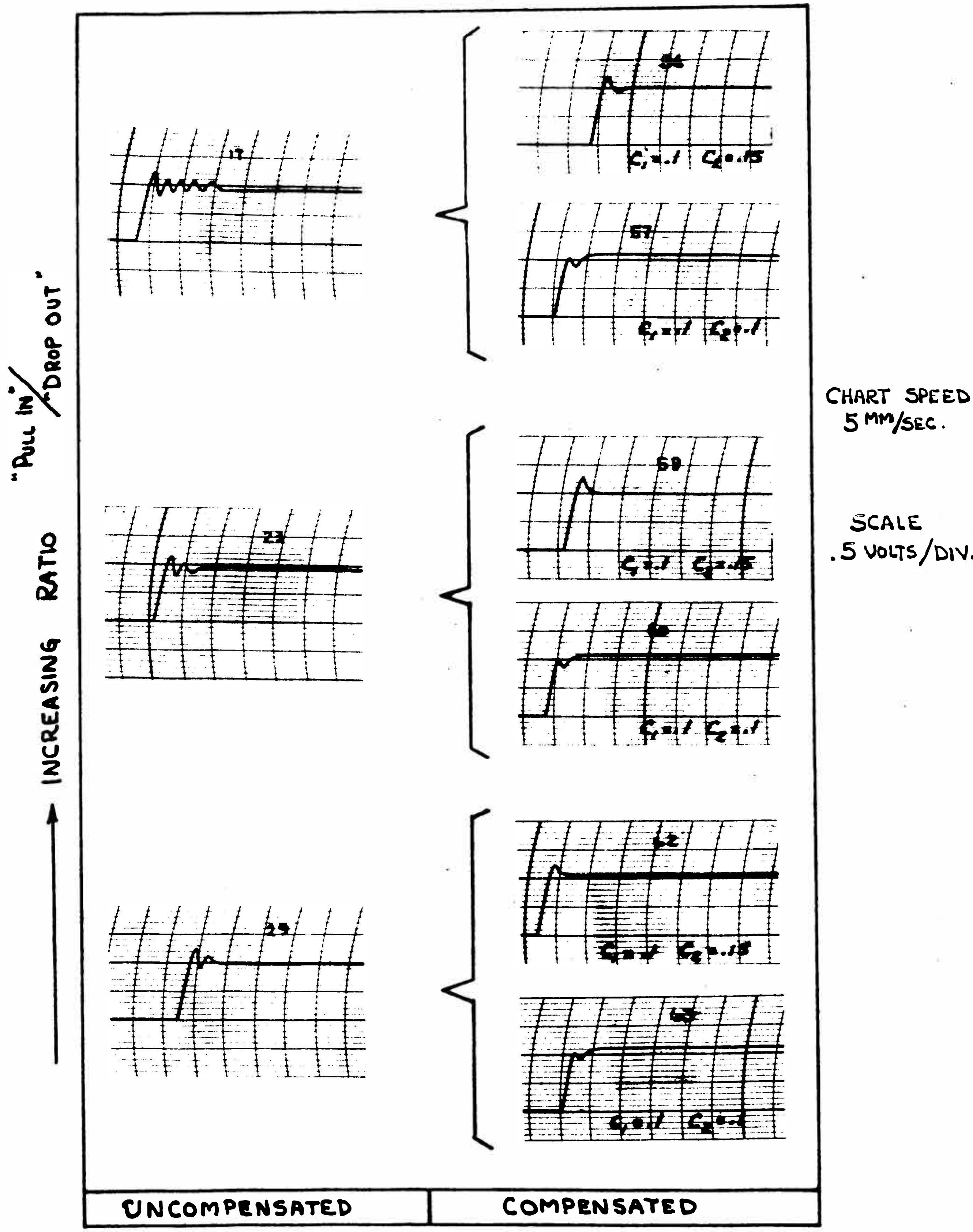


FIGURE 32. COMPARISON OF COMPENSATED AND UNCOMPENSATED RELAY SERVO HAVING VARIABLE HYSTERESIS ZONE IN RELAY.

VI. CONCLUSIONS

The role of the analog computer in the analysis and synthesis of nonlinear systems cannot be overemphasized. When parameter variations are considered from an engineering point of view, the study of nonlinear systems by analytical methods becomes so extensive and tedious as to be almost insurmountable. A major portion of this problem was solved by using the analog computer to simulate results, which otherwise could not have been done economically and efficiently. The approach for synthesis of relay servos found most desirable was:

1. Use describing functions to determine the relative stability of the basic system. The frequency response approach here is desirable to determine the relative stability and indicate if the use of compensation networks is necessary.

2. Predict the dynamic transient response by phase plane methods. This is limited to second order systems, however, it is sometimes possible to approximate higher order systems by second order systems. The maximum overshoot, number of transient oscillations and maximum velocity are readily obtained.

3. Perform actual test runs, either on actual equipment or by computer simulation for improving system performance

and studying different forms of driving functions. Compensation methods are best studied by using the computer or actual test runs.

A. VISCOUS FRICTION

Greater amounts of viscous damping in relay servos can be tolerated than in linear systems. Viscous friction was found to be very desirable as a stabilizing factor with very little sacrifice of performance for a compensated system. The reason is that the runaway velocity usually is never reached unless large errors occur and, therefore, the response time is not significantly affected. For the motor in question in this thesis, it was found that before velocity saturation was reached, the viscous friction on the output shaft had to be increased 100 per cent over the rated viscous friction parameter of 3 lb.ft./rad/sec. with a 5 volt step input.

It was found that 120 per cent more viscous friction could be tolerated in the system studied when compared to an equivalent system operating as a linear servo. The rise time for instance, in the case of the relay servo, was on an average of 25 per cent better than the linear servo. The overshoot, however, of the relay system was, in general, about 10 per cent larger without compensation. When the gain of the linear system (without compensation) was increased to give the same rise time as the relay system, the overshoot of the linear system was 5 per cent larger than the relay servo.

The settling time of the relay servo was, in general, longer than that of the linear system even with phase lead compensation. This is apparent because of the large corrective torque used in the relay system and the tendency toward oscillation is invited. However, the settling time can be improved considerably by phase lead compensation.

B. RELAY CHARACTERISTIC

The relay used in the relay servo is the predominate element in the system, affecting the position error, settling time and stability.

1. IDEAL RELAY

The ideal relay, with no dead zone, theoretically produces oscillation at zero amplitude and infinite frequency with viscous friction. Although this was not consistent with experimental results, the theoretical result seems to be consistent with the dynamics of the physical system. The settling time of the oscillation is determined by the degree of damping.

2. RELAY WITH DEAD ZONE

The dead zone width is most effective in determining whether the system oscillates in a limit cycle or reduces to a steady state error. Only a small dead zone, with a reasonable amount of viscous damping is sufficient to remove the tendency toward continuous oscillation. The larger the dead zone, the greater the stability. However, the possibility of steady state error in output position increases. This error can be any value within the dead zone range.

3. RELAY WITH DEAD ZONE AND HYSTERESIS

The hysteresis zone has a detrimental affect on any relay servo. The delay in the "drop out" point introduces a lag in the system which produces less stability. The larger the hysteresis zone, the less stable the relay servo becomes. Experimentally, it was found that a ratio of "pull in" to "drop out" voltage greater than 2 produces unstable operation so that it is impossible to compensate satisfactorily by phase lead networks.

C. LINEAR PHASE LEAD COMPENSATION

There is no doubt that phase lead compensation is desirable in the operation of a relay servo. I mprovement in stability and settling time can be accomplished and the rise time in general is not affected. The phase lead network used to satisfactorily compensate a system for one magnitude of input signal may not be the most desirable network when the input signal is changed. No correlation between the signal amplitude and phase lead compensation was observed.

D. FUTURE CONSIDERATIONS

1. A practical design technique for higher order nonlinear systems is not at all well defined. For instance, reasonable approximations for the study of the higher order systems in the phase-plane instead of phase space do not exist. There are no analytical or empirical rules for determining the limits of good approximations.

2. Further analytical work for the study in technique of how a change in input signal changes the response of a nonlinear system should be studied. This area is large and extensive and provides a stimulus for a considerable amount of further research. Figure 32 illustrates, for example, how a change in input affects the response of a relay system. In the figure, three responses are shown with and without compensation. The inputs in each case are steps of magnitudes 2, 5 and 10. A suitable response by compensation was secured for the 10 volt step and the same compensation

network was used as the compensation circuit for the other two inputs. These results are summarized below:

	UNCOMPENSATED			COMPENSATED		
Magnitude of step input	Over-shoot	Settling time	Steady State Error	Over-shoot	Settling Time	Steady State Error
2	50%	3 1/2 cycles	10%	80%	2 1/2 cycles	8%
5	25%	3 1/2 cycles	10%	20%	2 1/2 cycles	2%
10	25%	4 cycles	5%	13% under-shoot	2 cycles	0%

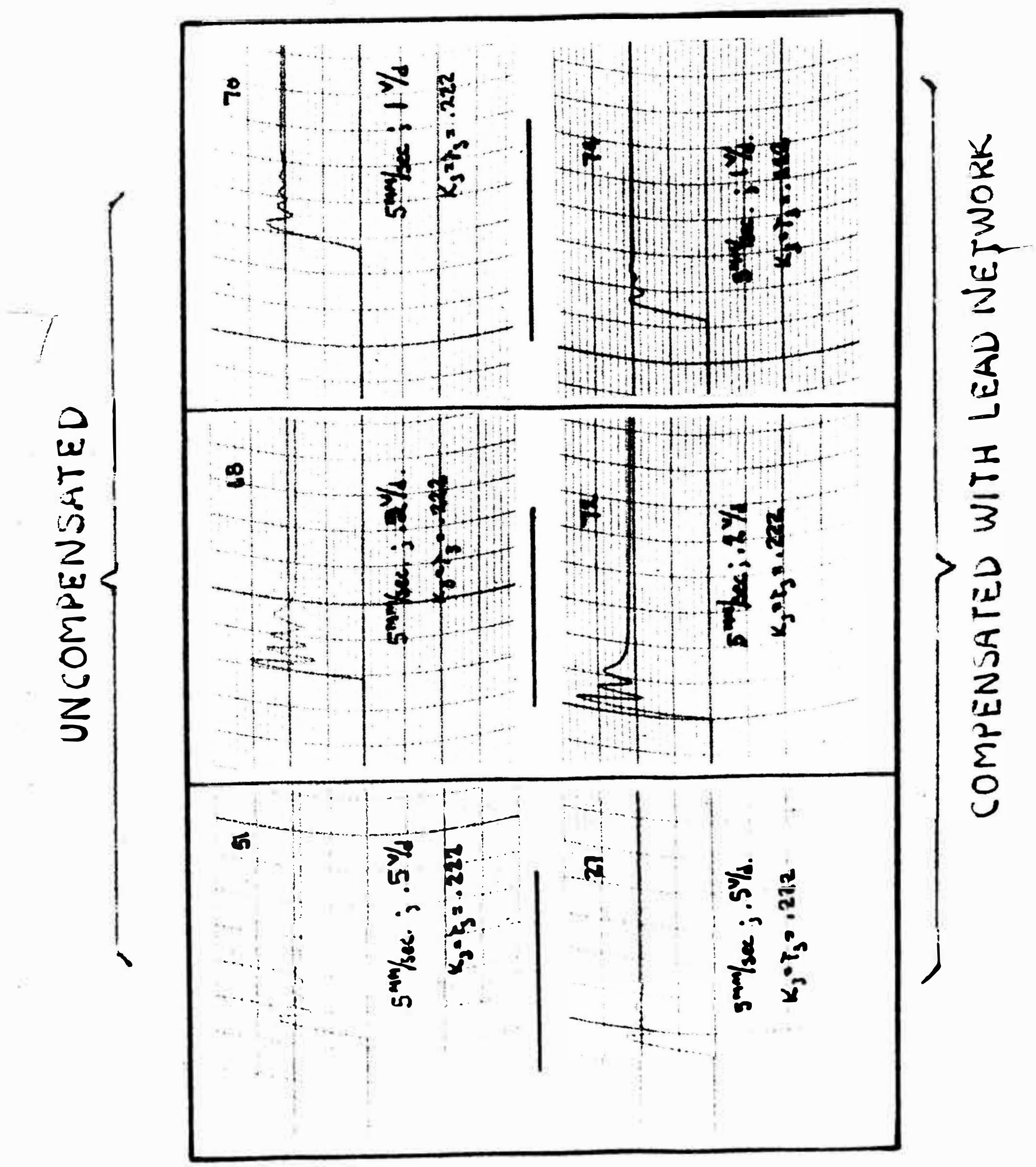


FIGURE 33. ANALOG COMPUTER SHOWING THE RESULT OF DRIVING-FUNCTION ON A RELAY SERVO WITH AND WITHOUT COMPENSATION.

APPENDIX I

THE TRANSFER FUNCTION OF SERVO-MOTOR AND LOAD

The schematic diagram of a dc servo-motor and load is shown in Fig. A-1. The symbols used for this derivation are as follows:

θ_m	-	Motor shaft position in radians.
θ_L	-	Load shaft position in radians.
J_L	-	Polar moment of inertia of load in slug-ft.^{2}
f_L	-	Viscous friction of load in lb.-ft./rad./sec.
J_m	-	Polar moment of inertia of motor in slug-ft^2
f_m	-	Viscous friction of motor in lb.-ft./rad./sec.
K_t	-	Motor torque constant lb.-ft./amp.
K_e	-	Counter emf of motor in volts/ rad./sec.
R_a	-	Resistance of motor armature in ohms.
N	-	Ratio of gear reduction between motor and load shafts.
e_a	-	Applied emf to armature circuit.
i_a	-	Resulting armature current with e_a applied.

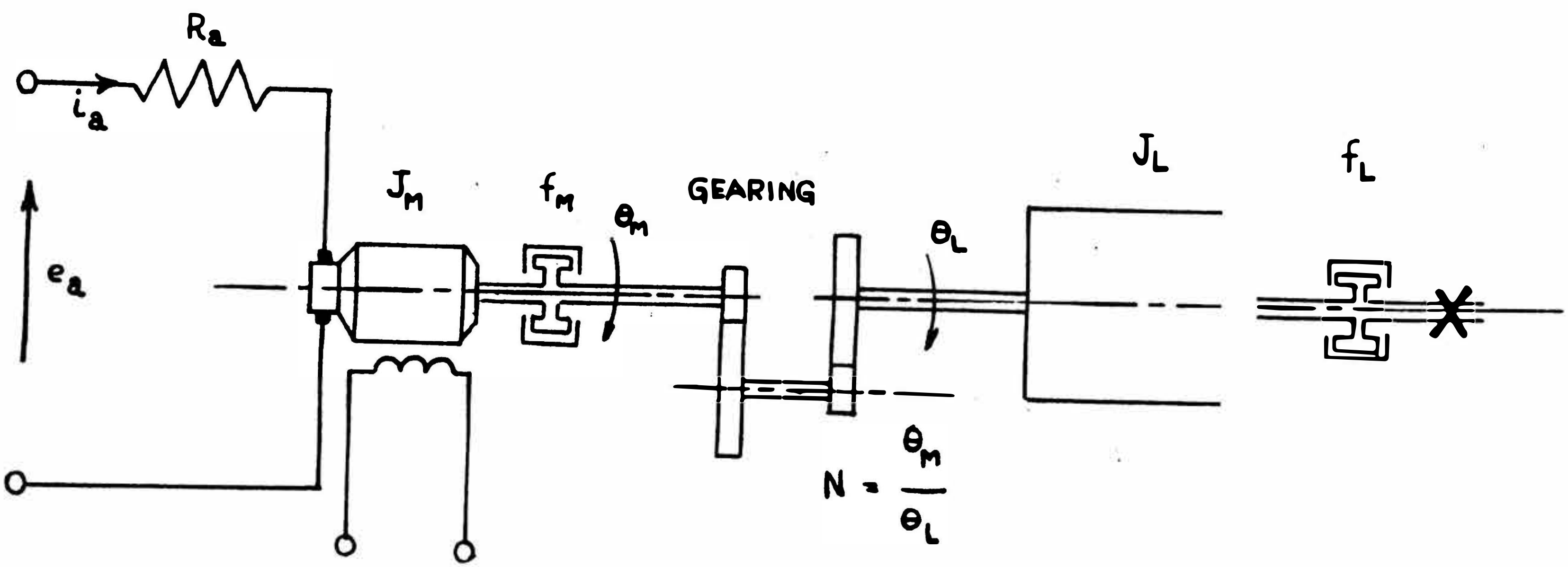


FIGURE A-1 SCHEMATIC DIAGRAM OF ARMATURE CONTROLLED DC SERVOMOTOR AND LOAD WITH CONSTANT MOTOR FIELD EXCITATION.

With negligible armature inductance and constant field excitation, the equation relating the applied emf and the motor armature circuit is:

$$e_a = i_a R_a + K_e \frac{d\theta_m}{dt} \qquad (1)$$

this can be related to output shaft position as:

$$e_a = i_a R_a + K_e N \frac{d\theta_L}{dt} \qquad (2)$$

Solving (2) in terms of i_a yields:

$$i_a = \frac{e_a}{R_a} - \frac{K_e N}{R_a} \frac{d\theta_L}{dt} \qquad (3)$$

The differential equation relating the torque-equilibrium conditions for the electro-mechanical system is:

$$i_a K_t = J_m \frac{d^2\theta_m}{dt^2} + f_m \frac{d\theta_m}{dt} + \frac{1}{N}\left[J_L \frac{d^2\theta_L}{dt^2} + f_L \frac{d\theta_L}{dt} \right] \qquad (4)$$

By considering the relationship between output and load shift position, equation (4) becomes:

$$i_a K_t N = (N^2 J_m + J_L) \frac{d^2\theta_L}{dt^2} + (N^2 f_m + f_L) \frac{d\theta_L}{dt} \qquad (5)$$

Let: $J_{eL} = N^2 J_m + J_L$ and $f_{eL} = N^2 f_m + f_L$

Then: $$i_a K_t N = J_{eL} \frac{d^2\theta_L}{dt^2} + f_{eL} \frac{d\theta_L}{dt} \qquad (6)$$

By introducing equation (3) into equation (6) the forcing function becomes:

$$\frac{e_a}{R_a} K_t N = J_{eL} \frac{d^2\theta_L}{dt^2} + \left(f_{eL} + \frac{K_e K_t N^2}{R_a} \right) \frac{d\theta_L}{dt} \qquad (7)$$

Solving by Laplace transformation methods and simplifying, establishes the output-input relation:

$$\frac{\bar{\theta}_L(s)}{e_a} \quad \frac{K_t \; N/R_a f_{el} + N^2 K_e K_t}{S\left[\frac{J_{eL} R_a}{R_a f_{eL} + N^2 K_e K_t} \quad S + 1 \right]}$$

Where: $\bar{\theta}_L(S) = \mathcal{L}\left[\theta_L \quad (t) \right]$ (8)

Simplification can be accomplished by the following substitutions which yield the transfer function of the motor and load with a viscous damped load and negligible armature inductance.

$$K_1 = \frac{K_t N}{R_a f_{eL} + N^2 K_e K_t} \quad \text{and} \quad \frac{R_a \; J_{eL}}{R_a f_{eL} + N^2 K_e K_t}$$

then:

$$\frac{\bar{\theta}_L(S)}{e_a} = \frac{K_1}{S(\lambda_1 \; S + 1)} \qquad (9)$$

Where K_1 is the gain constant and λ_1 is the time constant of the system

Equation (8), when rewritten in terms of the viscous friction is:

$$\frac{\bar{\theta}_L(s)}{e_a} \quad \frac{K_t\ N/R_a(f_m N^2 + f_L) + N^2 K_e K_t}{S\left[\dfrac{J_{eL}\ R_a}{R_a\ (f_m N^2 + f_L) + N^2 K_e K_t}\ S + 1\right]}$$

(10)

Two special cases of this differential equation are:

$$\frac{\bar{\theta}_L(S)}{e_a} \quad \frac{K_t/N(R_a f_m + K_e K_t)}{S\left[\dfrac{J_{eL}\ R_a}{N^2\ (R_a f_m + K_e K_t)}\ S+1\right]} \qquad \text{for } f_L = 0$$

(11)

$$\frac{\bar{\theta}_L(S)}{e_a} \quad \frac{1/K_e N}{S\left[\dfrac{J_{eL}\ R_a}{N^2 K_e K_t}\ S + 1\right]} \qquad \text{for } f_m = f_L = 0$$

(12)

APPENDIX II

SELECTION OF MOTOR AND LOAD FOR THE BASIC SYSTEM

As a basic system, a General Electric servo-motor type BBY29YA was selected which has the following parameters:

Commercial rating: 1 horse power, 200 volts.

J_m - Polar moment of inertia of motor - .0388 slug-ft.^{2}

f_m - Viscous friction of motor - .02 lb.ft./rad/sec.

R_a - Armature resistance - 3 ohms.

K_t - Motor torque constant - 0.31 lb.ft./amp.

K_e - Counter emf constant of motor - 0.45 volt/rad/sec.

Constants selected for the load were:

N - Gear reduction ratio - 10

f_L - Viscous friction of load - 3 lb.ft./rad./sec.

J_L - Polar moment of inertia of load - .645 slug-ft^2.

The typical values when used to evaluate the motor load constants of Appendix I with viscous friction are:

$$K_1 = \frac{K_t \ N}{R_a \ f_{eL} + N^2 \ K_e \ K_t} = 0.107 \text{ rad./sec.}$$

where

$$= \frac{R_a \ J_{eL}}{R_a \ f_{eL} + N^2 \ K_e \ K_t} = 0.107 \text{ sec.}$$

and

$$J_{el} = N^2 \ J_m + J_L = 1.033 \text{ slug - ft.}^2$$

$$f_{el} = N^2 \ f_m + f_L = 5 \text{ lb. ft./rad/sec.}$$

whence the transfer function of the motor and load with viscous friction is:

$$\frac{\bar{\theta}_L(s)}{e_a} = \frac{K_1}{S \ (\lambda_1 S+1)} = \frac{0.107}{S \ (.107S+1)} \tag{13}$$

The transfer function of the motor and load with no viscous friction in load when computed from equation 12 of the Appendix I is:

$$\frac{\bar{\theta}_L(s)}{e_a} = \frac{\dfrac{K_t}{N \ (R_a \ f_m + K_e \ K_t)}}{S \left[\dfrac{J_{eL} \ R_a}{N^2 \ (R_a \ f_m + K_e \ K_t} \ S + 1 \right]} = \frac{K_2}{S(\lambda_2 S+1)} = \frac{.1552}{S[.1552S+1]} \tag{14}$$

where:

$$K_2 = \lambda_2 = .1552$$

The transfer function of the motor and load with the viscous friction of both motor and load equal to zero is:

$$\frac{\bar{\theta}_L(s)}{e_a} = \frac{1/K_e\ N}{S\left[\frac{J_{eL}\ R_a}{N^2 K_e K_t} S + 1\right]} = \frac{K_3}{S(\lambda_3 S+1} = \frac{.222}{S(.222\ S+1)}$$

where $K_3 = \lambda_3 = .222$ (15)

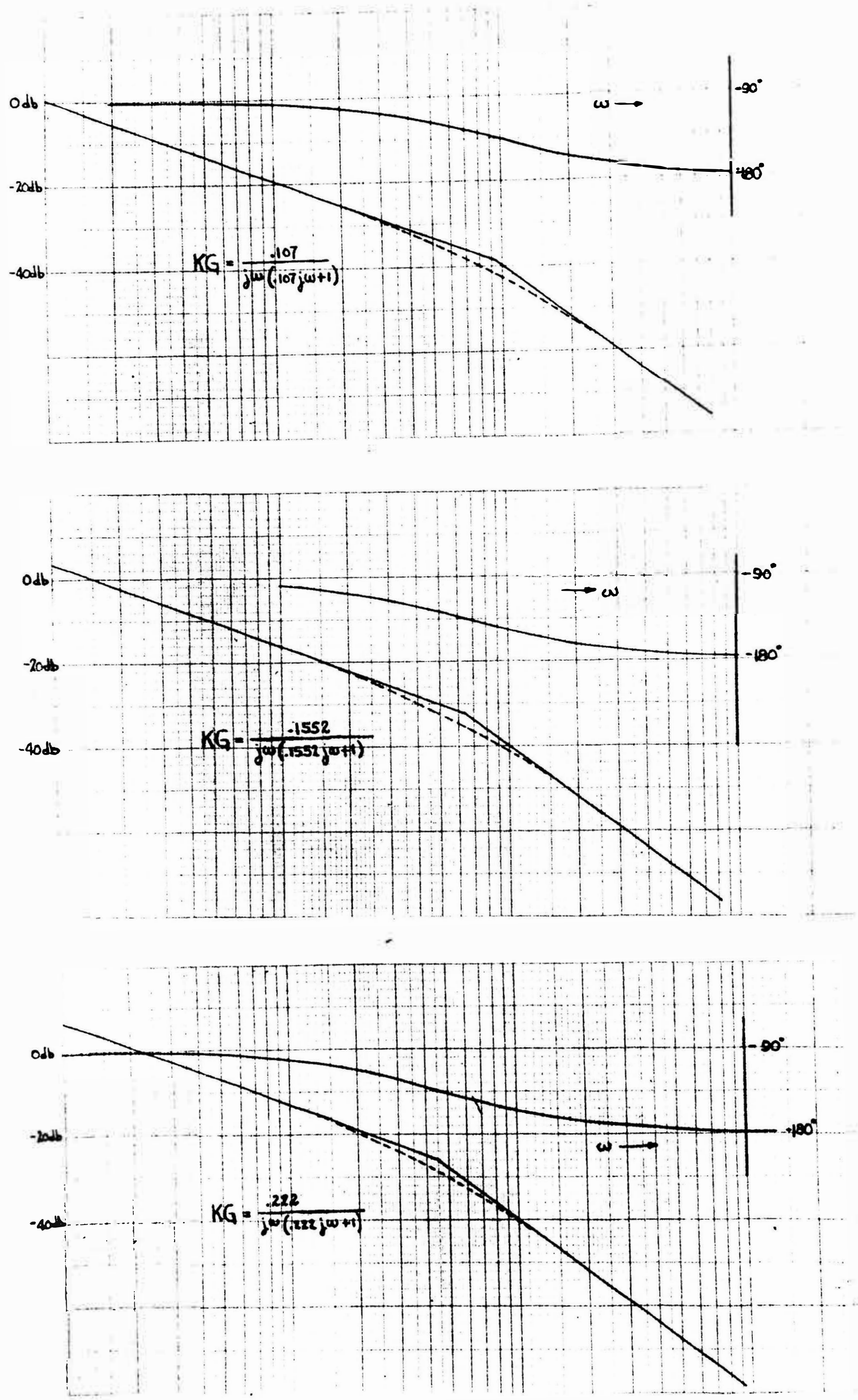


FIGURE A-2 FREQUENCY-RESPONSE AND PHASE ANGLE PLOT OF THE BASIC SYSTEM WITH THREE VALUES OF VISCOUS DAMPING.

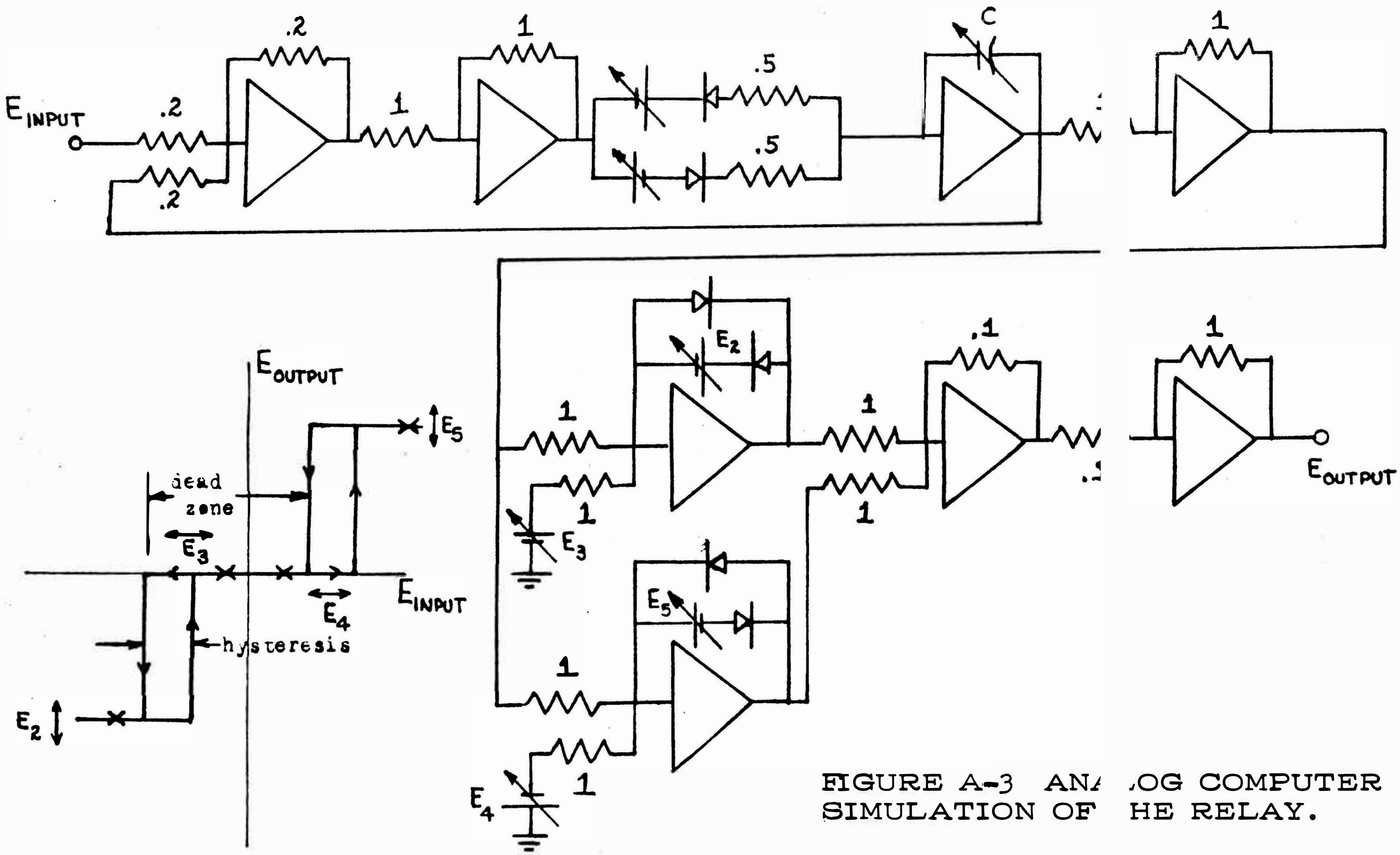


FIGURE A-3 ANALOG COMPUTER SIMULATION OF THE RELAY.

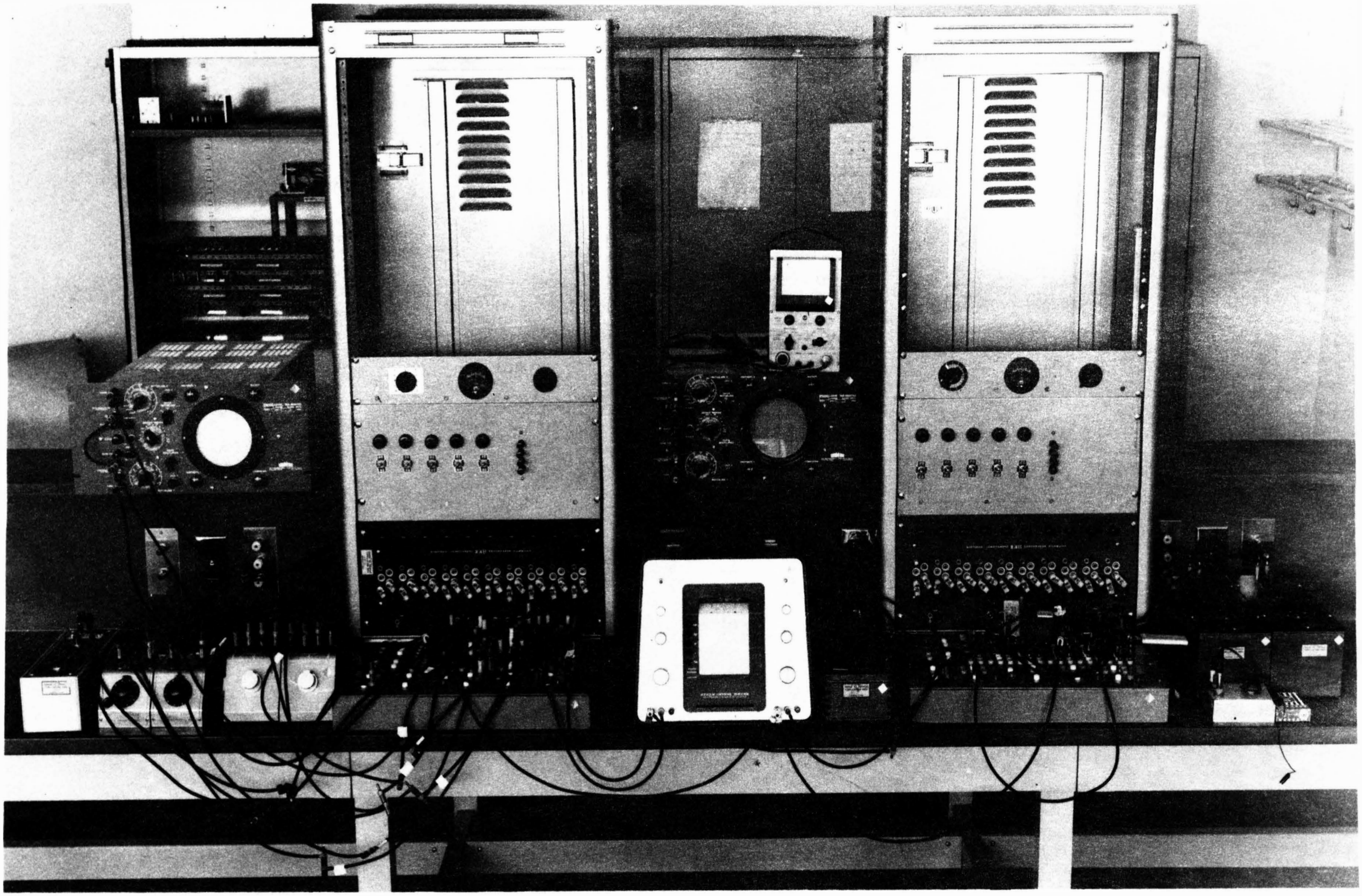


FIGURE A-4 PHOTOGRAPH OF THE ANALOG COMPUTER AND EQUIPMENT USED IN THE LABORATORY.

BIBLIOGRAPHY

1. WEISS, H. K. (1946) Analysis of Relay Servomechanisms. Journal of Aeronautical Sciences, Easton, Pa., Vol. 13, No. 7.

2. MINORSKY, Nicholas (1947) Introduction to Nonlinear Mechanics. J. W. Edwards Brothers, Ann Arbor, Michigan.

3. KAHN, David A. (1949) An Analysis of Relay Servomechanisms. A.I.E.E. Trans. Vol. 68, pp.1079-1088.

4. KOCHENBURGER, Ralph A. (1950) Frequency Response Method for Analyzing and Synthesizing Contactor Servomechanisms. A.I.E.E. Trans. vol. 69, pp. 270-284.

5. PARZIALE, TILTON (1950) Magnetic-Fluid Clutch Servomechanism. A.I.E.E. Trans. vol. 69, pp. 150-157.

6. STUELPNAGEL, T. R. and DELLOS, J. P. (1953) Considerations of off-on Modulated Reversing Clutch Servo Systems. A.I.E.E. Trans. vol. 71, pp. 406-410.

7. JOHNSON, E. Calvin (1952) Sinusoidal Analysis of Feedback Control Systems. A.I.E.E. Trans. vol. 71, pp. 169-81.

8. KAZDA, Louis F. (1953) Errors in Relay Servomechanism Systems. A.I.E.E. Trans. Vol. 72, pp. 323-328.

9. BOGNER, I. and KAZDA, L.F. (1954) An Investigation of the Switching Criteria for Higher Order Contactor Servomechanisms. A.I.E.E. Trans. Vol. 73, pp 118-127.

10. KUBA, R. and KAZDA, L. (1956) A Phase Space Method for Synthesis of Nonlinear Servomechanisms. A.I.E.E. Trans. vol. 75, pp. 282-290.

11. HART, John E. (1955) An Analytical Method for the Design of Relay Servomechanisms. A.I.E.E. Trans. Vol. 74, pp. 83-90.

12. CHAO, Stanley K. (1956) Design of a Contactor Servo Using Describing Function Theory. A.I.E.E. Trans. vol. 75, pp. 223-233.

13. McDONALD, D. (1950) Nonlinear Techniques for Improving Servo Performance. National Electronic Con. pp. 400-421.

14. EMBLER, J. and WEAVER, C. H. (1957) A New Method of Compensating Contactor Servomechanisms. A.I.E.E. Trans. vol. 76, pp. 165-170.

15. HARRIS, W. L. and THALER, McDonald G. (1957) Quasi Optimization of Relay Servos by use of Continuous Damping. A.I.E.E. Trans. vol. 87, pp. 292-296.

16. McDONALD, C. and THALER, G. J. (1959) Quasi Optimization of Relay Servomechanisms by use of Stored Energy for Braking. A.I.E.E. Trans. vol. 77.

17. BULAND, R. N. and FURUMOTO, N. (1960) Dual-Mode Relay Servos. A.I.E.E. Trans. vol. 78, pp. 405-411.

18. SAVANT, (book) Basic Feedback Control System Design. McGraw Hill, New York.

19. THALER & BROWN (book) Analysis and Design of Feedback Control Systems. McGraw Hill New York.

20. IZAWA, K. I. and WEAVER, L. (1959) Relay Type Feedback Control Systems With Dead Time and Sampling. A.I.E.E. Trans. vol. 78, pp. 49-54.

21. FICKEISEN, F. and STOUT, T. (1952) Analog Methods for Optimum Servomechanism Design. A.I.E.E. Trans. vol. 71, pp. 244-250.

22. COSGRIFF, (book) Nonlinear Control Systems. McGraw Hill.

23. SMITH and WOOD (book) Analog Computation. McGraw Hill.

24. MORRILL, C.D. and BAUM, R. V. (1952) Diode Limiters Simulate Mechanical Phenomena. Electronics 25; pp. 122.

25. BRUNS, R. (1952) Analog Computer for Feedback Control Systems. A.I.E.E. Trans. vol. 71, pp. 250-254.

26. BOGNER, I. and KAZDA, L. (July, 1954) Switching Criteria for Contactor Servos. A.I.E.E. Trans. vol. 73, pp. 118-127.

27. STOUT, T. E. (1953) Effects of Friction in an Optimum Relay Servomechanism. A.I.E.E. Trans. vol. 72, pp. 329-336.

28. NEISWONDER, R. and MAC NEAL, R. (1953) Optimization of Nonlinear Control Systems by Means of Nonlinear Feedbacks. A.I.E.E. Trans. vol. 72, pp. 262-272.

29. CLEGG, J. (1956) A Time Dependent Nonlinear Compensating Network. A.I.E.E. Trans. vol. 75, pp. 306-308

30. KU, Y. and WOLF, A. (1957) A Stability Criterion for Nonlinear Systems. A.I.E.E. trans. vol. 78, pp. 144-148.

31. KU, Y. (1957) Analysis of Servomechanisms With Non-Linear Feedback Control. A.I.E.E. Trans. vol, 75, pp. 402-406.

32. GIBSON, J. and TUTAUR, F. (1957) The Response of Relay Amplifiers With Feedback. A.I.E.E. Trans. vol. 76, pp.303-307.

VITA

The author, EARL FREDERICK RICHARDS was born on March 11, 1923, in Detroit, Michigan. His elementary education was obtained through the Detroit Public School System. He graduated from Western High School, Detroit, Michigan in 1941.

After graduating from High School, the author was employed with the Michigan Bell Telephone Company, Detroit, Michigan until 1941. He served with the Office of Strategic Services Command, U. S. Army, attending the Illinois Institute of Technology for a period of one year. He served with the Communications Division of the Offices of Strategic Services until his discharge in 1946. In 1947, he enrolled as a student at Wayne State University, graduating in January, 1951 with the degree of Bachelor of Science in Electrical Engineering.

From 1951 to 1958, the author was employed in industry as an Electrical Engineer with experience primarily in controls. In September of 1958 he entered the University of Missouri School of Mines and Metallurgy as a full time instructor and part time graduate student. He is a member of Eta Kappa Nu, and registered professional engineer in the State of Michigan.