What Are the Next Three Terms in This Sequence?

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Abstract: *Continuing a sequence when the first three terms are provided is a problem that students encounter at many levels of instruction. Consider the following:* $3, 6, 9, \_\_, \_\_, \_\_$. *When asked to fill in the blanks, many students (and more than a few teachers) mistakenly believe that there is only one correct answer. In the following article, the authors explore alternative solutions to such tasks, confirming that—in fact—such questions have many correct answers. Using a mix of by-hand and technology-based approaches, the authors explore three different types of sequences and show how to determine the explicit formula for each based on the term that is chosen next.*
Keywords: *Algebraic thinking, algebra, problem solving, technology*

1 Introduction

1.1 A Familiar Task

Students explore patterns and sequences at a very early age. Using toys, cubes, pattern blocks, beads, and even food, students learn to recognize and continue patterns. When they get to school, students build on their informal experiences using numbers. For instance, they recognize, continue, and generate patterns and sequences such as the following:

$$3, 6, 9, \_\_, \_\_, \_\_$$

Students of mathematics are often asked to continue this, or similar sequences, by finding the next three terms.

1.2 Connection to Standards

As early as 4th grade, teachers stress the importance of sequences under the Operations and Algebraic Thinking domain in the Common Core State Standards for Mathematics (CCSSM) which states that students should be able to generate and analyze patterns. Furthermore, CCSSM emphasizes that high school students should describe patterns recursively and explicitly (CCSSO, 2017). Thus, the type of question posed above is explored throughout the K-12 mathematics curriculum.

1.3 Common Student Solutions

Considering the above sequence, many—if not most—students will add three repeatedly and conclude that 12, 15, and 18 are the values that fill in the blanks. Moreover, many will claim that these are the only three numbers that satisfy the pattern. With more exploration, many high school students will define the sequence algebraically with the explicit formula $f(x) = 3x$ (x being the term number).

1.4 Questions to Consider

But, what if the sequence is not defined by this rule? Is it possible that another set of numbers come next? In other words, are the next three terms in the sequence unique? Is there a limit to the number of sets that can be derived? Consider these questions before reading on.

2 Exploring Alternatives

2.1 Terminology

We begin by defining a **sequence** as an ordered list of numbers. The numbers in the list are referred to as **elements** or **terms.** For instance, $3, 6, 9 \ldots$ is a sequence with terms 3, 6, and 9. A **term number** describes the placement of each term. For example, in the sequence $3, 6, 9 \ldots$, the second term is 6, and the third term is 9.

2.2 Modifying the Fourth Term

Now, let's consider the original sequence, $3, 6, 9 \ldots$, and choose an arbitrary number for the fourth term. For sake of illustration, let's use 25. Note that the explicit formula $f(x) = 3x$ no longer holds (since $f(4) = 12 \neq 25$), and the sequence is no longer arithmetic. This new term 25 has redefined the rule, and the next two terms will depend upon this new rule. *How do we find an explicit function that defines this new sequence and the next two terms?*

2.3 Finding a New Explicit Formula

Our goal is to find an explicit rule, say $f(x)$, such that $3, 6, 9$, and 25 are the outputs when the term numbers input are $x = 1, 2, 3, 4$, respectively. To accomplish this, one may begin by expressing the desired outputs as a sum:

$$f(x) = 3 + 6 + 9 + 25$$

When $x = 1$, we want our output to be 3, the first value. To eliminate the other addends (i.e., 6, 9, and 25), we multiply each of them by zero. Since the term number is 1, we multiply the terms we want to eliminate by the factor of $(x - 1)$:

$$f(x) = 3 + (x - 1) \cdot 6 + (x - 1) \cdot 9 + (x - 1) \cdot 25$$

and thus, $f(1) = 3$.

For $x = 2$, we follow a similar process, multiplying unwanted addends by the factor $(x - 2)$, further modifying our function as follows:

$$f(x) = (x - 2) \cdot 3 + (x - 1) \cdot 6 + (x - 2)(x - 1) \cdot 9 + (x - 2)(x - 1) \cdot 25$$

and thus, $f(2) = 6$. Using similar reasoning for $x = 3$ and $x = 4$, we introduce factors $(x - 3)$ and $(x - 4)$, modifying our rule as:

$$f(x) = (x-2)(x-3)(x-4)\cdot 3+(x-1)(x-3)(x-4)\cdot 6+(x-1)(x-2)(x-4)\cdot 9+(x-1)(x-2)(x-3)\cdot 25$$

As we continue to introduce factors, note that $f(1)$, $f(2)$, and $f(3)$ no longer assume desired values. For instance, $f(1) = (1 - 2)(1 - 3)(1 - 4) \cdot 3 = (-1)(-2)(-3)(3) = -18$ which is not equal to 3, our desired output. We need to adjust our function to account for the factors that we introduced at each stage of the modification process in order to generate desired outputs.

Notice that we want the coefficient in front of the desired output, 3 in the case of $x = 1$, to be 1. Since any number divided by itself is 1, we modify our function, dividing by the same factors we multiplied by:

$$f(x) = \frac{(x-2)(x-3)(x-4)}{(x-2)(x-3)(x-4)} \cdot 3 + \frac{(x-1)(x-3)(x-4)}{(x-1)(x-3)(x-4)} \cdot 6 + \frac{(x-1)(x-2)(x-4)}{(x-1)(x-2)(x-4)} \cdot 9 + \frac{(x-1)(x-2)(x-3)}{(x-1)(x-2)(x-3)} \cdot 25$$

Unfortunately, the function in this form yields no output for $x = 1, 2$, or 3 since division by 0 is undefined. For $f(x)$ to work, we need to replace the x's in the denominator with term numbers:

$$f(x) = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} \cdot 3 + \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} \cdot 6 + \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} \cdot 9 + \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \cdot 25$$

Notice when substituting 1 for x, the first term simplifies to 3, and the other terms simplify to zero. Continuing with the second number, we substitute 2 for x. Note that the second term simplifies to 6, and the others equal zero. Note, also, that the rule provides the desired outputs $f(1) = 3$, $f(2) = 6$, $f(3) = 9$, and $f(4) = 25$. Next, we expand numerators and simplify denominators of our formula:

$$f(x) = \frac{3x^3-27x^2+78x-72}{-6} + \frac{6x^3-48x^2+114x-72}{2} + \frac{9x^3-63x^2+126x-72}{-2} + \frac{25x^3-150x^2+275x-150}{6}$$

Finding a common denominator yields:

$$f(x) = \frac{-3x^3+27x^2-78x+72}{6} + \frac{18x^3-144x^2+342x-216}{6} + \frac{-27x^3+189x^2-378x+216}{6} + \frac{25x^3-150x^2+275x-150}{6}$$

A final simplification yields a simplified form of the explicit formula:

$$f(x) = \frac{13}{6}x^3 - 13x^2 + \frac{161}{6}x - 13$$

To find the fifth and sixth term, we evaluate $f(5)$ and $f(6)$, respectively, to find $f(5) = 67$ and $f(6) = 148$. Hence, $3, 6, 9, \underline{25}, \underline{67}, \underline{148}$ is an alternative valid pattern for the sequence $3, 6, 9, \_\_, \_\_, \_\_$. The terms of this alternative pattern are defined by the explicit formula $f(x) = \frac{13}{6}x^3 - 13x^2 + \frac{161}{6}x - 13$.

Alternatively, what if we picked 18 for the fourth term in the sequence? What would be the explicit formula in that case? What would be the next two terms? See if you can figure this out (Authors' note: Answers to these questions are provided at end of the article—labeled Question #1.)

3 Explorations with Technology

3.1 TI-Nspire Computer Algebra System (CAS)

The simplification process described in the previous section, although instructive, is tedious. Most high school students would have neither the desire, time, perseverance, nor curiosity to simplify the resulting polynomials with pencil and paper. Fortunately, a Computer Algebra System (CAS) can automate much of this work. Figure 1 illustrates the simplification of an explicit formula for the sequence $3, 6, 9, 25 \ldots$ using the TI-Nspire CAS.

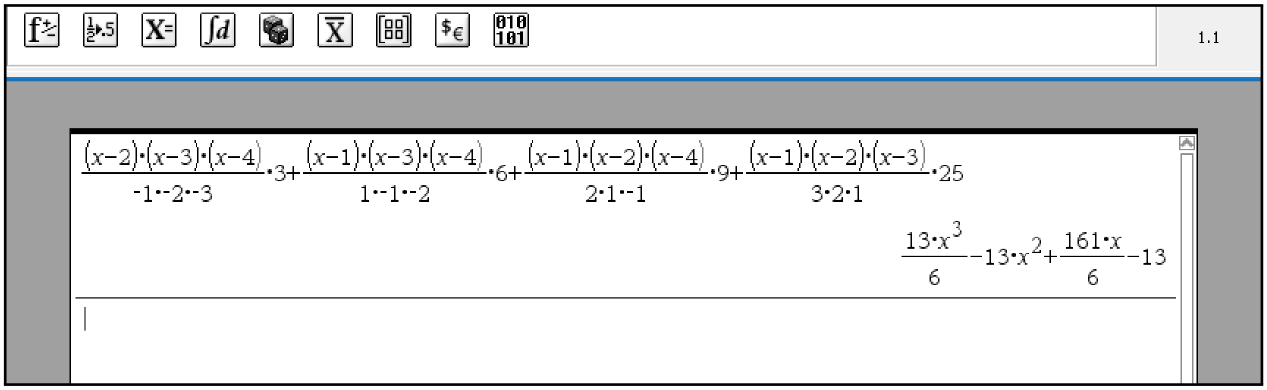
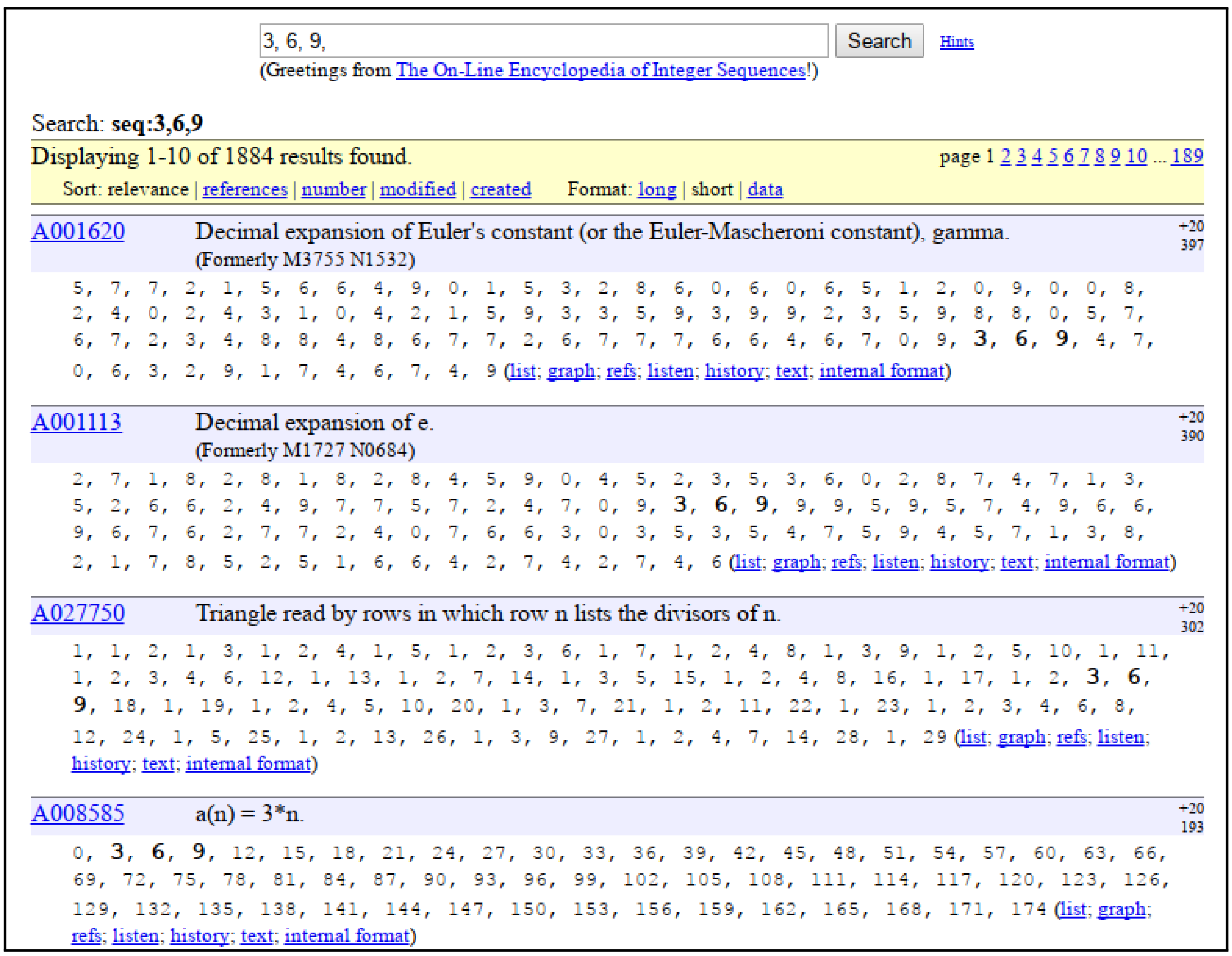


Fig. 1: *Simplying an explicit formula for the sequence* $3, 6, 9, 25, \ldots$

3.2 On-Line Encyclopedia of Integer Sequences (OEIS)

Once students have derived explicit formulas, either by hand or with technology, they can use the OEIS (Sloane, 2017) (`https://oeis.org/`) to check if their functions are listed as possible descriptors for the given sequence. Figure 2 illustrates partial results obtained by entering the terms 3, 6, 9 into the website.



3, 6, 9, Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:3,6,9**

Displaying 1-10 of 1884 results found. page 1 2 3 4 5 6 7 8 9 10 ... 189

Sort: relevance | references | number | modified | created Format: long | short | data

A001620 Decimal expansion of Euler's constant (or the Euler-Mascheroni constant), gamma. +20 397
(Formerly M3755 N1532)
5, 7, 7, 2, 1, 5, 6, 6, 4, 9, 0, 1, 5, 3, 2, 8, 6, 0, 6, 0, 6, 5, 1, 2, 0, 9, 0, 0, 8, 2, 4, 0, 2, 4, 3, 1, 0, 4, 2, 1, 5, 9, 3, 3, 5, 9, 3, 9, 9, 2, 3, 5, 9, 8, 8, 0, 5, 7, 6, 7, 2, 3, 4, 8, 8, 4, 8, 6, 7, 7, 2, 6, 7, 7, 7, 6, 6, 4, 6, 7, 0, 9, **3, 6, 9**, 4, 7, 0, 6, 3, 2, 9, 1, 7, 4, 6, 7, 4, 9 (list; graph; refs; listen; history; text; internal format)

A001113 Decimal expansion of e. +20 390
(Formerly M1727 N0684)
2, 7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, 9, 0, 4, 5, 2, 3, 5, 3, 6, 0, 2, 8, 7, 4, 7, 1, 3, 5, 2, 6, 6, 2, 4, 9, 7, 7, 5, 7, 2, 4, 7, 0, 9, **3, 6, 9**, 9, 9, 5, 9, 5, 7, 4, 9, 6, 6, 9, 6, 7, 6, 2, 7, 7, 2, 4, 0, 7, 6, 6, 3, 0, 3, 5, 3, 5, 4, 7, 5, 9, 4, 5, 7, 1, 3, 8, 2, 1, 7, 8, 5, 2, 5, 1, 6, 6, 4, 2, 7, 4, 2, 7, 4, 6 (list; graph; refs; listen; history; text; internal format)

A027750 Triangle read by rows in which row n lists the divisors of n. +20 302
1, 1, 2, 1, 3, 1, 2, 4, 1, 5, 1, 2, 3, 6, 1, 7, 1, 2, 4, 8, 1, 3, 9, 1, 2, 5, 10, 1, 11, 1, 2, 3, 4, 6, 12, 1, 13, 1, 2, 7, 14, 1, 3, 5, 15, 1, 2, 4, 8, 16, 1, 17, 1, 2, **3, 6, 9**, 18, 1, 19, 1, 2, 4, 5, 10, 20, 1, 3, 7, 21, 1, 2, 11, 22, 1, 23, 1, 2, 3, 4, 6, 8, 12, 24, 1, 5, 25, 1, 2, 13, 26, 1, 3, 9, 27, 1, 2, 4, 7, 14, 28, 1, 29 (list; graph; refs; listen; history; text; internal format)

A008585 a(n) = 3*n. +20 193
0, **3, 6, 9**, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75, 78, 81, 84, 87, 90, 93, 96, 99, 102, 105, 108, 111, 114, 117, 120, 123, 126, 129, 132, 135, 138, 141, 144, 147, 150, 153, 156, 159, 162, 165, 168, 171, 174 (list; graph; refs; listen; history; text; internal format)

Fig. 2: *Partial results obtained by entering terms* 3, 6, 9 *into OEIS website.*

Note that the results in Figure 2 indicate that 1,884 different sequences include the numbers 3, 6, 9 as consecutive terms. The fourth search result, listed as A008585, is the familiar explicit formula $a(n) = 3n$. Curiously, the explicit formula that we derived in the previous section—namely, $f(x) = \frac{13}{6}x^3 - 13x^2 + \frac{161}{6}x - 13$—is not provided as one of the nearly 2,000 results. Moreover, the terms $3, 6, 9, 25$ do not appear sequentially in any of the entries provided by the OEIS database. We found a new sequence! Students are encouraged to submit new sequences such as this using the short OEIS on-line form. As the site indicates, "it will (probably) be added to the OEIS! Include a brief description and if possible enough terms to fill 3 lines on the screen" (Sloane, 2017).

4 A New Example

Let's consider a second sequence that is different from our first:

$$2, 5, 9, 14, \_\_, \_\_, \_\_, \ldots$$

Notice that—unlike our first example—this is not an arithmetic sequence. A recognizable pattern is determined by adding 3 to the first term to get the second term, then adding four to the second term to get the third, and so on. Using this pattern, the next three terms in the sequence are 20, 27, and 35. What would be the explicit formula for this sequence? See if you can derive a formula using our previous procedure (Authors' note: We provide an answer at the end of this article—labeled Question #2). Alternatively, see if the answer is shown on the OEIS website.

Are the next three terms—$20, 27, 35$—unique? What if we let the fifth term be an arbitrary value? Let's use a number smaller than the fourth term (thus, creating a non-increasing sequence) and one that is not an integer, say 7.3, and see if the same procedure will yield a viable explicit formula—one that "works."

Since we have five terms in this example, we add an additional factor into the expression, $(x - 5)$. Then, substituting the term numbers into the denominators to eliminate the division by zero error, we get:

$$f(x) = \frac{(x-2)(x-3)(x-4)(x-5)}{(1-2)(1-3)(1-4)(1-5)} \cdot 2 + \frac{(x-1)(x-3)(x-4)(x-5)}{(2-1)(2-3)(2-4)(2-5)} \cdot 5 + \frac{(x-1)(x-2)(x-4)(x-5)}{(3-1)(3-2)(3-4)(3-5)} \cdot 9 + \frac{(x-1)(x-2)(x-3)(x-5)}{(4-1)(4-2)(4-3)(4-5)} \cdot 14 + \frac{(x-1)(x-2)(x-3)(x-4)}{(5-1)(5-2)(5-3)(5-4)} \cdot 7.3$$

If you are using the TI-Nspire CAS, you can simply copy the last entry into a new line on the calculator and edit it for the extra term and the multipliers as shown in Figure 3.

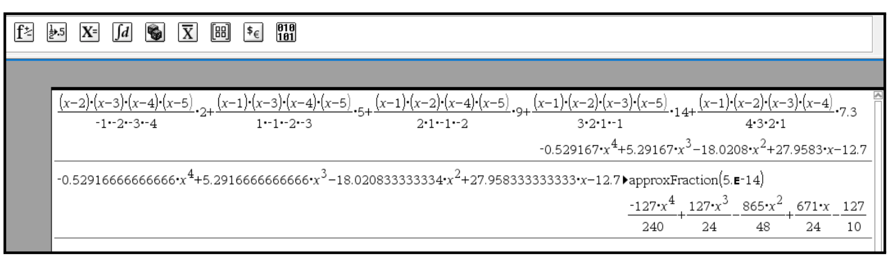


Fig. 3: *Simplifying an explicit formula for the sequence* $2, 5, 9, 14, 7.3 \ldots$.

As Figure 3 suggests, the explicit formula is $f(x) = \frac{-127}{240}x^4 + \frac{127}{24}x^3 - \frac{865}{48}x^2 + \frac{671}{24}x - \frac{127}{10}$. We can use this formula to determine the next three terms in the sequence: $f(6) = -36.5$, $f(7) = -155.5$, and $f(8) = -400.5$. These calculations suggest that our procedure for finding explicit formulae "works" for non-increasing sequences with non-integer terms.

5 Finding a Missing Term when the Explicit Formula is Known

Let's change the problem yet again and give students something different to think about. Consider a sequence of three terms, say $5, 9, 13, \ldots$. What is the fourth term if the explicit formula is given as $f(x) = \frac{25}{6}x^3 - 25x^2 + \frac{299}{6}x - 24$?

Since we are looking for the fourth term, we substitute 4 for x, giving the answer of 42.

Although a simpler problem, this scenario forms the foundation for an activity that encourages students to compare term numbers and terms (students often confuse these). For instance, we challenged our students by asking them to create a similar problem using their own sequence of three numbers, choosing a "secret" fourth number for a friend to determine. In the following passages, we share work from two of our students, Alex and Taylor.

5.1 Alex's Task

One student, Alex, decided to use the same sequence as above, but chose 71 to be his "secret" number for the fourth term and the function $f(x) = 9x^3 - 54x^2 + 103x - 53$. He then challenged another student to see if the "secret" term could be found. This was a good example of how the functions change each time a different "secret" number is chosen.

This activity helps students understand and distinguish between the term number and the value of that term or the term itself. In order to find the "secret" number, students must use the term number as the x-value in the explicit formula provided. This can be confusing for some students as illustrated in the following section.

5.2 Taylor's Misconception

Taylor created the sequence $3, 9, 21, 45, \ldots$, multiplying a given term by two and adding three to generate the next term. She mistakenly thought the explicit function $f(x) = 2x + 3$ would generate each term of this sequence, however that was not the case. When she was asked what the fifth term should be, her explicit function indicated 13, however, her recursive definition yielded $45 * 2 + 3 = 93$. Taylor's misconception stemmed from confusion between term number and term of the sequence as well as differences between recursive and explicit definitions. After some guidance, Taylor constructed the explicit function $f(x) = 6 \cdot 2^{(x-1)} - 3$ for her original sequence. Then we asked her to use her TI-Nspire CAS to determine the explicit formula for these same four terms. She was confused when the calculator suggested that $g(x) = x^3 - 3x^2 + 8x - 3$ was the explicit formula. Are $f(x)$ and $g(x)$ equivalent? Why or why not? If they're different, which is correct? *Is it possible that the two different formulas correctly describe sequences that contain the listed terms?*

6 Connections to Lagrange Polynomials

Students should have opportunities to analyze the explicit formulas that this method generates. They should notice that all of the generated formulas are polynomials, even though we didn't always start with polynomials (as in Taylor's example). Other worthwhile questions include the following:

1. Looking back at the first example (i.e., $3, 6, 9, \ldots$), what type of function did we have originally? (linear). Why? (It is an arithmetic sequence.)

2. When we chose a random fourth term in the first example (i.e., $3, 6, 9, 25, \ldots$), we no longer had a linear function. What type of function was determined for the given three terms and a random fourth term? (cubic)

3. In the second example (i.e., $2, 5, 9, 14, \ldots$) what type of function did we start with? (quadratic)

4. When we used the given four terms and chose a random fifth term in the second example (i.e., $2, 5, 9, 14, 7.3, \ldots$), what type of function was determined? (quartic)

5. Look at the third example and determine what type of function was found given three terms (i.e., $5, 9, 13, \ldots$) and choosing a "secret" fourth term. (cubic)

6. Make a conjecture about the number of terms given and the type of function that is determined based on those terms if an arbitrary term is selected next. (The number of terms that are given with an arbitrary value determines the degree of the polynomial for the explicit function. These are called Langrange Polynomials and can be determined using any number of terms (Celant & Broniatowski, 2016)).

6.1 Graphing Sequences

Student understanding of sequences may be strengthened by graphing terms as ordered pairs on a coordinate grid. Consider, for instance, our first example (i.e., $3, 6, 9 \ldots$). Points are plotted in the form (*term number, term*). For example, the first three terms of $3, 6, 9 \ldots$ are plotted as (1, 3), (2, 6), and (3, 9). Note that these points lie on a line; however, plotting our fourth term—namely, 25—as (4,25) reveals a non-linear relationship, as suggested in Figure 4.

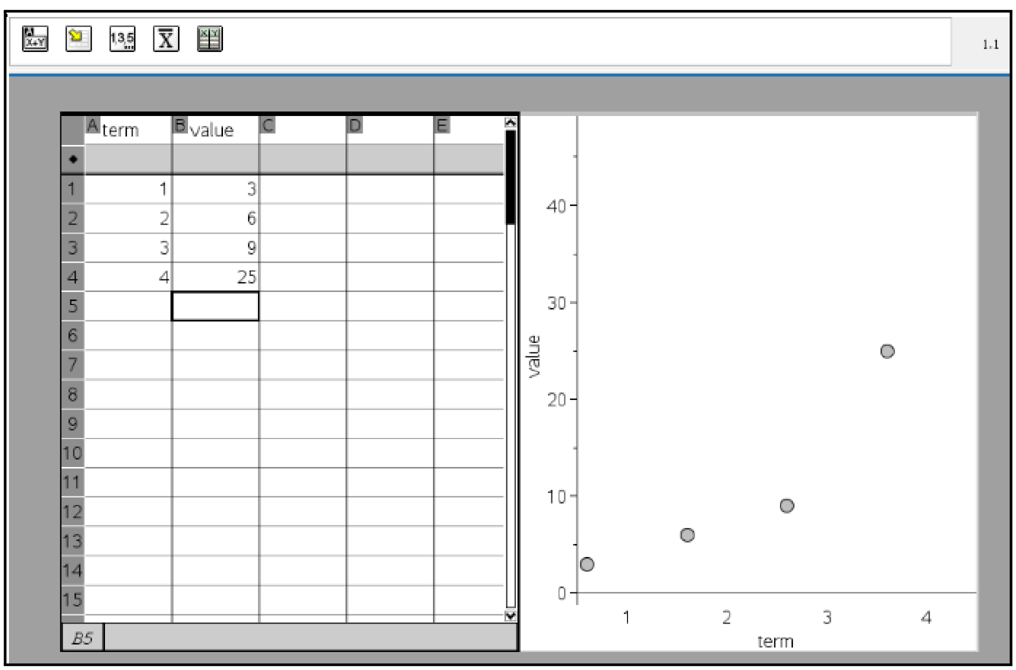


Fig. 4: *Graphical representation of sequence* $3, 6, 9, 25, \ldots$.

The graph helps students see that the fourth term renders the sequence non-linear while influencing the construction of a non-linear explicit formula.

7 Conclusion

We often have students find the next terms in a sequence or find the explicit formula for a sequence. The examples above show that students can choose any random number as the next term in a given sequence and then derive an explicit polynomial function for that sequence. For this reason, it is vital to ask students to list the rule they are using whenever asking them to find the next terms of a sequence. As demonstrated, without stating the rule, any answer could be justified as correct. Further exploration could be done by asking students if the polynomials determined are unique.

By using the TI-Nspire CAS, students can explore the patterns within the sequences in both an algebraic and a graphical manner. Through this process, students experience a broader understanding of how sequences are developed. Students can find this type of topic helpful at a more advanced level as well. With the methods described in this article, students should be able to fit a polynomial curve to a series of data points. Calculus students can easily take the derivative of a fitted polynomial function, as opposed to a complicated function that was originally defined. Additionally, algebra and probability & statistics students will find this an interesting application to various regression topics they use to describe, interpolate, and extrapolate data.

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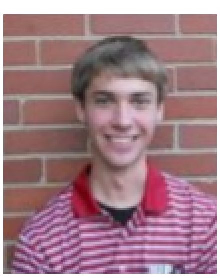

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Answers to questions

Question #1. The explicit formula for $3, 6, 9, 18, \_\_, \_\_, \_\_$ is $f(x) = x^3 - 6x^2 + 14x - 6$.

The next two terms are 39 and 78.

Question #2. Using the procedure described above and then the term numbers for the 5th, 6th, and 7th terms, the following output is generated by the TI-Nspire CAS.