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SYNOPSIS: We describe a class of viscoplastic constitutive models capable of simulating the monotonic and cyclic rate-dependent soil behavior. These models are developed by enriching their inviscid counterparts with a viscous character to model the irreversible deformation that develops with time. The viscoplastic strain rate is of the Duvaut-Lions type whose magnitude increases with the distance of the stress point from its projection onto the inviscid solution. With appropriate choice of an inviscid elasto-plastic soil model, one can generate quasipreconsolidation effects during creep and account for the influence of frequency on the shape and width of hysteresis loops formed during cyclic loading.

INTRODUCTION

Considerable attention has been directed in recent years to the study of stress-strain-time aspects of soils for investigating the effects of time and rate of strain on the soil's shearing strength and deformation behavior (Murayama and Shibata 1964; Singh and Mitchell 1968; Borja and Kavazanjian 1985, to name a few). The viscous response of soils manifests not only during monotonic loading but also, and perhaps more distinctively, during cyclic testing where the frequency of loading affects the shape and width of hysteresis loops as well as the number of cycles that a soil could sustain prior to failure (Mroz and Norris 1982). While rate-dependent theory appears to be most suitable for mathematically describing real soil behavior, viscoplastic models for soils are scarce and have not received the same level of treatment and sophistication that many robust rate-independent plasticity models enjoy.

In this paper, we explore an elasto-viscoplastic constitutive law originally proposed by Duvaut and Lions (1976) and elaborated further by Simo et al. (1988) for characterizing the rate-dependent soil behavior. The mathematical formulation involves a simple enrichment of an inviscid elasto-plastic law with a viscous character to model the time-dependent soil deformation behavior. In essence, the inviscid solution is viewed as one to which the viscoplastic solution will tend in the limit as time approaches infinity. The idea and the mathematical formulation are appealing since all of the features that pertain to the inviscid model upon which the viscoplastic model is based are captured automatically. We also describe a conceptual framework for capturing quasipreconsolidation effects in creeping soils in the context of bounding surface plasticity models. Finally, we show some preliminary results of numerical experiments designed to expand the capability of a deviatoric plasticity model with linear combination of isotropic and kinematic hardening to include viscous behavior.

ELASTO-VISCOPLASTIC CONSTITUTIVE LAW

Consider the following rate-constitutive equation:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{c}^e : \dot{\boldsymbol{\epsilon}}^e = \boldsymbol{c}^e : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^{vp}) \tag{1}$$

where $\boldsymbol{\sigma}$ = Cauchy stress tensor; $\boldsymbol{\epsilon}$ = small strain tensor; $\boldsymbol{\epsilon}^e$, $\boldsymbol{\epsilon}^{vp}$ = elastic and viscoplastic components of $\boldsymbol{\epsilon}$, respectively; $\boldsymbol{c}^e$ = rank-four elasticity tensor; and the superimposed dots imply material rates. The idea in the subject viscoplastic model is that the stress-point $\boldsymbol{\sigma}$ may fall outside a yield surface, but will get attracted to its projection $\bar{\boldsymbol{\sigma}}$ *on* the yield surface. The farther is $\boldsymbol{\sigma}$ from $\bar{\boldsymbol{\sigma}}$, the larger is the magnitude of the viscoplastic strain rate. In the context of the Prandtl-Reuss constitutive law, $\bar{\boldsymbol{\sigma}}$ is simply the orthogonal (or radial) projection of $\boldsymbol{\sigma}$ on the (convex) yield surface in the usual Euclidean sense (Duvaut and Lions 1976). More generally, $\bar{\boldsymbol{\sigma}}$ may be thought of as the closest point projection, in some specified metric, of $\boldsymbol{\sigma}$ on the yield surface. Clearly, $\bar{\boldsymbol{\sigma}}$ represents the inviscid solution of the rate-constitutive relation.

Based on a metric defined by the elasticity tensor $\boldsymbol{c}^e$, Simo et al. (1988) proposed a simple form for $\dot{\boldsymbol{\epsilon}}^{vp}$ as follows:

$$\dot{\boldsymbol{\epsilon}}^{vp} = \frac{1}{\eta}(\boldsymbol{c}^e)^{-1} : (\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}) \tag{2}$$

where η = fluidity-like parameter. For the Prandtl-Reuss model, (2) degenerates to the expression originally proposed by Duvaut and Lions (1976):

$$\dot{\boldsymbol{\epsilon}}^{vp} = \frac{1}{2\eta}(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}) \tag{3}$$

with η taking on a parallel meaning. In (3), $\dot{\boldsymbol{\epsilon}}^{vp}$ is assumed coaxial with the stress difference $\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}$. The foregoing idea allows the construction of more elaborate models to define $\dot{\boldsymbol{\epsilon}}^{vp}$, but for simplicity and illustration purposes, we consider the form (2) for now.

Substituting (2) in (1) yields

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}^{\mathrm{tr}} - \frac{1}{\eta}(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}) \tag{4}$$

where $\dot{\boldsymbol{\sigma}}^{\mathrm{tr}} = \boldsymbol{c}^e : \dot{\boldsymbol{\epsilon}}$ is the usual trial rate-of-stress. Equation (4) is a first order ordinary differential equation in $\boldsymbol{\sigma}$ which can be integrated, e.g., by the stable backward difference scheme to yield

$$\boldsymbol{\sigma}_{n+1} = \frac{\boldsymbol{\sigma}^{\mathrm{tr}}_{n+1} + (\Delta t/\eta)\bar{\boldsymbol{\sigma}}_{n+1}}{1 + (\Delta t/\eta)} \tag{5}$$

Clearly, all of the features of an inviscid model are captured in (5) via the projection stress $\bar{\boldsymbol{\sigma}}_{n+1}$. Since $\boldsymbol{\sigma}_{n+1}$ is a linear function of $\bar{\boldsymbol{\sigma}}_{n+1}$, it follows that all aspects of a stress point integration algorithm, including the attendant problem of linearization, are rendered trivial once the algorithm for the inviscid problem has been set up. Furthermore, one obtains the elastic solution from (5) as $\Delta t/\eta \to 0$, while the inviscid solution is recovered as $\Delta t/\eta \to \infty$.

RATE-DEPENDENT DEVIATORIC PLASTICITY

In this section we consider specializations of the previous ideas to the simple case of deviatoric viscoplasticity. The resulting formulation is not intended to model real soil behavior, but rather, to illustrate how one can incorporate other, more robust rate-independent models into the viscoplastic framework.

We consider in particular an elasto-plastic constitutive theory with Mises yield surface, associative flow rule, and linear combination of isotropic and kinematic hardening. The evolution equations for this theory are summarized in Hughes (1984). Specifically, the radius R of the yield surface is assumed to vary according to the effective plastic strain, while the back stress $\boldsymbol{\alpha}$ is assumed to evolve according to Prager's translation rule. An effective technique for integrating stresses with this model is embodied in the generalized radial-return concept of Krieg and Key (1976), which is also summarized in Hughes (1984). This integration algorithm produces the time-discrete evolution equations for the inviscid solutions $\bar{\boldsymbol{\sigma}}_{n+1}$, $\bar{R}_{n+1}$, and $\bar{\boldsymbol{\alpha}}_{n+1}$.

For the viscoplastic case, we now postulate the following specializations of the Duvaut-Lions model to deviatoric yielding with combined isotropic and kinematic hardening:

Time-discrete evolution equation for radius:

$$R_{n+1} = \frac{R_n + (\Delta t/\eta)\bar{R}_{n+1}}{1 + (\Delta t/\eta)} \tag{6}$$

Time-discrete evolution equation for back-stress:

$$\boldsymbol{\alpha}_{n+1} = \frac{\boldsymbol{\alpha}_n + (\Delta t/\eta)\bar{\boldsymbol{\alpha}}_{n+1}}{1 + (\Delta t/\eta)} \tag{7}$$

As before, the time-discrete evolution equation for $\boldsymbol{\sigma}_{n+1}$ is given in (5).

Note that the linearization of $\boldsymbol{\sigma}_{n+1}$ usually required to construct the consistent tangent operator is rendered trivial once the time-discrete evolution equation for $\bar{\boldsymbol{\sigma}}_{n+1}$ has been set up. For example, with application to deviatoric viscoplasticity, (5) can simply be differentiated with respect to total strain $\boldsymbol{\epsilon}_{n+1}$ to obtain the desired consistent tangential moduli:

$$\boldsymbol{\sigma}'_{n+1}(\boldsymbol{\epsilon}_{n+1}) = \frac{\mathbf{c}^e + (\Delta t/\eta)\,\bar{\boldsymbol{\sigma}}'_{n+1}(\boldsymbol{\epsilon}_{n+1})}{1 + (\Delta t/\eta)} \tag{8}$$

where $\bar{\boldsymbol{\sigma}}'_{n+1}(\boldsymbol{\epsilon}_{n+1})$ is simply the consistent rate-independent tangential moduli of Simo and Taylor (1985).

STRESS RELAXATION, CREEP, AND QUASIPRECONSOLIDATION

In order for the viscoplastic theory to be useful in practice, it must have the capability to model many of the most important rate-dependent features of soil behavior. Stress-relaxation is obviously contained in equation (5), as one may simply consider a soil subjected to sustained deformation over a long period of time. In this case, the stress intensity $\boldsymbol{\sigma}_{n+1}$ relaxes from its initial value $\boldsymbol{\sigma}^{tr}_{n+1}$, just after instantaneous loading, to the inviscid solution $\bar{\boldsymbol{\sigma}}_{n+1}$ as time approaches infinity (Fig. 1). Assuming that (5) is invertible, either explicitly or implicitly, so that the strain $\boldsymbol{\epsilon}_{n+1}$ can be expressed in terms of $\boldsymbol{\sigma}_{n+1}$, then one can also capture creep effects by imposing a sustained stress. In this case, the soil experiences delayed deformation during the time that the yield surface grows until it reaches a size in the limit where the projection stress $\bar{\boldsymbol{\sigma}}_{n+1}$ contacts the stress point $\boldsymbol{\sigma}_{n+1}$ (Fig. 2).

Quasipreconsolidation, or increase in soil strength with time, is another important rate-dependent feature of soil behavior that analysts have long failed to incorporate in their constitutive models until recently (Kavazanjian and Mitchell 1980; Borja and Kavazanjian 1985). In the present viscoplastic framework, what we seek to model is the growth of the yield surface with time while the stress point is still inside the yield surface. Since the growth of the yield surface is contingent upon the development of an irreversible deformation, quasipreconsolidation can be modeled only by allowing plastic deformation to take place inside the yield surface. This idea is embodied in so-called bounding surface plasticity models.

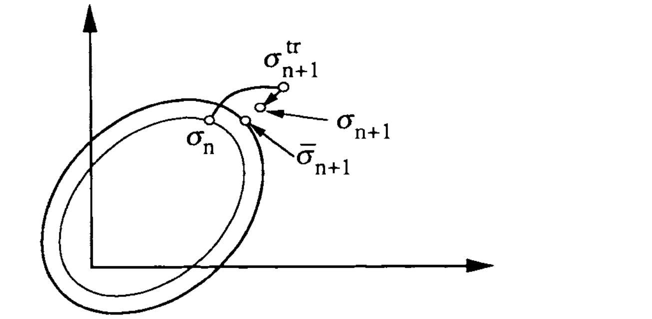

Fig. 1. Stress relaxation: stress relaxes from the trial elastic predictor to the inviscid elasto-plastic solution.

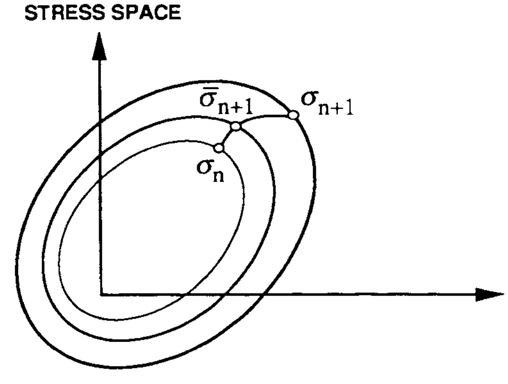

Fig. 2. Creep: projection stress approaches the sustained stress asymptotically while yield surface expands.

Assume that the inviscid model upon which the viscoplastic model is based is of the bounding-surface type. This includes the model of Dafalias and Herrmann (1982) and the two-yield-surface theory with a vanishing elastic region of Mroz and Norris (1982). To each stress point $\boldsymbol{\sigma}_{n+1}$ inside the bounding surface we associate an image stress $\boldsymbol{\sigma}^*_{n+1}$ on the bounding surface which can be determined, e.g., using the simple radial rule of Dafalias and Herrmann or the intersection rule of Mroz and Norris. Note that the image stress $\boldsymbol{\sigma}^*_{n+1}$ is *not* the same as the projection stress $\bar{\boldsymbol{\sigma}}_{n+1}$. In fact, one can clearly make the distinction between the two stress terms if the notion of two-yield-surface with a vanishing elastic region of Mroz and Norris is employed. In this case,

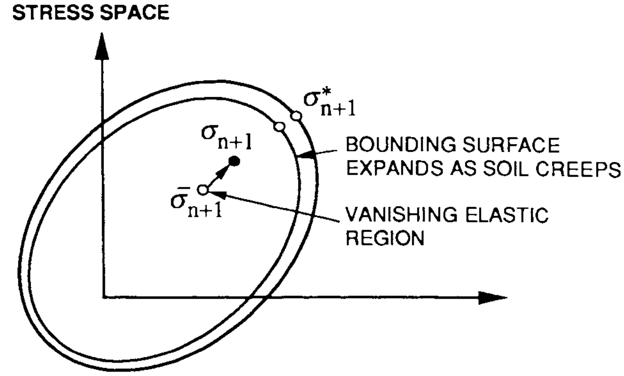

Fig. 3. Quasipreconsolidation: bounding surface expands as the vanishing elastic region approaches the stress point asymptotically.

$$\bar{\sigma}_{n+1} = \bar{\alpha}_{n+1} \tag{9}$$

i.e., the projection stress on the interior yield surface, which vanishes to a point, coincides with the back stress (see Fig. 3). Thus, during creep under a sustained stress, the projection stress $\bar{\sigma}_{n+1}$ approaches the fixed stress point σ_{n+1} asymptotically with time, as delayed deformation accumulates. The bounding surface then grows concurrently with plastic deformation so that the image stress σ^*_{n+1} moves away from σ_{n+1}, thus resembling the effects of quasipreconsolidation.

PRELIMINARY RESULTS

In this section we present some preliminary results of our numerical experiments with the simple rate-dependent deviatoric plasticity model. We show that despite a relatively simple formulation, one can generate curves that reflect the rate-dependent character of most soils.

The soil was assumed to have an undrained modulus $E_u = 4,000$ kPa; Poisson's ratio $\nu = 0.3$; and an initial undrained shear strength of $s_u = 20$ kPa (e.g., medium to soft clay). Thus, the equivalent initial radius of the yield surface is $R_0 = 1.63 s_u = 32.6$ kPa. The parameters we controlled in our study are the hardening modulus H' (hardening if $H' > 0$, perfect plasticity if $H' = 0$, and softening if $H' < 0$) and the yield surface translation parameter β (isotropic hardening if $\beta = 1$, kinematic hardening if $\beta = 0$, and combined isotropic/kinematic hardening if $0 < \beta < 1$). If we assume a constant strain rate, then a third parameter, $\Delta t/\eta$, describes the rate-dependent character of the model (very viscous if $\Delta t/\eta \to 0$ and nearly inviscid if $\Delta t/\eta \to \infty$).

Figure 4 shows the predicted response of an elastic, perfectly plastic inviscid soil ($H' = 0$ kPa; $\beta = 1$) subjected to a strain-controlled monotonic undrained triaxial compression. Superimposed in the figure are the viscoplastic predictions for various values of $\Delta t/\eta$. Unlike the inviscid response, note that the rate-dependent stress-strain responses exhibit a smooth character even at initial yield. Also, the soil strength increases as the viscosity parameter η increases. If η is viewed as fixed, then the rate-dependent responses suggest that the strength increases as the strain rate increases, for then, Δt decreases. A similar behavior is seen from the soil responses of Fig. 5 where a strain-softening inviscid soil behavior was assumed ($H' = -600$ kPa; $\beta = 1$). Note that the corresponding viscoplastic strength responses also diminish as the soil exhibits a strain-softening response for a reasonably large value of $\Delta t/\eta$.

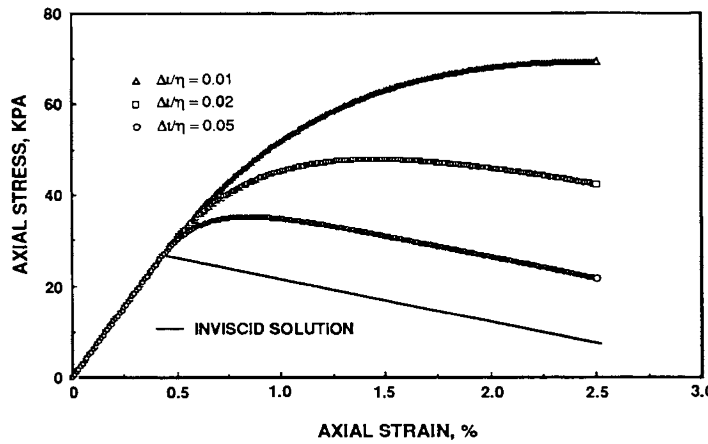

Fig. 5. Undrained axial stress-axial strain curves for a strain-softening soil.

Figure 6 shows the predicted inviscid and viscous load-unload responses of an elastic-perfectly plastic soil ($H' = 0$ kPa; $\beta = 1$). For comparison, Fig. 7 shows the load-unload-reload behavior of a soil exhibiting a combined isotropic-kinematic hardening response ($H' = 800$ kPa; $\beta = 1/2$). Note in both figures the clear dependence of the viscoplastic responses on the frequency of loading (i.e., on the ratio $\Delta t/\eta$), an important soil response feature central to the model. As shown in both figures, the inviscid solution is recovered when $\Delta t/\eta \to \infty$, while the elastic response is obtained in the limit as $\Delta t/\eta \to 0$.

CLOSING REMARKS

We have presented a conceptual framework for capturing the rate-dependent response of soils based on the simple viscoplastic theory of Duvaut and Lions. The formulation allows one to incorporate the robustness of many existing rate-independent plasticity models. This study is by no means complete, as we have yet to see all of the implications of incorporating the sophistication of many inviscid elasto-plastic models into the viscoplastic framework, both from the practical as well as from the computational standpoint. Results of the preliminary studies show promise, however, and the prognosis for more interesting and useful results is good.

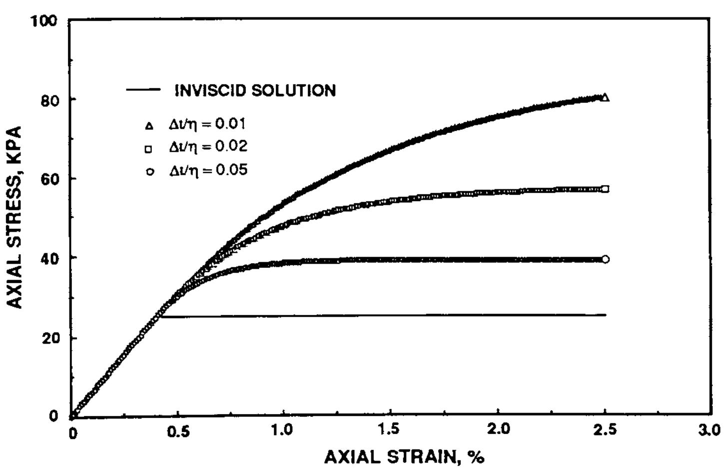

Fig. 4. Axial stress-axial strain curves for monotonic undrained compression tests.

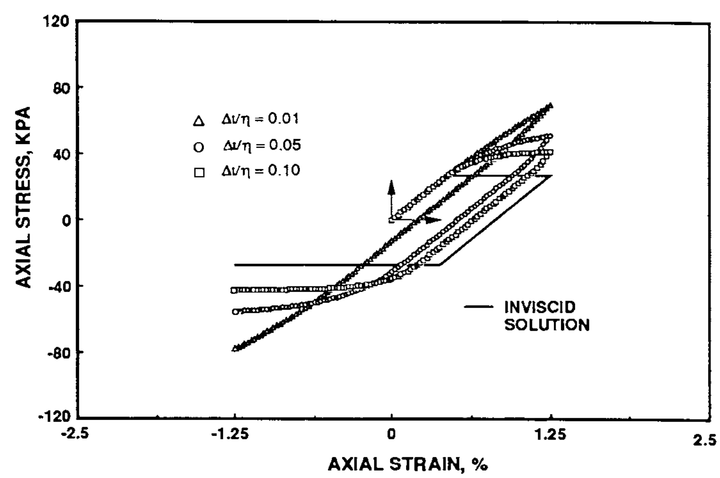

Fig. 6. Load-unload curves for elasto-plastic soil showing effect of strain rate.

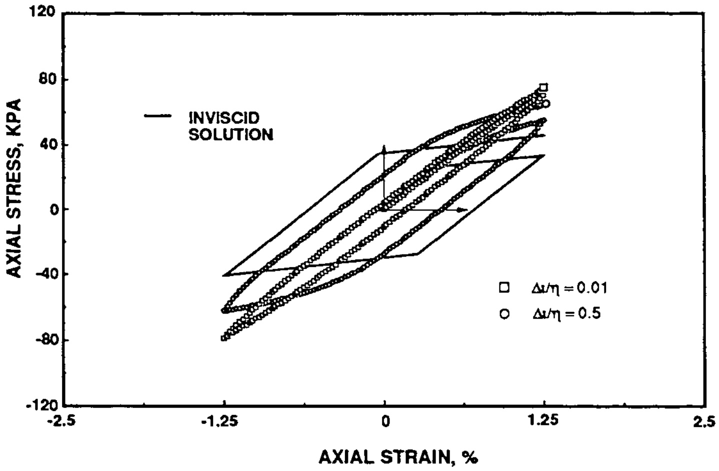

Fig. 7. Load-unload-reload curves for elasto-plastic soil with combined isotropic and kinematic hardening.

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REFERENCES

Borja, R. I., and Kavazanjian, E. Jr. (1985). "A constitutive model for the stress-strain-time behaviour of 'wet' clays." *Géotechnique*, 35(3), 283–298.

Dafalias, Y. F. and Herrmann, L. R. (1982). "Bounding surface formulation of soil plasticity." Chapter 10: *Soil Mechanics - Transient and Cyclic Loads*, G. N. Pande and O. C. Zienkiewicz, eds., John Wiley and Sons, 253–282.

Duvaut, G. and Lions, J. L. (1976). *Inequalities in Mechanics and Physics*, Springer-Verlag, New York, N. Y.

Hughes, T. J. R. (1984). "Numerical implementation of constutitive models: rate-independent deviatoric plasticity." Capter 2: in *Theoretical Foundations for Large-Scale Computations for Nonlinear Material Behavior*, S. Nemat-Nasser, R. J. Asaro and G. A. Hegemier, eds., Martinus Nijhoff Publishers, Boston, 29–57.

Kavazanjian, Jr. E. and Mitchell, J. K. (1980). "Time-dependent deformation behavior of clays." *J. Geotech. Engrg. Div, ASCE*, 106(GT6), 611–630.

Krieg, R. D. and Key, S. W. (1976). "Implementation of a time-independent plasticity theory into structural computer programs." in *Constitutive Equations in Viscoplasticity: Computational and Engineering Aspects*, J. A. Stricklin and K. J. Saczalski, eds., AMD-20, ASME, New York, N. Y.

Mroz, Z. and Norris, V. A. (1982). "Elastoplastic and viscoplastic constitutive models for soils with application to cyclic loading." Chapter 8: *Soil Mechanics - Transient and Cyclic Loads*, G. N. Pande and O. C. Zienkiewicz, eds., John Wiley and Sons, 173–217.

Murayama, S. and Shibata, T. (1964). "Flow and stress relaxation of clays." *Rheology and Soil Mechanics*, Symposium of the Int. Union of Theoretical and Applied Mechanics, Grenoble, France, 99–129.

Simo, J. C. and Taylor, R. L. (1985). "Consistent tangent operators for rate-independent elastoplasticity." *Comp. Meths. Appl. Mech. Engrg.*, 48, 101–118.

Simo, J. C., Kennedy, J. G., and Govindjee, S. (1988). "Non-smooth multisurface plasticity and viscoplasticity. Loading/unloading conditions and numerical algorithms." *Int. J. Num. Meths. Engrg.*, 26, 2161–2185.

Singh, A. and Mitchell, J. K. (1968). "General stress-strain-time function for soils." *J. Soil Mech. Found. Division, ASCE*, 94(1), 21–46.