Lumped Parameters in Permanent Displacement

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SYNOPSIS: New dynamic spring and damping values, which are dependent on the frequency, have been introduced into calculations of horizontal permanent displacement. The new dynamic damping value, which is inversely proportional to the circular frequency, has the same form as the material damping, while the new value of stiffness increases with increase in circular frequency. Similar methods can be used to obtain new dynamic spring and damping values for other modes of vibrations such as vertical, rocking and torsional.

An analytical treatment of the coupled rocking and sliding modes is presented which considers horizontal and vertical Coulomb friction forces, two-layered soil deposits and also the vertical vibration.

INTRODUCTION

Old lumped parameters developed by Arnold et al.(1955), Bycroft (1956) and (1959), Hsieh (1962) and Hall (1967), which are based on the periodic loading function of P_oe^{iwt}, have been used extensively in predicting resonant frequency and amplitude of vibration of foundations. In this paper, when a horizontal dynamic loading of the form $P_oT+P_o\sin nwt$ is applied, new lumped parameters are generated for use in calculations of horizontal permanent displacement. Therefore, various loading functions have different values of stiffness and damping which should be appropriately used in cyclic loading situations.

The treatment used by Hsieh (1962) has been repeated to find the new values of stiffness and damping which, surprisingly, have the old values of damping and stiffness (Richart, et al. 1970), respectively. For low frequencies of 0.1 Hz to 1.0 Hz, the new values of damping are much larger than the new values of stiffness.

HORIZONTAL VIBRATION

For waves radiated from a source at the surface of a semi-infinite, homogeneous, isotropic, elastic body and created by a weightless rigid circular disk of radius r_o under a vertical periodic loading $P(t)= P_oe^{iwt_o}$, it has been shown by Reissner (1936), Quinlan (1953), Sung (1953) and Arnold et al.(1955) that the displacement

$$x=\frac{1}{Gr_o}(f_1+if_2)P_oe^{iwt} \quad (1)$$

where P_o = Magnitude of cyclic load
w = Circular frequency of footing
t = Time variable
e = Base of Napierian logarithms
i = Imaginary number = $(-1)^{1/2}$
r_o = Radius of footing
G = Shear modulus of founding medium
and f_1 and f_2 = Functions (first introduced by Reissner,1936) which depend upon the Poisson's ratio μ of the medium and a dimensionless quantity, **a**, described by

$$a=wr_o\sqrt{\frac{\rho}{G}} \quad (2)$$

where ρ = Mass density of medium.

If the horizontal force has the following form:

$$P(t)=P_oT+P_o\sin nwt \quad (3)$$

using the symbol Re to mean "the real part of", then

$$P(t)=P_oT+Re(-iP_oe^{iwt}) \quad (4)$$

Since f_1 and f_2 are the values of the oscillatory part of Eq.1, so

$$P(t)=Re(-iP_oe^{iwt}) \quad (5)$$

Eq.1 can be written as follows:

$$x=\frac{1}{Gr_o}(if_1-f_2)(-iP_oe^{iwt}) \quad (6)$$

By differentiating Eq.6 with time t,

$$\frac{dx}{dt}=\frac{iw}{Gr_o}(if_1-f_2)(-iP_oe^{iwt}) \quad (7)$$

Thus, multiplying Eq.6 by f_2w and Eq.7 by f_1, a new relationship has been found as follows:

$$f_2wx+f_1\frac{dx}{dt}=-\frac{w}{Gr_o}(f_1^2+f_2^2)(-iP_oe^{iwt}) \quad (8)$$

$$Re(f_2wx+f_1\frac{dx}{dt})=Re(\frac{w}{Gr_o}-[f_1^2+f_2^2](-iP_oe^{iwt})) \quad (9)$$

$$P_o\sin nwt=-(\frac{Gr_o}{w})(\frac{f_1}{f_1^2+f_2^2})\frac{dx}{dt}-Gr_o(\frac{f_2}{f_1^2+f_2^2})x \quad (10)$$

Note that for vertical and horizontal vibrations, Hsieh (1962) found that:

$$P_o e^{iwt} = -\left(\frac{Gr_o}{w}\right)\left(\frac{f_2}{f_1^2+f_2^2}\right)\frac{dx}{dt} + Gr_o\left(\frac{f_1}{f_1^2+f_2^2}\right)x \quad \textbf{(11)}$$

Eq.11 was obtained by multiplying Eqs.9 and 10 by $f_1 w$ and f_2, respectively.

Another simple way to find the value of stiffness for the force P_o sinwt by using Eq.1 is as follows:

$$\frac{P_o(\cos wt + i\sin wt)}{x} = \frac{Gr_o(f_1 - if_2)}{f_1^2+f_2^2} \quad \textbf{(12)}$$

therefore,

$$\frac{P_o \cos wt}{x} = Gr_o\left(\frac{f_1}{f_1^2+f_2^2}\right) \quad \textbf{(13)}$$

and

$$\frac{P_o \sin wt}{x} = -Gr_o\left(\frac{f_2}{f_1^2+f_2^2}\right) \quad \textbf{(14)}$$

The values of stiffness for the functions $P_o$coswt and $P_o$sinwt are the right hand terms of Eqs.13 and 14, respectively.

Comparing Eqs.10 and 11, the old coefficients of stiffness and damping (Hsieh,1962) will remarkably be the new coefficients of damping and stiffness, respectively.

The values of f_1 and f_2 for various cases can be found in the papers by Arnold et al. (1955) and Bycroft (1956) and (1959).

Using

$$F_2' = F_1 = \left[-\frac{f_1}{f_1^2+f_2^2}\right] \quad \textbf{(15)}$$

and

$$F_1' = -F_2 a = \left[-\frac{f_2}{f_1^2+f_2^2}\right] \quad \textbf{(16)}$$

where F_1 and F_2 = Constants used by Hsieh (1962) and Hall (1967). Subscripts 1 and 2 of F and F' denote stiffness and damping, respectively.

The ratio of F_1 and F_2 is $-(f_1/f_2)a$, while the ratio of F'_2 and F'_1 is $+(f_1/f_2)$. f_2 has to be a negative value if the function (Eq.3) is used. Note that Toriumi(1955) found that f_2 is smaller than zero only for vertical vibration and the absolute value of f_2 is the same value as that of Reissner (1936). F_2 actually is a positive value (Hsieh,1962 & Hall,1967), therefore $F'_1=F_2a$ is used hereafter. The new values of dynamic stiffness and damping can be obtained by multiplying the static stiffness constant with F_2a and F_1 (Lysmer & Richart,1966), respectively.

Eq.10 can be written as follows:

$$P_o \sin wt = F_1\left(\frac{Gr_o}{w}\right)\frac{dx}{dt} + F_2 w r_o^2 \sqrt{\frac{\rho}{G}}\, x \quad \textbf{(17)}$$

NEW HORIZONTAL LUMPED PARAMETERS

(a) The new expressions for stiffness, K'_x, and damping, C'_x, derived from Bycroft's method are as follows:

$$K_x' = \frac{18.4(1-\mu)}{(7-8)} w r_o^2 \sqrt{\rho G} \quad \textbf{(18)}$$

and

$$C_x' = \frac{32(1-\mu)}{(7-8\mu)}\left(\frac{Gr_o}{w}\right) \quad \textbf{(19)}$$

Similar forms of K'_x and C'_x have been found by Yan (1981), and Elsabee et al.(1977) also found that dynamic stiffness depends upon the circular frequency. The new dynamic stiffness is equal to zero when the circular frequency becomes zero; this is called the static stiffness of the founding medium. The zero static stiffness for a strip footing was also mentioned by Roesset & Kausel (1976).

In order to take into account for the dynamic material damping, terms suggested by Dobry et al.(1985) and Sankaran et al.(1977) are (bK/w) and (b/w), respectively; these terms should be included in analyses of vibrations of foundations. Where K is the coefficient of stiffness and b is the material damping constant, which is twice the dynamic material damping ratio (Vierck,1967 and Wolf,1985). The new dynamic damping value is also inversely proportional to the circular frequency, w. The total damping is the sum of the geometric damping and material damping.

(b) The new expressions for dynamic stiffness and damping for the method of Veletsos and Wei (1971) are:

$$K_x' = \frac{4.6}{(2-\mu)} w r_o^2 \sqrt{\rho G} \quad \textbf{(20)}$$

$$C_x' = \frac{8}{(2-\mu)}\left(\frac{Gr_o}{w}\right) \quad \textbf{(21)}$$

ROCKING VIBRATION

Arnold et al.(1955) has shown that the rotation of a weightless circular disk excited by a couple $C(t)=Me^{iwt}$ about a vertical or horizontal axis is given by:

$$\theta(t) = Me^{iwt}(f_1 + if_2) \quad \textbf{(22)}$$

where M = Constant.

If a couple has the following form:

$$C(t) = h_r(P_o T + P_o \sin wt) \quad \textbf{(23)}$$

where h_r = Rocking level, defined as the vertical distance between the base of the footing and the horizontal force P(t) (Fig.1).

Then, following the same procedure as used for horizontal vibration, a similar relationship between the couple, the angular velocity and rotation is obtained as follows:

$$C(t)=-(\frac{Gr_o^3}{w})[\frac{f_1}{(f_1^2+f_2^2)}]\frac{d\theta}{dt}-Gr_o^3[\frac{f_2}{(f_1^2+f_2^2)}]\theta \quad (24)$$

New expression for stiffness has been derived based on the stiffness and damping coefficients of Hall (1967) as follows:

$$K'_\theta=\frac{0.8}{(1-\mu)(1+B'_\theta)}wr_o^4\sqrt{\rho G} \quad (25)$$

where B_θ' = New mass ratio

$$B'_\theta=\frac{(1-\mu)}{0.8}(\frac{I_\theta}{\rho r_o^5}) \quad (26)$$

where I_θ = Mass moment of inertia of the footing in rotation.

The new expression for damping is:

$$C'_\theta=(\frac{8}{3(1-\mu)})(\frac{Gr_o^3}{w}) \quad (27)$$

VERTICAL VIBRATION

A similar procedure was carried out for vertical oscillation, the new stiffness and damping values using the Timoshenko's static value are as follows:

$$K'_z=(\frac{3.4}{(1-\mu)})wr_o^2\sqrt{\rho G} \quad (28)$$

$$C'_z=(\frac{4}{(1-\mu)})(\frac{Gr_o}{w}) \quad (29)$$

TORSIONAL VIBRATION

The new stiffness and damping values based on Richart et al.(1970) are, respectively,

$$K'_\phi=4(\frac{B_\phi}{(1+2B_\phi)})wr_o^4\sqrt{\rho G} \quad (30)$$

where the mass ratio B_ϕ for torsional vibration is

$$C'_\phi=(\frac{16}{3})(\frac{Gr_o^3}{w}) \quad (31)$$

$$B_\phi=\frac{I_\phi}{\rho r_o^5} \quad (32)$$

in which I_ϕ = Mass moment of inertia about the axis of rotation.

COUPLED ROCKING AND SLIDING OF THE RIGID FOOTING ON THE ELASTIC HALF-SPACE

The rocking and sliding modes are frequently encountered as coupled motions. The analytical treatment of this coupled mode of vibration can be handled using the equations of motion of the centre of gravity (C.G.) of the footing given by Ratay (1971), Kuppusamy (1977), Moore (1985), Richart et al. (1970) and also including the Coulomb friction forces for horizontal and vertical directions denoted as F_x and F_z, respectively. The resulting equations are as follows:

$$m\frac{d^2z}{dt^2}+C_z\frac{dz}{dt}+K_zz+e(C_x\frac{d\theta}{dt}+K_z\theta)+F_z=mg \quad (33)$$

$$m\frac{d^2x}{dt}+C_x\frac{dx}{dt}+K_xx-h(K_x\theta+C_x\frac{d\theta}{dt})+F_x=P_oT+P_o\sin wt \quad (34)$$

$$I_c\frac{d^2\theta}{dt^2}+\frac{d\theta}{dt}[C_\theta+e^2C_x+h^2C_x]+\theta[K_\theta+e^2K_x+h^2K_x-Wh]$$

$$+eK_xz+eC_x\frac{dz}{dt}-hK_xx-hC_x\frac{dx}{dt}$$

$$=(h_r-h)(P_oT+P_o\sin wt) \quad (36)$$

where d^2z/dt, dz/dt, d^2x/dt, dx/dt, $d^2\theta/dt$ and $d\theta/dt$ = Second and first derivatives of z, x and θ with time t, respectively.

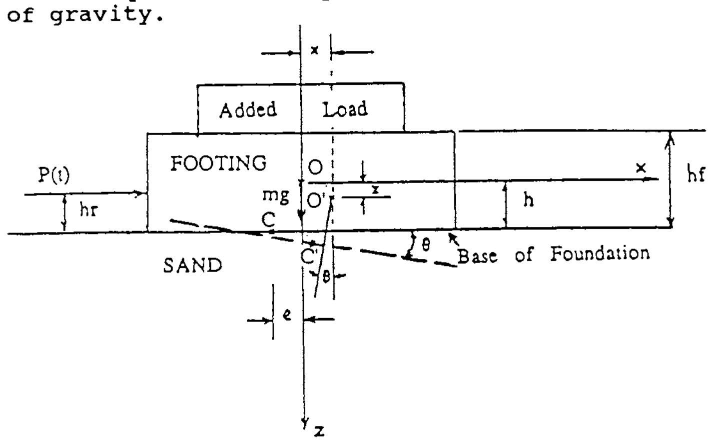
C_z, K_z, C_x, K_x, C_θ and K_θ = Coefficients of damping and stiffness of vertical, horizontal and rocking vibrations, respectively.

h = Vertical distance between the base of the footing and the centre of gravity of the footing system, which is composed of the footing and the known added dead loads on the footing (Fig. 1).

e = Horizontal distance between the centroid of the base and the centre gravity of the footing, which can have any positive or negative value (Fig.1).

W = Weight of the footing system

I_c = Mass moment of inertia of the footing with respect to the y-axis through the centre of gravity.



O is the original centre of gravity (C.G.)
O' is the later centre of gravity
C is the original centroid of base
C' is the later centroid of base

Fig.1 Notation For Rocking-Sliding Vibration (After Moore, 1985)

For a two-layered system, Kagawa and Kraft (1981) has found that the value of stiffness must increase with a decrease in thickness ratio and with a decrease in relative stiffness of the two layers. The effect of a two-layered system has been considered in the companion paper (Truong, 1991).

CONCLUSIONS

A new periodic loading function, $P_oT+P_o\sin wt$, has been introduced and new expressions have been generated for stiffness and damping for different modes of vibrations, especially horizontal vibration. Generally, the value of stiffness is less than that of damping due to the relationship with the value of the shear modulus G. The value of damping tends to decrease with an increase in frequency of the cyclic loading, while the value of stiffness increases linearly with the frequency. The new values of stiffness and damping have been used to calculate the horizontal permanent displacement, and the results agree very well with the experimental results (Truong, 1991).

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