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SYNOPSIS: Galerkin's procedure and the modified energy expression of Banerjee are used to obtain the central deflection of circular plates with linearly varying thicknesses, resting on a linear as well as a nonlinear elastic foundation. A new series formula for the deflection has been considered. The accuracy of the method has been tested for clamped plates with movable and immovable ends under uniform static patch loading. Graphical results are drawn for the central deflection for different end conditions, different plate thicknesses and different foundation properties. Comparison of the results are made with other known results and shows a good agreement in most tested cases.

1. INTRODUCTION

Plates under flexure have found a wide spread application as structural elements in mechanical, civil, marine, aeronautical, and other fields of engineering. When the deflection is quite large, that it is of the same order of magnitude as the thickn ss of the plat , it beco es necessary to apply nonlinear plate theories, which take into account the interaction between the bending and membrane stresses.

Timoshenko and Woinowsky (1959) have solved different plates with various boundary conditions using the small deflection theory. They also included problems of plates resting on elastic supports. Wang (1934), Stippes (1952), Schmidt (1968), Gordon, Mah (1969) and others have discussed various plates of uniform and nonlinear thickness of analysis.

Berger (1955) suggested an approximate method for solving the problem of plates in large deflection by neglecting the strain energy due to the second invariant of the middle surface strains. Sinha (1963) used the technique of Berger to solve the problem of the large static deflection of uniformly loaded circular as well as rectangular plates, resting on linear elastic foundation and with immovable edges.

Nowinski and Ohnabe (1958) concluded that Berger's line of thought leads to meaningless results for movable edge conditions. This is due to the fact that the neglect of second invariant for movable edges fails to imply freedom of rotation in the meridian planes.

B. Banerjee (1981) has included a new approach to the analysis of large deflections of thin elastic plates. This approach make it possible to decouple the nonlinear governing equations of the problem.

In this paper the new approach of Banerjee has been considered to solve the equations representing the plates of linear varying thickness. The load distribution is taken axisymmetric, uniform, static patch loading, and the variational principle is employed to drive th equation of equilibr um. The approxima e Galerkin's procedure is applied and the nonlinear algebraic equations obtained has been solved using the modified Newton - Raphson's iterative technique. Graphical results for the central deflections are represented for different boundary conditions.

2. BASIC EQUATIONS

The total potential energy of thin isotropic circular plates in large deflection resting on a nonlinear elastic foundation is given by

$$\pi = \int_0^a \{ \frac{D}{2} [(\frac{d^2w}{dr^2})^2 + \frac{2\nu}{r} \frac{dw}{dr} \frac{d^2w}{dr^2} + \frac{1}{r^2} (\frac{dw}{dr})^2 + \frac{12}{h^2} (\bar{e}_1^2 + (1-\nu^2) \frac{u^2}{r^2})] + \frac{k}{n+1} w^{n+1} - q(r)w \} r dr \quad (1)$$

Where

- a = radius of the plate
- $D(r)$ = $D_o(1-\alpha r/a)^3$ = flexural rigidity of the plate, where $D_o = Eh_o^3 / 12(1-\nu^2)$
- h = $h_o(1-\alpha r/a)$ = thickness of the plate
- α = taper ratio
- w = transverse displacement
- u = enplane displacement
- ν = poisson's ratio
- $\bar{e}_1 = \frac{1-\nu^2}{Eh} N_r$

N_r = membrane forces in the radial direction.
ξ = non-linearity coefficient
n = index of non-linearity (n=1 for linear elastic foundation, n=3 for nonlinear elastic foundation)
$q(r)$ = arbitrary lateral load

Banerjee (1981) suggested a modified energy expression in place of the expression given in equation (1). This comes true when the term $(1-\nu^2)(u^2/r^2)$ is replaced by $(\lambda/4)(dw/dr)^4$, where λ is a factor depending on poisson's ratio. This replacement makes the decoupling of the governing equilibriums equation becomes possible.

Thus the total potential energy of the plate becomes ,

$$\pi = \int_0^a \{ \frac{D}{2} [(\frac{d^2w}{dr^2})^2 + \frac{2\nu}{r} \frac{dw}{dr} \frac{d^2w}{dr^2} + \frac{1}{r^2} (\frac{dw}{dr})^2 + \frac{12}{h^2} (\bar{e}_1{}^2 + \frac{\lambda}{4} (\frac{dw}{dr})^4)] + \frac{\xi}{n+1} w^{n+1} - q(r)w \} r dr \quad (2)$$

Applying the dimensionless quantities,

$$\rho = \frac{r}{a}, \; y = \frac{w}{a}, \; U = \frac{u}{a}, \; t = \frac{h}{a}, \; P = \frac{qa^3}{D},$$

$$\eta = (\xi\, G) a^n \quad (3)$$

and minimizing equation (2), the governing equilibrium equation of the plate becomes,

$$(1-\alpha\rho)^3 \nabla^4 y - 3\alpha (1-\alpha\rho)^2 [2 \frac{d^3y}{d\rho^3} + \frac{2+\nu}{\rho} \frac{d^2y}{d\rho^2} - \frac{1}{\rho^2} \frac{dy}{d\rho}] + 6\alpha^2 (1-\alpha\rho) (\frac{d^2y}{d\rho^2} + \frac{\nu}{\rho} \frac{dy}{d\rho}) - \frac{6\lambda}{t_o^2} (1-\alpha\rho) (\frac{dy}{d\rho})^2 (\nabla^2 y + 2\frac{d^2y}{d\rho^2}) + \frac{6\lambda}{t_o^2} \alpha (\frac{dy}{d\rho})^3 - \frac{12}{t_o^2} B\rho^{\nu-1} (\frac{\nu}{\rho} \frac{dy}{d\rho} + \frac{d^2y}{d\rho^2}) + \eta y^n - P(\rho) = 0 \quad (4)$$

determined from,

$$\frac{dU}{d\rho} + \frac{1}{2} (\frac{dy}{d\rho})^2 + \nu \frac{U}{\rho} = B \frac{\rho^{\nu-1}}{(1-\alpha\rho)} \quad (5)$$

3. SOLUTION OF THE PROBLEM

The deflection of the plate is assumed as a series function of the central distance, as

$$y(\rho) = \sum_{i=1}^{l} a_i \phi_i$$

where the coordinate function ϕ_i must satisfy all of the essential boundary conditions. These boundary conditions may be stated as:

1- Clamped with immovable edge

$$y(1) = \frac{dy}{d\rho}(1) = U(1) = 0 \quad (6)$$

2- Clamped with movable edge

$$y(1) = \frac{dy}{d\rho}(1) = N(1) = 0 \quad \text{or } B = 0 \quad (7)$$

This coordinate functions ϕ_i may be represented in the form

$$\phi_i = \frac{1}{4i} (1 - 2\rho^{2i} + \rho^{4i}) \quad (8)$$

Substituting equation (8) into equation (4) we obtain the non-vanishing error function as

$$E(\rho) = a_1 [16 - 9\alpha\rho(11+\nu) + 24\alpha^2\rho^2(7+\nu) - 5\alpha^3\rho^3 (17+3\nu) + (1+\nu)(\frac{3\alpha}{\rho} - 12\alpha^2 + 9\alpha^3\rho)] + \sum_{i=2}^{l} a_i [-8i(i-1)^2 \rho^{2i-4} + 16i(2i-1)^2\rho^{4i-4}] - 3\alpha \sum_{i=2}^{l} a_i [(2i-1) (-4i^2+2i+1-\nu) \rho^{2i-3} + (4i-1) (16i^2-4i-1+\nu) \rho^{4i-3}] + 3\alpha^2 \sum_{i=2}^{l} a_i [4i (-2i^2+1-\nu) \rho^{2i-2} + 8i(8i^2-1+\nu) \rho^{4i-2}] - \alpha^3 \sum a_i [(2i+1) (-4i^2-2i+3-3\nu) \rho^{2i-1} + (4i+1) (16i^2+4i-3+3\nu) \rho^{4i-1}] - \frac{6\lambda}{t_o^2} \sum_{i,j,s=1}^{l} a_i a_j a_s [(-6i+2) (\rho^{2i+2j+2s-4} - \rho^{2i+2j+4s-4} - \rho^{2i+4j+2s-4} + \rho^{2i+4j+4s-4}) + (12i-2) (\rho^{4i+2j+2s-4} - \rho^{4i+2j+4s-4} - \rho^{4i+4j+2s-4} + \rho^{4i+4j+4s-4})] + \frac{6\lambda}{t_o^2} \alpha \sum_{i,j,s=1}^{l} a_i a_j a_s [(-6i+1) (\rho^{2i+2j+2s-3} - \rho^{2i+4j+2s-3} - \rho^{2i+2j+4s-3} + \rho^{2i+4j+4s-3}) + (12i-1) (\rho^{4i+2j+2s-3} - \rho^{4i+4j+2s-3} - \rho^{4i+2j+4s-3} + \rho^{4i+4j+4s-3})] - \frac{12}{t_o^2} B \rho^{\nu-1} \sum_{i=1}^{l} a_i [(-2i+1-\nu)\rho^{2i-2} + (4i-1+\nu)\rho^{4i-2}] + \eta [\sum_{i=1}^{l} \frac{a_i}{4i} (1-2\rho^{2i}+\rho^{4i})]^n - P(\rho) \quad (9)$$

The Galerkin procedure requires that the error function $E(\rho)$ should be orthogonal to the coordinate functions $\phi_K(\rho)$ over the domain of the plate, i.e

$$I = \int_{\Omega} E(\rho)\ \phi_K(\rho)\ d\Omega = 0, \quad (K = 1, 2, 3, \dots l) \qquad (10)$$

This integral can be divided into three parts, i.e

$$I = I^D + I^W + I^P = 0 \qquad (11)$$

Where

$$I^D = 2K\, a_i\, [\, A_1 - 3\alpha A_2 + 6\alpha^2 A_3 - \alpha^3 A_4 + 3\alpha A_5 - 6\alpha^2 A_6 + 3\alpha^3 A_7\,] + 2K \sum_{i=2}^{l} a_i\, [\, B_1 + B_2 - 3\alpha (B_3 + B_4) + 6\alpha^2 (B_5 + B_6) - \alpha^3 (B_7 + B_8)\,] - 2K \frac{6\lambda}{t_o^2} \sum_{i,j,s=1}^{l} a_i a_j a_s [(-6i+2)(C_1 - C_2 - C_3 + C_4) + (12i-2)(C_5 - C_6 - C_7 + C_8)] + 2K \frac{6\lambda}{t_o^2} \alpha \sum_{i,j,s=1}^{l} a_i a_j a_s\, [(-6i+1)(D_1 - D_2 - D_3 + D_4) + (12i-1)(D_5 - D_6 - D_7 + D_8)\,] - \frac{12}{t_o^2} B\, 2K \sum_{i=1}^{l} a_i\, [\, -E_1 + E_2\,]$$

Where

$A_1 = 8/(2+2K)/(2+4K)$, $A_2 = (11+\nu)/(3+2K)/(3+4K)$

$A_3 = (7+\nu)/(4+2K)/(4+4K)$, $A_4 = (17+3\nu)/(5+2K)/(5+4K)$

$A_5 = (1+\nu)/(1+2K)/(1+4K)$, $A_6 = (1+\nu)/(2+2K)/(2+4K)$

$A_7 = (1+\nu)/(3+2K)/(3+4K)$

$B_1 = -2i(2i-2)/(2i-2+2K)/(2i-2+4K)$

$B_2 = 4i(4i-2)/(4i-2+2K)/(4i-2+4K)$

$B_3 = (-4i^2+2i+1-\nu)/(2i-1+2K)/(2i-1+4K)$

$B_4 = (16i^2-4i-1+\nu)/(4i-1+2K)/(4i-1+4K)$

$B_5 = (-2i^2+1-\nu)/(2i+2K)/(2i+4K)$

$B_6 = (8i^2-1+\nu)/(4i+2K)/(4i+4K)$

$B_7 = (-4i^2-2i+3-3\nu)/(2i+1+2K)/(2i+1+4K)$

$B_8 = (16i^2+4i-3+3\nu)/(4i+1+2K)/(4i+1+4K)$

$C_1 = 1/(2i+2j+2s-2)/(2i+2j+2s-2+2K)/(2i+2j+2s-2+4K)$

$C_2 = 1/(2i+2j+4s-2)/(2i+2j+4s-2+2K)/(2i+2j+4s-2+4K)$

$C_3 = 1/(2i+4j+2s-2)/(2i+4j+2s-2+2K)/(2i+4j+2s-2+4K)$

$C_4 = 1/(2i+4j+4s-2)/(2i+4j+4s-2+2K)/(2i+4j+4s-2+4K)$

$C_5 = 1/(4i+2j+2s-2)/(4i+2j+2s-2+2K)/(4i+2j+2s-2+4K)$

$C_6 = 1/(4i+2j+4s-2)/(4i+2j+4s-2+2K)/(4i+2j+4s-2+4K)$

$C_7 = 1/(4i+4j+2s-2)/(4i+4j+2s-2+2K)/(4i+4j+2s-2+4K)$

$C_8 = 1/(4i+4j+4s-2)/(4i+4j+4s-2+2K)/(4i+4j+4s-2+4K)$

$D_1 = 1/(2i+2j+2s-2)/(2i+2j+2s-2+2K)/(2i+2j+2s-2+4K)$

$D_2 = 1/(2i+2j+4s-1)/(2i+2j+4s-1+2K)/(2i+2j+4s-1+4K)$

$D_3 = 1/(2i+4j+2s-1)/(2i+4j+2s-1+2K)/(2i+4j+2s-1+4K)$

$D_4 = 1/(2i+4j+4s-1)/(2i+4j+4s-1+2K)/(2i+4j+4s-1+4K)$

$D_5 = 1/(4i+2j+2s-1)/(4i+2j+2s-1+2K)/(4i+2j+2s-1+4K)$

$D_6 = 1/(4i+2j+4s-1)/(4i+2j+4s-1+2K)/(4i+2j+4s-1+4K)$

$D_7 = 1/(4i+4j+2s-1)/(4i+4j+2s-1+2K)/(4i+4j+2s-1+4K)$

$D_8 = 1/(4i+4j+4s-1)/(4i+4j+4s-1+2K)/(4i+4j+4s-1+4K)$

$E_1 = 1/(2i+\nu-1+2K)/(2i+\nu-1+4K)$

$E_2 = 1/(4i+\nu-1+2K)/(4i+\nu-1+4K)$

The constant B is determined for each end condition as:

B=0 for clamped movable edge, and

$$B = \frac{1}{2} \sum_{i,j=1}^{l} a_i a_j \left(\frac{1}{(2i+2j-1+\nu)} - \frac{1}{(2i+4j-1+\nu)} - \frac{1}{(4i+2j-1+\nu)} - \frac{1}{(4i+4j-1+\nu)} \right) \Big/ \left(\sum_{n=1}^{\infty} \frac{\alpha^{n-1}}{2\nu - 1 + n} \right)$$

for clamped immovable edge (13)

$$I^W = \eta \sum_{i=1}^{l} \frac{a_i}{4i}\, 2K\, (\, F_1 - F_2 + F_3\,) \qquad (14)$$

for n=1 (linear elastic foundation), and

$$I^W = \eta \sum_{i,j,s=1}^{l} \frac{a_i a_j a_s}{4i.4j.4s}\, 2K\, [\, G_1 - G_2 - G_3 - G_4 + G_5 + G_6 + G_7 + G_8 + G_9 + G_{10} - G_{11} - G_{12} - G_{13} - G_{14} -$$

[illegible] G_{15} [illegible] G_{16} [illegible] G_{17} [illegible] G_{18} [illegible] G_{19} $-G_{20} + G_{21} + G_{22} +$

$$G_{23} - G_{24} - G_{25} - G_{26} + G_{27}\,] \qquad (15)$$

for n=3 (nonlinear elastic foundation)

Where

$F_1 = 1/2/(2+2K)/(2+4K)$

$F_2 = 2/(2i+2)/(2i+2+2K)/(2i+2+4K)$

$F_3 = 1/(4i+2)/(4i+2+2K)/(4i+2+4K)$

$G_1 = 1/2/(2+2K)/(2+4K)$

$G_2 = 2/(2i+2)/(2i+2+2K)/(2i+2+4K)$

$G_3 = 2/(2i+2)/(2i+2+2K)/(2i+2+4K)$

$G_4 = 2/(2s+2)/(2s+2+2K)/(2s+2+4K)$

$G_5 = 1/(4i+2)/(4i+2+2K)/(4i+2+4K)$

$G_6 = 1/(4j+2)/(4j+2+2K)/(4j+2+4K)$

$G_7 = 1/(4s+2)/(4s+2+2K)/(4s+2+4K)$

$G_8 = 4/(2i+2j+2)/(2i+2j+2+2K)/(2i+2j+2+4K)$

$G_9 = 4/(2i+2s+2)/(2i+2s+2+2K)/(2i+2s+2+4K)$

$G_{10} = 4/(2s+2i+2)/(2s+2i+2+2K)/(2s+2s+2+4K)$

$G_{11} = 2/(2i+4j+2)/(2i+4j+2+2K)/(2i+4j+2+4K)$

$G_{12} = 2/(2i+4s+2)/(2i+4s+2+2K)/(2i+4s+2+4K)$

$G_{13} = 2/(4i+2j+2)/(4i+2j+2+2K)/(4i+2j+2+4K)$

$G_{14} = 2/(4i+2s+2)/(4i+2s+2+2K)/(4i+2s+2+4K)$

$G_{15} = 2/(2i+4s+2)/(2i+4s+2+2K)/(2i+4s+2+4K)$

$G_{16} = 2/(4j+2s+2)/(4j+2s+2+2K)/(4j+2s+2+4K)$

$G_{17} = 1/(4i+4j+2)/(4i+4j+2+2K)/(4i+4j+2+4K)$

$G_{18} = 1/(4j+4s+2)/(4j+4s+2+2K)/(4j+4s+2+4K)$

$G_{19} = 1/(4s+4i+2)/(4s+4i+2+2K)/(4s+4i+2+4K)$

$G_{20} = 8/(2i+2j+2s+2)/(2i+2j+2s+2+2K)/(2i+2j+2s+2+4K)$

$G_{21} = 4/(2i+2j+4s+2)/(2i+2j+4s+2+2K)/(2i+2j+4s+2+4K)$

$G_{22} = 4/(2i+4j+2s+2)/(2i+4j+2s+2+2K)/(2i+4j+2s+2+4K)$

$G_{23} = 4/(4i+2j+2s+2)/(4i+2j+2s+2+2K)/(4i+2j+2s+2+4K)$

$G_{24} = 2/(2i+4j+4s+2)/(2i+4j+4s+2+2K)/(2i+4j+4s+2+4K)$

$G_{25} = 2/(4i+2j+4s+2)/(4i+2j+4s+2+2K)/(4i+2j+4s+2+4K)$

$G_{26} = 2/(4i+4j+2s+2)/(4i+4j+2s+2+2K)/(4i+4j+2s+2+4K)$

$G_{27} = 1/(4i+4j+4s+2)/(4i+4j+4s+2+2K)/(4i+4j+4s+2+4K)$

If the applied load is distributed over a central area of radius b, with uniform intensity P_o, then the load can be represented as (1988)

$$P(\rho) = P_o\, m \int_0^1 J_1(sm)\, J_0(s\rho)\, ds = \begin{cases} P_o & \rho<m \\ P_o/2 & \rho=m \\ 0 & \rho>m \end{cases}$$

Consequently,

$$I^P = \frac{1}{4K}\left(\frac{m^2}{2} - \frac{2m^{2+2K}}{2+2K} + \frac{m^{2+4K}}{2+4K}\right) P_o \qquad (16)$$

Where m = b/a

The L nonlinear algebraic equations obtained from equations (10) through (16) are solved using the iterative Newton - Raphson technique. The nonlinear terms in equation (12) depends on the factor λ. This factor equals $2\nu^2$ for clamped edges (1981). The applied load is considered as uniform patch loading at central ratio m = 0.8. Numerical results are calculated assuming linear elastic supports in some cases and nonlinear elastic supports in other cases.

The relationship between the central deflection of the plate and the applied load is illustrated in Fig. 1,2 for different values for the tape ratio α and for different end conditions. The number of terms of the series function of the deflection is tested so that the error in the central deflection for successive number of terms does not exceed 0.5%. This number is found to be 4 for movable ends and 3 for immovable ends.

FIG. 1

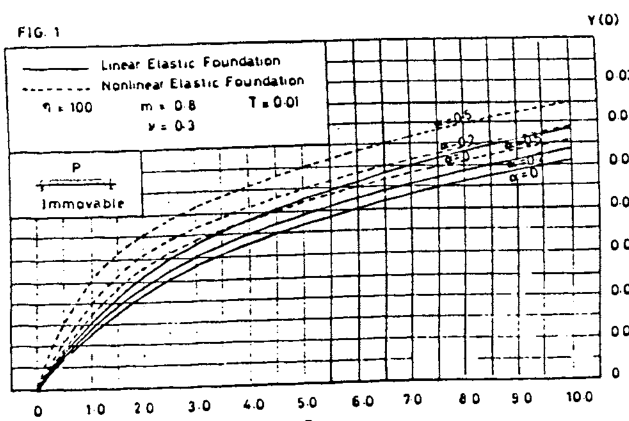
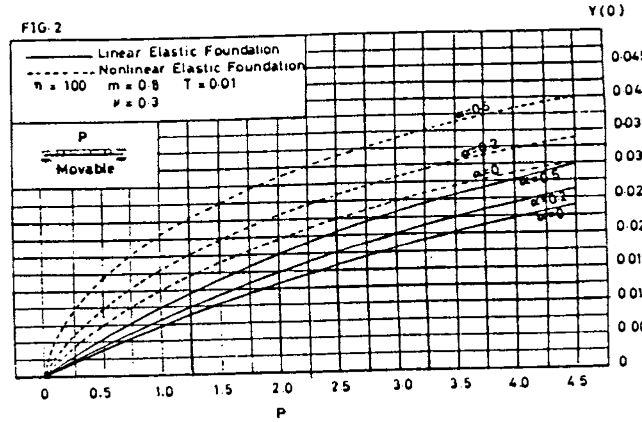


FIG. 2



4. Comparison of the results

The central deflection is calculated using the new proposed deflection formula. This deflection is compared with the different existing formulas.

FIG. 3

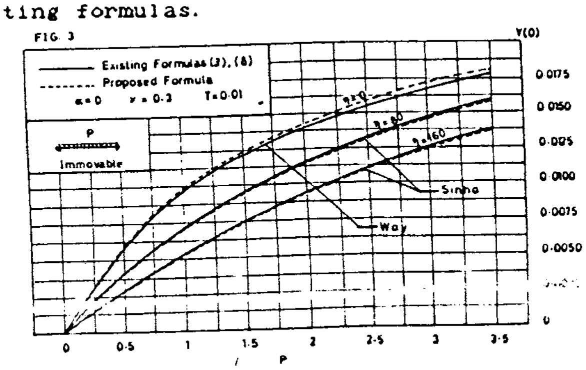
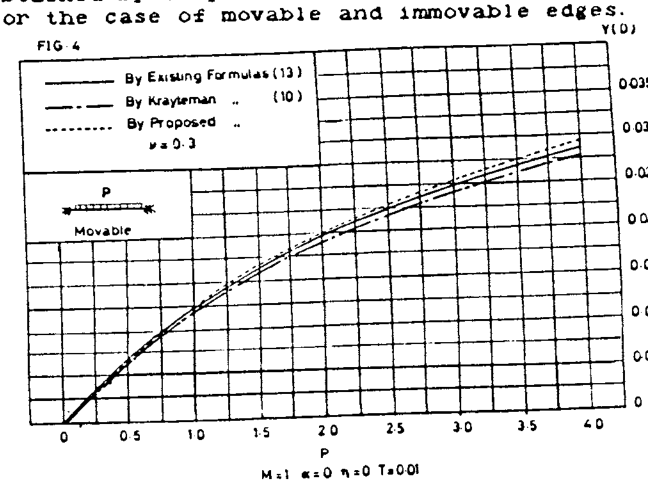
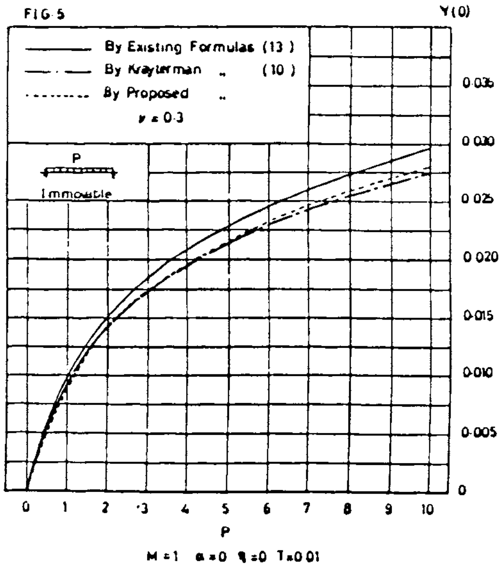


Figure 3 shows the central deflections obtained by Way (1934) and Sinha (1963) together with the central deflection obtained by the proposed formula for the case of immovable clamped edges. It is clear that there is a good agreement for all tested cases.

Figures 4 and 5 show the proposed central deflection as well as the central deflection obtained by Krayterman (1985) and Roark (1975) for the case of movable and immovable edges.

FIG. 4





It can be noticed that there are some discrepancies between the results obtained in this paper and the results obtained by Krayterman and Roark. The discrepancy with Krayterman is attributed to the violation of his deflection function to some of the boundary conditions. Beside, the first term of his series function does not belong to the domain of the operator of the governing equilibrium equation.

Higher [illegible] comparison is made with the deflections given by Roark (1975). These discrepancies are due to the assumption of using the deflection function obtained by the small deflection theory.

5. CONCLUSIONS

From the analysis and results presented herein, the following conclusions are drawn:

1-The governing equations obtained in this paper are decoupled, and hence they can be solved without difficulties.

2-Results obtained for immovable edge conditions are in good agreement with the results obtained by Sinha and Way.

3-The deflection formula proposed in this paper is more exact and gives better accuracy than the formulas proposed by Krayterman and Roark.

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