12 Mar 1991, 10:30 am - 12:00 pm

Mechanical Properties of Cemented Sands Based on Inter-Particle Contact Behavior

Anil Misra
University of Missouri-Kansas City (UMKC), Independence, MO

Ching S. Chang
University of Massachusetts, Amherst, MA

Proceedings: Second International Conference on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics, March 11-15, 1991, St. Louis, Missouri, Paper No. 1.34

Mechanical Properties of Cemented Sands Based on Inter-Particle Contact Behavior

Anil Misra
Assistant Professor of Civil Engineering, UMC-UMKC Coordinated Engineering Program, UMKC Truman Campus, Independence, Missouri

Ching S. Chang
Professor of Civil Engineering, University of Massachusetts, Amherst, MA

SYNOPSIS: The mechanical properties of cemented sands are modelled using a micromechanical approach. The derived model accounts for the packing structure, the particle properties and the contact properties.

INTRODUCTION

The mechanical properties of cemented sands under small deformations are of interest in problems involving low amplitude oscillatory loading, such as, seismic ground excitation, traffic loadings and vibration of machine on soil foundations. In recent literature, a number of experimental results are available on the initial moduli, secant moduli and damping ratios of cemented sands measured from tests conducted in resonant column device. However, there is a clear lack of mathematical models available for theoretically understanding the behavior of mechanical properties of cemented sands.

Cemented sands, like other granular materials, are significantly influenced by micro-scale interactions between particles. Cemented sands are made of sand particles bonded together at the inter-particle contacts by cementing agents like silica, hydrous silicates, hydrous ion oxides, calcite cement, carbonates etc.. The cement at the inter-particle contacts creates adhesive forces between particles. Thus, unlike clean uncemented sands which can resist only compressive and shear forces at contacts, the cemented sands can resist tensile forces in addition to compression and shear. To account for the load transference and deformation mechanism of sands, it is rational to obtain the mechanical properties of an element of cemented sand by considering the inter-particle behavior. This type of approach has been successfully applied to model the mechanical properties of uncemented sands under small deformations (Chang et al. 1989), considering sand to be a random packing of spheres with frictional contacts (Chang 1987). In the present work, an extension of the 'micro-structural' model which includes the effect of inhomogeneity of deformation fields in sands is presented. The inter-particle contact interaction with effect of adhesion is discussed.

THEORETICAL DEVELOPMENT

The theoretical development of the stress-strain modelling for cemented sands is broadly discussed under two categories, namely: (a) the inter-particle behavior considering effect of adhesion in addition to friction at a contact; and (b) the overall behavior of a collection particles forming an element of the cemented sand.

Inter-Particle Behavior

Cemented sand is considered to be a collection of a large number of spherical particles with both frictional and adhesive resistance at the inter-particle contacts. The imposed load at the boundary of an element of cemented sand is supported through this resistance at inter-particle contacts. Under the imposed load, the particles move relative to each other leading to distortion at the inter-particle contacts. It is conceptually easier to represent this distortion at the inter-particle contacts by deformable springs connecting rigid particles. These springs carry forces in the tangential and the normal direction at a contact. It is assumed for simplicity that no moment is transmitted at the inter-particle contacts.

The contact spring deformation δ_i^{nm}, as mentioned previously, is due to the relative movement between particles. Considering each particle to have six degrees of freedom, namely: three translational and three rotational, the relative displacement between two particles, say n and m, is written as

$$\delta_i^{nm} = u_i^m - u_i^n + e_{ijk}(\omega_j^m r_k^m - \omega_j^n r_k^n) \qquad (1)$$

where u_i denotes the particle displacement, ω_k denotes the particle rotation, r_j is the vector joining the centroid of a particle to the contact point, superscripts refer to the particles and e_{ijk} is the permutation symbol. Tensor summation convention is assumed for the subscripts throughout the paper.

The spring stiffness K_{ij}^{nm} relates the contact force f_j^{nm} to the contact spring deformation δ_j^{nm} as

$$f_i^{nm} = K_{ij}^{nm}\,\delta_j^{nm} \tag{2}$$

where K_{ij}^{nm} is the contact stiffness tensor for the contact between particles n and m.

In this work, we assume a simple form of the K_{ij} given by

$$K_{ij}^{nm} = K_n n_i^{nm} n_j^{nm} + K_r (s_i^{nm} s_j^{nm} + t_i^{nm} t_j^{nm}) \tag{3}$$

where K_n and K_r are the contact stiffnesses along the normal and tangential direction of the contact surface respectively. The unit vector **n** is normal to the contact surface and vectors **s** and t are arbitrarily chosen such that **nst** forms a local cartesian coordinate system. This simplified contact stiffness tensor is fairly reasonable for two elastic non-conforming bodies in contact (Mindlin and Deresiewicz 1953).

In the present study we model two sand particles bonded together with light amount of cement as frictional elastic spheres with adhesive surface. The contact stiffnesses K_n and K_r are dependent on the surface (such as friction and adhesion) and stiffness (modulus and Poisson's ratio) properties of the two particles in contact and therefore, will be different for the various contacts. All the particles are assumed to have same surface and stiffness properties. The contact stiffnesses are also dependent on the force at the inter-particle contact.

The effect of contact force and particle stiffness on contact behavior in the normal direction for topographically smooth, non-conforming, elastic bodies was studied by Hertz (Mindlin and Deresiewicz 1953). The Hertzian theory was extended by Johnson (1976) to include the effect of adhesion at the contact. Based on the extension by Johnson, the normal stiffness K_n is given by (Chang et al. 1990)

$$K_n = \frac{96E^2a^4(\chi_c + f_n)}{40E\chi_c a^3 + 32Ef_n a^3 - 3\rho f_n^2} \tag{4}$$

where a is the radius of the contact surface given by

$$a = \left[\frac{3\rho}{8E}\,[f_n + 2\chi_c + 2\sqrt{(f_n\chi_c + \chi_c^2)}\,]\right]^{1/3} \tag{5}$$

where $E=G/(1-\nu)$, G is the particle shear modulus, ν is the particle Poisson's ratio, ρ is the particle radius, χ_c is the contact adhesion force, and f_n is the contact force in the normal direction to the contact surface. When the tensile force at contact exceeds the contact adhesion, that is $f_n > -\chi_c$, the contact separates (i.e. $K_n=0$).

The contact stiffness in the tangential direction has been investigated by Mindlin and Deresiewicz (1953) for frictional contacts. For contacts with adhesion, the expression for tangential contact stiffness K_r are obtained by modifying those for frictional contacts (Chang et al. 1990). The tangential contact stiffness K_r under various loading condition is given by

$$K_r = C_2 K_n \left(\frac{1}{\theta} + \frac{\Delta f_n \tan\phi_\mu}{\Delta f_r}\left[\frac{1}{\theta} - 1\right]\right)^{-1} \tag{6}$$

where C_2 is a material constant, K_n is the normal contact stiffness given in Eq. 5, ϕ_μ is the inter-particle friction, f_r is the tangential force at the contact, Δ indicates incremental values, positive sign is invoked during unloading only and θ takes various value depending on first time loading, unloading or reloading given by

$$\theta^3 = 1 - \left(\frac{f_r}{2(\chi_c + f_n)\tan\phi_\mu} + \frac{\Delta f_n}{f_n}\right) \quad \text{(virgin loading)} \tag{7}$$

$$\theta^3 = 1 - \left(\frac{f_r^* - f_r}{2(\chi_c + f_n)\tan\phi_\mu} + \frac{\Delta f_n}{f_n}\right) \quad \text{(unloading)} \tag{8}$$

$$\theta^3 = 1 - \left(\frac{f_r^{**} - f_r}{2(\chi_c + f_n)\tan\phi_\mu} + \frac{\Delta f_n}{f_n}\right) \quad \text{(reloading)} \tag{9}$$

where f_r^* and f_r^{**} are the loading and unloading reversal points.

STRESS-STRAIN RELATIONSHIP FOR A PACKING

In what follows, we consider a given volume of the granular solid. This given volume is assumed to contain enough number of particles in order to be representative of the material behavior of the granular solid. The representative volume is conceptually equivalent to a point in a continuum material. Thus by defining the stress, strain and stiffness tensors for this representative volume we endeavor to homogenize the granular media. Note that the stress, strain and stiffness fields (which are referred to as local or micro field quantities) within the representative volume itself are highly heterogeneous. Therefore the stress, strain and stiffness tensors defined for the representative volume are chosen to be some 'self consistent' average of the local quantities.

The overall stress and strain, denoted by $\bar{\sigma}_{ij}$ and $\bar{e}_{ij}$ associated with this volume of granular solid are obtained as volume averages of the corresponding local quantities as

$$\bar{\sigma}_{ij} = \frac{1}{V} \sum_n V^n \sigma^n_{ij} \tag{10}$$

$$\bar{\varepsilon}_{ij} = \frac{1}{V} \sum_n V^n \varepsilon^n_{ij} \tag{11}$$

where σ^n_{ij} and ε^n_{ij} are the stresses and strains at a particle level and $V = \sum_n V^n$ and the summation is carried over all the particles in the volume. This definition of overall stress and strain has been shown to hold under both displacement or traction boundary conditions provided the displacements or tractions are compatible with a uniform overall strain or stress (Hill 1967).

The stress σ^n_{ij} for a particle in a volume of granular solid can be obtained in terms of the forces at the contacts f^{nm}_j as (Christoffersen et al. 1981)

$$\sigma^n_{ij} = \frac{1}{2V^n} \sum_m l^{nm}_i f^{nm}_j \tag{12}$$

where the superscript nm refers to the m-th contact of the n-th particle, V^n is the volume associated with the nth particle, and $l^{nm}_i = X^m_i - X^n_i$ is the branch vector joining the centroid of n-th particle with its neighbor at the m-th contact.

Under the assumption that the strain field in the immediate neighborhood of a particle is affine we can relate the relative displacement δ^{nm}_l at a contact to the strain ε^n_{kl} at a particle level by

$$\delta^{nm}_l = \varepsilon^n_{kl} \, l^{nm}_k \tag{13}$$

In Eq. 13, the strain tensor ε^n_{kl} is defined as

$$\varepsilon^n_{kl} = \phi_{l,k} = u^n_{l,k} + e_{lkm} \, \omega^n_m \tag{14}$$

where $u_{l,k}$ is the displacement gradient, ω_m is the rotation of the particles in the packing, and ϕ_l is a 'generalized' displacement function introduced for convenience.

The local stiffness tensor relating the stress-strain at a particle level is obtained as (from Eqs. 2, 12, and 13)

$$\sigma^n_{ij} = C^n_{ijkl} \, \varepsilon^n_{kl} \tag{15}$$

where

$$C^n_{ijkl} = \frac{1}{2V^n} \sum_m l^{nm}_i K^{nm}_{jl} l^{nm}_k \tag{16}$$

Note that the strain tensor defined in Eq. 15 is, in general, asymmetric. Therefore, the stiffness tensor C^n_{ijkl} is, in general, not symmetric with respect to interchange of the terminal pair of indices.

The aim of the subsequent discussion is to relate the overall stress and strain in order to derive an expression for the effective stiffness tensor, such that

$$\bar{\sigma}_{ij} = C_{ijkl} \, \bar{\varepsilon}_{kl} \tag{17}$$

where C_{ijkl} is the effective stiffness tensor of the homogenized equivalent continuum.

In view of Eq. 11 and 17, a connection between the overall strain $\bar{\varepsilon}_{ij}$ and the strain at a particle level ε^n_{ij} is sought. In this regard, it is postulated that the two strains are related through a, as yet unknown, 'concentration' tensor H^n_{mnkl} such that

$$\varepsilon^n_{mn} = H^n_{mnkl} \, \bar{\varepsilon}_{kl} \tag{18}$$

From Eq. 11, it can be seen that the volume averaging requires

$$I_{mnkl} = \frac{1}{V} \sum_n V^n H^n_{ijkl} \tag{19}$$

where I_{mnkl} is a fourth rank identity tensor.

Thus, the effective stiffness tensor C_{ijkl} can be written in terms of the particle stiffness tensor C^n_{ijkl} as (from Eqs. 10, 11, 17 and 18)

$$C_{ijkl} = \frac{1}{V} \sum_n V^n C^n_{ijmn} H^n_{mnkl} \tag{20}$$

It now remains to obtain the unknown 'concentration' tensor H^n_{mnkl} to completely define the stiffness tensor C_{ijkl} in terms of known quantities. In order to obtain the unknown 'concentration' tensor H^n_{mnkl}, the particle is considered to be an inhomogeneity with moduli C^n_{ijkl} embedded in an infinite medium of moduli C_{ijkl} (see Misra 1990). Thus, Eqs. 16, 17 and 20 define the stress-strain relationship for a packing.

If the 'concentration' tensor is assumed to be identity, that is the strain field in the material is assumed to be homogeneous, which is reasonable assumption at small strain levels (Chang and Misra 1990), the overall stiffness tensor is written as

$$C_{ijkl} = \frac{1}{V} \sum_n V^n C^n_{ijkl} \tag{21}$$

and for a large number of contacts in the assembly Eq. 21 can be converted into an integral form (see Chang et al. 1990) written as (using Eq. 16)

$$C_{ijkl} = \frac{N}{V} \int_{\Omega} l_i \, K_{kl} \, l_k \, \xi(\Omega) \, d\Omega \qquad (22)$$

where N is the total number of contacts in the volume V (each contact counted twice), and $d\Omega = \sin\gamma \, d\gamma \, d\beta$ with the integration limits of $0 \le \gamma \le \pi$ and $0 \le \beta \le 2\pi$ (Chang et al. 1990).

Considering an isotropic packing structure of equal particle size such that

$$\xi(\Omega) = \frac{1}{4\pi} \qquad (23)$$

and replacing for $\frac{N}{V}$ by

$$\frac{N}{V} = \frac{3\bar{n}}{4\pi\rho^3(1+e)} \qquad (24)$$

where $\bar{n}$ is the coordination number, ρ is the particle radius, and e is the void ratio, the initial shear modulus of the packing is derived to be

$$G_{max} = \frac{4.8E^2 a^4 \bar{n}(2+3C_2)(\chi_c + f_n)}{\pi\rho(1+e)(40E\chi_c a^3 + 32Ef_n a^3 - 3\rho f_n^2)} \qquad (25)$$

where the normal force f_n is given as (Chang et al. 1990)

$$f_n = (4\pi\rho^2(1+e)/\bar{n})\sigma_o \qquad (26)$$

The derived expression for the shear modulus captures the effect of confining stress σ_o, contact adhesion χ_c, particle stiffness E and C_2, and the packing stucture characterised by the particle radius ρ, void ratio of the packing e and the coordination number $\bar{n}$.

CONCLUSIONS

A mathematical model is presented for modelling behavior of cemented sands from a micro-mechanical point of view. The model accounts for particle properties, contact properties (friction and adhesion), and packing properties. Based on the model an expression is presented for the shear modulus of isotropic cemented sands.

REFERENCES

Chang, C.S., Sundaram, S.S., and Misra, A., "Initial moduli of particulated mass with frictional contacts," International Journal for Numerical and Analytical Methods in Geomechanics, 1989, 13(6), 629-644.

Chang, C.S., "Micromechanical modelling of constitutive relations for granular material," Micromechanics of Granular Materials, Eds. M. Satake and J.T. Jenkins, Elsevier, Amsterdam, 1987, 271-278.

Chang, C.S., Misra, A. and Sundaram, S.S., "Micromechanical modelling of cemented sands under low amplitude oscillations," Geotechnique, 1990, 40(2), 251-263.

Chang, C.S., and Misra, A., "Application of uniform strain theory to heterogeneous granular solids," J. of Engr. Mech., ASCE, 1990, 116(10), 2310-2328.

Hill, R., "The essential structure of constitutive laws for metal composites and polycrystals," J. Mech. Phy. Solids, 1967, 15(2), 79-95.

Misra, A., "Constitutive relationships for granular solids with particle slidings and fabric changes," Ph.D. dissertation, University of Massachusetts at Amherst, 1990