On Surface Waves in a Gibson Half-Space

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SYNOPSIS Harmonic Rayleigh-type and transverse surface waves in a half-space of incompressible material with constant density and with shear modulus linearly increasing with depth (Gibson half-space) are discussed. Under certain hypotheses a discrete spectrum yielding polynomial eigenfunctions is obtained, a fact which makes the eigenvalue problem more tractable. The dispersion laws are presented and evaluated numerically.

INTRODUCTION

Surface waves in a nonhomogeneous half-space were first investigated by Meissner (1921), who analyzed transverse seismic surface waves for quadratic and linear variations of the shear modulus and for density varying linearly with depth. Stoneley (1934) analyzed Rayleigh-type waves in an incompressible half-space of constant density and with shear modulus depending linearly on depth. Stoneley's solution is given in terms of Whittaker functions. As it is recognized by Meissner, under certain hypotheses, there exists a discrete spectrum with associated polynomial eigenfunctions related to the Laguerre polynomials. As it will be shown elsewhere these polynomial solutions are the only physically meaningfull ones (layer effect). The polynomial formulation followed here makes the eigenvalue problem and the formulation of the dispersion law more tractable. The present analysis is mainly based on a recent paper by Vardoulakis (1981).

HARMONIC WAVES

We shall consider the propagation of a plane harmonic surface wave through an incompressible Gibson half-space. The boundary of the half-space coincides with the xy-plane, with z positive towards the interior of the half-space. The surface wave is travelling through the half-space in the x-direction.

In a Gibson half-space the shear modulus μ is assumed to be a linear function of the depth coordinate:

$$\mu = \mu_o + \overset{*}{\mu}\gamma z \ , \tag{1}$$

where μ_o is the shear modulus at the surface, $\overset{*}{\mu}$ is a dimensionless shear modulus and γ is the unit weight of the considered body.

Let $2\pi L$ be the wave length and ω the circular frequency. The dimensionless depth coordinate ξ and the phase angle ϕ are defined by the following equations.

$$\xi := \xi_o + \frac{z}{L};\quad \xi_o := \frac{1}{\overset{*}{\mu}}\frac{\mu_o}{\gamma L};\quad \phi := \frac{x}{L} - \omega t. \tag{2}$$

The displacement field and the mean pressure increment considered here are of the form:

$$u = \hat{u}\,(\xi)\sin\phi,\quad v = \hat{v}\,(\xi)\sin\phi,\quad w = \hat{w}\,(\xi)\cos\phi, \tag{3}$$

$$\Delta p = \hat{p}\,(\xi)\cos\phi. \tag{4}$$

Vardoulakis (1981) has recently shown that *Rayleigh-waves* are described by the following displacement field:

$$\hat{u} = -\hat{w}';\quad \hat{v} = 0;\quad \hat{w} = \{C_1 + C_2\,\xi W\}e^{-\xi} \tag{5}$$

where $(\cdot)' \equiv d/d\xi$, C_1, C_2 are constants and $W(\xi)$ satisfies the following differential equation

$$\xi W'' + 2(1 - \xi)W' - (2 - \alpha)W = 0. \tag{6}$$

α is a dimensionless acceleration defined by the relation

$$\alpha := \frac{1}{\overset{*}{\mu}}\frac{\omega^2 L}{g} \tag{7}$$

and g the acceleration of gravity.

Transverse waves are described by the following displacement field (Vardoulakis, 1981):

$$\hat{u} = \hat{w} = 0;\quad \hat{v} = C_3 V e^{-\xi}\ , \tag{8}$$

where C_3 is a constant and $V(\xi)$ satisfies the following differential equation:

$$\xi V'' + (1 - 2\xi)V' - (1 - \alpha)V = 0\ . \tag{9}$$

A simple change of variables $\zeta = 2\xi$ shows that the equations (6) and (9) are special cases of the confluent hypergeometric equation (cf. Erdélyi et al., Vol. 1, 1953) which, depending on α, possesses in general non-polynomial solutions. However, as will be shown elsewhere, the assumption that the amplitudes $\hat{u}(\xi)$, $\hat{v}(\xi)$, $\hat{w}(\xi)$ be square-integrable on $[0,\infty)$ (a fact which corresponds to finite elastic energy of the traveling wave front), leads to *polynomial eigenfunctions* corresponding to integer values of α in (6) and (9) (layer effect). Specifically, it is easy to see that if $\alpha = 2n$ and $\alpha = 2n-1$ ($n = 1, 2, \ldots$) the solutions of (6) and (9), respectively, are Laguerre polynomials of degree $n - 1$ (cf. Erdélyi et al., Vol. 2, 1953):

$$W_n(\xi) = L^1_{n-1}(2\xi) \ ; \quad V_n(\xi) = L^0_{n-1}(2\xi) \quad . \tag{10}$$

BOUNDARY CONDITIONS

The boundary conditions indicating that the half-space surface is stress free can be expressed in terms of the considered displacement amplitudes $\hat{v}$ and $\hat{w}$ (see $(5)_1$ and Vardoulakis, 1981). These boundary conditions are as follows:

$$\mu_o(w''+w) = 0 \ ; \qquad \xi_o\hat{w}''' + \hat{w}'' - (3\xi_o - \alpha)\hat{w}' + \hat{w} = 0 \ ; \tag{11}$$

$$\mu_o v' = 0 \ , \tag{12}$$

at $\xi = \xi_o$. Note that regular behavior is assumed; i.e., $\mu_o \neq 0$ and consequently $\xi_o \neq 0$. For the representations $(5)_3$ and $(10)_1$ of the vertical displacement amplitude the boundary conditions (11) for Rayleigh-type surface waves yield the following characterisitc equation:

$$2\xi_o L^0_{n-1}(2\xi_o) + \frac{n(n-1-2\xi_o)}{\xi_o}[L^0_{n-1}(2\xi_o) - L^0_n(2\xi_o)] = 0 \ . \tag{13}$$

which is a polynomial equation of degree n in ξ_o. For transverse surface waves equations (12), $(8)_2$ and $(10)_2$ yield the characteristic equation

$$L^0_{n-1}(2\xi_o) - \frac{(n-1)}{\xi_o}[L^0_{n-1}(2\xi_o) - L^0_{n-2}(2\xi_o)] = 0 \ , \tag{14}$$

a polynomial equation of degree n-1. As will be shown elsewhere (13) and (14) have all their roots positive and distinct.

DISPERSION LAW AND NUMERICAL RESULTS

The roots of the characteristic equations (13) and (14) are denoted by ξ^i_{on}. As it is shown in Vardoulakis (1981) if the roots ξ^i_{on} are known then the relation between the dimensionless wave propagation velocity and the dimensionless wave length can be easily deduced. This relation will be the required dispersion law for the corresponding surface-wave modes. In (Vardoulakis, 1981) the first two pairs of the characteristic relations (13), (14) were solved numerically. To extend these results and obtain more information about the dispersion law we found numerically the roots of (13) for $2 \leq n \leq 30$ and the roots of (14) for $3 \leq n \leq 31$. We generated the coefficients and the values of the Laguerre polynomials in (13), (14) using the well-known recursion relations in n that these polynomials satisfy. We then found the roots by using the root-finding routine ZPOLR of the ISML Subroutine Library, based on the Laguerre method for finding the roots of a real polynomial. Treating the values found by ZPOLR as first approximations we then refined the roots using Newton's method. All computations were done in double precision on the University of Tennessee IBM 370/3031 computer with code compiled by the FORTRAN G level compiler. To check our computations, since the problem of finding the roots of (13) or (14) is of comparable difficulty to that of finding the roots of Laguerre polynomials $L^0_n(\zeta)$, we also compared the values of the roots of $L^0_n(\zeta)$ for $2 \leq n \leq 32$ found by our computer program to the known tabulated values of these roots (cf. Rabinowitz and Weiss, 1959), and found ten-decimal places agreement for all tabulated roots, consistent with our stopping criterion for the Newton iterations in our program. Fig. 1, resp. Fig. 2, show the largest four roots $\xi^4_{on} < \xi^3_{on} < \xi^2_{on} < \xi^1_{on}$, of (13), resp. (14), as functions of n. We also computed the least squares straight line fits to the largest two roots as functions of n, using the values corresponding to $5 \leq n \leq 30$ for (13) and $6 \leq n \leq 31$ for (14). With three decimal places accuracy we found:

$$\text{for (13):} \quad \xi^1_{on} \simeq 2.183n - 2.575 \ ; \quad \xi^2_{on} \simeq 1.794n - 5.879 \tag{15}$$

$$\text{for (14):} \quad \xi^1_{on} \simeq 1.934n - 3.087 \ ; \quad \xi^2_{on} \simeq 1.772n - 6.597 \ . \tag{16}$$

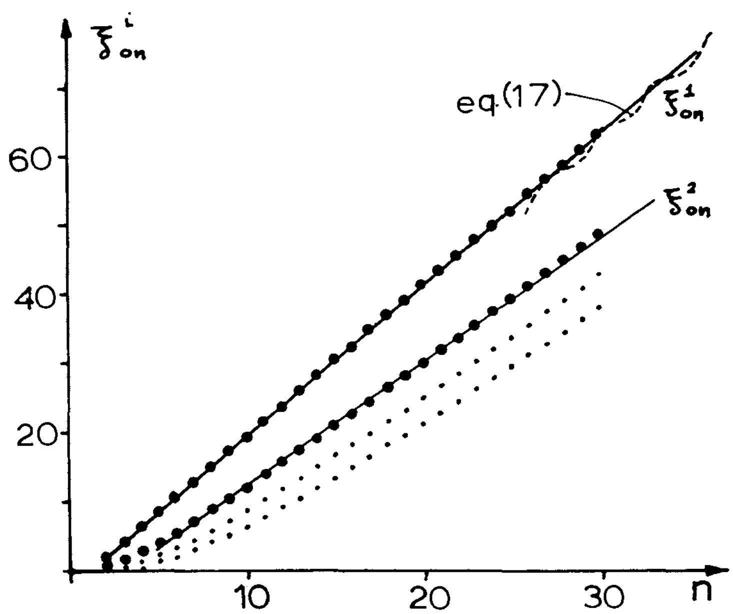

Fig. 1 Dispersion law for Rayleigh-type waves

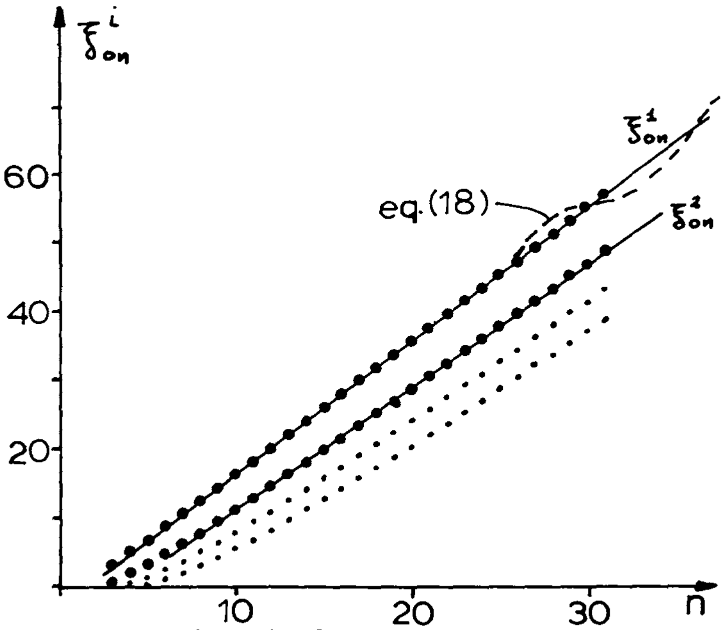

Fig. 2 Dispersion law for transverse waves

We also observed that for $2 \leq n \leq 30$, the largest root ξ^1_{on} of (13) is allways larger than the largest root of the corresponding $W_n(\xi)$. The same is true in the case of (14) with $V_n(\xi)$ replacing $W_n(\xi)$. Hence, if $\xi_{on} = \max \xi^j_{on}$ since $z = (\xi - \xi_{on})L$, the wave mode corresponding to ξ_{on} is node-free on the positive z-axis for both the cases of Rayleigh-type and transverse surface waves.

As a note of computational interest we note that the convenient asymptotic expression (1) in (Erdélyi et al., Vol. 2, 1953, p. 199) for $L^m_n(\zeta)$, valid for large n and positive ζ in a fixed internal [a, b], if substituted in (13), resp. (14), yields transcedental equations with an infinite number of roots for each n:

$$(13): \frac{\sin\psi_{n-1}}{\cos\psi_{n-2}} = \sqrt[4]{\frac{4(n-2)^3}{n-1}}\,\frac{\xi\sqrt{\xi}}{(2n-1)\xi - n(n-1)}\ , \qquad (17)$$

$$(14): \frac{\sin\psi_{n-2}}{\cos\psi_{n-1}} = -\frac{\sqrt{\xi}}{\sqrt[4]{4(n-1)(n-2)}}\ , \qquad (18)$$

where

$$\psi_n = \sqrt{8n\xi} - \pi/4 \quad . \qquad (19)$$

By taking in both cases the linear least-squares estimate for the largest root as an initial approximation for an iterative scheme to solve these transcedental equations we found that the iterations converged to values that were in fairly good agreement with the exact values of the largest roots (see Fig. 1 and Fig. 2). Of course, we do not recommend this approach for very large n since the asymptotic expression cited above is valid for large n and bounded x; we anticipate that the largest root will vary linearly with n. The last fact along with a more accurate asymptotic description of the roots will be given elsewhere.

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