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# Longitudinal Waves in a Rod in an Elastic Medium

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**SYNOPSIS** The general equations for the propagation of longitudinal waves in an infinite cylindrical rod including the effect of a surrounding elastic medium have been derived in order to obtain the relationship between the circular frequency and the wave number. A comparison has then been made with the Smith model in order to investigate the limits of the model and to refine the expressions for the spring and damping constants.

## INTRODUCTION

The scope of this study has been to derive the general equations for the propagation of longitudinal waves in an infinite cylindrical rod including the effect of a surrounding elastic medium. The relationship between the circular frequency and the wave number has been obtained. This relationship can be used to give a rheologic model of the soil to be used in pile analysis. It can also be used to understand the limits of the Smith model.

## WAVES IN A ROD WITH A SURROUNDING MEDIUM

A rod with radius,  $a$ , in an infinite elastic medium is subjected to a propagating wave. See figure 1. In order to solve the wave equations for the longitudinal elastic stress waves, expressed in the potential functions  $\phi$  and  $\bar{H}$  in cylindrical coordinates, the following expressions are introduced;

$$\phi(r,z,t) = f(r) e^{i(kz - \omega t)}, \quad H_{\theta}(r,z,t) = h(r) e^{i(kz - \omega t)} \quad (1)-(2)$$

where  $k$  is the wave number in the axial direction in the rod and  $\omega$  the angular frequency.

After substitution the wave equations then appear like;

$$\frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} + \left( \frac{\omega^2}{c_p^2} - k^2 \right) f = 0 \quad (3)$$

$$\frac{d^2 h}{dr^2} + \frac{1}{r} \frac{dh}{dr} + \left( \frac{\omega^2}{c_s^2} - k^2 - \frac{1}{r^2} \right) h = 0 \quad (4)$$

where  $c_p$  and  $c_s$  are the velocity of propagation of the P-wave and S-wave, respectively.

Equation (3) and (4) are the Bessel differential equations with the following solutions,

$$f(r) = A J_0(lr) + B Y_0(lr); \quad l^2 = \frac{\omega^2}{c_p^2} - k^2 \quad (5)$$

$$h(r) = C J_1(mr) + D Y_1(mr); \quad m^2 = \frac{\omega^2}{c_s^2} - k^2 \quad (6)$$

where,  $A$ ,  $B$ ,  $C$  and  $D$  are constants.  $J_0$  and  $J_1$  are Bessels functions of the first kind and of the zero and first order, respectively, and  $Y_0$  and  $Y_1$  Bessel functions of the second kind (also called Weber's functions) of zero and first order.  $l$  and  $m$  are the wave number in the radial direction, in the rod and the surrounding medium, respectively.

Substituting equation (5) and (6) into (1) and (2) gives the following expression for the displacements;

$$u_r = \begin{bmatrix} -A l J_1(lr) - B l Y_1(lr) - \\ -C i k J_1(mr) - D i k Y_1(mr) \end{bmatrix} \quad (7)$$

$$u_z = \begin{bmatrix} A i k J_0(lr) + B i k Y_0(lr) + \\ + C m J_0(mr) + D m Y_0(mr) \end{bmatrix} \quad (8)$$

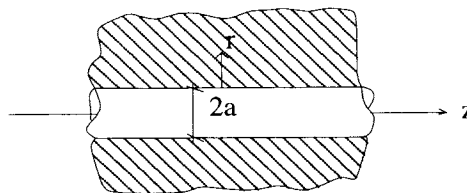


Figure 1. A rod with a surrounding medium.

With the use of Hooke's law in three dimensions, the normal and shear stresses can be expressed as;

$$\sigma_{rr} = 2G \left[ \begin{aligned} & \left[ -\frac{1}{2}(m^2 - k^2) J_0(1r) + \frac{1}{r} J_1(1r) \right] A + \\ & + \left[ -\frac{1}{2}(m^2 - k^2) Y_0(1r) + \frac{1}{r} Y_1(1r) \right] B + \\ & + i \left[ -k m J_0(mr) + \frac{k}{r} J_1(mr) \right] C + \\ & + i \left[ -k m Y_0(mr) + \frac{k}{r} Y_1(mr) \right] D \end{aligned} \right] \quad (9)$$

$$\tau_{rz} = 2G \left[ \begin{aligned} & -2ikl J_1(1r) A - 2ikl Y_1(1r) B + \\ & + (k^2 - m^2) J_1(mr) C + (k^2 - m^2) Y_1(mr) D \end{aligned} \right] \quad (10)$$

Equations (9) and (10) hold for both the rod and the surrounding medium, i.e. totally four expressions,  $\sigma_{rr1}$ ,  $\sigma_{rr2}$ ,  $\tau_{rz1}$  and  $\tau_{rz2}$ . (The entities with subscripts 1 and 2 refer to the rod and the surrounding medium, respectively.)

The constants A and C are in combinations of Bessel functions  $J_0$  and  $J_1$  and the constants B and D combinations of Weber functions,  $Y_0$  and  $Y_1$ . As the origo is set within the rod,  $B_1$  and  $D_1$  must be zero as  $Y_0$  and  $Y_1$  go to infinity. The expressions for the surrounding medium contain the whole set of constants,  $A_2$ ,  $B_2$ ,  $C_2$ ,  $D_2$ , i.e. there are totally six unknown constants, which means that six boundary conditions will be needed for the sytem of equations to be solved.

The boundary conditions, valid for this case, are the following,

$$\sigma_{rr1}(r=a) = \sigma_{rr2}(r=a) , \quad \tau_{rz1}(r=a) = \tau_{rz2}(r=a) \quad (11)-(12)$$

$$u_{r1}(r=a) = u_{r2}(r=a) , \quad u_{z1}(r=a) = u_{z2}(r=a) \quad (13)-(14)$$

Finally, the radiation condition, (only outgoing waves in infinity)

$$\begin{aligned} u_r(r \rightarrow \infty, t) &\approx e^{i\left(\frac{1}{m}r - \omega t\right)} \text{ where } \frac{1}{m} > 0 \\ u_z(r \rightarrow \infty, t) &\approx e^{i\left(\frac{1}{m}r - \omega t\right)} \end{aligned} \quad (15)$$

For the solution the Hankel function needs to be introduced;

$$H_n(r) = J_n(r) + i Y_n(r), \quad n = 0, 1 \quad (16)$$

which corresponds to a Bessel function of the third order.

The expression for stresses and displacements can be written as;

$$\sigma_{rr} = G \left[ \begin{aligned} & \left[ -\frac{1}{2}(m^2 - k^2) H_0^{(1)}(1r) + \frac{1}{r} H_1^{(1)}(r) \right] \mathcal{A} + \\ & + \left[ -\frac{1}{2}(m^2 - k^2) H_0^{(2)}(1r) + \frac{1}{r} H_1^{(2)}(r) \right] \mathcal{B} + \\ & + i \left[ -k m H_0^{(1)}(mr) + \frac{k}{r} H_1^{(1)}(mr) \right] \mathcal{C} + \\ & + i \left[ -k m H_0^{(2)}(mr) + \frac{k}{r} H_1^{(2)}(mr) \right] \mathcal{D} \end{aligned} \right] \quad (17)$$

$$\tau_{rz} = G \left[ \begin{aligned} & -2ikl \left[ H_1^{(1)}(1r) \mathcal{A} + H_1^{(2)}(1r) \mathcal{B} \right] + \\ & + (k^2 - m^2) \left[ H_1^{(1)}(mr) \mathcal{C} + H_1^{(2)}(mr) \mathcal{D} \right] \end{aligned} \right] \quad (18)$$

$$u_r = \frac{1}{2} \left[ \begin{aligned} & -1 H_1^{(1)}(1r) \mathcal{A} - 1 H_1^{(2)}(1r) \mathcal{B} - \\ & - i k H_1^{(1)}(mr) \mathcal{C} - i k H_1^{(2)}(mr) \mathcal{D} \end{aligned} \right] \quad (19)$$

$$u_z = \frac{1}{2} \left[ \begin{aligned} & i k H_0^{(1)}(1r) \mathcal{A} + i k H_0^{(2)}(1r) \mathcal{B} + \\ & + m H_0^{(1)}(mr) \mathcal{C} + m H_0^{(2)}(mr) \mathcal{D} \end{aligned} \right] \quad (20)$$

where

$$\mathcal{A} = A + \frac{B}{i} = A - iB , \quad \mathcal{B} = A - \frac{B}{i} = A + iB \quad (21)-(22)$$

$$\mathcal{C} = C + \frac{D}{i} = C - iD , \quad \mathcal{D} = C - \frac{D}{i} = C + iD \quad (23)-(24)$$

In the rod,  $B_1 = D_1 = 0$ , as explained above, i.e.  $A_1 = \mathcal{A}_1 = \mathcal{B}_1$  and  $C_1 = \mathcal{C}_1 = \mathcal{D}_1$ . The radiation condition (15) means that  $\mathcal{B}_2 = \mathcal{D}_2 = 0$  as

$$H_n^{(2)}\left(\frac{1}{m}r\right) \rightarrow \sqrt{\frac{2}{\pi r}} e^{i\left(\frac{1}{m}r + \frac{\pi}{4} + \frac{n\pi}{2}\right)} \text{ where } r \rightarrow \infty$$

Consequently  $B_2 = i A_2$  and  $D_2 = i C_2$ .

The boundary conditions (11)-(15) can be written as a homogeneous sytem of linear equations. For the system to be solvable the determinant of the matrix must be zero.

By introducing the following notations,

$$\gamma = \frac{G_1}{G_2}, \quad \kappa = k a \quad (25)-(26)$$

$$\lambda_i = l_i a, \quad \mu_i = m_i a \quad i = 1, 2 \quad (27)-(28)$$

$$(\kappa^2 + \lambda_i^2) = \left(\frac{\omega a}{c_{pi}}\right)^2 \quad i = 1, 2 \quad (29)$$

$$(\kappa^2 + \mu_i^2) = \left(\frac{\omega a}{c_{si}}\right)^2 \quad i = 1, 2 \quad (30)$$

$$H_i \equiv H_i^{(1)} = J_i + i J_i \quad i = 1, 2 \quad (31)$$

where  $\gamma$  is the stiffness ratio,  $G_1$  and  $G_2$  the shear modulus of the rod and the surrounding medium, respectively,  $\kappa$  the dimensionless wave number in the axiell direction of the rod,  $\lambda$  and  $\mu$  dimensionless radial wave numbers the following equation is obtained;

$$\det \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{bmatrix} = 0 \quad (32)$$

where  $a_{1,1} = \gamma \left[ \frac{1}{2}(\kappa^2 - \mu_1^2) J_0(\lambda_1) + \lambda_1 J_1(\lambda_1) \right]$   $a_{3,1} = -\lambda_1 J_1(\lambda_1)$   
 $a_{1,2} = i\gamma \left[ -\kappa \mu_1 J_0(\mu_1) + \kappa J_1(\mu_1) \right]$   $a_{3,2} = -i\kappa J_1(\mu_1)$   
 $a_{1,3} = \frac{1}{2}(\kappa^2 - \mu_2^2) H_0(\lambda_2) + \lambda_2 H_1(\lambda_2)$   $a_{3,3} = -\lambda_2 H_1(\lambda_2)$   
 $a_{1,4} = i \left[ -\kappa \mu_2 H_0(\mu_2) + \kappa H_1(\mu_2) \right]$   $a_{3,4} = -i\kappa H_1(\mu_2)$   
 $a_{2,1} = -2i\gamma\kappa\lambda_1 J_1(\lambda_1)$   $a_{4,1} = i\kappa J_0(\lambda_1)$   
 $a_{2,2} = \gamma(\kappa^2 - \mu_1^2) J_1(\mu_1)$   $a_{4,2} = \mu_1 J_0(\mu_1)$   
 $a_{2,3} = -2i\kappa\lambda_2 H_1(\lambda_2)$   $a_{4,3} = i\kappa H_0(\lambda_2)$   
 $a_{2,4} = (\kappa^2 - \mu_2^2) H_1(\mu_2)$   $a_{4,4} = \mu_2 H_0(\mu_2)$

The solution of this equation gives the relationship between the wave number,  $k$ , and the angular frequency,  $\omega$ , or rewritten,  $\kappa$  (which is a complex function) as a function of the dimensionless angular frequency,  $\Omega$ . These relationships plotted, see figure 2-5, give the values of the parameters for a homogeneous elastic soil medium as a function of the angular frequency.

————	$\Sigma^2 = 8.75, \gamma = 100$
-----	$\Sigma^2 = 17.5, \gamma = 400$
- - - - -	$\Sigma^2 = 17.5^2, \gamma = 1600$
.....	$\Sigma^2 = 17.5^4, \gamma = 6400$

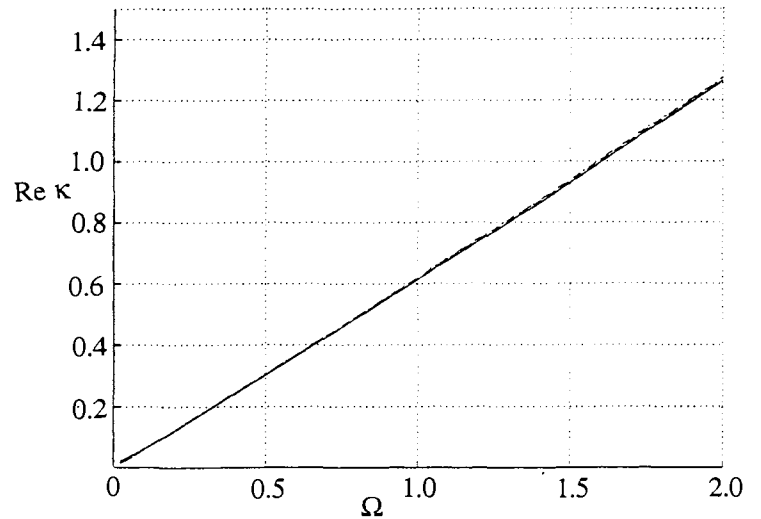


Figure 2. Relationship between  $\Omega$  and  $Re \kappa$ .

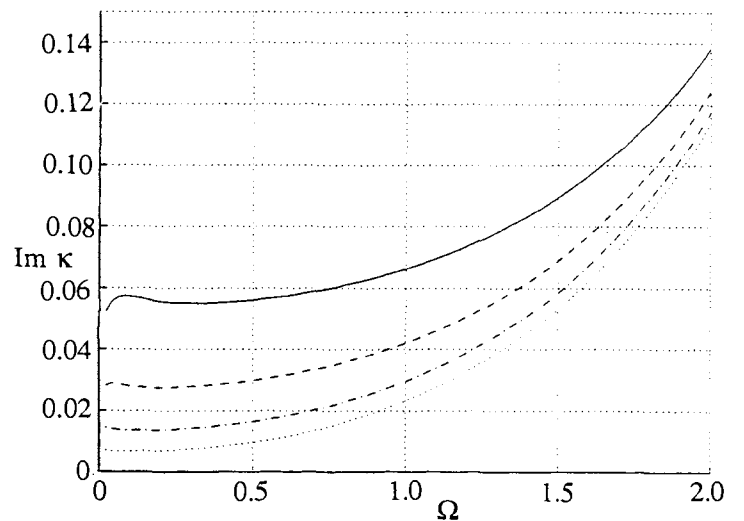


Figure 3. Relationship between  $\Omega$  and  $Im \kappa$ .

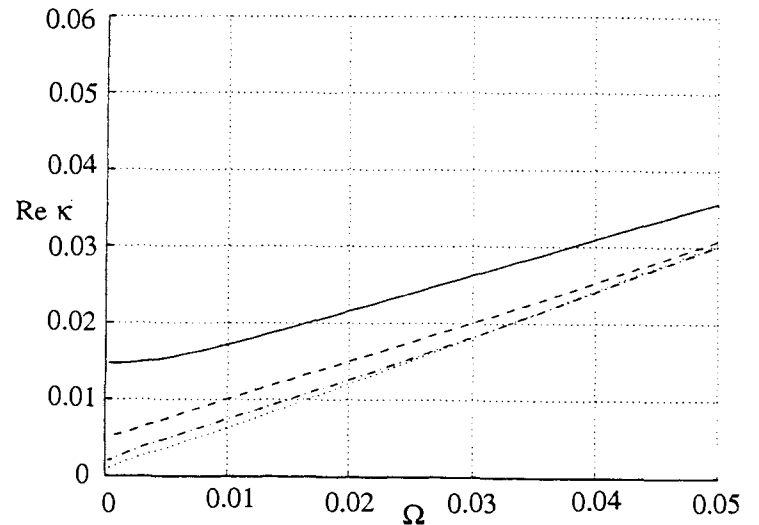


Figure 4. Relationship between  $\Omega$  and  $Re \kappa$ .

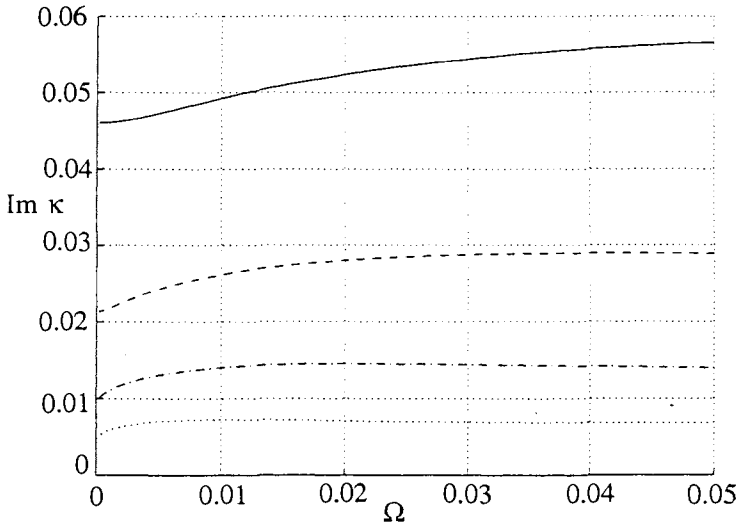


Figure 5. Relationship between  $\Omega$  and  $Im \kappa$ .

### SMITH'S SOIL MODEL

Having derived the general equation for the propagation of longitudinal waves in a cylindrical rod including the effect of a surrounding elastic medium, we now want to compare this relationship between the excited circular frequency and the wave number with the one derived from the Smith soil model.

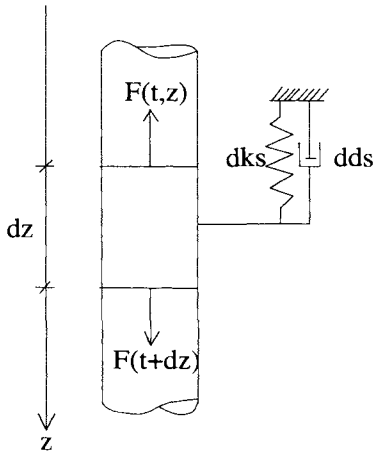


Figure 6. Smith's model. Spring stiffness and damping coefficient per unit of length.

$$dk_s = k_s dz, \quad dd_s = d_s dz \quad (33)-(34)$$

where  $k_s$  is the spring constant per unit of length and  $d_s$  the damping coefficient per unit of length.

The equation of motion of the rod when it has been disturbed through a distance  $w$ , in the  $z$ -direction, can be written from Newton's second law of motion;

$$\ddot{w} + \beta \dot{w} + \alpha^2 w = \frac{d^2 w}{dz^2} c_b^2 \quad (35)$$

Where  $\alpha$  is the root of the spring constant,  $\beta$  the damping coefficient and  $c_b$  the velocity of the wave propagation in a cylindrical bar.

$$\alpha = \sqrt{\frac{k_s}{A \rho}}, \quad \beta = \frac{d_s}{A \rho}, \quad c_b = \sqrt{\frac{E}{\rho}} \quad (36)-(38)$$

Where  $A$  is the cross-sectional area of the cylindrical rod,  $\rho$  the density of the rod and  $E$ , Young's modulus.

Suppose that,

$$w = \hat{w} e^{i(k_s z - \omega t)} \quad (39)$$

Derivations of equation (39) and substitution into equation (35) gives us the following general relation,

$$\omega^2 - \alpha^2 + i \omega \beta = k_s^2 c_b^2 \quad (40)$$

As  $\omega$  is real, excited by a vibrator,  $k_s$  must be imaginary.

A dimensionless expression of the wave number as a function of the frequency is

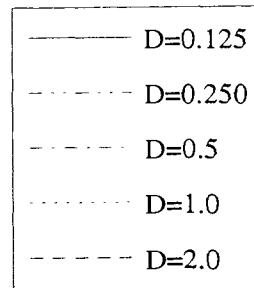
$$\kappa_s = \pm \left( \frac{\Omega_s^2 - 1^2 + \left( (\Omega_s^2 - 1^2)^2 + (2\Omega_s D)^2 \right)^{0.5}}{2} \right)^{0.5} + i \left( \frac{-\Omega_s^2 + 1^2 + \left( (\Omega_s^2 - 1^2)^2 + (2\Omega_s D)^2 \right)^{0.5}}{2} \right)^{0.5} \quad (41)$$

where

$$\kappa = k_s c_b, \quad \kappa_s = \frac{\kappa}{\alpha}, \quad \Omega_s = \frac{\omega}{\alpha}, \quad D = \frac{\beta}{2\alpha} \quad (42)-(45)$$

Where  $\kappa_s$  is the normalized wave number,  $\Omega_s$  the dimensionless angular frequency in the Smith model, and  $D$  the damping ratio.

Expression (41) plotted, for different values of the damping ratio,  $D$ , gives us the curves in figure 7 and 8.



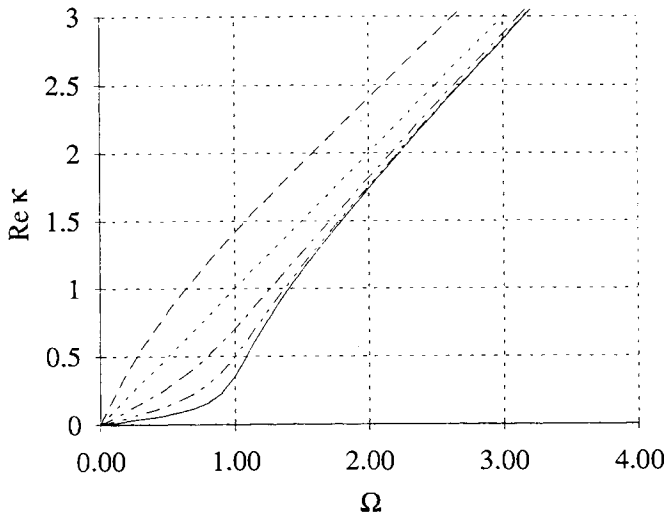


Figure 7. Relationship between  $\Omega$  and  $Re \kappa$  according to Smith's model.

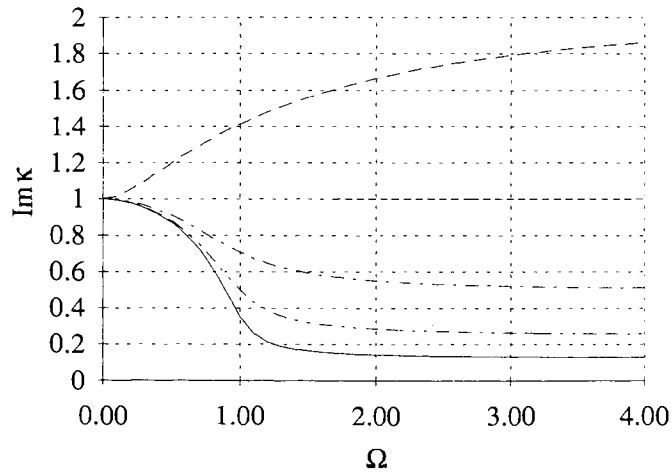


Figure 8. Relationship between  $\Omega$  and  $Im \kappa$  according to Smith's model.

## SUMMARY

When comparing the Smith model with the derived general case, regarding the relationship between  $\Omega$  and  $Re \kappa$ , see figure 2 and 7, we find that the Smith model well corresponds to the general case, when a damping ratio,  $D=1$ , is chosen. However, when the area around origo is enlarged, see figure 4, (which, concerning piling activities, is the area of interest), something unexpected is revealed, namely that the curves in the general case do not pass origo.

The correspondence between the Smith model and the derived general case, regarding the relationship between  $\Omega$  and  $Im \kappa$ , see figure 3 and 8, is on the other hand very rough, but for small values of  $\Omega$ , see figure 5, the correspondence is good.

To improve the expression for the damping effect in the Smith model, let us compare figure 5 with the curve,  $D=1$  in figure 8, to find a relationship between the imaginary

part of  $\kappa$  and  $\gamma = G_1/G_2$ . This leads us to the conclusion that when the Smith model is used, the following values of the spring and damping constants should be chosen;

$$k_S = 1.184 (1 + \nu_1)^{0.252} E_1^{0.748} G_2^{0.252} \quad (46)$$

$$d_S = 2\sqrt{\rho_1 A_1} \sqrt{1.184 (1 + \nu_1)^{0.252} E_1^{0.748} G_2^{0.252}} \quad (47)$$

where  $\nu_1$  is Poisson's ratio of the rod.

Table 1. A numerical example.

Pile material	$\rho_1$ [t/m <sup>3</sup> ]	$E_1$ [GPa]	$\nu_1$ [-]	$G_2^*$ [MPa]	$k_S$ [kg/s <sup>2</sup> m]	$d_S$ [kg/sm]
Steel	7.0-7.8	210	0.3	3.5-54	(16.6-33.1)E9	(8.78-13.09)E6
Concrete	2.4	30	0.2	3.5-54	(3.80-7.56)E9	(2.46-3.47)E6

\* Range from saturated clay to saturated till.

## CONCLUSION

When having compared the general equations for the propagation of longitudinal waves in a cylindrical rod including the effect of a surrounding elastic medium with the equations in the Smith model, we have found that the expression for the damping coefficient is very rough. In order to get a better correspondence to the general equations we have proposed a new approximate expression for the spring and damping constants in the Smith model, as functions of the shear modulus of the soil and the modulus of elasticity of the pile, see equation (46) and (47).

## SYMBOLS

### Roman letters

- A= cross-sectional area
- A, B, C, D,  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $\mathcal{C}$ ,  $\mathcal{D}$  = constants
- $c_b$  = velocity of the wave propagation in a cylindrical rod
- $c_p$  = velocity of propagation of the P-wave
- $c_s$  = velocity of propagation of the S-wave
- D = damping ratio
- $d_S$  = damping coefficient per unit of length, Smith model
- E = Young's modulus
- $f(r)$  = solution of the Bessel function of the zero order
- G = shear modulus
- $\bar{H}$  = potential for the S-wave
- $H_n$  = Hankel function of the n order
- $h(r)$  = solution of the Bessel function of the first order
- $J_n$  = Bessel function of the first kind and n order
- k = scale factor
- k = wave number in the axiell direction of the rod
- $k_S$  = spring constant per unit of length, Smith model
- l = wave number in the radiell direction of the rod
- m = wave number in the radiell direction of the rod

$u_r$  = displacement in the radial direction  
 $u_z$  = displacement in the axial direction  
 $r$  = distance along radial axis  
 $t$  = time  
 $w$  = displacement  
 $Y_n$  = Bessel function of the second kind and order  $n$   
 $z$  = distance along  $Z$ -axis

#### Greek letters

$\alpha$  = the root of the spring constant in the Smith model  
 $\beta$  = damping coefficient in the Smith model  
 $\gamma$  = stiffness ratio  
 $\theta$  = spherical coordinate  
 $\kappa$  = dimensionless wave number in the axial direction of the rod  
 $\kappa_S$  = normalized wave number, Smith model  
 $\lambda$  = dimensionless radial wave number  
 $\mu$  = dimensionless radial wave number  
 $\nu$  = Poisson's ratio  
 $\rho$  = density  
 $\Sigma$  = constant  
 $\sigma_{rr}$  = normal stress in the  $r$ -direction  
 $\tau_{rz}$  = shear stress in the  $z$ - $r$  plane  
 $\phi$  = potential for the P-wave  
 $\Omega$  = dimensionless angular frequency  
 $\Omega_S$  = dimensionless angular frequency, Smith model  
 $\omega$  = angular frequency

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