1972

Statistical studies of various time-to-fail distributions

James Addison Eastman

STATISTICAL STUDIES OF VARIOUS TIME-TO-FAIL DISTRIBUTIONS

by

JAMES ADDISON EASTMAN, 1943-

A DISSERTATION

Presented to the Faculty of the Graduate School of the

UNIVERSITY OF MISSOURI-ROLLA

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

1972

Lee J. Bain
Advisor

Maxwell E. Engelhardt

Dallas E. Johnson

Glen Haddock

C. Y. Ho

ABSTRACT

Three models are considered that have U-shaped hazard functions, and a fourth model is considered that has a linear hazard function. Several methods for estimating the parameters are given for each of these models. Also, various tests of hypotheses are considered in the case of the model with the linear hazard function. One of the models with a U-shaped hazard function has a location and a scale parameter, and it is proved in general that any other parameters in a distribution of this type are distributed independently of the location and scale parameters.

A new method used to estimate the parameters in the preceding distributions is also employed to estimate the parameters in the Logistic distribution, and comparisons based on Monte Carlo methods are made between these estimators and the Maximum Likelihood estimators for n = 10, 20, 40, 80 and for complete samples and censoring from the right for r/n = .1, .3, .5 and .7. The distributions of the pivotal quantities $\sqrt{n}(\hat{\mu} - \mu)/\hat{\sigma}$, $\sqrt{n}(\hat{\sigma}/\sigma - 1)$, and $(\hat{\mu} - \mu)/\hat{\sigma} + k\sigma/\hat{\sigma}$, where the estimates are the Maximum Likelihood estimates, are obtained by Monte Carlo simulation for the sample sizes and level of censoring given above, so that confidence intervals and tolerance limits can be found. The means and variances of the estimators of reliability are given.

ACKNOWLEDGEMENT

The author wishes to express his sincere appreciation to his advisor Dr. Lee J. Bain and to Dr. Maxwell E. Engelhardt both of the Department of Mathematics for their assistance and advice.

The author also expresses his gratitude to his wife for her patience through the research and the graduate studies which preceded it.

TABLE OF CONTENTS

Table of Contents (continued) Page

LIST OF TABLES

I. INTRODUCTION

There are a number of distributions that have proven useful in describing the distribution of the time-to-failure of an item. Typically, the random variable x, which is the time it takes the item to fail is of the continuous type and has a range from zero to infinity. Given that the probability density function is f(x), $0 \leq x < \infty$, the cumulative distribution function is given by

$$F(x) = \int_0^x f(u)du$$

and is the probability that the item has failed by time x. The probability of an item failing in the interval $(x, x + \Delta x)$ is given by $F(x + \Delta x) - F(x)$, so the average rate of failure in this interval is

$$\frac{F(x + \Delta x) - F(x)}{\Delta x}$$

and the average rate of failure given that the item has survived to time x is

$$\frac{F(x + \Delta x) - F(x)}{\Delta x[1 - F(x)]} .$$

The instantaneous rate of failure is

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x[1 - F(x)]} = \frac{f(x)}{1 - F(x)} . \tag{1}$$

The probability that an item survives to time x is called the reliability and is given by $R(x) = 1 - F(x)$. Therefore, (1) can be rewritten as

$$\frac{f(x)}{R(x)} = \frac{-R'(x)}{R(x)} .$$

If we let h(x) be the instantaneous rate of failure, also referred to as the failure rate or the hazard function, then

$$h(x) = \frac{-R'(x)}{R(x)} ,$$

$$- \int_0^x h(u)du = \ln R(x),$$

$$R(x) = \exp\left[- \int_0^x h(u)du\right]$$

or

$$F(x) = 1 - \exp\left[- \int_0^x h(u)du\right], \qquad (2)$$

and

$$f(x) = h(x) \exp\left[- \int_0^x h(u)du\right]. \qquad (3)$$

The most popular time-to-failure models have either a constant failure rate, which is the exponential model, or a monotonically increasing or monotonically decreasing failure rate, such as with the Gamma or Weibull models. In this paper, more general models are considered so as to include models that have failure rates with U or bath tub shapes. The purpose of this shape is to better describe items which initially have a high rate of failure caused by some phenomenon such as faulty manufacturing, then have a useful life period in which the failure rate is at a minimum, and then have an increasing failure rate caused by wear out.

Some general methods for constructing models with bath tub shaped failure rates will first be considered. The first method produces a model of the type referred to by Kao [1] as a mixed model (n fold). If $F_i(x), i=1,\ldots,n$ is a cumulative distribution function for some time-to-fail variable, then a new distribution function can be generated by forming the linear combination

$$F(x) = \sum p_i F_i(x), \quad 0 \le p_i \le 1, \quad \sum p_i = 1, \quad 0 \le x < \infty$$

or

$$F(x) = 1 - \sum p_i R_i(x) .$$

In this case,

$$f(x) = \sum p_i f_i(x)$$

and

$$h(x) = \frac{-\sum p_i R_i'(x)}{\sum p_i R_i(x)}$$

$$= \frac{\sum p_i R_i(x) h_i(x)}{\sum p_i R_i(x)} .$$

Therefore, the failure rate is a linear combination of the original failure rates.

Thus, one way to construct a distribution that has a U-shaped failure rate, using this general method, is to let $F_1(x)$ have a decreasing failure rate, and let $F_2(x)$ have an increasing failure rate, but such that $F_2(x) = 0, \ x < \gamma$, where $\gamma > 0$ is the guarantee time, or the time before which the item will not fail. Kao [1] has used this method with two Weibull distributions, where

$$F_1(x) = 1 - \exp\left\{-(x/\alpha_1)^{\beta_1}\right\}, \quad x > 0, \quad \alpha_1 > 0, \quad 0 < \beta_1 < 1,$$

and

$$F_2(x) = 1 - \exp\left\{-[(x-\gamma)/\alpha_2]^{\beta_2}\right\}, \quad x > \gamma, \quad \alpha_2 > 0, \quad \beta_2 > 1.$$

His purpose for choosing this model was to let $F_1(x)$ describe catastrophic or sudden failures and let $F_2(x)$ describe wear out or delayed failures in electron tubes.

There are several difficulties with this model, including the fact that in general there would be many parameters to estimate, particularly since the p_i would ordinarily be parameters. Also, since the probability density function is in general a sum of terms, standard methods involving the likelihood function, such as maximum likelihood estimation or the likelihood ratio for testing hypotheses become next to impossible to use. The method of moments has been used by Rider [2] to estimate parameters in a model of this type that is a mixture of two exponentials. He suggests that a person try to avoid this model and that the results are mostly of an academic nature. In a later paper, Rider [3] gives the equations for estimating the parameters of mixed Poisson, Binomial, and Weibull distributions by the method of moments. Kao [1] has given a graphical method for estimating the parameters in the model that is a mixture of two Weibulls, and has applied this method to some empirical data on the time to failure of electron tubes.

The second general method of construction gives what Kao [1] refers to as the composite model. In this model, the time interval is broken up, and different distributions are used for each interval. This gives

$$F(x) = F_i(x) , \quad \delta_{i-1} \le x < \delta_i , \quad i=1,\cdots,n , \quad \delta_0 = 0 , \quad \delta_n = \infty ,$$

$$f(x) = f_i(x) ,$$

and

$$h(x) = h_i(x) , \quad \delta_{i-1} \le x < \delta_i , \quad i=1,\cdots,n , \quad \delta_0 = 0 , \quad \delta_n = \infty .$$

To make these functions continuous, let

$$F_i(\delta_i) = F_{i+1}(\delta_i), \quad i = 1, \ldots, n-1.$$

Then the δ_i's will be expressed in terms of the parameters of the distribution functions, and so will not be independent parameters.

As an example of a distribution constructed by this method with a U-shaped failure rate, consider the composite model of three Weibulls, with the respective scale parameters α_1, α_2, and α_3, and the first with $\beta_1 = \frac{1}{2}$, the second with $\beta_2 = 1$, and the third with $\beta_3 = 2$, to give a failure rate that is first decreasing, then constant, and finally increasing. To determine δ_1, let

$$F_1(\delta_1) = F_2(\delta_1),$$

or

$$1 - \exp\{-(\delta_1/\alpha_1)^{\frac{1}{2}}\} = 1 - \exp\{-(\delta_1/\alpha_2)\}.$$

This gives $\delta_1 = \alpha_2^2/\alpha_1$. Similarily, to obtain δ_2, let

$$1 - \exp\{-(\delta_2/\alpha_2)\} = 1 - \exp\{-(\delta_2/\alpha_3)^2\}.$$

This gives $\delta_2 = \alpha_3^2/\alpha_2$.

This model shares with the first model the problem of a large number of parameters. Kao [1] has given a graphical method for estimating the parameters in a model of this type consisting of two Weibulls and applies this method to the same data as he did the first model.

A third general type is the components-in-series model. Suppose there is a system consisting of n independent components in series such that the failure of one component causes the failure of the entire system. Let x_i be the random variable that characterizes the time-to-failure of the i^{th} component. Then the time-to-failure of the entire system is $Y = \min \{x_1, x_2, \ldots, x_n\}$. To find

$$F(y) = P[Y \leq y]$$

$$= P[\min \{x_1, x_2, \ldots, x_n\} \leq y],$$

we first note that

$$1 - F(y) = P[Y > y]$$

$$= P[\min \{x_1, x_2, \ldots x_n\} > y]$$

$$= P[x_1 > y, x_2 > y, \ldots x_n > y]$$

$$= \Pi \; P[x_i > y]$$

$$= \Pi \; [1 - F_i(y)],$$

and so

$$F(y) = 1 - \Pi \; [1 - F_i(y)]$$

$$= 1 - \Pi \; R_i(y), \qquad (4)$$

where $F_i(y)$ is the cumulative distribution function of the i^{th} random variable. If the cumulative distribution function of each random variable is of the form

$$F_i(x) = 1 - \exp\left[- \int_0^x h_i(u)du\right],$$

then (4) becomes

$$F(x) = 1 - \exp\left[- \sum \int_0^x h_i(u)du\right]$$

$$= 1 - \exp\left[- \int_0^x \sum h_i(u)du\right] ,$$

and so the hazard function $h(x) = \sum h_i(x)$.

Murthy and Swartz [4] consider two models, one of which may be considered as being of the component-in-series type. In this one, called Bath-Tub Model I,

$$h(x) = \frac{\alpha}{1 + \beta x} + \gamma x^\delta ; \quad \alpha,\ \beta,\ \gamma, \quad \delta \geq 0 \ , \quad x \geq 0 \ .$$

This corresponds to two components in series, the one component failing according to the Parets distribution, $f(x) = \alpha(1+\beta x)^{-(1+\alpha/\beta)}$, and the other according to the Weibull distribution. They give graphs of this failure rate function for various values of the parameters.

A fourth general type is the components-in-parallel model. In this case there is assumed a system of independent components in parallel such that the system fails when all components fail. Let x_i be the random variable that characterizes the time-to-failure of the i^{th} component. Then the time-to-failure of the entire system is $Y = \max \{x_1, x_2, \cdots, x_n\}$. To find

$$F(y) = P[Y \leq y]$$

$$= P[\max \{x_1, x_2, \cdots, x_n\} \leq y]$$

we note that for the maximum to be less than y, each must be less than y, and so we have

$$F(y) = P[x_1 \le y] \cdot P[x_2 \le y] \cdots P[x_n \le y]$$
$$= \Pi F_i(y)$$

and

$$R(y) = 1 - \Pi F_i(y) \quad .$$

This model would not ordinarily produce a distribution with a U-shaped failure rate, however. To see this, consider a system of two components in parallel, where the system fails when both components fail. In this case, $F(x) = F_1(x)F_2(x)$, and

$$h(x) = \frac{f_1(x)F_2(x) + f_2(x)F_1(x)}{1 - F_1(x)F_2(x)} \quad . \tag{5}$$

For h(x) to be U-shaped, $h(0) \neq 0$, which as can be seen from (5), would generally require that either $F_1(0) \neq 0$ or $F_2(0) \neq 0$, which appears to be an unrealistic assumption. However, a distribution with a U-shaped failure rate can be constructed if either $F_1(0) \neq 0$ or $F_2(0) \neq 0$.

A fifth general method for finding a time-to-failure distribution with a U-shaped failure rate is simply to find a function h(u) that has a U-shape in the first quadrant, and express

$$F(x) = 1 - \exp\left[-\int_0^x h(u)du\right] \quad . \tag{6}$$

This is a general method that gives many other time-to-fail distributions, not just those with U-shaped failure rates. For instance, if $h(x) = \theta$, the exponential distribution is the result, whereas if

$$h(x) = \frac{\alpha}{\beta} \left(\frac{x}{\beta}\right)^{\alpha-1}, \quad 0 \leq x < \infty$$

the Weibull distribution is the result. Not all functions h(x) are eligible, however. It is necessary for $\int_0^\infty h(x)dx = \infty$ and $h(x) \geq 0$, $0 < x < \infty$. Both of these conditions are met when h(x) is U-shaped in the first quadrant.

Murthy and Swartz [4] consider a model of this general type with their model called Bath-Tub Model II, where

$$h(x) = \alpha \exp [-(\beta x)] + \gamma x^{\delta}; \ \alpha, \beta, \gamma, \delta > 0, \ x \geq 0.$$

They give the graph of this failure-rate function for $\alpha = 10$, $\beta = 10$, $\gamma = .3$, and $\delta = 2.5$. Although this appears to be a component failing according to the Gompertz distribution [5] and the second according to the Weibull distribution, this is not the case, since the hazard function of the Gompertz distribution is given by $h(x) = a \exp (bx)$. The function $g(x) = \alpha \exp [-(\beta x)]; \alpha, \beta > 0$ cannot be a hazard function since it does not meet the condition mentioned above that $\int_0^\infty h(x)dx = \infty$. However, the integral of the entire function given above by Murthy and Swartz in their Model II is equal to infinity, and it is also positive over the entire interval. Therefore, the entire function satisfies the conditions of being a hazard function.

The three U-shaped hazard functions considered in this paper are

$$h(x) = ax^2 + bx + c; \ x \geq 0, \ a \geq 0, \ c \geq 0, \ b > -2\sqrt{ac}$$

$$h(x) = bc \cosh b(x - a); \ x \geq 0, \ |a| < \infty, \ b, c > 0,$$

and

$$h(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left[1 + \gamma e^{(x/\beta)^{\alpha}}\right], \; x \geq 0, \; \alpha, \; \beta > 0, \; 0 < \gamma < 1 \; .$$

From (3), the corresponding density functions are

$$f(x) = (ax^2 + bx + c) \exp\left[-\left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right)\right],$$

$$f(x) = [bc \cosh b(x - a)] \exp \; [-c \sinh b(x - a)],$$

and

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left[1 + \gamma e^{(x/\beta)^{\gamma}}\right] \exp \{-[(\tfrac{x}{\beta})^{\alpha} + \gamma(e^{(x/\beta)^{\alpha}} - 1)]\}$$

respectively, where the ranges on the variable and the parameters are the same as above.

The model with the polynomial as its hazard function is of the components-in-series type, where in this case, there are three components, two failing according to the Weibull distribution, and the third according to the exponential distribution. It seems reasonable to consider polynomial hazard functions since in general nice functions can be approximated by truncated Taylor series.

The model with the hyperbolic cosine as the hazard function fits into the fifth general category for constructing these time-to-fail distributions. Although this model did not develop from physical considerations, it appears to have an appropriately shaped hazard function. The third model was developed because it has a location and a scale parameter. This distribution is used to illustrate a generalized result about location and scale parameters in maximum likelihood estimation and maximum agreement estimation.

A special case of the polynomial model mentioned above in which a, the coefficient of x^2, is zero is also considered in detail. This model has a linear failure rate

$$h(x) = ax + b; \; x \geq 0, \; a > 0, \; b \geq 0$$

and this would appear to be a useful generalization of the exponential model. It is of the components-in-series type, where the one component fails according to the Weibull distribution and the other component fails according to the exponential distribution. This model appears useful for describing data that has an initial non-zero failure rate or has a hazard function that is increasing at a rate directly proportional to time. In this case, using (3), the density function is given by

$$f(x) = (ax + b) \exp\left[-\left(\frac{ax^2}{2} + bx\right)\right], x \geq 0, \; a \geq 0, \; b \geq 0.$$

Neither the second or the third model appears to have been considered before. However, the first model has been considered by Krane [5] in the more general form $h(x) = \sum_{i=1}^{n} \beta_i x^i$ and the unknown parameters are estimated by regression techniques. Krane stipulates that $\beta_i > 0$ for all i, so that h(x) cannot be U-shaped when it is a quadratic function. The model with the linear failure-rate has been independently investigated by Kodlin [7] and Farmelo [8]. Kodlin gives a theoretical explanation for the usefulness of this model, and uses the method of maximum likelihood to estimate the parameters. He applies this model to some empirical medical data. Farmelo compares several methods for estimating the parameters of this distribution. Among the methods considered is the method of moments and the method of maximum likelihood.

Particular attention is given here to the estimation of parameters in these four models using the maximum likelihood procedure, and particular application of an estimation procedure refered to in this paper as agreement estimation, considered by Bain and Antle [9]. Also, various tests of hypotheses are considered in the case of the model with the linear failure-rate.

The method of estimation refered to above as agreement estimation is also applied to estimating the parameters in the Logistic distribution. Comparisons are made between these estimates and the maximum likelihood estimates using Monte Carlo techniques. Furthermore, the distributions of the pivotal quantities $\sqrt{n}(\hat{\mu} - \mu)/\hat{\sigma}$, $\sqrt{n}(\hat{\sigma}/\sigma - 1)$ and $(\hat{\mu} - \mu)/\hat{\sigma} + k\ \sigma/\hat{\sigma}$ are obtained by Monte Carlo techniques for n = 10, 20, 40, 80 and censoring from the right for r/n = .1, .3, .5, .7, 1., where the estimates are the maximum likelihood estimates, so that confidence intervals can be made on the parameters, and tolerance limits can be made on the distribution. Antle, Klimko, and Harkness [10] have considered this problem but found the distributions only for the case of complete samples.

II. LINEAR FAILURE-RATE MODEL

A. Range for the Parameters

In this chapter the model with the failure-rate $h(x) = ax + b$ will be considered. In this case

$$f(x) = (ax + b) \exp [-(\frac{ax^2}{2} + bx)]; \qquad x\geq 0,\ a>0,\ b\geq 0 \ ; \quad (1)$$

$$F(x) = 1 - \exp [-(\frac{ax^2}{2} + bx)]; \qquad x\geq 0,\ a>0,\ b\geq 0 \ ;$$

and

$$R(x) = \exp [-(\frac{ax^2}{2} + bx)]; \qquad x\geq 0,\ a>0,\ b\geq 0 \ .$$

It can be seen from (1) that $b \geq 0$, or else $f(x)$ will be negative for $x = 0$, which is an impossibility. If $b = 0$, the model degenerates to the Weibull model.

Although ordinarily one would be concerned with $a \geq 0$, it is possible to have $a < 0$. In this case, the rate of failure is decreasing, which does not seem to lend itself to real-life situations. Furthermore, with $a < 0$, the range becomes $0 \leq x \leq -b/a$, and it is necessary to multiply the probability density function by the constant

$$k = \{1 - \exp [-(\frac{b^2}{2a} - \frac{b^2}{a})]\}^{-1},$$

so that $\int f(x)\, dx = 1$. Of course, if $a = 0$, the model degenerates to the exponential model. The only case that will be considered will be with $a > 0$.

The first two moments of this model are

$$E(x) = \sqrt{\frac{2\pi}{a}} \exp\left[(\frac{b^2}{2a}) \{1 - P[Y \leq \frac{b}{\sqrt{a}}]\}\right],$$

and

$$E(x^2) = \frac{2}{a} - \frac{2b}{a} E(x) \ ,$$

where $Y \sim N(0,1)$.

B. Maximum Likelihood Estimates

The likelihood function in this case is

$$L(x;a,b) = \{\Pi(ax_i + b)\} \exp - \sum\left(\frac{ax_i^2}{2} + bx_i\right) ,$$

and

$$\ln L(x;a,b) = \sum \ln(ax_i + b) - \sum\left(\frac{ax_i^2}{2} + bx_i\right) . \qquad (2)$$

Differentiating (2) with respect to a and b gives

$$\frac{\partial \ln L}{\partial a} = \sum \frac{x_i}{ax_i + b} - \frac{\sum x_i^2}{2} \qquad (3)$$

and

$$\frac{\partial \ln L}{\partial b} = \sum \frac{1}{ax_i + b} - \sum x_i \ . \qquad (4)$$

The values $\hat{a}$ and $\hat{b}$ that satisfy

$$\sum \frac{x_i}{\hat{a}x_i + \hat{b}} - \sum \frac{x_i^2}{2} = 0 \qquad (5)$$

and

$$\sum \frac{1}{\hat{a}x_i + \hat{b}} - \sum x_i = 0 \qquad (6)$$

are the maximum likelihood estimates, if the values are in the parameter space and maximize $L(x;a,b)$. Rewriting (5) as

$$n - \hat{b} \sum \frac{1}{\hat{a}x_i + \hat{b}} - \hat{a} \sum \frac{x_i^2}{2} = 0 \qquad (7)$$

and substituting from (6)

$$\sum \frac{1}{\hat{a}x_i + \hat{b}} = \sum x_i$$

into (7), gives

$$n - \frac{\hat{a}}{2} \sum x_i^2 - \hat{b} \sum x_i = 0 , \qquad (8)$$

which can be rewritten as an explicit function of the one estimate in terms of the other.

Rewriting (8) as

$$\hat{b} = \frac{1}{\sum x_i} [n - \frac{\hat{a}}{2} \sum x_i^2]$$

and substituting this into (6), gives

$$\sum \frac{\hat{a}[2x_i(\sum x_i/n) - (\sum x_i^2/n)]}{\{\hat{a}[2x_i(\sum x_i/n) - (\sum x_i^2/n)]\} + 2} = 0 , \qquad (9)$$

which can be rewritten as a polynomial in a of degree n [8]. It should be noted that a = 0 as in general not a solution to (9), since this would imply from (5) and (6) that

$$(\sum x_i)^2 - \frac{n}{2} \sum x_i^2 = 0 ,$$

which is not usually the case. Also, since (9) is a polynomial, it will not always have a unique solution. In this case the estimates must be substituted back into the likelihood function to see which estimates produce the maximum.

Another problem that can develop from this method of estimation, is for certain estimates of a and b to be inadmissible. As an example, consider a sample of size three with $x_1 = 1.0$, $x_2 = 2.0$, and $x_3 = 3.0$. In this case, (9) becomes

$$-48.8\overline{8}\ \hat{a}^2 + 69.3\overline{3}\ \hat{a} + 40 = 0 ,$$

and so (1.86, -1.67) and (-.44, 1.01) are the estimates for (a,b). It can be seen that (1.86, -1.67) is inadmissible since b must be greater than or equal to zero. The second pair is inadmissible since if $a < 0$, then $0 \leq x \leq -b/a$ or $0 \leq x < 2.3$, which is not the case since $x_3 = 3$.

In general, the admissible estimates must lie in the region bounded by $\hat{b} = 0$ and $\hat{a} = -\hat{b}/\max\{x_i\}$ in order to make $\hat{a}x + \hat{b} \geq 0$, thus making $f(x) \geq 0$. If the estimates that maximize the likelihood function lie outside this region, then the estimates that are admissible lie on the boundary, since the likelihood function goes to zero as a and b become large. Since the likelihood function is zero if $a = -b/\max\{x_i\}$, the boundary that will yield the maximum is along $b = 0$. In this case the likelihood function is

$$L(x;a) = a^n \Pi\, x_i \exp - \frac{a}{2} \sum x_i^2 ,$$

$$\ln L(x;a) = n \ln a + \sum \ln x_i - \frac{a}{2} \sum x_i^2 ,$$

and upon setting the partial of ln L(x;a) with respect to a equal to zero,

$$\hat{a} = 2n/\sum x_i^2 . \qquad (10)$$

In the particular example that we have been considering, $L(x;1.86, -1.67) = .0758$, $L(x;.44, 1.01) < 0$, and using (10) gives $\hat{a} = .429$ with $L(x; .429, 0) = .0024$.

As another example, consider the sample with $x_1 = .9$, $x_2 = 1.0$, and $x_3 = 1.1$. In this case the polynomial becomes

$$2.82\ \hat{a}^2 + 11.68\ \hat{a} + 11.92 = 0\ ,$$

and the estimates for (a,b) are (2.32, -1.66) and (1.83, .08) with the corresponding likelihood function values L(x; 2.32, -1.66) = .489 and L(x; 1.83, .08) = .342. In this case also, the maximum is outside the admissible region and so (10) is used to give $\hat{a} = 1.99$ and L(x; 1.99, 0) = .387.

Since complex roots of polynomials come in conjugate pairs, if n is any odd value, (9) will have two or more real solutions, and therefore the likelihood function will need to be evaluated for the various estimates to determine the maximum likelihood estimate. If one of the estimates is inadmissible because $\hat{b} < 0$, the likelihood function should be evaluated for this estimate to see if this point is a maximum so that the admissible values that maximize the likelihood function are on the boundary, that is, when $\hat{b} = 0$.

Monte Carlo work has led the author to believe that for large even n, there is a unique solution to (9). However, even if (9) has a unique solution, it is still a very difficult equation to solve. A systematic search procedure has proven to be the only method that always works.

If $x_1, \ldots, x_r$ are the r smallest values from a random sample of size n, then the simultaneous equations that the maximum likelihood estaimtes $\hat{a}_r$ and $\hat{b}_r$ satisfy are

$$\sum_{i=1}^{n-r} \frac{x_i}{\hat{a}_r x_i + \hat{b}_r} - \left[\sum_{i=1}^{n-r} \frac{x_i^2}{2} + \frac{r x_{n-r}^2}{2} \right] = 0 \tag{11}$$

and

$$\sum_{i=1}^{n-r} \frac{1}{\hat{a}_r x_i + \hat{b}_r} - \left[\sum_{i=1}^{n-r} \frac{x_i}{2} + r x_{n-r} \right] = 0 \ , \tag{12}$$

if the estimates are in the parameter space and maximize $L(x;\ \hat{a}_r,\ \hat{b}_r)$. The case in which the censoring is from the smallest values, besides not appearing to be a very likely situation, does not yield very usable results, and so will be omitted.

C. Maximum Agreement Estimates

The method of estimation to maximize agreement according to some criterion between u(x) which is some function of the random variable x and E(u(x)) has been proposed by Bain and Antle [9]. In this case, in order to make the estimates computationally easy, we let u(x) = F(x), and use least squares as the criterion for maximizing agreement.

Since u(x) is distributed uniformly, $E(u(x_i)) = \frac{i}{n+1}$, and so the problem reduces to minimizing

$$A^*(x;a,b) = \sum \{1 - \exp\left[-\left(\frac{a x_i^2}{2} + b x_i\right)\right] - \frac{i}{n+1} \}^2 \ . \tag{13}$$

Equation (13) is not linear in the parameters and so, from a computational standpoint, would yield undesirable estimates of a and b. Therefore, since ln is a monotone function, we minimize instead

$$A(x;a,b) = \sum [(\frac{ax_i^2}{2} + bx_i) + \ln(1 - \frac{i}{n+1})]^2 . \tag{14}$$

Differentiating (14) with respect to a and b gives

$$\frac{\partial A}{\partial a} = 2 \sum \frac{x_i^2}{2} [(\frac{ax_i^2}{2} + bx_i) + \ln(1 - \frac{i}{n+1})] \tag{15}$$

and

$$\frac{\partial A}{\partial b} = 2 \sum x_i[(\frac{ax_i^2}{2} + bx_i) + \ln(1 - \frac{i}{n+1})] . \tag{16}$$

The values $\hat{a}$ and $\hat{b}$ that satisfy

$$\frac{\hat{a}}{2} \sum x_i^4 + \hat{b} \sum x_i^3 + \sum x_i^2 \ln (\frac{n+1-i}{n+1}) = 0 \tag{17}$$

and

$$\frac{\hat{a}}{2} \sum x_i^3 + \hat{b} \sum x_i^2 + \sum x_i \ln (\frac{n+1-i}{n+1}) = 0 \quad , \tag{18}$$

are maximum agreement estimates, which will be referred to as the first method. Using Cramer's method to solve (17) and (18) gives

$$\frac{\hat{a}}{2} = \frac{\sum x_i^3 \sum x_i \ln(\frac{n+1-i}{n+1}) - \sum x_i^2 \sum x_i^2 \ln(\frac{n+1-i}{n+1})}{\sum x_i^2 \sum x_i^4 - \sum x_i^3 \sum x_i^3} \tag{19}$$

and

$$\frac{\hat{b}}{2} = \frac{\sum x_i^3 \sum x_i^2 \ln(\frac{n+1-i}{n+1}) - \sum x_i^4 \sum x_i \ln(\frac{n+1-i}{n+1})}{\sum x_i^2 \sum x_i^4 - \sum x_i^3 \sum x_i^3}$$

Another approach to obtaining maximum agreement estimates, which will be referred to as the second method, is to let $u(x) = \frac{ax^2}{2} + bx$ and then minimize

$$A(x;a,b) = \sum [(\frac{ax^2}{2} + bx) - E(\frac{ax^2}{2} + bx)]^2 . \tag{21}$$

To find $E[\frac{ax^2}{2} + bx]$, first note that

$$v = 1 - e^{-(\frac{ax^2}{2} + bx)}$$

is distributed uniformly. Therefore,

$$h(v) = 1 , \qquad 0 \le v \le 1 ,$$

and so be letting $w = 1 - v$, and noting that $\left|\frac{\partial v}{\partial w}\right| = 1$, we have

$$h(w) = 1 , \qquad 0 \le w \le 1 ,$$

and so,

$$w = e^{-(\frac{ax^2}{2} + bx)}$$

is distributed uniformly. Therefore, $-\ln(w(x)) = \frac{ax^2}{2} + bx$ is distributed as a standard exponential variable.

The expected value of the i^{th} order statistic from a sample of standard exponentials of size n is given by Epstein and Sobel [11] to be

$$E(i,n) = \sum_{j=1}^{i} [1/(n-j+1)] .$$

Harter [11] has calculated these values for $n = 1(1)120$ and all values of i, $1 \le i \le n$.

Differentiating (21) with respect to a and b, and setting the partials equal to zero gives

$$\frac{\hat{a}}{2} \sum x_i^4 + \hat{b} \sum x_i^3 - \sum x_i^2(E(x_i,n)) = 0 \tag{22}$$

and

$$\frac{\hat{a}}{2} \sum x_i^3 + \hat{b} \sum x_i^2 - \sum x_i(E(i,n)) = 0 \ , \qquad (23)$$

where $\hat{a}$ and $\hat{b}$ are the maximum agreement estimates of the second method. Using Cramér's method to solve (22) and (23) gives

$$\frac{\hat{a}}{2} = \frac{\sum x_i^2 \ \sum x_i^2(E(i,n)) - \sum x_i^3 \ \sum x_i(E(i,n))}{\sum x_i^2 \ \sum x_i^4 - \sum x_i^3 \ \sum x_i^3}$$

and

$$\hat{b} = \frac{\sum x_i^4 \ \sum x_i(E(i,n)) - \sum x_i^3 \ \sum x_i^2(E(i,n))}{\sum x_i^4 \ \sum x_i^2 - \sum x_i^3 \ \sum x_i^3} \ .$$

D. Cramér-Rao Lower Bound(CRLB)

The Cramér-Rao Lower Bound Matrix is given by

$$\frac{1}{n} \begin{bmatrix} -E[\frac{\partial^2 \ln L}{\partial a^2}] & -E[\frac{\partial^2 \ln L}{\partial a \partial b}] \\ -E[\frac{\partial^2 \ln L}{\partial a \partial b}] & -E[\frac{\partial \ln L}{\partial b^2}] \end{bmatrix}^{-1} ,$$

where L denotes the likelihood function. Differentiation of ln f yields

$$\frac{\partial^2 \ln f}{\partial a^2} = \frac{-x^2}{(ax+b)^2} \ ,$$

$$\frac{\partial^2 \ln f}{\partial b^2} = \frac{-1}{(ax+b)^2} \ , \qquad (24)$$

and

$$\frac{\partial^2 \ln f}{\partial a \partial b} = \frac{-x}{(ax+b)^2} \ .$$

Now,

$$-E\left[\frac{\partial^2 \ln f}{\partial a^2}\right] = \int_0^\infty \frac{x^2(ax+b)}{(ax+b)^2} \exp\left[-\left(\frac{ax^2}{2} + bx\right)\right]dx$$

$$= \frac{\exp(b^2/2a)}{a^2} \left\{ \int_0^\infty \frac{(a^2x^2+2abx+b^2)}{ax+b} \exp\left[-\left(\sqrt{\frac{a}{2}}\, x + \frac{b}{\sqrt{2a}}\right)^2\right]dx \right.$$

$$-2b \int_0^\infty \frac{(ax+b)}{(ax+b)} \exp\left[-\left(\sqrt{\frac{a}{2}} + \frac{b}{\sqrt{2a}}\right)^2\right]dx$$

$$\left. +b^2 \int_0^\infty \frac{\exp\left[-\left(\sqrt{\frac{a}{2}}\, x + \frac{b}{\sqrt{2a}}\right)^2\right]}{(ax+b)} dx \right\}$$

$$= \frac{\exp(b^2/2a)}{a^2} \left\{ \int_{\frac{b^2}{2a}}^\infty \exp(-y)dy - 2b \int_{\frac{b}{\sqrt{2a}}}^\infty \sqrt{\frac{2}{a}} \exp(-y^2)dy \right.$$

$$\left. + \frac{b^2}{2a} \int_{\frac{b^2}{2a}}^\infty \frac{\exp(-y)}{y} dy \right\}$$

$$= \frac{\exp(b^2/2a)}{a^2} \left\{ -\exp(-y) \Big|_{\frac{b^2}{2a}}^\infty - \frac{2b\sqrt{2\pi}}{\sqrt{a}} \int_{\frac{b}{\sqrt{a}}}^\infty \frac{\exp\left[-\left(\frac{z^2}{2}\right)\right]}{\sqrt{2\pi}} dz \right.$$

$$\left. - \frac{b^2}{2a} \int_\infty^1 \frac{\exp\left[-\left(\frac{b^2 z}{2a}\right)\right]}{z} dz \right\}$$

$$= \frac{\exp(b^2/2a)}{a^2}\left\{\exp\left[-\left(\frac{b^2}{2a}\right)\right] - 2b\sqrt{\frac{2\pi}{a}}\left[1 - P\left[Z \le \frac{b}{\sqrt{a}}\right]\right]\right.$$

$$\left. - \frac{b^2}{2a}\left[\gamma + \log\left(\frac{b^2}{2a}\right) - \frac{b^2}{2a} + \frac{\left(\frac{b^2}{2a}\right)^2}{2\cdot 2!} - \frac{\left(\frac{b^2}{2a}\right)^3}{3\cdot 3!} + \cdots\right]\right\}$$

$$= \frac{\exp k}{a^2}\ \{\exp(-k) - 4\sqrt{\pi k}\ [1-P(Z\le\sqrt{2k})]$$

$$-k[\gamma + \log(k) - k + \frac{k^2}{2\cdot 2!} - \frac{k^3}{3\cdot 3!} + \cdots]\}\ ,$$

where $\gamma \doteq .5772157$, $Z \sim N(0,1)$, and $k = b^2/2a$. The negative expected value of (24) is

$$-E\left[\frac{\partial^2 \ln f}{\partial b^2}\right] = \int_0^\infty \frac{ax+b}{(ax+b)^2}\exp\left[-\left(\frac{ax^2}{2} + bx\right)\right]dx$$

$$= \exp\left(\frac{b^2}{2a}\right)\int_0^\infty \frac{(ax+b)}{(ax+b)^2}\exp\left[-\left(\sqrt{\frac{a}{2}}\,x + \frac{b}{\sqrt{2a}}\right)^2\right]dx$$

$$= \frac{-\exp\left(\frac{b^2}{2a}\right)}{2a}\int_\infty^1 \frac{\exp\left[-\left(\frac{b^2 z}{2a}\right)\right]}{z}\,dz$$

$$= \frac{-\exp\left(\frac{b^2}{2a}\right)}{2a}\left[\gamma + \log\left(\frac{b^2}{2a}\right) - \frac{b^2}{2a} + \frac{\left(\frac{b^2}{2a}\right)^2}{2\cdot 2!} - \frac{\left(\frac{b^2}{2a}\right)^3}{3\cdot 3!} + \cdots\right]$$

$$= \frac{-\exp k}{2a}\ [\gamma + \log k - k + \frac{k^2}{2\cdot 2!} - \frac{k^3}{3\cdot 3!} + \cdots]$$

where $\gamma \doteq .5772157$ and $k = b^2/2a$.

Finally,

$$-E\left[\frac{\partial^2 \ln f}{\partial a \partial b}\right] = \int_0^\infty \frac{x(ax+b)}{(ax+b)^2} \exp\left[-\left(\frac{ax^2}{2} + bx\right)\right]dx$$

$$= \frac{\exp(b^2/2a)}{a} \left\{ \int_0^\infty \frac{(ax+b)^2}{(ax+b)^2} \exp\left[-\left(\sqrt{\frac{a}{2}}\, x + \frac{b}{\sqrt{2a}}\right)^2\right]dx \right.$$

$$\left. -b \int_0^\infty \frac{ax+b}{(ax+b)^2} \exp\left[-\left(\sqrt{\frac{a}{2}}\, x + \frac{b}{\sqrt{2a}}\right)^2\right]dx \right\}$$

$$= \frac{\exp(b^2/2a)}{a} \left\{ \sqrt{\frac{2}{a}} \int_{\frac{b}{\sqrt{2a}}}^\infty \exp(-y^2)\, dy + \frac{b}{2a} \int_\infty^{\frac{b^2}{2a}} \frac{e^{-y}}{y}\, dy \right\}$$

$$= \frac{\exp(b^2/2a)}{a} \left\{ \sqrt{\frac{2\pi}{a}} \int_{\frac{b}{\sqrt{a}}}^\infty \frac{\exp - \left(\frac{z^2}{2}\right)}{\sqrt{2\pi}}\, dz \right.$$

$$\left. + \frac{b}{2a} \int_\infty^1 \frac{\exp - \left(\frac{b^2}{2a}\right)}{z}\, dz \right\}$$

$$= \frac{\exp(b^2/2a)}{a} \left\{ \sqrt{\frac{2\pi}{a}} \left[1 - P\left(Z \le \frac{b}{\sqrt{a}}\right)\right] \right.$$

$$\left. + \frac{b}{2a} \quad \gamma + \ln\left(\frac{b^2}{2a}\right) - \frac{b^2}{2a} + \frac{\left(\frac{b^2}{2a}\right)^2}{2\cdot 2!} - \frac{\left(\frac{b^2}{2a}\right)^3}{3\cdot 3!} + \cdots \right\}$$

$$= \frac{\sqrt{a}\,\exp(k)}{a^2}\left\{\sqrt{2\pi}\,[1 - P(Z \le \sqrt{2k})]\right.$$

$$\left. + \sqrt{\frac{k}{2}}\left[\gamma + \log k - k + \frac{k^2}{2\cdot 2!} - \frac{k^3}{3\cdot 3!} + \cdots\right]\right\},$$

where $\gamma \doteq .5772157$, $Z \sim N(0,1)$, and $k = b^2/2a$. Table 1 gives a comparison of the maximum likelihood estimates and the two maximum agreement estimators with the CRLB for a = 5, b = 1, and n = 20, 40.

TABLE 1. Monte Carlo study comparing various estimates for n = 20, 40 with CRLB for a = 5, b = 1.

	20	40
Maximum likelihood		
$E(\hat{a})$	6.58	5.36
$Var(\hat{a})$	11.08	3.38
$E(\hat{b})$	.75	.88
$Var(\hat{b})$	.70	.36
Maximum agreement - 1^{st} method		
$E(\hat{a})$	4.21	4.06
$Var(\hat{a})$	11.14	5.16
$E(\hat{b})$	1.18	1.24
$Var(\hat{b})$	.84	.47
Maximum agreement - 2^{nd} method		
$E(\hat{a})$	5.65	5.10
$Var(\hat{a})$	15.24	6.39
$E(\hat{b})$	.95	1.04
$Var(\hat{b})$	1.04	.54
CRLB $(\hat{a})$	6.64	3.32
CRLB $(\hat{b})$	.52	.25

E. Reparameterization of the Model

If

$$f(x) = (a'x + b') \exp [-(\frac{a'x^2}{2} + b'x)] ,$$

then letting $a^2 = a'$ and $c = b'/\sqrt{a'}$ gives

$$f(x) = a(ax + c) \exp [-a(\frac{ax^2}{2} + cx)] .$$

In this case,

$$L(x;a,c) = [\ \Pi\ a(ax_i + c)] \exp [-a \sum (\frac{ax_i^2}{2} + cx_i)] ,$$

$$\ln L(x;a,c) = \sum \ln a(ax_i + c) - a \sum (\frac{ax_i^2}{2} + cx_i) ,$$

$$\frac{\partial \ln L(x;a,c)}{\partial a} = \sum \frac{2ax_i + c}{a^2 x_i + ac} - \sum ax_i^2 - \sum cx_i , \qquad (25)$$

and

$$\frac{\partial \ln L(x;a,c)}{\partial c} = \sum \frac{a}{a^2 x_i + ac} - \sum ax_i . \qquad (26)$$

Setting (25) and (26) equal to zero gives the maximum likelihood estimates $\hat{a}$ and $\hat{c}$, and be letting $y = ax$, these equations can be rewritten as

$$\frac{2}{a} \sum \frac{y_i}{\frac{\hat{a}}{a} y_i + \hat{c}} - \frac{1}{a}\frac{\hat{a}}{a} \sum y_i^2 - \frac{\hat{c}}{a} \sum y_i + \frac{\hat{c}}{a} \sum \frac{1}{(\frac{\hat{a}}{a})^2 y_i + \frac{\hat{a}}{a}\hat{c}} = 0$$

and

$$\sum \frac{1}{\frac{\hat{a}}{a} y_i + \hat{c}} - \frac{\hat{a}}{a} \sum y_i = 0 .$$

Since y is distributed independently of a, this proves the following

Theorem: $\frac{\hat{a}}{a}$ is distributed independently of a and has the same distribution as $\hat{a}_{1,c}$ which is the maximum likelihood estimator of a when the sampling is from a distribution with the probability density function

$$f(y) = (y + c) \exp [-(\frac{y^2}{2} + cy)].$$

Although it has not been done here, with this theorem it becomes possible to find the distribution of $\frac{\hat{a}}{a}$ for various values of c and n so that confidence intervals could be established for a. Furthermore, any scale invariant statistic, such as $\sum x_i^2/(\sum x_i)^2$, will be distributed independently of a. Therefore, since the distribution of any such statistic depends only on c and n, by finding the distribution of the statistic either analytically or by Monte Carlo techniques, tests can be made on the parameter c. A theorem in Chapter III shows that the maximum likelihood estimator $\hat{c}$ is distributed independently of a, so that $\hat{c}$ could also be used as the test statistic.

F. Tests of Hypotheses

1. Tests on the Parameter a with b Known

A test for $H_0 : a = a_0$ versus $H_1 : a = a_1$ can be made using the statistic

$$[\frac{a_0}{2} x_1^2 + bx_1]/[\frac{a_0}{2} x_2^2 + bx_2] , \tag{27}$$

where x_1 and x_2 are the first two ordered observations. This test has the property that it can be made, even when a large amount of censoring has taken place. Also, under the special case when the test is $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$, the test can be made with b unknown.

If a and b are known, then $y = (\frac{a}{2} x_i^2 + bx_i)$ id distributed as the i^{th} observation from the standard exponential distribution, if x_i is the i^{th} ordered observation from the distribution under consideration. Since

$$g_{12}(y_1, y_2) = n(n-1) \exp[-(n-2)y_2] \exp[-y_1] \exp[-y_2]$$
$$= n(n-1) \exp[-(n-1)y_2] \exp[-y_1] ,$$

letting $z_1 = y_1/y_2$ and $z_2 = y_2$ gives

$$g(z_1, z_2) = n(n-1)z_2 \exp[-(n-1)z_2] \exp[-z_1 z_2]$$

since $|J| = z_2$. Integrating $g_{12}(z_1, z_2)$ with respect to z_2 by parts by letting $u = z_2$ and $dv = \exp\{-[(n-1)+z_1]z_2\}$ gives

$$g_1(z_1) = \frac{n(n-1)}{[(n-1) + z_1]^2} , \qquad 0 \le z_1 \le 1 ,$$

and

$$G_1(z_1) = \frac{n\ z_1}{[(n-1) + z_1]} , \qquad 0 \le z_1 \le 1 .$$

Monte Carlo simulation showed that the statistic (27) is larger under H_1 than under H_0, and so the critical region is taken from the right and

$$C(\alpha) = \frac{(n-1)(1-\alpha)}{\alpha + (n-1)} .$$

Thus H_0 is accepted if (27) is less than or equal to $C(\alpha)$ and is rejected if (27) is greater than $C(\alpha)$.

Another statistic that can be used for testing $H_0 : a = a_0$ versus $H_1 : a = a_1 > a_0$ is

$$2\left[\sum_{i=1}^{r} \left(\frac{a_0}{2} x_i^2 + bx_i\right) + (n - r) \left(\frac{a_0}{2} x_r^2 + bx_r\right)\right], \qquad (28)$$

where x_i is the i^{th} ordered observation. This statistic has the advantage over the preceding statistic of being better suited for various levels of censoring as well as for complete samples.

As in the preceding case, it is noted that $y = (\frac{a}{2} x^2 + bx)$ is distributed as a standard exponential. Since

$$2\left[\sum_{i=1}^{r} y_i + (n - r) y_r\right] \sim \chi^2 (2r)$$

where y_i is the i^{th} ordered observation from a standard exponential [12], a test can be made using this statistic. Chi-Square tables need to be used for obtaining the critical region, and Monte Carlo simulation has shown that the statistic is smaller under H_1 than under H_0 and so the critical region is taken from the left and H_0 is rejected if the statistic is less than $\chi^2_\alpha(2r)$.

A special case of the above test occurs when r = 1, since in this case, it is possible to compute the power of the test. When r = 1, it is simpler to use as the statistic $(\frac{a_0}{2} x_1^2 + bx_1)$. If $a = a_0$,

$$f_1(x_1) = n(a_0 x_1 + b) \exp\left[-n \left(\frac{a_0}{2} x_1^2 + bx_1\right)\right], \quad 0 \leq x_1 \leq \infty$$

and the critical point $C(\alpha)$ can be found by setting

$$\int_0^{C(\alpha)} n(a_0 x_1 + b) \exp\left[-n \left(\frac{a_0}{2} x_1^2 + bx_1\right)\right] dx_1$$

equal to α, and solving for $C(\alpha)$. This gives $C(\alpha) = [(-nb + \sqrt{n^2b^2 - 2\, a_0\, n \ln (1 - \alpha)}\,)/n\, a_0]$. The null hypothesis is accepted if the statistic is greater than $C(\alpha)$, and is rejected otherwise. The power of this test is given by

$$\int_{C(\alpha)}^{\infty} n(a_1 x + b) \exp \left[-n\left\{\frac{a_1 x^2}{2} + bx\right\}\right]dx$$

$$= 1 - \exp \left[-n\left\{\frac{a_1}{2} [C(\alpha)]^2 + b\, C(\alpha)\right\}\right] .$$

In the special case where the test is $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$, $C(\alpha) = -\ln(1-\alpha)/nb$.

The best test for the hypothesis $H_0 : a = a_0$ versus $H_1 : a = a_1 > a_0$ is the test given by the Neyman-Pearson Lemma. This test is considered here to give a basis for comparison of the various tests. Under this test, H_0 is accepted with probability α of a type I error if

$$\frac{\Pi\, f(x_i;\, a_0,\, b)}{\Pi\, f(x_i;\, a_1,\, b)} = \frac{\Pi\, [(a_0 x_i + b) \exp - (\frac{a_0}{2} x_i^2 + bx_i)]}{\Pi\, [(a_1 x_i + b) \exp - (\frac{a_1}{2} x_i^2 + bx_i)]} \qquad (29)$$

$$\leq k \ ,$$

and is rejected otherwise, where k is suitably chosen.

The problem with this test is finding k, and the only way seems to be by Monte Carlo simulation, finding k for various values of n, a, b, and α. This can be eased somewhat so that k need be determined for various values of the ratio a / b^2, n, and α.

In order to do this, let $y = bx$, $c_0 = a_0/b^2$, and $c_1 = a_1/b^2$. The ratio (29) can be rewritten as

$$\frac{\Pi\, b\, [\frac{a_0}{b^2}(bx_i) + 1]\ \exp\{-\sum [\frac{a_0}{b^2}\frac{b^2x_i^2}{2}] + bx_i\}}{\Pi\, b\, [\frac{a_1}{b^2}(bx_i) + 1]\ \exp\{-\sum [\frac{a_1}{b^2}\frac{b^2x_i^2}{2}] + bx_i\}},$$

which upon making the substitutions becomes

$$\frac{\Pi\, [c_0y_i + 1]\ \exp\{-\sum (c_0\frac{y_i^2}{2} + y_i)\}}{\Pi\, [c_1y_i + 1]\ \exp\{-\sum (c_1\frac{y_i^2}{2} + y_i)\}},$$

which simplifies to

$$\frac{\Pi\,(c_0y_i + 1)}{\Pi\,(c_1y_i + 1)}\exp\{-(c_0-c_1)\sum \frac{y_i^2}{2}\}\ . \tag{30}$$

The random samples would come from a distribution with a probability density function

$$f(x) = (c_0x + 1)\exp - (\frac{c_0x^2}{2} + x)\ ,$$

and k would be chosen such that only $\alpha(100)$ percent of the time (30) is greater than k.

2. Tests on the Parameter a with b Unknown.

A test for a more specific hypothesis $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$ with b unknown is to use

$$\hat{b}_s = + \sum_{i=1}^{r} |y_i - y_s| / n\, k_{s,r,n}\ , \tag{31}$$

where s is chosen to minimize the variance of $\hat{b}_s$, $k_{s,r,n}$ is an unbiasing constant, $y_i = \ln x_i$, and $x_1, x_2, \cdots x_r$ are the first r ordered observations. The estimator

$$\hat{b} = - \sum_{i=1}^{r-1} (y_i - y_r) \; / \; n \, k_{r,n}$$

was first proposed by Bain [13] for estimating the shape parameter in the Weibull distribution or the scale parameter in the Extreme Value distribution. This estimator was later modified by Bain and Engelhardt [14] to $\hat{b}_s$ so that the unbiased estimator would be acceptable for all levels of censoring. The original estimator $\hat{b}$ had been good only for a high amount of censoring.

Under H_0, x_i is distributed as an exponential, and $y_i = \ln x_i$ is distributed as a variable from the extreme value distribution with scale parameter $b = 1$. Since $\hat{b}_s$ is distributed independently of all parameters in the extreme value distribution, under H_0, $\hat{b}/b = \hat{b}_s$ is distributed independently of all parameters, and approximately $h \, \hat{b}_s \sim \chi^2(h)$ where $h = 2/\mathrm{Var}\,(\hat{b}_s)$. Table 3 in [14] gives values for h for the complete sample case, and Table 5 gives values for h in the censored sample case. Monte Carlo work was performed to determine that H_0 is rejected if $h \, \hat{b}_s < \chi^2_\alpha\,(h)$.

The exact distribution of

$$s = \exp[-n \, k_{r,n} \, \hat{b}/b]$$

$$= \prod_{i=1}^{r-1} (x_i/x_r)^{1/b}$$

is derived for $r = 2$, 3 by Bain [13]. For $r = 2$, and $b = 1$,

$$s = x_1/x_2 \ ,$$

which is the test statistic (27).

Since the test $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$ is a test for exponentiality versus a linearly increasing failure rate, another logical choice for a statistic would be one that has proved good for testing exponentiality versus any distribution with an increasing failure rate, such as the Weibull when the exponent $b > 1$. Hager, Bain, and Antle [15] have considered this problem and concluded that both $M = -2 [\sum \ln x_i - n \ln \overline{x}]$ and $\hat{b}$, the maximum liklihood estimator of the parameter b from the Weibull distribution, suggested by Thoman, Antle, and Bain [16], perform better than the other statistics they considered for the case mentioned above and also for certain Gamma alternatives. Table 2 in [15] gives the critical values for the M' statistic for $\gamma = .90, .95, .98, .99$ and $n = 10, 20, 30, 50, 100$ where $M' = \exp [-(M/2n)]$. The same tail is used for rejecting H_0 as was used by Hager, Bain, and Antle [15] for their tests.

Another statistic for testing $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$ is

$$S_1(n) = \frac{\sum x_i^2}{(\sum x_i)^2} . \tag{32}$$

Under H_0, $f(x) = b \exp(-bx)$, and if the transformation $y = bx$ is made, the random variable γ is distributed as a standard exponential. Since

$$S_1(n) = \frac{\sum b^2 x_i^2}{(\sum b x_i)^2}$$

the statistic can be expressed in terms of the standard exponential, and so is distributed only as a function of n. Table 2 gives critical values for $n = 5(1)10, 10(2)30, 30(5)60$ and $\alpha = .01, .05, .1$ based on 1000 Monte Carloed samples.

TABLE 2. Cumulative percentage points of $\sum x_i^2$ and $\sum x_i^2/(\sum x_i)^2$ values of $c_{1-\alpha,n}$ such that

$P[\sum x_i^2/(\sum x_i)^2 \leq c_{1-\alpha,n}] = \alpha$ and $P[b^2 \sum x_i^2 \leq c_{1-\alpha,n}] = \alpha.$

1-α / n	.90	.95	.99	.90	.95	.99
5	.2429	.2311	.2115	1.9611	1.2192	.5922
6	.2139	.1990	.1839	2.6124	1.6435	.9518
7	.1871	.1783	.1669	4.080	2.7147	.9747
8	.1670	.1582	.1426	4.6694	3.6470	1.9629
9	.1507	.1432	.1293	5.9466	4.2175	2.2829
10	.1415	.1342	.1267	6.6394	4.5573	2.3425
12	.1166	.1121	.1060	9.3048	6.6974	3.9529
14	.1048	.1010	.0931	11.7848	9.1271	5.0136
16	.0927	.0883	.0798	13.5005	11.1445	6.9672
18	.0827	.0790	.0737	14.8530	12.0681	7.9450
20	.0764	.0736	.0675	18.3924	14.9023	11.1801
22	.0710	.0680	.0632	20.1671	17.5530	12.6446
24	.0658	.0622	.0554	24.8621	20.5711	13.2549
26	.0510	.0576	.0527	27.4664	23.7874	15.4541
28	.0564	.0543	.0508	28.9050	26.0775	19.7788
30	.0534	.0512	.0480	32.5475	28.7153	22.8365
35	.0458	.0443	.0415	39.7205	33.8480	25.4995
40	.0410	.0390	.0368	46.6766	42.6221	32.4567
45	.0365	.0353	.0334	57.4834	50.0298	38.5423
50	.03320	.0321	.0298	61.2942	52.9768	38.8960
55	.0305	.0295	.0281	71.1136	62.4577	51.1092
60	.02838	.0274	.0254	78.6230	70.3753	52.2778

A similar statistic to $S_1(n)$ is

$$S_2(n) = b^2 \sum x_i^2 \quad , \tag{33}$$

for testing $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$. Clearly, $S_2(n)$ is also distributed independently of b. However, to compute $S_2(n)$, b must be known. Table 2 gives critical values for n = 5(1)10, 10(2)30, 30(5)60 and α = .01, .05, .1 based on 1000 Monte Carloed samples. As might be expected, Monte Carlo studies show that the test using S_2 is more powerful than the test using S_1.

A more general test than $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$ would be a test for a constant failure-rate versus an increasing or decreasing failure-rate that is not necessarily linear. Gnedenko, et al [17] consider such a test and suggest the statistic

$$(n-r) \sum_{i=1}^{r} [(n-i+1)(x_i - x_{i-1})]/[r \sum_{i=r+1}^{n} (n-i+1)(x_i - x_{i-1})] \quad ,$$

where the x_i are ordered observations, $x_i \leq x_{i+1}$, and the statistic is distributed as an F with 2r and 2(n-r) degrees of freedom under H_0. Monte Carlo investigations by Fercho and Ringer [17] show this test with $\frac{r}{n} \doteq .5$ to be most powerful, in general, among four tests for constant failure rate that they considered. Therefore, it would seem best to choose r such that $\frac{r}{n} \doteq 0.5$, where it is noted that in this case r does not refer to censoring, but instead refers to the computation of the statistic.

3. Tests on the Parameter b

A test for $H_0 : b = b_0$ versus $H_1 : b = b_1 > b_0$ can be made using either of the statistics

$$[\frac{a}{2} x_1^2 + b_0 x_1] / [\frac{a}{2} x_2^2 + b_0 x_2]$$

or

$$2 [\sum_{i=1}^{r} x_i + (n-r) x_r]$$

suggested for testing a, since the distribution of either of these statistics depends only on the fact that $y = \frac{a}{2} x^2 + bx$ is distributed as a standard exponential variable. Therefore, the same critical value is used, and the test is made in the same manner.

A test for $H_0 : b = 0$ versus $H_1 : b = b_1 > 0$ with a known can be made using any of the statistics mentioned for testing $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$ with b known simply by letting $y_i = x_i^2/2$, since under H_0, y_i is distributed as an exponential variable just as x_i is under H_0 when testing $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$. The power of the test is of course changed.

The test for $H_0 : b = b_0$ versus $H_1 : b = b_1 > b_0$ using the Neyman-Pearson Lemma has no simple conversion from the test on a, unless $b_0 = 0$, in which case the methods of the proceeding paragraph apply. In the case that $b_0 \neq 0$, H_0 is accepted with probability α of a type I error if

$$\frac{\Pi (ax_i + b_0) \exp [-\sum(\frac{a}{2} x_i^2 + b_0 x_i)]}{\Pi (ax_i + b_1) \exp [-\sum(\frac{a}{2} x_i^2 + b_1 x_i)]} \leq k , \qquad (34)$$

and is rejected otherwise, where k is chosen most likely after some Monte Carlo simulations. In order to make k a function of four instead of five (n, α, a, b_0, b_1) variables, let $d^2 = a$ and rewrite

(33) as

$$\frac{\Pi\ d(dx + \frac{b_0}{d})\ \exp\left\{-\sum\left[\frac{(dx)^2}{2} + \frac{b_0(dx)}{d}\right]\right\}}{\Pi\ d(dx + \frac{b_1}{d})\ \exp\left\{-\sum\left[\frac{(dx)^2}{2} + \frac{b_1(dx)}{d}\right]\right\}} ,$$

which upon letting $y = dx$, $c_0 = b_0/d$, and $c_1 = b_1/d$ becomes

$$\frac{\Pi(y_i + c_0)\ \exp\left\{-\sum\left[\frac{y_i^2}{2} + c_0 y_i\right]\right\}}{\Pi(y_i + c_1)\ \exp\left\{-\sum\left[\frac{y_i^2}{2} + c_1 y_i\right]\right\}}$$

which simplifies to

$$\frac{\Pi(y_i + c_0)}{\Pi(y_i + c_1)}\ \exp\left\{-\frac{(c_0 - c_1)}{2}\sum y_i\right\} .$$

The random sample would come from a distribution with a probability density function

$$f(x) = (c_0\ x + 1)\ \exp\ [-(\frac{c_0}{2}\ x^2 + x)] .$$

A more general test than $H_0 : b = 0$ versus $H_1 : b = b_1 > 0$ would be a goodness of fit test for the variable coming from the Weibull distribution versus some other distribution. N. Marm and E. Schuer [17] present such a statistic with

$$L(r, s, m, u) = \frac{1}{r}\sum_{j=m-r}^{m-1} \ell_j \Big/ \left[\frac{1}{s}\sum_{j=1}^{s} \ell_j\right]$$

where

$$\ell_i = (x_{i+1,n} - x_{i,n}) / E(x_{i+1,n} - x_{i,n}) , \quad i = 1, \cdots n-1 ,$$

the x_i are ordered observations, the sample is censored at the r^{th} of n observations, and $r + x \leq m \leq n$. They give values for the expected values for n = 3(1)25 and critical regions for various levels of significance, and censoring for m = 3(1)n.

4. Comparisons of the Various Test Statistics

Table 3 gives the resulting powers of various tests obtained from a Monte Carlo simulation of 500 samples, each of size 20. The powers are for the test $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$ with the significance level $\alpha = .05$. To compare the power for various alternatives, $a_1 = .01, .1, 1.0$ and $b = .01, .1, 1.0$ were considered. From the results obtained from 250 samples which were generated under H_0, table 3 appears to be accurate to $\pm$ 3%.

From table 3 it can be seen that for all tests, as a is increased, the power of the test increases. Furthermore, if a is held fixed, the power decreases as b is increased.

For all alternatives, the test which uses the statistic (27) appears to have the lowest power of all the tests. The test statistic (28) with only the first two observations has better power than the proceeding. Of course, (27) has the advantage when $H_0 : a = 0$ that the test can be made with b unknown.

Of the two statistics (28) and (31) that can be readily used for all levels of censoring, (28) was found to consistently give the higher power. The test (28) can be used for testing $H_0 : a = a_0$ versus $H_1 : a = a_1$ with b known, whereas the test (31) can only be used for testing $H_0 : a = 0$ versus $H_1 : a = a_1 > 0$.

In complete sampling, the tests (28), (29), (32) and (33) were considered. The test (33) seems to be preferable to (28) but has the disadvantage of requiring special tables. For b unknown, the test (32) is quite satisfactory.

TABLE 3. Power for various tests of $H_0 : a = 0$ versus $H_1 : a > 0$, $a = .01, .1, 1.0$ and $b = .01, .1, 1.0$ with $\alpha = .05$ and $n = 20$ based on 500 Monte Carloed samples*

Test Statistic	a = .01			a = .1			a = 1.0		
	b = .01	b = .1	b = 1.0	b = .01	b = .1	b = 1.0	b = .01	b = .1	b = 1.0
(27)	10	7	5	7	5	4	9	7	5
(28) $r = 2$	14	4	5	85	8	4	100	17	3
(28) $\frac{r}{n} = .5$	100	11	4	100	88	5	100	100	8
(28) $r = n$	100	53	5	100	100	8	100	100	55
(29)	100	70	6	100	100	9	100	100	83
(31) $\frac{r}{n} = .5$	43	8	5	58	25	4	64	45	8
(31) $r = n$	88	27	6	95	69	10	96	87	24
(32)	93	30	6	96	75	10	97	90	40
(33)	100	66	3	100	100	8	100	100	68

*Power expressed as percent.

III. MODELS WITH U-SHAPED FAILURE-RATES

A. Quadratic Failure-Rate Model

1. Range for the Parameters

The model with the failure-rate function $h(x) = ax^2 + bx + c$ can yield U-shaped failure rates. Since $h(x)$ must be non-negative for all positive x, the parameters a and c must be non-negative, and b can be non-negative. However, if $h(x)$ is to be U-shaped in the first quadrant, then it must have a minimum in this quadrant, which is equivalent to $h'(x) = 0$ for some non-negative x. In this case, $h'(x) = 0$ when $x = -b/2a$, so $b < 0$ when $h(x)$ is U-shaped. However, when $b < 0$, $b^2 - 4ac < 0$ or else there would be at least one root on the positive x-axis which would mean $h(x) < 0$ for some non-negative interval. This results in $b > -2\sqrt{ac}$. Therefore,

$$f(x) = (ax^2 + bx + c)\exp[-(\frac{ax^3}{3} + \frac{bx^2}{2} + cx)]; x > 0,\ a \geq 0,\ c \geq 0,$$

$$b > -2\sqrt{ac};$$

$$F(x) = 1 - \exp[-(\frac{ax^3}{3} + \frac{bx^2}{2} + cx)]; x > 0,\ a \geq 0,\ c \geq 0,\ b > -2\sqrt{ac};$$

and

$$R(x) = \exp[-(\frac{ax^3}{3} + \frac{bx^2}{2} + cx)];\ x > 0,\ a \geq 0,\ c \geq 0,\ b > -2\sqrt{ac}$$

2. Maximum Likelihood Estimates

The likelihood function in this case is

$$L(x;a,b,c) = \{\Pi(ax_i^2 + bx_i + c)\}\ \exp\ [-\Sigma(\frac{ax_i^3}{3} + \frac{bx_i^2}{2} + cx_i)]$$

and

$$\ln L(x;a,b,c) = \Sigma \ln(ax_i^2 + bx_i + c) - \Sigma(\frac{ax_i^3}{3} + \frac{bx_i^2}{2} + cx_i). \tag{1}$$

Differentiating (1) with respect to a, b, and c gives

$$\frac{\partial \ln L(x;a,b,c)}{\partial a} = \sum \frac{x_i^2}{ax_i^2 + bx_i + c} - \frac{\Sigma x_i^3}{3}, \tag{2}$$

$$\frac{\partial \ln L(x;a,b,c)}{\partial b} = \sum \frac{x_i}{ax_i^2 + bx_i + c} - \frac{\Sigma x_i^2}{2} \tag{3}$$

and $$\frac{\partial \ln L(x;a,b,c)}{\partial c} = \sum \frac{1}{ax_i^2 + bx_i + c} - \Sigma x_i . \tag{4}$$

The values $\hat{a}$, $\hat{b}$ and $\hat{c}$ that satisfy (2), (3) and (4) when the partials are equated with zero are the maximum likelihood estimates.

The above set of three equations in three unknowns can be reduced to a set of two equations in two unknowns by rewriting (2) as

$$\frac{1}{a} [\sum \frac{ax_i^2 + bx_i + c}{ax_i^2 + bx_i + c} - \sum \frac{bx_i + c}{ax_i^2 + bx_i + c}] - \frac{\Sigma x_i^3}{3} = 0, \tag{5}$$

and substituting from (3)

$$\sum \frac{x_i}{ax_i^2 + bx_i + c} = \frac{\Sigma x_i^2}{2}$$

into (5) to give

$$\frac{1}{\hat{a}} [n - \hat{b} \frac{\Sigma x_i^2}{2} - \hat{c}\Sigma x_i] - \frac{\Sigma x_i^3}{3} = 0,$$

which can be rewritten as

$$n - \frac{\hat{b}\Sigma x_i^2}{2} - \hat{c}\Sigma x_i - \frac{a\Sigma x_i^3}{3} = 0,$$

or

$$\hat{c} = \frac{n - \frac{\hat{b}}{2}\Sigma x_i{}^2 - \frac{\hat{a}}{3}\Sigma x_i^3}{\Sigma x_i}. \quad (6)$$

Substituting (6) into (4) gives

$$\sum \frac{1}{\hat{a}((\Sigma x_i)x_i^2 - \frac{1}{3}\Sigma x_i^3) + \hat{b}((\Sigma x_i)x_i - \frac{1}{2}\Sigma x_i^2) + n} - 1 = 0 \quad (7)$$

and substituting (6) into (3) gives

$$\sum \frac{x_i}{\hat{a}((\Sigma x_i)x_i^2 - \frac{1}{3}\Sigma x_i^3) + \hat{b}((\Sigma x_i)x_i - \frac{1}{2}\Sigma x_i^2) + n} - \frac{\Sigma x_i^2/2}{\Sigma x_i} = 0. \quad (8)$$

An iterative procedure between (7) and (8), modifying first $\hat{a}$ and the $\hat{b}$ until the equalities are sufficiently close and evaluating $\hat{c}$ from (6) seems to be the best procedure for obtaining the maximum likelihood estimates.

3. Maximum Agreement Estimates

Using the same technique as in the case of the model with the linear failure rate, the problem of estimating the parameters in this case becomes the problem of minimizing

$$A(x;a,b,c) = \sum \left[\frac{ax_i^3}{3} + \frac{bx_i^2}{2} + cx_i + \ln(1 - \frac{i}{n+1})\right]^2. \quad (9)$$

Differentiating (9) with respect to a, b, and c gives

$$\frac{\partial A}{\partial a} = \frac{2}{3} \sum x_i^3 \left[\left(\frac{a}{3} x_i^3 + \frac{b}{2} x_i^2 + cx_i\right) + \ln\left(1 - \frac{i}{n+1}\right)\right] ,$$

$$\frac{\partial A}{\partial b} = \frac{2}{2} \sum x_i^2 \left[\left(\frac{a}{3} x_i^3 + \frac{b}{2} x_i^2 + cx_i\right) + \ln\left(1 - \frac{i}{n+1}\right)\right] ,$$

and

$$\frac{\partial A}{\partial c} = 2 \sum x_i \left[\left(\frac{a}{3} x_i^3 + \frac{b}{2} x_i^2 + cx_i\right) + \ln\left(1 - \frac{i}{n+1}\right)\right] .$$

Therefore, the values $\hat{a}$, $\hat{b}$, and $\hat{c}$ that satisfy

$$\frac{\hat{a}}{3} \sum x_i^6 + \frac{\hat{b}}{2} \sum x_i^5 + \hat{c} \sum x_i^4 + \sum x_i^3 \ln\left(1 - \frac{i}{n+1}\right) = 0 ,$$

$$\frac{\hat{a}}{3} \sum x_i^5 + \frac{\hat{b}}{2} \sum x_i^4 + \hat{c} \sum x_i^3 + \sum x_i^2 \ln\left(1 - \frac{i}{n+1}\right) = 0 ,$$

and

$$\frac{\hat{a}}{3} \sum x_i^4 + \frac{\hat{b}}{2} \sum x_i^3 + \hat{c} \sum x_i^2 + \sum x_i \ln\left(1 - \frac{i}{n+1}\right) = 0$$

are the maximum agreement estimators.

This system can be solved for $\hat{a}$, $\hat{b}$, and $\hat{c}$ by various methods, including Cramer's method.

Another approach to obtaining maximum agreement estimates, which was referred to in Chapter II, Section C as the second method, is to let $u(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$ and then minimize

$$A(x;a,b,c) = \sum \left[\left(\frac{ax_i^3}{3} + \frac{bx_i^2}{2} + cx_i\right) - E\left(\frac{ax_i^3}{3} + \frac{bx_i^2}{2} + cx_i\right)\right]^2 .$$

Following the steps of Chapter II, Section C yields the system

$$\frac{\hat{a}}{3} \sum x_i^6 + \frac{\hat{b}}{2} \sum x_i^5 + \hat{c} \sum x_i^4 - \sum [x_i^3 \sum_{j=1}^{i} \frac{1}{n-j+1}] = 0 ,$$

$$\frac{\hat{a}}{3} \sum x_i^5 + \frac{\hat{b}}{2} \sum x_i^4 + \hat{c} \sum x_i^3 - \sum [x_i^2 \sum_{j=1}^{i} \frac{1}{n-j+1}] = 0 ,$$

and

$$\frac{\hat{a}}{3} \sum x_i^4 + \frac{\hat{b}}{2} \sum x_i^3 + \hat{c} \sum x_i^2 - \sum [x_i \sum_{j=1}^{i} \frac{1}{n-j+1}] = 0 .$$

The values $\hat{a}$, $\hat{b}$, and $\hat{c}$ which satisfy the above system are the maximum agreement estimate by the second method.

B. Model with Failure-Rate $h(x) = c \cosh [b(x-a)]$

1. Maximum Likelihood Estimators

If

$$h(x) = bc \cosh [b(x-a)]; \quad x \geq 0 , \quad a \geq 0 , \quad |b| < \infty , \quad c > 0,$$

then since

$$\int_0^x h(t)dt = c \sinh [b(x-a)] ,$$

$$F(x) = 1 - \exp \{-c \sinh [b(x-a)]\} ,$$

and

$$f(x) = bc \cosh [b(x-a)] \exp \{-c \sinh [b(x-a)]\} .$$

The estimates of the parameters which maximize

$$L(x;a,b,c) = (bc)^n \{\Pi \cosh [b(x_i-a)]\} \exp \{-c \sum [b(x_i-a)]\}$$

satisfy

$$\sum_{j=1}^{n} \sinh \hat{b}(x_j-\hat{a}) \prod_{\substack{i=1 \\ i \neq j}}^{n} \cosh \hat{b}(x_i-a) - \hat{c} \prod_{i=1}^{n} \cosh \hat{b} (x_i-\hat{a}) \sum_{i=1}^{n} \cosh \hat{b}(x_i-\hat{a}) = 0,$$

$$n \prod_{i=1}^{n} \cosh \hat{b}(x_i-\hat{a}) + \hat{b} \prod_{j=1}^{n} (x_j-\hat{a}) \sinh \hat{b}(x_j-\hat{a}) \prod_{\substack{i=1 \\ i \neq j}}^{n} \cosh \hat{b}(x_i-\hat{a})$$

$$+ \hat{b}\hat{c} \cosh \hat{b}(x_i-\hat{a}) \sum_{i=1}^{n} (x_i-\hat{a}) \cosh \hat{b}(x_i-\hat{a}) = 0$$

and

$$n - \hat{c} \sum_{i=1}^{n} \sinh \hat{b}(x_i-\hat{a}) = 0 .$$

2. Agreement Estimates

Using the same technique as in the case of the linear failure rate, we let

$$u(x_i) = 1 - \exp \{-c \sinh[b(x_i-a)]\}$$

and the function that we want to minimize is

$$\sum [\exp \{-c \sinh[b(x_i-a)]\} - (1 - \frac{i}{n+1})]^2 .$$

As in the previous cases, since the function is not linear in the parameters and therefore does not yield estimates of the parameters in closed form, we consider

$$A(x;a,b,c) = \sum [b(x_i-a) + \sinh^{-1}(\frac{1}{c} \ln(1 - \frac{1}{n+1})]^2 \quad (10)$$

since $\sinh^{-1}$ is a monotone function. Since

$$\sinh^{-1}(\frac{y}{a}) = \ln(\frac{y + \sqrt{y^2 + a^2}}{a}),$$

(10) becomes

$$\sum [b(x_i-a) + \ln\{\frac{\ln(\frac{n+1-i}{n+1}) + \sqrt{\ln(\frac{n+1-i}{n+1}) + c^2}}{c}\}]^2 . \quad (11)$$

Differentiating (10) with respect to a and b gives

$$\frac{\partial A}{\partial a} = -2b \sum [b(x_i-a) + y_i], \quad (12)$$

$$\frac{\partial A}{\partial b} = 2 \sum x_i[b(x_i-a) + y_i]. \quad (13)$$

The values $\hat{a}$ and $\hat{b}$ that satisfy

$$\hat{b} \sum [\hat{b}(x_i-\hat{a}) + y_i] = 0 \quad (14)$$

and

$$\sum x_i[\hat{b}(x_i-\hat{a}) + y_i] = 0 \quad (15)$$

where

$$y_i = \ln\{\frac{\ln(\frac{n+1-i}{n+1}) + \sqrt{(\ln(\frac{n+1-i}{n+1}))^2 + c'^2}}{c'}\} \quad (16)$$

are the maximum agreement estimators.

If $b \neq 0$, (14) can be rewritten as

$$\hat{b} = \Sigma y_i/(n\hat{a} - \Sigma x_i) \quad (17)$$

and then substituting (17) into (15) gives

$$\hat{a} = \frac{\Sigma x_i^2 \, \Sigma y_i - \Sigma x_i \, \Sigma x_i y_i}{\Sigma x_i \, \Sigma y_i - n \, \Sigma x_i y_i} . \quad (18)$$

Now for a given value of c', which is not necessarily the true value of c, $\hat{a}$ can be obtained explicitly from (18), and then $\hat{b}$ can be obtained from (17).

Two estimates of c,

$$\hat{c}_1 = \frac{-\sum [\sinh (\hat{b}(x_i-\hat{a}))\ \ln(\frac{n+1-i}{n+1})]}{\sum [\sinh (\hat{b}(x_i-\hat{a}))]^2} \qquad (19)$$

and

$$\hat{c}_2 = \frac{n}{\sum \sinh (\hat{b}(x_i-\hat{a}))} \qquad (20)$$

were considered. The first estimator, (19), minimizes

$$\sum[-c \sinh b(x_i-a) - \ln(\frac{n+1-i}{n+1})]^2$$

with respect to c, and the second estimator (20) is the maximum likelihood estimator. Some Monte Carlo runs were made to determine which estimator was preferable. The results were inconclusive, so (20) is probably the better choice since it is easier to compute.

C. Model with Failure-Rate $h(x) = \frac{\alpha}{\beta} (\frac{x}{\beta})^{\alpha-1} + \frac{\alpha\gamma}{\beta} (\frac{x}{\beta})^{\alpha-1} \exp (\frac{x}{\beta})^{\alpha}$

1. Range for the Parameters

Another possibility for a hazard function that can have a U-shape is

$$h(x) = \frac{\alpha}{\beta} (\frac{x}{\beta})^{\alpha-1} + \frac{\alpha\gamma}{\beta} (\frac{x}{\beta})^{\alpha-1} \exp (\frac{x}{\beta}) \ ; \ \alpha, \beta, \gamma > 0, \ x > 0.$$

The corresponding density function is

$$F(x) = 1 - \exp \{-(\frac{x}{\beta})^{\alpha} - \gamma[\exp (\frac{x}{\beta})^{\alpha} - 1]\}; \ \alpha, \beta, \gamma > 0, \ x > 0.$$

This distribution can be thought of as describing two components in series where the failure of either component causes the failure of the system. In this case, one component fails according to the Weibull distribution where

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]; \alpha, \beta > 0, x > 0,$$

and the other component fails according to the distribution whose density function is

$$F(x) = 1 - \exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]; \alpha, \beta > 0, x > 0,$$

and the other component fails according to the distribution whose density function is

$$F(x) = 1 - \exp\left\{-\gamma\left[\exp\left(\frac{x}{\beta}\right)^{\alpha} - 1\right]\right\}; \alpha, \beta, \gamma > 0, x > 0.$$

When $\alpha = 1.0$, this becomes the Gompertz distribution [5]. Therefore, this distribution will be referred to as a Gompertz type.

The parameter γ serves the purpose of determining the mixture between the Weibull and the Gompertz type failure-rates. The closer γ is to zero, the more the failure rate behaves like a Weibull, where as with γ large, the failure rate behaves more like that from a Gompertz type distribution. Finally, it should be noted that $1/\alpha$ and $\ln\beta$ are scale and location parameters respectively if the transformation $y = \ln x$ is made.

In order for $h(x)$ to have a U-shape, it is necessary for $0 < \alpha < 1$ so that the first term decreases as x increases. Furthermore, γ must be positive, and the larger γ, the faster the curve will increase. The value of x for which

$$\exp\left(\frac{x}{\beta}\right)^{\alpha} = \left(\frac{x}{\beta}\right)^{\alpha-1}\left[1 + \gamma \exp\left(\frac{x}{\beta}\right)^{\alpha}\right]$$

is the minimum point of the curve.

2. Maximum Likelihood Estimators

The likelihood function is given by

$$L(x;\alpha,\beta,\gamma) = \left(\frac{\alpha}{\beta}\right)^{n} \Pi\left(\frac{x_i}{\beta}\right)^{-1}\left[1 + \gamma e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}\right] \exp - \sum\left\{\left(\frac{x_i}{\beta}\right)^{\alpha} + \left[e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right]\right\}\Big]$$

and the log of the likelihood function is given by

$$\ln L(x;\alpha,\beta,\gamma) = n \ln \alpha - n \ln \beta + (\alpha-1) \sum \ln x_i - n(\alpha-1) \ln \beta +$$

$$\sum \ln \left[1 + \gamma e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}\right] - \sum \left[\left(\frac{x_i}{\beta}\right)^{\alpha} + \gamma\left(e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)\right] \quad . \tag{21}$$

The values $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\gamma}$ which maximize (21) satisfy the equations

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum \ln x_i - n \ln \beta + \sum \frac{\gamma\left(\frac{x_i}{\beta}\right)^{\alpha+1} e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}}{1 + \gamma e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}}$$

$$- \sum \left\{\ln\left(\frac{x_i}{\beta}\right)\left(\frac{x_i}{\beta}\right)^{\alpha} + \gamma\left[\left(\frac{x_i}{\beta}\right)^{\alpha+1} e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right]\right\} = 0 \tag{22}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{-n}{\beta} - \frac{n(\alpha-1)}{\beta} + \sum \frac{\left(-\frac{\alpha}{\beta}\right)\left(\frac{x_i}{\beta}\right)^{\alpha} e^{\left(\frac{x_i}{\beta}\right)^{\alpha}}}{1 + \gamma e^{\left(\frac{x}{\beta}\right)^{\alpha}}}$$

$$- \sum \left[- \frac{\alpha}{\beta} \left(\frac{x_i}{\beta}\right)^{\alpha} + \gamma\left(- \frac{\alpha}{\beta} \left(\frac{x_i}{\beta}\right)^{\alpha} e^{\left(\frac{x_i}{\beta}\right)^{\alpha}} - 1\right)\right] \tag{23}$$

and

$$\frac{\partial \ln L}{\partial \gamma} = \sum \frac{e^{(\frac{x_i}{\beta})^\alpha}}{1 + \gamma e^{(\frac{x_i}{\beta})^\alpha}} - \sum (e^{(\frac{x_i}{\beta})^\alpha} - 1) = 0 \qquad (24)$$

and are called the maximum likelihood estimators.

Since $\ln \beta$ and $1/\alpha$ may be regarded as location and scale parameters, respectively, if the transformation $y = \ln x$ is made, by employing the theorem of Antle and Bain [18], it follows that $(\widehat{1/\alpha})/(1/\alpha)$, $(\widehat{\ln \beta} - \ln \beta)\alpha$ and $(\widehat{\ln \beta} - \ln \beta)\hat{\alpha}$ have distributions that are independent of the parameters. By the invariance property of maximum likelihood estimators, $\alpha/\hat{\alpha}$, $(\hat{\beta}/\beta)^{\hat{\alpha}}$, and $(\hat{\beta}/\beta)^{\hat{\alpha}}$ are also distributed independently of all parameters. Therefore, it is possible to Monte Carlo random samples from this distribution and compute the maximum likelihood estimates so that the distributions of $\alpha/\hat{\alpha}$, $(\hat{\beta}/\beta)^{\hat{\alpha}}$ and $(\hat{\beta}/\beta)^{\hat{\alpha}}$ can be approximated to enable one to make various confidence intervals and tests of hypotheses.

If α and β are known, then the distribution of $\hat{\gamma}$ could be approximated by Monte Carlo techniques as a function of γ and n, and confidence intervals and tests of hypotheses could be made on γ. Furthermore, as the following general theorem shows, $\hat{\gamma}$ is distributed independently of α and β even if they are unknown, so that confidence intervals can be determined for γ even in this case by approximating the distribution of $\hat{\gamma}$.

Theorem: Let $x_1, \cdots, x_n$ be a random sample from a distribution where x is a continuous variate whose density is of the form

$$f(x;\alpha,\beta,\overline{\gamma}) = \frac{1}{\beta}\, g((x-\alpha)/\beta,\overline{\gamma}), \qquad (\alpha,\beta,\overline{\gamma}) \in \Omega$$

$$-\infty < \alpha < \infty, \quad 0 < \beta$$

$$\overline{\gamma} \in \Gamma \text{ is a vector}$$

(i.e., α and β are location and scale parameters, respectively). For the likelihood function given by

$$L(x;\alpha,\beta,\overline{\gamma}) = \Pi\, f(x_i;\alpha,\beta,\overline{\gamma})$$

$$= \frac{1}{\beta^n}\, \Pi\, g((x_i-\alpha)/\beta,\overline{\gamma}) \ .$$

the maximum likelihood estimator $\hat{\overline{\gamma}}$ is distributed independently of α and β.

Proof: The maximum likelihood estimators of α, β, $\overline{\gamma}$ are the values $\hat{\alpha}$, $\hat{\beta}$, $\hat{\overline{\gamma}}$ which satisfy

$$L(x;\hat{\alpha},\hat{\beta},\hat{\overline{\gamma}}) = \max_{\Omega}\, L(x;\alpha,\beta,\overline{\gamma}) \quad ,$$

or

$$(\hat{\beta})^{-n}\, \Pi\, g((x_i-\hat{\alpha})/\hat{\beta},\hat{\overline{\gamma}}) = \max_{\Omega}\, \beta^{-n}\, \Pi\, g((x_i-\alpha)/\beta,\overline{\gamma}) \ .$$

This is the same as

$$\beta_0^{-n}\, \left(\frac{\hat{\beta}}{\beta_0}\right)^{-n}\, \Pi\, g\left(\frac{\beta_0}{\hat{\beta}}\, (x_i - \alpha_0 - (\hat{\alpha}-\alpha_0))/\beta_0,\hat{\overline{\gamma}}\right) \ ,$$

which is

$$\max_{\Omega} \beta_o^{-n} \left(\frac{\beta}{\beta_o}\right)^{-n} \Pi\, g\left(\frac{\beta_o}{\beta} (x_i-\alpha_o- (\alpha-\alpha_o))/\beta_o, \bar{\gamma}\right) ,$$

or

$$(\hat{\beta}_s)^{-n} \Pi\, g((z_i-\hat{\alpha}_s)/\hat{\beta}_s, \hat{\bar{\gamma}}) = \max_{\Omega^*} (\beta^*)^{-n} \Pi\, g((z_i-\alpha^*)/\beta^*, \bar{\gamma})$$

where $\alpha^* = (\alpha-\alpha_o)/\beta_o$, $\beta^* = \beta/\beta_o$ and $z_i = (x_i-\alpha_o)/\beta_o$, and α_o and β_o are the true values of the parameters. Since $\Omega^* = \Omega$, $\hat{\alpha}_s$ and $\hat{\beta}_s$ correspond to the maximum likelihood estimators of α and β when the sampling is actually on the standardized variate z_i. Therefore $\hat{\bar{\gamma}}$ is distributed independently of α and β.

IV. ESTIMATION AND INFERENCES ON THE LOGISTIC DISTRIBUTION

A. Estimation of the Parameters of the Logistic Distribution

The problem of estimating the parameters of the Logistic distribution has been considered by a number of authors. Harter and Moore [19] have considered the problem of obtaining the maximum likelihood estimates from censored samples using a computer. Gupta, Qureishi, and Shah [20] have considered best linear unbiased estimators for samples of $n \leq 25$.

Here the method of maximum agreement will be considered. This method has the desirable features of being very easy to compute, of not requiring any tables, and of working for censored sampling.

One possible set of maximum agreement estimators would be the particular values which minimized

$$A = \Sigma \{F(x_i) - E[F(x_i)]\}^2 \qquad (1)$$

where

$$F(x) = \frac{1}{\{1 + \exp[-(x-\mu)/\sigma']}, \quad \sigma' = \frac{\sqrt{3}}{\Pi}\sigma .$$

Since finding these particular values would prove very difficult, (1) is modified by first taking the reciprocal and then the natural logarithm of each term to obtain

$$A = \Sigma \{\frac{x-\mu}{\sigma'} + \ln(\frac{n+1-i}{i})\}^2 . \qquad (2)$$

The estimators which minimize (2) are not actually maximum agreement estimators in that the agreement function is not of the form $u(x) - E(u(x))$ but it will be shown that these estimators compare very favorably with one maximum agreement estimator.

To find the values which minimize A with respect to μ and σ', the partials

$$\frac{\partial A}{\partial \mu} = \frac{-2}{\sigma'} [\frac{(x_i-\mu)}{\sigma'} + \ln (\frac{n+1-i}{i})]$$

and

$$\frac{\partial A}{\partial \sigma'} = \frac{-2}{\sigma'^2} \sum (x_i-\mu) [\frac{(x_i-\mu)}{\sigma'} + \ln (\frac{n+1-i}{i})]$$

are set to zero to give the two simultaneous equations

$$\sum x_i - n\hat{\mu} + \sigma' \sum \ln (\frac{n+1-i}{i}) = 0 \qquad (3)$$

and

$$\sum (x_i-\hat{\mu})^2 + \hat{\sigma'} \sum x_i \ln (\frac{n+1-i}{i}) - \hat{\mu}\hat{\sigma'} \sum \ln (\frac{n+1-i}{i}) = 0 . \qquad (4)$$

Solving (4) for σ' gives

$$\hat{\sigma'} = \frac{\sum (x_i-\hat{\mu})^2}{[\hat{\mu} \sum \ln (\frac{n+1-i}{i}) - \sum x_i \ln (\frac{n+1-i}{i})]} \qquad (5)$$

and substituting this into the first equation gives

$$\sum x_i - n\hat{\mu} + [\frac{\sum (x_i-\hat{\mu})^2}{\hat{\mu} \sum \ln (\frac{n+1-i}{i}) - \sum x_i \ln (\frac{n+1-i}{i})}] \sum \ln (\frac{n+1-i}{i}) = 0 .$$

This can be simplified to

$$- \mu \sum x_i \sum \ln (\frac{n+1-i}{i}) - \sum x_i \sum x_i \ln (\frac{n+1-i}{i}) + n \hat{\mu} \sum x_i \ln (\frac{n+1-i}{i})$$

$$+ \sum x_i^2 \sum \ln (\frac{n+1-i}{i}) = 0 .$$

Solving for $\hat{\mu}$ gives

$$\hat{\mu} = \frac{\sum x_i \sum x_i \ln\left(\frac{n+1-i}{i}\right) - \sum x_i^2 \sum \ln\left(\frac{n+1-i}{i}\right)}{n \sum x_i \ln\left(\frac{n+1-i}{i}\right) - \sum x_i \sum \ln\left(\frac{n+1-i}{i}\right)} . \qquad (6)$$

Upon obtaining $\hat{\mu}$ from (6), $\hat{\sigma}'$ can be obtained from (5).

To compare the above modified maximum agreement estimators with true maximum agreement estimators, the agreement function

$$A = \sum \left\{\frac{x_i-\mu}{\sigma'} - E\left(\frac{x_i-\mu}{\sigma'}\right)\right\}^2 \qquad (7)$$

was considered. Taking the partials of (7) with respect to μ and σ', setting these equal to zero, and solving the two equations simultaneously in a manner similar to the above, gives

$$\hat{\sigma}' = \frac{\sum x_i^2 - 2\,\hat{\mu} \sum x_i + n\,\hat{\mu}^2}{\sum x_i\, E\left(\frac{x_i-\hat{\mu}}{\hat{\sigma}'}\right) - \mu \sum E\left(\frac{x_i-\hat{\mu}}{\hat{\sigma}'}\right)}$$

and

$$\hat{\mu} = \frac{\sum x_i^2 \sum E\left(\frac{x_i-\hat{\mu}}{\hat{\sigma}'}\right) - \sum x_i \sum x_i\, E\left(\frac{x_i-\hat{\mu}}{\hat{\sigma}'}\right)}{\sum x_i \sum E\left(\frac{x_i-\hat{\mu}}{\hat{\sigma}'}\right) - n \sum x_i\, E\left(\frac{x_i-\hat{\mu}}{\hat{\sigma}}\right)} .$$

Gupta and Shah [21] have given the exact moments of the k^{th} order statistic for $n \leq 10$ from a standard Logistic distribution so that $E\left(\frac{x_i-\hat{\mu}}{\hat{\sigma}'}\right)$ can be obtained.

The need for tables is one disadvantage of this estimator. Furthermore, several Monte Carlo studies were undertaken to compare these estimates with the modified maximum agreement estimators, and it was found that the modified maximum agreement estimator for σ had both a smaller variance and less bias than the agreement estimator, and the variance and bias for μ was the same for both estimators. Table 4 gives a comparison between the modified maximum agreement estimators and the maximum likelihood estimators based on a Monte Carlo study of 950 samples for sample size n = 10, 20, 40, 80 and censoring from the right for r/n = 1.0, .7, .5, .3.

B. Statistical Inferences for the Logistic Distribution Based on Maximum Likelihood Estimators.

1. Confidence Intervals for μ and σ

For both complete and censored sampling $(\hat{\mu}-\mu)/\hat{\sigma}$ and $\hat{\sigma}/\sigma$ are pivotal quantities whose distributions are independent of unknown parameters. Therefore, the percentage points m_γ, where

$$P[\sqrt{n}(\hat{\mu}-\mu)/\hat{\sigma} < m_\gamma] = \gamma$$

and the percentage points s_γ where

$$P[\sqrt{n}(\hat{\sigma}/\sigma - 1) < S_\gamma] = \gamma$$

can be determined by Monte Carlo simulation. Table A1 and Table A2 respectively, give these percentage points for n = 10, 20, 40, 80 for censored sampling from the right for r/n = .3, .5, .7, .9, 1., and for γ = .01, .025, .05, .10, .25, .5, .75, .90, .95, .975 and .99. All the Monte Carlo results are based on 4000 samples for n = 80 and 8000 samples for n = 10, 20 and 40.

These tables can be used to determine confidence intervals for both μ and σ. For example, to obtain a 95% confidence interval for μ,

$$P[m_{.025} < \sqrt{n}(\hat{\mu}-\mu)/\hat{\sigma} < m_{.975}] = .95$$

TABLE 4. Monte Carlo study comparing maximum agreement estimators with maximum likelihood estimators for the standard Logistic distribution

	r/n n	1.0				.7			
		$E(\hat{\mu})$	$Var(\hat{\mu})$	$E(\hat{\sigma})$	$Var(\hat{\sigma})$	$E(\hat{\mu})$	$Var(\hat{\mu})$	$E(\hat{\sigma})$	$Var(\hat{\sigma})$
Max. Lik.	10	-.01	.10	.95	.07	-.03	.10	.90	.09
Max. Agree.		.01	.10	1.31	.14	.05	.04	1.40	.23
Max. Lik.	20	.00	.05	.97	.03	-.01	.05	.94	.05
Max. Agree.		.00	.05	1.19	.06	.03	.02	1.22	.10
Max. Lik.	40	-.00	.02	.99	.02	-.01	.02	.98	.03
Max. Agree.		-.00	.03	1.13	.03	.02	.01	1.15	.05
Max. Lik.	80	.00	.01	.99	.01	.00	.01	.99	.01
Max. Agree.		.00	.01	1.08	.01	.02	.00	1.09	.02

	r/n n	.5				.3			
		$E(\hat{\mu})$	$Var(\hat{\mu})$	$E(\hat{\sigma})$	$Var(\hat{\sigma})$	$E(\hat{\mu})$	$Var(\hat{\mu})$	$E(\hat{\sigma})$	$Var(\hat{\sigma})$
Max. Lik.	10	-.09	.13	.84	.13	-.26	.23	.71	.22
Max. Agree.		.06	.57	1.41	.31	.03	.01	1.32	.33
Max. Lik.	20	-.04	.06	.91	.07	-.11	.12	.85	.12
Max. Agree.		.04	.01	1.23	.13	.03	.00	1.19	.13
Max. Lik.	40	-.02	.03	.96	.04	-.06	.06	.93	.06
Max. Agree.		.03	.00	1.16	.06	.02	.00	1.14	.06
Max. Lik.	80	-.01	.01	.98	.02	-.02	.03	.97	.03
Max. Agree.		.02	.00	1.09	.03	.01	.00	1.01	.03

and

$$P[\hat{\mu} - m_{.975}\sigma/\sqrt{n} < \mu < \hat{\mu} - m_{.025}\sigma/\sqrt{n}] = .95.$$

Similarily, to obtain a 95% confidence interval for σ,

$$P[S_{.025} < \sqrt{n}(\hat{\sigma}/\sigma - 1) < S_{.975}] = .95 .$$

and

$$P[\frac{\hat{\sigma}}{1 + S_{.975}/\sqrt{n}} < \sigma < \frac{\hat{\sigma}}{1 + S_{.025}/\sqrt{n}}] = .95$$

2. Point Estimation of μ, σ, and R(t)

The means and variances of the maximum likelihood estimates, $\hat{\mu}_0$ and $\hat{\sigma}_0$, were obtained from the Monte Carlo simulation of 4000 samples for n = 80 and 8000 samples for n = 40, 20, 10 for $\mu = 0$ and $\sigma = 1$ and these are in Table A3. Since $E(\hat{\mu}_0) = E(\hat{\mu}-\mu)/\sigma$, the bias of $\hat{\mu}$ is $\sigma\, E(\hat{\mu}_0)$. Also, $E(\hat{\sigma}/\sigma) = E(\hat{\sigma}_0)$, so that $E(\hat{\sigma})/E(\hat{\sigma}_0)$ is an unbiased estimator of σ.

The means and variances of the point estimator $\hat{R}(t)$ were obtained from the same Monte Carloed simulations for R(t) = .5, .7, .9, .95 and are presented in Table A4.

3. Tolerance Limits

Let $x_1, \ldots, x_n$ be a random sample from a distribution with cumulative distribution function F(x) and let ζ_β be the point such that $F(\zeta_\beta) = 1 - \beta$. A function $L(x_1, \ldots, x_n)$ is a lower γ tolerance limit for proportion β if $P[L(x_1, \ldots, x_n) \leq \zeta_\beta] = \gamma$. If the distribution has a location and scale parameter, then ζ_β can be expressed in the form $\mu - k(\beta)\sigma$. In the case of the Logistic distribution, $k(\beta) =$

$\frac{\sqrt{3}}{\pi} \ln (\frac{\beta}{1-\beta})$. Haas [22] shows that for a distribution with a location and scale parameter, there always exists a function $t_\gamma(\beta,n)$ such that

$$P[\hat{\mu} - t_\gamma(\beta,n)\hat{\sigma} \leq \mu - k(\beta)\sigma] = \gamma$$

for all μ and σ. The function $t_\gamma(\beta,n)$ is chosen such that

$$P[\frac{\hat{\mu}-\mu}{\hat{\sigma}} + k(\beta) \frac{\sigma}{\hat{\sigma}} \leq t_\gamma(\beta,n)] = \gamma$$

Simple manipulation shows that these two probability statements are equivalent. The importance of this is that $\frac{\hat{\mu}-\mu}{\hat{\sigma}} + k(\beta) \frac{\sigma}{\hat{\sigma}}$ is distributed independently of μ and σ and so depends only on β and n, thus making it convenient to obtain $t_\gamma(\beta,n)$ by Monte Carlo simulation. The lower (γ,β) confidence limit therefore is given by $L(x) = \hat{\mu} - t_\gamma(\beta,n)\hat{\sigma}$ and by $U(x) = \hat{\mu} + t_\gamma(\beta,n)\hat{\sigma}$. Values of $t_\gamma(\beta,n)$ based on a Monte Carlo simulation of 1000 samples for n = 80 and of 2000 samples for n = 40, 20, 10 is given in Table A5 for β = .500(.025).975; γ = .75, .85, .90, .95, .99 and r/n = 1.0, .7, .5 and .3.

V. NUMERICAL TECHNIQUES

A. Generation of Random Samples

It is well known that if x is a continuous random variable with cumulative distribution function F(x), then U = F(x) has a uniform distribution over the interval (0,1) [23]. Using the random number generator subroutine found in the IBM Scientific Subroutine Package for the IBM 360 to obtain a random sample of size n from the uniform distribution, and then solving to obtain $x_i = F^{-1}(u_i)$ gives a sample from the desired distribution.

In the case of the model with the linear failure-rate,

$$u_i = 1 - \exp\left[-\left(\frac{ax_i^2}{2} + bx_i\right)\right]$$

so that x_i is the positive root to the quadratic equation

$$\frac{ax_i^2}{2} + bx_i + \ln(1 - u_i) = 0,$$

since $x_i > 0$.

In the case of the model with the quadratic failure-rate,

$$u_i = 1 - \exp\left[-\left(\frac{ax_i^3}{3} + \frac{bx_i^2}{2} + cx_i\right)\right],$$

so that x_i is the positive real root to the cubic equation

$$\frac{ax_i^3}{3} + \frac{bx_i^2}{2} + cx_i + \ln(1-u_i) = 0$$

It can be shown that there is only one positive real root to this equation.

For the case when the failure-rate function

$$h(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} + \frac{\alpha\gamma}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(\frac{x}{\beta}\right)^{\alpha},$$

$$u_i = 1 - \exp - \left[\frac{\alpha}{\beta}\left(\frac{x}{\beta}i\right)^{\alpha-1} + \frac{\alpha\gamma}{\beta}\left(\frac{x_i}{\beta}\right)^{\alpha-1} \exp\left(\frac{x_i}{\beta}\right)^{\alpha}\right]$$

so that x_i satisfies

$$\frac{\alpha}{\beta}\left(\frac{x_i}{\beta}\right)^{\alpha-1} + \frac{\alpha\gamma}{\beta}\left(\frac{x_i}{\beta}\right)^{\alpha-1} \exp\left(\frac{x_i}{\beta}\right)^{\alpha} + \ln(1 - \mu_i) = 0$$

Many simulations have shown that the positive root to the above equation can almost always be obtained using the Newton-Raphson method with an initial guess for x_i equal to zero.

B. Numerical Solution of the Maximum Likelihood Equations for the Logistic Distribution

Let $x_1, \ldots, x_r$ be the r smallest ordered observations from a sample of size n from the Logistic distribution. Harter and Moore [19] give the maximum likelihood function for censoring from both sides and the first and second partial derivatives with respect to μ and σ of this function. These equations can be simplified by using the fact that the hazard function $h(x) = f(x)/(1-F(x)) = (\Pi/\sqrt{3}\,\sigma)\,F(x)$ to give

$$\frac{\partial \ln L}{\partial \mu} = (\Pi/\sqrt{3}\,\sigma)\left[2\sum_{i=1}^{n-r} F(z_i) + rF(z_{n-r}) - (n-r)\right], \quad (1)$$

$$\frac{\partial \ln L}{\partial \sigma} = (1/\sigma)\left[2\sum_{i=1}^{n-r} z_i F(z_i) + rz_{n-r}F(z_{n-r}) - \sum_{i=1}^{n-r} z_i - (n-r)\right]$$

$$\frac{\partial^2 \ln L}{\partial \mu^2} = \frac{-\Pi^2}{3\sigma^2}\left\{2\sum_{i=1}^{n-r} F(x_i)[1-F(z_i)] + rF(z_{n-r})[1-F(z_{n-r})]\right\}, \quad (2)$$

$$\frac{\partial^2 \ln L}{\partial\sigma^2} = \frac{1}{\sigma^2}\{-2\sum_{i=1}^{n-r} z_i^2 F(z_i)[1-F(z_i)] - rz_{n-r}^2 F(z_{n-r})[1-F(z_{n-r})]$$

$$-4\sum_{i=1}^{n-r} z_i F(z_i) - 2rz_{n-r}F(z_{n-r}) + 2\sum_{i=1}^{n-r} z_i + (n-r)\} \qquad (3)$$

and

$$\frac{\partial^2 \ln L}{\partial\mu\partial\sigma} = \frac{-\pi}{\sqrt{3}\sigma^2}\{2\sum_{i=1}^{n-r} z_i F(z_i)[1-F(z_i)] + rz_{n-r}F(z_{n-r})$$

$$[1-F(z_{n-r})] + 2\sum_{i=1}^{n-r} F(z_i) + rF(z_{n-r}) - (n-r)\}\ , \qquad (4)$$

where $z_i = \pi(x-\mu)/\sqrt{3}\ \sigma$ and $F(z_i) = 1/[1 + \exp(-z_i)]$.

The maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$ are the values which satisfy the equations $\partial \ln L/\partial\sigma = 0$ and $\partial \ln L/\partial\mu = 0$. There does not appear to be a closed form solution to these equations. Harter and Moore [19] suggest an iterative estimation procedure estimating the parameters one at a time, in the cyclic order μ, σ. At each step, the one parameter is estimated by the method of false position while the latest estimate of the other parameter is substituted in the equation. This method always converges, and usually in a reasonable number of iteration. However, care must be taken to prevent successive estimates of $\hat{\sigma}$ from getting too close to zero since this causes the equations to blow up because of division by a very small number. Also, it is possible for a negative value of $\hat{\sigma}$ to satisfy the equations, which is of course, a meaningless result.

An alternative method which converges considerably faster to the estimates $\hat{\mu}$ and $\hat{\sigma}$ when it converges is the Newton-Raphson method for two equations in two unknowns. If $\hat{\mu}_{i+1} = \hat{\mu}_i + h$ and $\hat{\sigma}_{i+1} = \hat{\sigma}_i + k$ are the i^{th} and $(i + 1)^{th}$ iterations of $\hat{\mu}$ and $\hat{\sigma}$ then the problem is to find h and k from the simultaneous linear equations

$$\left.\frac{\partial \ln L}{\partial \mu}\right|_{\substack{\mu=\hat{\mu}_i \\ \sigma=\hat{\sigma}_i}} + \left.\frac{\partial^2 \ln L}{\partial \mu^2}\right|_{\substack{\mu=\hat{\mu}_i \\ \sigma=\hat{\sigma}_i}}(h) + \left.\frac{\partial^2 \ln L}{\partial \mu \partial \sigma}\right|_{\substack{\mu=\hat{\mu}_i \\ \sigma=\hat{\sigma}_i}}(k) = 0$$

and

$$\left.\frac{\partial \ln L}{\partial \sigma}\right|_{\substack{\mu=\hat{\mu}_i \\ \sigma=\hat{\sigma}_i}} + \left.\frac{{}^2 \ln L}{\partial \mu \partial \sigma}\right|_{\substack{\mu=\hat{\mu}_i \\ \sigma=\hat{\sigma}_i}}(h) + \left.\frac{{}^2 \ln L}{\partial \sigma^2}\right|_{\substack{\mu=\hat{\mu}_i \\ \sigma=\hat{\sigma}_i}}(k) = 0,$$

where the partials are given by (1), (2), (3), and (4). Table 5 gives the number of times this method diverged for 4000 Monte Carloed samples of size n = 80 with r/n = 1.0, .9, .7, .5, .3 and for 8000 Monte Carloed samples of size n = 40, 20, 10 with r/n = 1.0, .9, .7, .5, .3. The starting values for n = 80 and r/n = 1.0 were the true values for μ and σ. The starting values for the other cases were the proceeding Maximum Likelihood estimates. As might be expected, the method is better for larger more complete samples. Another possibility for starting values would be the Maximum Agreement estimates since these are easy to obtain.

TABLE 5. Number of times Newton-Raphson method failed to converge

n	number of samples	r/n = 1.0	.9	.7	.5	.3
80	8000	0	0	0	0	82
40	4000	67	0	14	106	741
20	4000	509	2	129	613	1811
10	4000	650	81	626	1560	3344

VI. SUMMARY, CONCLUSIONS AND FURTHER PROBLEMS

Although various methods for estimating the parameters of the model with the linear failure-rate have been suggested, none of the methods appear to be as easy to use as does the method of maximum agreement. Furthermore, this method of estimation gives easy to use estimators for other distributions, such as the Logistic distribution, and as Table 3 indicates, these estimators appear to be relatively unbiased with reasonable variance. Besides providing easy to use estimators for the linear failure-rate model, tests for various hypotheses are now available.

Various specific models with U-shaped failure rates have been presented so that the person with reason to believe that his data is best described with a distribution of this type, now has several more alternatives to try. Besides, relatively simple estimators are presented for each model.

The Logistic distribution has been made more useful for the applied statitician, first by presenting easier to use estimators, in that the maximum agreement estimators are easy to compute and do not require tables. Also, the unbiasing constants for the maximum likelihood estimators for various sample sizes and levels of censoring are presented as are confidence intervals and tolerance limits.

Tests of hypotheses, confidence intervals, and tolerance limits are still largely unavailable for a model with a U-shaped failure-rate. One approach to this problem would be to continue development on a model with a location and scale parameter along the lines of the work done on the Logistic distribution.

REFERENCES

1. Kao, John H. K. (1959). A Graphical Estimation of Mixed Weibull Parameters in Life-Testing of Electron Tubes, Technometrics, 1, 389-407.

2. Rider, Paul R. (1961). The Method of Moments Applied to a Mixture of Two Exponential Distributions, Annals of Mathematical Statistics, 32, 143-147.

3. Rider, Paul R. (1962). Estimating the Parameters of Mixed Poisson, Binomial, and Weibull Distributions by the Method of Moments, Institut International de Statistique, Bulletin, 39, 225-232.

4. Murthy, V. K. and Swartz, G. Boyd (1971). Contributions to Reliability, Aerospace Research Laboratories Report ARL 71-0060, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio.

5. Garg, Mohan L.; Rao, Raja B.; and Redmond, Carol K. (1970). Maximum-liklihood Estimation of the Parameters of the Gompertz Survival Function, Journal of the Royal Statistical Society (Series C), 19, 152-159.

6. Krane, Scott A. (1963). Analysis of Survival Data by Regression Techniques, Technometrics, 5, 161-174.

7. Kodlin, D. (1967). A New Response Time Distribution, Biometrics, 2, 227-239.

8. Farmelo, Gene R. (1970). Methods of Estimating the Two Parameters of a Life Distribution Characterized by a Linear Increasing Failure Rate, a + 2bt, Thesis, United States Naval Postgraduate School.

9. Bain, Lee J. and Antle, Charles E. (1967). Estimation of Parameters in the Weibull Distribution, Technometrics, 9, 621-627.

10. Antle, C., Klimko, L. and Harkness, W. (1970). Confidence Intervals for the Parameters of the Logistic Distribution, Biometrika, 57, 397-402.

11. Harter, H. Leon (1969). Order Statistics and Their Use in Testing and Estimation, Volume 2, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio.

12. Lehman, E. L. (1959). Testing Statistical Hypotheses, John Wiley and Sons, Inc., New York.

13. Bain, Lee J. (1972). Inferences Based on Censored Sampling from the Weibull or Extreme-Value Distribution, Technometrics, 14, 693-702.

14. Bain, Lee J. and Engelhardt, Max. Some Complete and Censored Sampling Results for the Weibull or Extreme-Value Distribution (To appear in Technometrics).

15. Hager, Harold W.; Bain, Lee J.; and Antle, Charles E. (1971). Reliability Estimation for the Generalized Gamma Distribution and Robustness of the Weibull Model, 13, 547-557.

16. Thoman, D. R.; Bain, Lee J.; and Antle, C. E. (1969). Inferences on the Parameters of the Weibull Distribution, Technometrics, 7, 349-358.

17. Mann, Nancy R.; Fertig, Kenneth W.; and Scheur, Ernest M. (1971). Confidence and Tolerance Bounds and a New Goodness-of-Fit Test for Two-Parameter Weibull or Extreme-Value Distributions (With Tables for Censored Samples of Size 3 (1) 25), Aerospace Research Laboratories Report ARL 71-0077, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio.

18. Antle, Charles E. and Bain, Lee J. (1969). A Property of Maximum Likelihood Estimators of Location and Scale Parameters, SIAM Review, 11, 251-253.

19. Harter, H. L. and Moore, A. H. (1967). Maximum Likelihood Estimation from Censored Samples of the Parameters of a Logistic Distribution, Journal of the American Statistical Association, 62, 675-684.

20. Gupta, S. S.; Qureishi, A. S.; and Shah, B. K. (1967). Best Linear Unbiased Estimators of the Parameters of the Logistic Distribution Using Order Statistics, Technometrics, 9, 43-56.

21. Gupta, S. S. and Shah, B. K. (1965). Exact Moments and Percentage Points of the Order Statistics and the Distribution of the Range of the Logistic Distribution, Annals of Mathematical Statistics, 36, 907-920.

22. Haas, Gerald N. (1969). Statistical Inferences for the Cauchy Distribution Based on Maximum Likelihood Estimators, Thesis, University of Missouri-Rolla, Rolla, Missouri.

23. Hogg, R. V. and Craig, A. T. (1965). Introduction to Mathematical Statistics, The Macmillan Book Co., Inc., New York.

VITA

James Addison Eastman was born June 2, 1943, at New York, New York. He graduated from Northern Valley Regional High School in Demerast, New Jersey in June 1961. He attended William Jewell College, Liberty, Missouri, from September 1961 to June 1962 and then attended Drew University, Madison, New Jersey, where he graduated in June 1965 with a B.A. in mathematics. From September 1965 to June 1967 he taught junior and senior high school mathematics at Community School, Teheran, Iran. He then returned to school, first attending Kansas State College at Pittsburg, Kansas, where he received the M.S. degree in mathematics in August 1968 and then attending the University of Missouri at Rolla. At the University of Missouri at Rolla he has been employed both as a part-time instructor and under Air Force research contracts. Since September 1971 he has been an instructor in the mathematics department at Lincoln University, Jefferson City, Missouri.

On March 16, 1968, he was married to the former Karel Marie Carstensen.

APPENDIX A

TABLES FOR INFERENCES OF THE LOGISTIC DISTRIBUTION

TABLE A1. Values of m_γ such that $P[\sqrt{n}(\hat{\mu} - \mu)/\hat{\sigma} < m_\gamma] = \gamma$

r/n	n	.01	.025	.05	.10	.25	.50	.75	.90	.95	.975	.99
1.0	10	-2.78	-2.24	-1.81	-1.37	- .70	.00	.70	1.37	1.81	2.24	2.78
	20	-2.51	-2.08	-1.70	-1.30	- .68	.00	.68	1.30	1.70	2.08	2.51
	40	-2.39	-1.96	-1.64	-1.25	- .65	.00	.65	1.25	1.64	1.96	2.39
	80	-2.28	-1.93	-1.60	-1.23	- .65	.00	.65	1.23	1.60	1.93	2.28
	∞	-2.22	-1.87	-1.57	-1.22	- .64	.00	.64	1.22	1.57	1.87	2.22
.9	10	-2.89	-2.35	-1.90	-1.40	- .70	.00	.72	1.42	1.84	2.28	2.83
	20	-2.61	-2.14	-1.74	-1.32	- .68	.01	.69	1.31	1.71	2.06	2.48
	40	-2.43	-2.00	-1.66	-1.26	- .65	- .01	.65	1.26	1.63	1.95	2.39
	80	-2.30	-1.96	-1.60	-1.23	- .64	.00	.67	1.24	1.62	1.90	2.31
	∞	-2.22	-1.87	-1.57	-1.22	- .64	.00	.64	1.22	1.57	1.87	2.22
.7	10	-3.98	-3.04	-2.40	-1.70	- .85	- .08	.68	1.40	1.86	2.33	2.84
	20	-3.08	-2.47	-2.00	-1.48	- .77	- .04	.66	1.29	1.69	2.05	2.48
	40	-2.64	-2.19	-1.80	-1.37	- .70	- .04	.63	1.24	1.63	1.93	2.38
	80	-2.38	-2.04	-1.67	-1.31	- .68	- .01	.66	1.24	1.60	1.90	2.33
	∞	-2.26	-1.91	-1.60	-1.25	- .66	.00	.66	1.25	1.60	1.91	2.26
.5	10	-7.84	-5.71	-4.08	-2.92	-1.39	- .28	.57	1.30	1.77	2.28	2.83
	20	-4.63	-3.68	-2.88	-2.10	-1.07	- .17	.59	1.26	1.66	2.03	2.42
	40	-3.60	-2.90	-2.38	-1.81	- .93	- .12	.61	1.24	1.61	1.96	2.37
	80	-3.16	-2.55	-2.06	-1.61	- .84	- .04	.66	1.26	1.66	1.97	2.30
	∞	-2.51	-2.11	-1.77	-1.38	- .73	.00	.73	1.38	1.77	2.11	2.51
.3	10			-16.19	-9.78	-4.43	-1.30	.25	1.14	1.63	2.12	2.64
	20	-12.55	-9.04	-6.76	-4.70	-2.34	- .69	.51	1.34	1.75	2.08	2.43
	40	-7.40	-5.58	-4.56	-3.33	-1.73	- .44	.63	1.43	1.85	2.19	2.57
	80	-5.60	-4.43	-3.67	-2.78	-1.45	- .21	.78	1.56	1.96	2.36	2.73
	∞	-3.59	-3.03	-2.54	-1.98	-1.04	.00	1.04	1.98	2.54	3.03	3.59

TABLE A2. Values of s_γ such that $P[\sqrt{n}(\hat{\sigma}/\sigma - 1) < s_\gamma] = \gamma$.

r/n	n	γ										
		.01	.025	.05	.10	.25	.50	.75	.90	.95	.975	.99
	10	-1.81	-1.59	-1.40	-1.18	- .76	- .24	.33	.89	1.28	1.61	2.02
	20	-1.86	-1.62	-1.40	-1.13	- .71	- .18	.39	.95	1.31	1.62	2.01
1.0	40	-1.91	-1.63	-1.41	-1.12	- .67	- .12	.45	1.00	1.34	1.66	2.05
	80	-1.93	-1.66	-1.42	-1.13	- .67	- .08	.48	1.03	1.35	1.65	1.94
	∞	-1.95	-1.64	-1.38	-1.07	- .56	.00	.56	1.07	1.38	1.64	1.95
	10	-1.91	-1.69	-1.49	-1.26	- .81	- .26	.32	.91	1.29	1.69	2.11
	20	-1.96	-1.70	-1.47	-1.22	- .76	- .20	.39	.99	1.37	1.74	2.18
.9	40	-1.97	-1.71	-1.49	-1.20	- .71	- .14	.45	1.04	1.38	1.74	2.11
	80	-2.04	-1.68	-1.46	-1.17	- .69	- .09	.50	1.06	1.39	1.65	1.97
	∞	-2.04	-1.72	-1.44	-1.12	- .59	.00	.59	1.12	1.44	1.72	2.04
	10	-2.16	-1.96	-1.75	-1.50	-1.03	- .41	.30	.97	1.43	1.81	2.36
	20	-2.25	-1.98	-1.75	-1.47	- .92	- .29	.41	1.08	1.56	1.93	2.43
.7	40	-2.28	-2.03	-1.75	-1.43	- .87	- .20	.50	1.16	1.59	1.98	2.42
	80	-2.28	-1.97	-1.72	-1.39	- .82	- .14	.55	1.24	1.65	1.98	2.28
	∞	-2.36	-1.99	-1.67	-1.30	- .69	.00	.69	1.30	1.67	1.99	2.36
	10	-2.53	-2.34	-2.14	-1.87	-1.36	- .65	.19	1.04	1.62	2.04	2.79
	20	-2.70	-2.40	-2.15	-1.84	-1.22	- .46	.38	1.27	1.78	2.26	2.99
.5	40	-2.77	-2.46	-2.18	-1.81	-1.12	- .32	.54	1.36	1.88	2.35	2.95
	80	-2.81	-2.45	-2.09	-1.71	-1.03	- .24	.64	1.48	1.98	2.43	3.01
	∞	-2.91	-2.45	-2.06	-1.60	- .84	.00	.84	1.60	2.06	2.45	2.91
	10	-2.98	-2.87	-2.73	-2.52	-2.00	-1.23	- .18	1.00	1.78	2.58	3.70
	20	-3.41	-3.16	-2.89	-2.51	-1.78	- .83	.29	1.47	2.25	3.04	3.78
.3	40	-3.62	-3.27	-2.88	-2.42	-1.62	- .60	.57	1.72	2.46	3.12	3.97
	80	-3.68	-3.24	-2.88	-2.34	-1.48	- .39	.75	1.89	2.65	3.22	3.89
	∞	-3.94	-3.32	-2.78	-2.17	-1.14	.00	1.14	2.17	2.78	3.32	3.94

TABLE A3. Means and variances of the maximum likelihood estimators of the parameters of the Logistic distribution ($E(\hat{\mu}_0)$ denotes the mean of $\hat{\mu}$ for the standard logistic, $\mu = 0$, $\sigma = 1$).

		$E(\hat{\mu}_0)$					$E(\hat{\sigma}/\sigma)$				
n	r/n	1.0	.9	.7	.5	.3	1.0	.9	.7	.5	.3
10		.000	.000	-.021	-.078	-.256	.943	.943	.899	.839	.702
20		.000	-.001	-.010	-.039	-.120	.971	.966	.951	.920	.858
40		.000	-.001	-.006	-.020	-.060	.986	.983	.975	.960	.929
80		.000	.000	-.001	-.007	-.026	.992	.991	.988	.981	.967
		$nV(\hat{\mu})/\sigma^2$					$nV(\hat{\sigma})/\sigma^2$				
10		.93	.93	.98	1.22	2.36	.67	.73	.95	1.35	2.10
20		.94	.94	.99	1.21	2.43	.68	.76	1.00	1.46	2.50
40		.91	.91	.95	1.18	2.38	.70	.77	1.03	1.52	2.68
80		.91	.91	.94	1.18	2.37	.70	.76	1.03	1.56	2.76
∞		.91	.91	.95	1.16	2.39	.70	.77	1.03	1.56	2.87

TABLE A4. Means and variances of $\hat{R}$

		$E(\hat{R})$				$V(\hat{R})$			
	R	.5	.7	.9	.95	.5	.7	.9	.95
r/n	n								
	10	.502	.707	.902	.948	.0208	.0160	.0049	.0021
	20	.501	.704	.901	.949	.0103	.0083	.0026	.0011
1.0	40	.500	.702	.900	.949	.0049	.0040	.0013	.0006
	80	.501	.702	.901	.950	.0024	.0020	.0007	.0003
	10	.485	.705	.906	.951	.0259	.0193	.0054	.0023
	20	.493	.703	.903	.951	.0118	.0090	.0029	.0013
.7	40	.496	.702	.902	.950	.0053	.0042	.0015	.0007
	80	.499	.702	.901	.950	.0025	.0020	.0008	.0004
	10	.441	.683	.909	.954	.0374	.0294	.0062	.0025
	20	.471	.695	.905	.952	.0163	.0112	.0032	.0014
.5	40	.485	.698	.903	.951	.0072	.0048	.0016	.0007
	80	.494	.700	.902	.951	.0033	.0022	.0008	.0004
	10	.324	.572	.892	.954	.0614	.0753	.0192	.0054
	20	.407	.647	.903	.954	.0340	.0266	.0047	.0016
.3	40	.452	.678	.903	.953	.0158	.0087	.0019	.0008
	80	.478	.691	.902	.951	.0071	.0034	.0009	.0005

TABLE A5. Tolerance factors t_γ such that $L(x) = \hat{\mu} - t_\gamma \hat{\sigma}$, $U(x) = \hat{\mu} + t_\gamma \hat{\sigma}$,

$\gamma = .75$

n	10				20				40				80			
r/n	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3
β																
.500	.22	.20	.16	.06	.15	.15	.13	.12	.096	.093	.089	.092	.077	.072	.076	.088
.525	.28	.26	.22	.12	.21	.21	.19	.17	.153	.149	.146	.149	.133	.129	.131	.142
.550	.34	.33	.29	.19	.27	.27	.25	.23	.209	.205	.203	.205	.188	.187	.187	.195
.575	.41	.40	.35	.26	.33	.33	.31	.29	.266	.265	.260	.262	.245	.244	.243	.249
.600	.47	.47	.43	.33	.39	.39	.37	.35	.325	.325	.319	.319	.304	.303	.300	.303
.625	.54	.54	.52	.41	.45	.46	.44	.41	.384	.384	.380	.377	.364	.363	.359	.363
.650	.61	.62	.59	.49	.52	.52	.51	.48	.446	.447	.443	.440	.427	.425	.420	.424
.675	.68	.69	.67	.59	.59	.59	.58	.55	.513	.516	.509	.506	.490	.490	.485	.484
.700	.75	.77	.76	.68	.65	.66	.65	.63	.581	.585	.578	.575	.557	.557	.553	.550
.725	.84	.86	.85	.79	.73	.74	.73	.71	.654	.656	.651	.647	.627	.628	.625	.623
.750	.92	.95	.95	.90	.81	.82	.82	.80	.732	.735	.731	.722	.703	.704	.702	.699
.775	1.01	1.05	1.06	1.02	.89	.91	.92	.89	.812	.817	.819	.804	.784	.783	.786	.781
.800	1.10	1.16	1.19	1.15	.99	1.01	1.02	1.00	.900	.910	.915	.901	.869	.871	.876	.872
.825	1.21	1.28	1.32	1.34	1.09	1.12	1.14	1.11	.997	1.009	1.018	1.005	.961	.970	.971	.972
.850	1.34	1.42	1.48	1.57	1.20	1.24	1.26	1.26	1.109	1.121	1.139	1.129	1.070	1.077	1.082	1.079
.875	1.48	1.59	1.66	1.82	1.34	1.38	1.41	1.43	1.236	1.252	1.274	1.272	1.193	1.200	1.211	1.210
.900	1.66	1.79	1.89	2.14	1.50	1.56	1.59	1.64	1.385	1.407	1.438	1.447	1.338	1.349	1.365	1.358
.925	1.88	2.04	2.18	2.61	1.70	1.77	1.82	1.90	1.571	1.603	1.642	1.667	1.519	1.534	1.551	1.551
.950	2.19	2.39	2.60	3.18	1.97	2.06	2.15	2.28	1.835	1.872	1.926	1.980	1.770	1.786	1.816	1.830
.975	2.71	2.96	3.28	4.25	2.44	2.54	2.69	2.92	2.265	2.321	2.404	2.507	2.190	2.206	2.254	2.286

TABLE A5. (continued)

γ = .85

n	10				20				40				80			
r/n	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3
β																
.500	.33	.32	.30	.21	.24	.23	.21	.21	.153	.151	.147	.176	.114	.115	.113	.139
.525	.39	.38	.36	.28	.30	.29	.28	.27	.210	.209	.201	.228	.171	.170	.167	.189
.550	.46	.45	.43	.36	.36	.35	.34	.33	.267	.266	.259	.282	.227	.227	.225	.243
.575	.53	.52	.50	.44	.42	.41	.40	.39	.327	.325	.318	.336	.286	.285	.283	.297
.600	.60	.60	.58	.51	.48	.48	.47	.45	.387	.386	.380	.390	.345	.344	.343	.353
.625	.66	.67	.65	.58	.55	.55	.53	.52	.449	.449	.443	.449	.405	.405	.401	.411
.650	.73	.75	.74	.67	.61	.61	.60	.58	.514	.515	.507	.509	.467	.468	.468	.472
.675	.81	.84	.83	.76	.68	.69	.68	.66	.579	.580	.577	.575	.530	.534	.531	.535
.700	.90	.93	.94	.87	.75	.77	.76	.74	.646	.652	.648	.648	.597	.600	.600	.599
.725	.98	1.03	1.06	1.01	.83	.85	.85	.83	.719	.726	.729	.717	.669	.672	.670	.671
.750	1.08	1.14	1.18	1.16	.91	.94	.94	.91	.796	.808	.812	.800	.742	.747	.746	.746
.775	1.18	1.26	1.30	1.34	1.00	1.03	1.04	1.02	.879	.896	.900	.891	.822	.827	.828	.826
.800	1.29	1.38	1.44	1.51	1.10	1.13	1.15	1.15	.970	.993	.999	.992	.911	.917	.918	.917
.825	1.42	1.51	1.59	1.72	1.21	1.24	1.28	1.29	1.069	1.096	1.107	1.108	1.008	1.015	1.026	1.021
.850	1.55	1.66	1.78	2.01	1.33	1.38	1.42	1.45	1.181	1.215	1.228	1.235	1.116	1.125	1.146	1.137
.875	1.72	1.84	2.02	2.36	1.47	1.53	1.60	1.66	1.311	1.345	1.373	1.386	1.243	1.254	1.274	1.275
.900	1.91	2.07	2.29	2.79	1.64	1.70	1.80	1.90	1.463	1.512	1.553	1.573	1.394	1.406	1.431	1.435
.925	2.15	2.35	2.65	3.38	1.85	1.93	2.06	2.22	1.658	1.716	1.765	1.811	1.582	1.599	1.631	1.648
.950	2.48	2.74	3.11	4.24	2.15	2.27	2.43	2.68	1.936	2.003	2.080	2.144	1.842	1.866	1.912	1.936
.975	3.05	3.39	3.91	5.69	2.65	2.82	3.05	3.43	2.398	2.484	2.595	2.725	2.275	2.317	2.373	2.432

TABLE A5. (continued)

$\gamma = .90$

n	10				20				40				80			
r/n	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3
β																
.500	.44	.43	.39	.33	.30	.30	.28	.29	.190	.190	.189	.229	.146	.140	.140	.177
.525	.50	.50	.46	.40	.35	.35	.34	.35	.251	.249	.248	.280	.202	.198	.195	.228
.550	.56	.58	.54	.46	.41	.41	.40	.40	.314	.308	.304	.328	.258	.257	.250	.280
.575	.63	.65	.62	.53	.48	.48	.46	.46	.371	.369	.363	.380	.314	.313	.309	.334
.600	.70	.72	.70	.61	.54	.54	.53	.52	.430	.430	.427	.434	.371	.373	.367	.389
.625	.77	.80	.79	.71	.60	.61	.61	.59	.493	.495	.488	.492	.432	.434	.430	.443
.650	.85	.89	.89	.82	.67	.68	.68	.66	.556	.562	.555	.553	.495	.496	.494	.503
.675	.93	.98	.98	.93	.75	.76	.76	.74	.620	.633	.630	.618	.561	.560	.559	.566
.700	1.01	1.07	1.09	1.05	.82	.84	.84	.82	.690	.707	.705	.691	.626	.628	.627	.635
.725	1.10	1.16	1.21	1.22	.91	.93	.93	.91	.762	.783	.784	.772	.699	.702	.699	.703
.750	1.19	1.28	1.35	1.42	.99	1.02	1.03	1.01	.842	.864	.871	.859	.772	.777	.776	.776
.775	1.30	1.40	1.50	1.60	1.08	1.13	1.15	1.13	.929	.951	.963	.953	.850	.858	.865	.861
.800	1.41	1.53	1.68	1.82	1.18	1.24	1.27	1.27	1.024	1.050	1.065	1.058	.938	.950	.956	.950
.825	1.54	1.67	1.87	2.11	1.29	1.36	1.40	1.43	1.126	1.160	1.179	1.172	1.033	1.049	1.056	1.052
.850	1.69	1.86	2.07	2.46	1.42	1.50	1.55	1.61	1.242	1.290	1.308	1.309	1.144	1.164	1.173	1.171
.875	1.86	2.05	2.29	2.86	1.56	1.66	1.73	1.83	1.378	1.434	1.461	1.469	1.272	1.292	1.313	1.316
.900	2.07	2.31	2.60	3.43	1.73	1.85	1.95	2.09	1.542	1.609	1.642	1.669	1.426	1.450	1.479	1.487
.925	2.33	2.61	2.98	4.18	1.95	2.09	2.23	2.45	1.744	1.827	1.872	1.927	1.617	1.653	1.687	1.698
.950	2.70	3.02	3.53	5.29	2.27	2.42	2.60	2.96	2.023	2.133	2.195	2.288	1.887	1.926	1.974	2.017
.975	3.32	3.74	4.46	7.13	2.78	2.99	3.25	3.85	2.493	2.640	2.740	2.909	2.333	2.385	2.459	2.546

TABLE A5. (continued)

γ = .95

n	10				20				40				80			
r/n	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3
β																
.500	.58	.58	.54	.51	.37	.37	.36	.39	.262	.260	.252	.292	.183	.183	.184	.228
.525	.64	.64	.62	.58	.43	.43	.43	.44	.321	.321	.313	.341	.239	.239	.235	.277
.550	.71	.72	.71	.65	.49	.50	.49	.50	.378	.377	.370	.391	.297	.297	.290	.329
.575	.79	.80	.80	.75	.56	.57	.56	.56	.438	.441	.430	.447	.354	.355	.349	.378
.600	.86	.90	.89	.84	.63	.64	.63	.62	.499	.498	.490	.505	.412	.413	.409	.433
.625	.94	1.00	1.01	.96	.70	.72	.71	.69	.561	.562	.554	.562	.473	.472	.469	.487
.650	1.03	1.10	1.11	1.10	.77	.79	.79	.76	.628	.628	.623	.622	.536	.537	.528	.544
.675	1.11	1.20	1.24	1.27	.85	.88	.88	.85	.693	.699	.697	.696	.602	.602	.596	.608
.700	1.20	1.32	1.38	1.42	.92	.96	.98	.95	.766	.775	.777	.766	.671	.667	.666	.675
.725	1.30	1.43	1.52	1.62	1.00	1.05	1.08	1.07	.842	.863	.859	.851	.737	.739	.737	.742
.750	1.41	1.55	1.69	1.88	1.10	1.15	1.20	1.20	.925	.954	.949	.940	.815	.814	.816	.816
.775	1.54	1.69	1.85	2.20	1.20	1.26	1.32	1.35	1.014	1.050	1.046	1.042	.899	.901	.907	.903
.800	1.68	1.86	2.05	2.59	1.30	1.39	1.46	1.53	1.108	1.154	1.157	1.161	.991	1.001	1.004	1.002
.825	1.82	2.03	2.23	3.03	1.43	1.51	1.62	1.70	1.217	1.266	1.278	1.303	1.091	1.101	1.112	1.106
.850	1.96	2.21	2.45		1.56	1.65	1.79	1.92	1.345	1.395	1.424	1.447	1.202	1.219	1.233	1.235
.875	2.14	2.43	2.76		1.71	1.83	2.01	2.19	1.490	1.544	1.582	1.637	1.335	1.356	1.375	1.385
.900	2.37	2.71	3.13		1.90	2.04	2.26	2.53	1.665	1.726	1.778	1.862	1.494	1.517	1.547	1.582
.925	2.64	3.06	3.60		2.15	2.34	2.58	2.93	1.880	1.952	2.014	2.143	1.691	1.722	1.769	1.813
.950	3.03	3.56	4.23		2.48	2.73	3.00	3.57	2.186	2.263	2.360	2.561	1.965	2.005	2.066	2.140
.975	3.70	4.36	5.42		3.02	3.38	3.71	4.65	2.689	2.786	2.954	3.224	2.417	2.477	2.562	2.717

TABLE A5. (continued)

$\gamma = .99$

n																
r/n	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3	1.0	.7	.5	.3
β																
.500	.89	.91	.95	.84	.56	.55	.54	.53	.362	.369	.375	.398	.240	.239	.246	.328
.525	.96	1.01	1.04	.99	.62	.64	.61	.60	.428	.432	.432	.455	.285	.294	.296	.375
.550	1.04	1.09	1.15	1.11	.69	.71	.69	.68	.497	.491	.490	.507	.343	.343	.352	.425
.575	1.12	1.20	1.24	1.29	.77	.79	.78	.77	.556	.558	.550	.559	.406	.405	.411	.470
.580	1.21	1.30	1.37	1.50	.85	.88	.87	.87	.624	.621	.620	.622	.470	.469	.472	.514
.625	1.30	1.41	1.52	1.83	.92	.97	.97	.97	.694	.692	.693	.699	.529	.535	.536	.566
.650	1.39	1.51	1.67	2.04	1.01	1.07	1.07	1.07	.764	.772	.778	.779	.592	.598	.599	.621
.675	1.49	1.62	1.91	2.25	1.09	1.14	1.17	1.18	.835	.857	.860	.856	.658	.676	.672	.683
.700	1.60	1.76	2.13		1.20	1.23	1.29	1.29	.916	.946	.939	.935	.738	.752	.750	.755
.725	1.70	1.96	2.37		1.31	1.36	1.42	1.45	1.014	1.036	1.025	1.016	.819	.834	.825	.827
.750	1.86	2.12	2.58		1.41	1.50	1.55	1.62	1.104	1.122	1.121	1.127	.899	.916	.913	.915
.775	2.00	2.30			1.52	1.63	1.70	1.82	1.198	1.210	1.231	1.266	.984	1.004	1.006	1.010
.800	2.18	2.48			1.66	1.81	1.87	2.12	1.300	1.320	1.359	1.420	1.085	1.099	1.106	1.111
.825	2.34	2.69			1.80	1.95	2.06	2.41	1.416	1.447	1.487	1.609	1.191	1.208	1.218	1.226
.850	2.56	2.99			1.96	2.15	2.30	2.68	1.542	1.592	1.643	1.809	1.308	1.328	1.342	1.371
.875	2.81	3.29			2.13	2.36	2.53		1.692	1.774	1.838	2.031	1.435	1.476	1.495	1.546
.900	3.08	3.68			2.35	2.60	2.87		1.875	1.977	2.081	2.310	1.581	1.644	1.688	1.779
.925	3.46	4.19			2.60	2.93	3.28		2.109	2.247	2.358	2.644	1.788	1.867	1.923	2.035
.950	3.98	4.80			2.97	3.38	3.89		2.437	2.620	2.758	3.150	2.069	2.190	2.255	2.401
.975	4.84	5.84			3.63	4.19	4.90		2.979	3.209	3.432	4.063	2.530	2.714	2.819	3.058