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Structure of zero divisors, and other algebraic structures, in higher dimensional real Cayley-Dickson algebras

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STRUCTURE OF ZERO DIVISORS, AND OTHER ALGEBRAIC STRUCTURES, IN HIGHER DIMENSIONAL REAL CAYLEY-DICKSON ALGEBRAS

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ABSTRACT

Real Cayley-Dickson algebras are a class of 2^n-dimensional real algebras containing the real numbers, complex numbers, quaternions, and the octonions (Cayley numbers) as special cases. Each real Cayley-Dickson algebra of dimension greater than eight (a higher dimensional real Cayley-Dickson algebra) is a real normed algebra containing a multiplicative identity and an inverse for each nonzero element. In addition, each element a in the algebra has defined for it a conjugate element $\bar{a}$ analogous to the conjugate in the complex numbers. These algebras are not alternative, but are flexible and satisfy the noncommutative Jordan identity. Each element in these algebras can be written $A = a_1+ea_2$ where e is a basis element and a_1, a_2 are elements of the Cayley-Dickson algebra of next lower dimension.

Results include the facts that for each real Cayley-Dickson algebra $a^i a^j = a^{i+j}$ and $(a^i b)a^j = a^i(ba^j)$ for all integers i,j and any a,b in the algebra. The major result concerns zero divisors.

MAJOR THEOREM. *Let* $A = a_1+ea_2$, $B = b_1+eb_2$, $A,B \neq 0$ *be elements of any real higher dimensional Cayley-Dickson algebra. Let* a_1 *and* a_2 *not be divisors of zero. Then* $AB = 0$ *if and only if*

1) $)a_1,b_1,a_2(= -(a_1,a_1,b_2)+(a_2,a_2,b_2)+[N(a_2)-N(a_1)]b_2$,

2) $(a_1,b_1,a_2) = (a_1,a_1,b_2)+(a_2,a_2,b_2)+[N(a_2)+N(a_1)]b_2$,

where $)A,B,C(= (AB)C+A(BC)$, $(A,B,C) = (AB)C-A(BC)$, *and* $N(a) = a\bar{a}$.

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I. Introduction.

This paper is a study of the properties of higher dimensional real Cayley-Dickson algebras. The real Cayley-Dickson algebras are an infinite class of 2^n-dimensional algebras over the real field. They include all alternative finite dimensional division algebras over the reals: The real numbers, complex numbers, quaternions, and the octonions. The quaternions are a four dimensional noncommutative real algebra devised by W. Hamilton. The octonions (usually called the Cayley numbers) are an eight dimensional noncommutative and nonassociative real algebra devised by A. Cayley. L. E. Dickson in 1919 [11] devised the process whereby each of these algebras generates the algebra of next larger dimension. This class of algebras is thus called the Cayley-Dickson algebras.

Cayley-Dickson algebras up to dimension eight over arbitrary fields (and particularly over the real field) have been extensively studied. It appears that Cayley-Dickson algebras of dimension higher than eight (higher dimensional Cayley-Dickson algebras) have been studied relatively little.

There are several reasons why a study of higher dimensional real Cayley-Dickson algebras could be valuable. Each such algebra is a normed algebra over the reals with the added property that each nonzero element has an inverse. Moreover, the elements of each such algebra satisfy the conditions necessary to permit their use in relativistic quantum mechanics, yielding perhaps, more insight into quantum mechanical phenomena than is presently available. Lastly, such a study could help classify types of algebras in the large class of nonassociative algebras, the noncommutative Jordan algebras.

This paper examines, in particular: The algebraic properties of the zero divisors of higher dimensional real Cayley-Dickson algebras; certain identities which hold (or fail to hold) in these algebras; and the properties of negative integer exponents in these nonassociative algebras.

We indicate the end of a proof in this paper by the symbol //, after the notation of K. G. Kurosh [20].

II. Review of the Literature.

L. E. Dickson [11] in the Annals of Math in 1919, devised a scheme whereby an infinite class of algebras of dimension 2^n could be constructed which contains the real and complex numbers, the quaternions, and the octonions. These algebras are called Cayley-Dickson algebras in honor of Dickson and Arthur Cayley who developed the octonions. Dickson was writing about the "Eight Square Problem", the solution to which was given by the following theorem due to A. Hurwitz in 1898.

> The identity $(x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) = (z_1^2 + \ldots + z_n^2)$ where the z_i are linear in x_i and y_i can hold only for $n = 1,2,4,8$.

For examples of this identity, for each specified n, Dickson used the real numbers, the complex numbers, the quaternions, and the octonions. One of the consequences of Hurwitz's theorem is that the only Cayley-Dickson algebras in which the norm of a product equals the product of the norms are of dimension 1,2,4,8.

An excellent account of the search for division algebras over the real field and the relation of that search to the eight square problem is given by Charles Curtis [10]. This vein of work was completed in 1958 by R. Bott and J. Milnor [8], [23]. They proved that the only division algebras over the reals are of dimension 1,2,4 and 8. Hence the first four Cayley-Dickson algebras over the real numbers are of primary importance.

A. A. Albert examined Hurwitz's proof and Dickson's process in 1941, and generalized these ideas to the idea of quadratic forms [1]. In 1946 [2] he considered the idea of an absolute-valued algebra, and showed that all real algebras are normed algebras. In 1948 [3] he examined the property of power associativity in rings as well as the properties of flexibility and trace-admissibility. Each Cayley-Dickson algebra enjoys these properties. In 1948 [4] he studied trace-admissible algebras and displayed several more properties of the trace operator which hold in any Cayley-Dickson algebra.

It was R. D. Schafer in 1954 [26], however, who came back to examine more closely the Cayley-Dickson algebras themselves. Indeed, the chief references for the subject are his 1954 paper and his 1966 book *An Introduction to Nonassociative Algebras* [28]. In his paper, he derives certain elementary properties of these algebras and examines chiefly their derivation algebras. He shows, for example, that all Cayley-Dickson algebras are flexible. He also shows that the basis elements of all such algebras are alternative, even though the algebras are not alternative if the dimension is greater than eight.

From here, the investigation seems to follow two widely separating paths. One path of study is the investigation and classification of the nonassociative algebras in general. In 1955 [27], Schafer was able to classify simple noncommutative Jordan algebras of characteristic 0 into three classes: simple (commutative) Jordan algebras, simple quasiassociative algebras, and simple flexible algebras of degree two. He then commented that much remains to be learned about that last classification inasmuch as it contains the Cayley-Dickson algebras for which relatively little is known. Most of the work in these areas of late, has been in trying to find identities characterizing certain classes of nonassociative algebras. These have included the concepts of standard algebras, generalized standard algebras, accessible algebras, generalized accessible algebras, and algebras with alternativity conditions. These have been studied by such people as A. J. Penico [24], R. D. Schafer [29], [30], R. E. Block [7], E. Kleinfeld [18], M. H. Kleinfeld, J. F. Kosier [19], and K. McCrimmon [21].

The other path has been to study the quaternions and octonions extensively with some of the techniques of other areas of mathematics. In particular, the ideas of functional analysis have been applied to them by H. H. Goldstine and L. P. Horwitz [14], [15], [16], and more recently by J. Jamison [17]. Number theory and basic algebra methods have been applied to the quaternions and octonions by S. Eilenberg and I. Niven [12], M. J. Wonenburger [32], and H. S. Coxeter [9]. The octonions and their relation to the Dirac wave equation in physics have been studied by R. Penney [25].

The last paper to date, to the author's knowledge, specifically mentioning the Cayley-Dickson algebras is R. D. Schafer's paper in 1970 concerning forms permitting composition [31]. In this paper, he does not mention any significant new results about the Cayley-Dickson algebras.

III. Basic Concepts.

A. Definitions.

The following ideas, which can be found in [21] and in [28], will be used frequently in our study.

Definition 3.1. A ring R is an additive abelian group with a multiplication satisfying the distributive laws $(x + y)z = xz + yz$ and $z(x + y) = zx + zy$ for any x,y,z in R. (Note that $(xy)z = x(yz)$ is not assumed.)

Definition 3.2. An algebra A is a ring which is also a vector space over a field F with a bilinear scalar multiplication satisfying the scalar associative law: $r(xy) = (rx)y = x(ry)$, for all r in F and for all x,y in A.

Generally, what we have defined is called a nonassociative algebra to emphasize that the associative law is not assumed. We will restrict our attention to algebras with a multiplicative identity element over the real field. It is important to note that the real field is of characteristic zero. In the following, we consider algebras rather than rings; but obviously, many definitions and results do not depend on scalars and hence are true for rings as well.

Definition 3.3. An element a of an algebra A is a zero divisor if $a \neq 0$ and there exists some element $b \neq 0$ of A for which $ab = 0$ or $ba = 0$. The elements a and b will be referred to as mutual zero divisors if they are zero divisors and $ab = 0$ or $ba = 0$.

Definition 3.4. A division algebra is an algebra in which every element has an inverse and there are no divisors of zero.

Definition 3.5. The commutator $[x,y]$ is given by $[x,y] = xy - yx$.

Definition 3.6. The associator (x,y,z) is given by $(x,y,z) = (xy)z - x(yz)$.

Definition 3.7. The nucleus of an algebra is the set of all elements x of an algebra A for which $(x,y,z) = (y,x,z) = (y,z,x) = 0$ for all y,z in A.

Definition 3.8. The center of an algebra is the set of all elements in the nucleus of the algebra which also commute with all elements of the algebra.

Definition 3.9. An involution (involutorial anti-automorphism) is a linear operator $x \to \bar{x}$ such that $\overline{xy} = \bar{y}\bar{x}$ and $\bar{\bar{x}} = x$. We call $\bar{x}$ the conjugate of x.

In this paper, we will only be concerned with involutions for which $x + \bar{x}$ and $x\bar{x}$ are in the center of the algebra. Dr. A. J. Penico has suggested that these might best be called "centered involutions".

Definition 3.10. The trace $T(x)$ of the element x is defined by $T(x) = x + \bar{x}$.

Definition 3.11. The norm $N(x)$ of the element x is defined by $N(x) = x\bar{x} = \bar{x}x$.

Definition 3.12. The flexible property is $(xy)x = x(yx)$. If the flexible property is satisfied for all elements of an algebra, the algebra is said to be flexible.

Definition 3.13. The right alternative property is

$(yx)x = yx^2$, and the <u>left alternative</u> property is $x(xy) = x^2y$. If both properties are satisfied by all elements of an algebra, the algebra is said to be <u>alternative</u>.

B. Basic consequences of the definitions.

In this section, we observe basic properties of the terms defined previously so that efficient algebraic manipulations will be available to examine the Cayley-Dickson algebras. Many of these observations are well known, and are included to aid the reader.

The commutator is defined by $[x,y] = xy - yx$. If $[x,y] = 0$, then $xy = yx$. Clearly, $[x,x] = 0$ and $[x,y] = -[y,x]$. In an algebra, a scalar r may be thought of as $r \cdot 1$ where 1 is the identity element of the algebra, so we have $[r,x] = 0$ for all x in the algebra. Moreover, $r[x,y] = [rx,y] = [x,ry]$. Commutators are also additive in each argument, i.e., $[x + y,z] = [x,z] + [y,z]$ and $[x,y + z] = [x,y] + [x,z]$. Thus commutators are linear in each argument.

Since the trace is in the center of an algebra, we have the following additional properties of the commutator and the conjugate.

$[x,y] = -[\bar{x},y] = -[x,\bar{y}] = [\bar{x},\bar{y}] = -\overline{[x,y]}$. These equalities follow from $[\bar{x},y] = [T(x) - x,y] = [T(x),y] + [-x,y] = 0 - [x,y]$; $[x,\bar{y}] = [x,T(y) - y] = [x,T(y)] + [x,-y] = -[x,y]$; and $\overline{[x,y]} = \overline{xy - yx} = \overline{xy} - \overline{yx} = \bar{y}\bar{x} - \bar{x}\bar{y} = [\bar{y},\bar{x}] = [y,x] = -[x,y]$.

The associator is defined by $(x,y,z) = (xy)z - x(yz)$: $(x,y,z) = 0$ if and only if $(xy)z = x(yz)$. Moreover, if r is an element of the center, then $(r,x,y) = (x,r,y) = (x,y,r) = 0$. The associator is also additive in each argument, i.e., for example, $(x + y,z,w) = (x,z,w) + (y,z,w)$. Thus the associator is linear in each argument.

If an algebra A is flexible, then we have $(xy)x = x(yx)$; or, in terms of associators, $(x,y,x) = 0$ for all x,y in A. By the additive property this means $0 = (x + w,y,x + w) = (x,y,x) + (x,y,w) + (w,y,x) + (w,y,w) = (x,y,w) + (w,y,x)$. Thus in a flexible algebra $(x,y,w) = -(w,y,x)$, for all x,y,w in A.

The following relations between conjugates and associators hold in algebras with involution.

$$\begin{aligned}(x,y,z) &= -(\bar{x},y,z) = -(x,\bar{y},z) = -(x,y,\bar{z}) \\ &= (\bar{x},\bar{y},z) = (\bar{x},y,\bar{z}) = (x,\bar{y},\bar{z}) \\ &= -(\bar{x},\bar{y},\bar{z}) \\ &= \overline{(z,y,x)}\end{aligned}$$

These properties follow from the observation that $(\bar{x},y,z) = (T(x) - x,y,z) = (T(x),y,z) + (-x,y,z) = 0 - (x,y,z)$; and $\overline{(z,y,x)} = \overline{(zy)x - z(yx)} = \bar{x}(\bar{y}\bar{z}) - (\bar{x}\bar{y})\bar{z} = -(\bar{x},\bar{y},\bar{z}) = (x,y,z)$.

The "Four Identity" is: $(xy,z,w) - (x,yz,w) + (x,y,zw) = x(y,z,w) + (x,y,z)w$. This is easily verified as being true in any ring, and will be useful in certain calculations.

Just as the flexible property can be expressed in terms of associators, the alternative properties can be as well. The right alternative property $(yx)x = yx^2$ can be written as $(y,x,x) = 0$, and the left alternative property $x(xy) = x^2y$ can be written as $(x,x,y) = 0$. If an algebra is alternative, i.e., both the left and right alternative properties hold, then $0 = (x + y,x + y,z) = (x,x,y) + (x,y,z) + (y,x,z) + (y,y,z) = (x,y,z) + (y,x,z)$. Thus in an alternative algebra $(x,y,z) = -(y,x,z)$. Similarly, in an alternative algebra $(x,y,z) = -(x,z,y)$. Moreover, any two of $(x,x,y) = 0$, $(y,x,x) = 0$, or $(x,y,x) = 0$ imply the third. For example, if the alternative laws hold, then $0 = (x,x + y,x + y) = (x,x,x) + (x,y,x) + (x,x,y) + (x,y,y) = (x,y,x)$.

The trace $T(x)$ has been defined to be in the center of the algebra. Thus if $r = \bar{r}$, then r is in the center since $T(r) = r + \bar{r} = 2r$. For such an r we have $T(rx) = rT(x)$, since $rx + \overline{rx} = rx + \bar{x}\bar{r} = rx + r\bar{x} = r(x + \bar{x})$. It is also clear from the definition that $T(\bar{x}) = T(x)$.

THEOREM 3.1. _In any flexible algebra with involution,_ $T(xy) = T(yx)$ _and_ $T([xy]z) = T(x[yz])$.

Proof. Since $[x,y] = [\bar{x},\bar{y}]$, we have $xy - yx = \bar{x}\bar{y} - \bar{y}\bar{x}$ or $xy + \bar{y}\bar{x} = \bar{x}\bar{y} + yx$. Therefore, $xy + \overline{xy} = yx + \overline{yx}$ and so $T(xy) = T(yx)$. Also since $\overline{(x,y,z)} = -(x,y,z)$, we have $(x,y,z) + \overline{(x,y,z)} = 0$, i.e., $T((x,y,z)) = 0$. Thus $T([xy]z - x[yz]) = T([xy]z) - T(x[yz]) = 0$.//

We also note that $T(x\bar{y}) = T(y\bar{x})$ since $x\bar{y} + \overline{x\bar{y}} = x\bar{y} + y\bar{x} = y\bar{x} + \overline{y\bar{x}}$. Hence $T(x\bar{y}) = T(\bar{x}y)$ because $T(xy) = T(yx)$. It is not true, however, that $T(\bar{x}y) = T(xy)$, as can be seen from the complex numbers with the usual conjugate. There, trace corresponds to twice the real part of each complex number. Thus $T([1 + i]i) = -2$, but $T([1 - i]i) = +2$. Finally, we observe a relationship that exists between $T(x\bar{y})$ and $T(xy)$.

LEMMA 3.2. $T(x)T(y) = T(xy) + T(x\bar{y})$.

<u>Proof</u>. $T(x)T(y) = (x + \bar{x})(y + \bar{y}) = xy + x\bar{y} + \bar{x}y + \bar{x}\bar{y}$ $= (xy + \bar{y}\bar{x}) + (\bar{x}y + x\bar{y} + \bar{x}\bar{y} - \bar{y}\bar{x}) = T(xy) +$ $(x\bar{y} + \bar{x}[y + \bar{y}] - \bar{y}\bar{x}) = T(xy) + (x\bar{y} + T(y)\bar{x} - \bar{y}\bar{x}) = T(xy) +$ $(x\bar{y} + [T(y) - \bar{y}]\bar{x}) = T(xy) + (x\bar{y} + y\bar{x}) = T(xy) + T(x\bar{y})$.//

In an algebra with involution, several useful relationships exist between the conjugate and the operation of association. The following will illustrate.

THEOREM 3.3. <u>In an algebra with involution, the following hold</u>:

i) $\bar{x}(xy) = x(\bar{x}y)$

ii) $(y\bar{x})x = (yx)\bar{x}$

iii) $x(\bar{x}^2 y) = \bar{x}^2(xy)$.

<u>If the algebra is also flexible, then</u>:

iv) $x(\bar{x}y) = (yx)\bar{x}$

v) $x^2(yx) = (x^2 y)x$.

<u>Proof</u>. We use the properties of trace to make the calculations:

i) $\bar{x}(xy) = (T(x) - x)(xy) = T(x)(xy) - x(xy) = x[T(x)y - xy] = x([T(x) - x]y) = x(\bar{x}y)$.

ii) is shown in a similar manner. For iii), consider, $x(\bar{x}^2y) = x[\bar{x}(T(x) - x)y] = x[T(x)\bar{x} - N(x))y] = x[T(x)(\bar{x}y) - N(x)y] = T(x)[x(\bar{x}y)] - N(x)(xy)$. Now by i) this equals $T(x)[\bar{x}(xy)] - N(x)(xy) = [T(x)\bar{x} - N(x)](xy) = [N(x) + \bar{x}^2 - N(x)](xy) = \bar{x}^2(xy)$.

Now assuming the flexible property, iv) follows from $x(\bar{x}y) = -(x,\bar{x},y) + (x\bar{x})y = -(\bar{x},x,y) + N(x)y = (y,x,\bar{x}) + N(x)y = (yx)\bar{x} - yN(x) + N(x)y = (yx)\bar{x}$.

To show v) holds, recall that an associator is zero if any entry is in the center. Then $(x^2,y,x) = (x[T(x) - \bar{x}],y,x) = (T(x)x - N(x),y,x) = (T(x)x,y,x) - (N(x),y,x) = T(x)(x,y,x) - 0 = 0$.//

<u>Definition</u> 3.14. A <u>noncommutative</u> <u>Jordan</u> <u>algebra</u> is a noncommutative flexible algebra satisfying $(x^2,y,x) = 0$.

COROLLARY 3.4. <u>Any flexible noncommutative algebra with involution is a noncommutative Jordan algebra</u>.

<u>Proof</u>. This follows from the proof of v) above.//

The following is given by Schafer [28] as an exercise.

LEMMA 3.5. <u>In any flexible algebra</u>: $(x^2,y,x) = x(x^2y) - x^2(xy) = (yx^2)x - (yx)x^2$.

<u>Proof</u>. $x(x^2y) - x^2(xy) = x[(xx)y] - (xx)(xy) = -(x,x,xy) + x[x(xy)] + x[(x,x,y) - x(xy)] = (xy,x,x) - x(y,x,x)$ by the flexible property. Now, this last expression becomes $[(xy)x]x - (xy)x^2 - x[(yx)x] + x(yx^2) = -(xy)x^2 + x(yx^2) - x[(yx)x] + [(xy)x]x = -(x,y,x^2) -$

$x[(yx)x] + [x(yx)]x = -(x,y,x^2) + (x,yx,x) = (x^2,y,x)$.

Likewise $(yx^2)x - (yx)x^2 = [y(xx)]x - (yx)(xx) = (yx,x,x) - [(yx)x]x + [(yx)x - (y,x,x)]x = (x,x,y)x - (x,x,xy)$ by the flexible property. This then becomes $(x^2y)x - [x(xy)]x - x^2(xy) + x[x(xy)] = (x^2,y,x) + x[x(xy)] - x[(xy)x] = (x^2,y,x) - x(x,y,x) = (x^2,y,x)$.//

LEMMA 3.6. *In any noncommutative Jordan algebra we have* $(x,x,yx) = (x,x,y)x$ *and* $(xy,x,x) = x(y,x,x)$.

Proof. We use the "Four Identity" which is valid in any algebra.

$$(xx,y,x) - (x,xy,x) + (x,x,yx) = x(x,y,x) + (x,x,y)x.$$

Now by the noncommutative Jordan identity and the flexible property, we have $(x,x,yx) = (x,x,y)x$.

Similarly, $(xy,x,x) - (x,yx,x) + (x,y,x^2) = x(y,x,x) + (x,y,x)x$. Thus, $(xy,x,x) = x(y,x,x)$.//

COROLLARY 3.7. *In any noncommutative Jordan algebra we have* $[x(xy)]x = x[x(yx)]$ *and* $[(xy)x]x = x[(yx)x]$.

Proof. $0 = (x,x,yx) - (x,x,y)x = x^2(yx) - x[x(yx)] - (x^2y)x + [x(xy)]x = -(x^2,y,x) + [x(xy)]x - x[x(yx)] = [x(xy)]x - x[x(yx)]$. The other part is similar.//

We have defined the norm of x to be $N(x) = x\bar{x}$, and have specified that $N(x)$ is in the center. The presence of norm and trace lead to the next lemma.

LEMMA 3.7. *Every element in an algebra with involution satisfies the quadratic equation:* $x^2 - T(x)x + N(x)1 = 0$.

Proof. $x^2 - T(x)x + N(x)1 = x^2 - (x + \bar{x})x + x\bar{x} = 0$.//

LEMMA 3.8. The following hold in any algebra with involution:

i) $N(r) = r^2$, if $r = \bar{r}$

ii) $N(rx) = r^2 N(x)$, if $r = \bar{r}$

iii) $N(x) = N(\bar{x})$.

Proof. Each is clear from the definition of norm.//

THEOREM 3.9. In any flexible algebra with involution the following hold:

i) $N(x\bar{y}) = N(xy)$

ii) $N(\bar{x}y) = N(xy)$

iii) $N(xy) = N(yx)$.

Proof. The proof relies on the fact that the trace is in the center of the algebra.

i) $N(x\bar{y}) = (x\bar{y})(y\bar{x}) = [x(T(y) - y)](y\bar{x}) = [T(y)x - xy](y\bar{x}) = [T(y)x](y\bar{x}) - (xy)(y\bar{x}) = [T(y)x](y\bar{x}) - (xy)[(T(y) - \bar{y})\bar{x}] = [T(y)x](y\bar{x}) - T(y)[(xy)\bar{x}] + (xy)(\bar{y}\bar{x}) = T(y)[x(y\bar{x}) - (xy)\bar{x}] + N(xy) = -T(x)(x,y,\bar{x}) + N(xy) = T(x)(x,y,x) + N(xy) = N(xy)$.

Now, to prove iii), we note that i) implies $N(xy) = N(\bar{\bar{x}}\bar{y}) = N(\overline{y\bar{x}}) = N(y\bar{x}) = N(yx)$.

From this ii) follows easily.//

We note here that the fact that $N(xy) = N(yx)$ proved to be very useful in studying zero divisors in Cayley-Dickson algebras.

LEMMA 3.10. In any algebra with involution we have: $N(x \pm y) = N(x) + N(y) \pm T(x\bar{y})$.

Proof. $N(x \pm y) = (x \pm y)\overline{(x \pm y)} = (x \pm y)(\bar{x} \pm \bar{y}) = x\bar{x} \pm x\bar{y} \pm y\bar{x} + y\bar{y} = N(x) + N(y) \pm (x\bar{y} + \overline{x\bar{y}}) = N(x) + N(y) \pm T(x\bar{y})$.//

IV. The Cayley-Dickson Algebras.

R. D. Schafer's book [28] gives a clear description of the process that Dickson devised to generate the Cayley-Dickson algebras. Let A be an algebra with identity 1 and with an involution. Let A have dimension n. Then we construct an algebra B of dimension 2n over the same field as A and having A isomorphic to a subalgebra of B. Let B consist of all ordered pairs (a_1,a_2) where a_1 and a_2 are elements of A. Let addition and multiplication by scalars be defined componentwise. Moreover define a multiplication in B by $(a_1,a_2)(a_3,a_4) = (a_1a_3 + va_4\bar{a}_2, \bar{a}_1a_4 + a_3a_2)$, where v is some nonzero scalar. Defining (1,0), the identity element of B, as 1, then: $A' = \{(a,0) | a \text{ in } A\}$ is a subalgebra isomorphic to A; $e = (0,1)$ is an element of B such that $e^2 = v1$; and B is the vector space direct sum $B = A' \oplus A'$.

For our purposes, we restrict our field to be the real numbers, and v to be -1. It will be handier to consider elements of B to be of the form $x = a_1 + ea_2$ with multiplication given by $(a_1 + ea_2)(a_3 + ea_4) = (a_1a_3 - a_4\bar{a}_2) + e(\bar{a}_1a_4 + a_3a_2)$. This notation will be consistently used. Define $\bar{x}$ by $\bar{x} = \bar{a}_1 - ea_2$ if $x = a_1 + ea_2$. Then it is easy to see that $\overline{xy} = \bar{y}\bar{x}$ since $a \to \bar{a}$ is an involution for A. Thus $x \to \bar{x}$ is an involution for B. Trace and norm are defined as before with the observation that for $x = a_1 + ea_2$, $T(x) = T(a_1)$ and $N(x) = x\bar{x} = (a_1 + ea_2)(\bar{a}_1 - ea_2) = (a_1\bar{a}_1 + a_2\bar{a}_2) + e(-\bar{a}_1a_2 + \bar{a}_1a_2) = N(a_1) + N(a_2)$. In any

Cayley-Dickson algebra, the inverse of any nonzero element exists and is defined by $x^{-1} = \bar{x}/N(x)$. Then $xx^{-1} = x[\bar{x}/N(x)] = x\bar{x}/N(x) = 1$. Since this is the case, there are no non-trivial proper ideals in any of these algebras. The reader should take note that all of the material presented above is used repeatedly in the remainder of this paper.

If A is the real numbers, then the B given by the Cayley-Dickson process is the complex numbers, e is represented by i, and the involution is the familiar conjugate. Moreover, $T(x) = 2\,\mathrm{Re}(x)$ and $N(x) = a_1^2 + a_2^2$ where $x = a_1 + ia_2$. We note that in going from the real to the complex field, order is lost, i.e., the complex numbers are not ordered.

If A is the complex numbers, B will be the quaternions, e will usually be represented by j, and $ij = k$. In going from the complex numbers to the quaternions, commutativity is lost since $ji = -k$.

If A is the quaternions, B will be the octonions (usually called the Cayley numbers). The octonions are not associative, but are alternative and are a division algebra.

If A is the octonions, the 2^4-dimensional algebra obtained is the smallest Cayley-Dickson algebra which is not a division algebra since it has zero divisors. It is not alternative for the same reason. It has no name, and will be referred to in this paper as A_4 in keeping with the notation of Schafer [26]. To the author's knowledge, A_4

has not been studied extensively, and very little is known about its properties. This paper is an attempt to answer some questions about A_4; in particular, questions regarding zero divisors. In our investigation, however, many other facts were discovered which are true for all Cayley-Dickson algebras.

There is an alternate way of looking at each of these algebras. On can consider the complex numbers as the vector space generated by {1,i} over the real numbers. Likewise the quaternions can be thought of as the vector space over the real numbers having basis elements {1,i,j,k} with multiplication of basis elements defined by the following table:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

An artibrary element thus has the form $r_0 + r_1 i + r_2 j + r_3 k$, where the r's are real numbers.

Similarly one can view each of the Cayley-Dickson algebras as a vector space over the reals. In order to facilitate notation, let the basis of A_n be denoted $\{1{=}e_0, e_1, \ldots, e_{2^n-1}\}$, where $e_0^2 = e_0$ and $e_i^2 = -e_0$ for $i \neq 0$. Thus an element of the algebra can be represented as the linear combination $x = \sum_i r_i e_i$, $i = 0, \ldots, 2^n-1$.

With this representation $\bar{x} = r_0e_0 - \sum_i r_ie_i$, $i = 0,\ldots,2^n-1$. Also $T(x) = 2r_0$, and $N(x) = \sum_i r_i^2$, $i = 0,\ldots,2^n-1$. Note also that $\bar{e}_0 = e_0$, but $\bar{e}_i = -e_i$ if $i \neq 0$. In the remainder of this paper, we shall refer to the subspace re_0 as simply "the reals r" with the meaning being clear from context.

LEMMA 4.1. $N(x) = 0$ _if and only if_ $x = 0$.

Proof. $N(x) = 0$ if and only if $r_i = 0$ for all i. This is equivalent to saying $x = 0$.//

Tables for the basis element multiplication for A_3, and for A_4 will be found in Appendix A and Appendix B.

It would seem that as the Cayley-Dickson process continues, some algebraic property would be lost at each step just as in the first three steps. This paper will indicate several ways that A_5 differs from A_4, but what is lost in higher dimensions remains largely unknown.

Another viewpoint is to consider what properties, if any, are enjoyed by all Cayley-Dickson algebras. This last question is easily answered at least in part. For example, if a new norm is defined by $n(x) = \sqrt{N(x)}$, then each Cayley-Dickson algebra becomes a normed linear space over the real numbers with the added property of having a multiplication and an inverse for each nonzero element. This explains why the techniques of functional analysis have been useful in the past in studying the quaternions and the octonions.

Since it has been known for many years that the octonions are the only alternative, nonassociative division algebra over the real numbers, an interesting question answered by Schafer is whether or not any higher dimensional Cayley-Dickson algebras are alternative. The answer is no, but they are all flexible as the following shows.

In 1954, R. D. Schafer [26] proved the following theorem. We shall give an alternate proof here.

THEOREM 4.2. *Any Cayley-Dickson algebra is flexible.*

Proof. This is obvious for any associative algebra. It therefore suffices to show that if A_n is flexible, then A_{n+1} is flexible. Let $A = a_1 + ea_2$ and $B = b_1 + eb_2$ be elements of A_{n+1}. Then $(A,B,A) = (\bar{A},\bar{B},A) =$
$[(\bar{a}_1-ea_2)(\bar{b}_1-eb_2)](a_1+ea_2) - (\bar{a}_1-ea_2)[(\bar{b}_1-eb_2)(a_1+ea_2)] =$
$[(\bar{a}_1\bar{b}_1-b_2\bar{a}_2)+e(-a_1b_2-\bar{b}_1a_2)](a_1+ea_2) +$
$(-\bar{a}_1+ea_2)[(\bar{b}_1a_1+a_2\bar{b}_2)+e(b_1a_2-a_1b_2)] =$
$[(\bar{a}_1\bar{b}_1)a_1-(b_2\bar{a}_2)a_1+a_2(\bar{b}_2\bar{a}_1)+a_2(\bar{a}_2b_1)-\bar{a}_1(\bar{b}_1a_1)-\bar{a}_1(a_2\bar{b}_2) -$
$(b_1a_2)\bar{a}_2+(a_1b_2)\bar{a}_2] + e[(b_1a_1)a_2-(a_2\bar{b}_2)a_2-a_1(a_1b_2) -$
$a_1(\bar{b}_1a_2)-a_1(b_1a_2)+a_1(a_1b_2)+(\bar{b}_1a_1)a_2+(a_2\bar{b}_2)a_2] =$
$[(\bar{a}_1,\bar{b}_1,a_1) + T(a_2(\bar{b}_2\bar{a}_1)) - T(\bar{a}_1(a_2\bar{b}_2)) + a_2(\bar{a}_2b_1) -$
$(b_1a_2)\bar{a}_2] + e[(b_1a_1 + \bar{b}_1a_1)a_2 - a_1(\bar{b}_1a_2+b_1a_2)] =$
$[(a_1,b_1,a_1)+T(\bar{a}_1(a_2\bar{b}_2))-T(\bar{a}_1(a_2\bar{b}_2))-(a_2,\bar{a}_2,b_1)-(b_1,a_2,\bar{a}_2)]$
$+ e[T(b_1)(a_1a_2) - T(b_1)(a_1a_2)] = (a_1,b_1,a_1)+[(a_2,a_2,b_1) -$
$(b_1,a_2,a_2)]$ which is zero if A_n is flexible.//

We note that xyx is now unambiguous since $(xy)x = x(yx)$.

THEOREM 4.3. *Let* $A = a_1+ea_2$ *and* $B = b_1+eb_2$ *be elements of a Cayley-Dickson algebra of any dimension.*

Then $(A,A,B) = [(a_1,b_2,a_2)+(a_2,a_2,b_1)+(a_1,a_1,b_1)] + e[(a_1,a_1,b_2)-(a_1,b_1,a_2)+(a_2,a_2,b_2)]$.

Proof. We first observe that $(A,A,B) = -(A,\bar{A},B) = -N(A)B + A(\bar{A}B)$.

Now $A(\bar{A}B) = (a_1+ea_2)[(\bar{a}_1-ea_2)(b_1+eb_2)] = (a_1+ea_2)[(\bar{a}_1b_1+b_2\bar{a}_2)+e(a_1b_2-b_1a_2)] = [a_1(\bar{a}_1b_1)+a_1(b_2\bar{a}_2)-(a_1b_2)\bar{a}_2 + (b_1a_2)\bar{a}_2] + e[\bar{a}_1(a_1b_2)-\bar{a}_1(b_1a_2)+(\bar{a}_1b_1)a_2+(b_2\bar{a}_2)a_2] = [-(a_1,b_2,\bar{a}_2)+a_1(\bar{a}_1b_1)+(b_1a_2)\bar{a}_2]+e[(\bar{a}_1,b_1,a_2)+\bar{a}_1(a_1b_2) + (b_2\bar{a}_2)a_2]$.

Also $-N(A)B = -(a_1\bar{a}_1 + a_2\bar{a}_2)(b_1 + eb_2) = [-(a_1\bar{a}_1)b_1-(a_2\bar{a}_2)b_1] + e[-(a_1\bar{a}_1)b_2-(a_2\bar{a}_2)b_2]$.

So $(A,A,B) = [-(a_1,b_2,\bar{a}_2)+a_1(\bar{a}_1b_1)-(a_1\bar{a}_1)b_1+(b_1a_2)\bar{a}_2-(a_2\bar{a}_2)b_1] + e[(\bar{a}_1,b_1,a_2)+\bar{a}_1(a_1b_2)-(a_1\bar{a}_1)b_2 + (b_2\bar{a}_2)a_2-(a_2\bar{a}_2)b_2]$. Now since $N(a) = a\bar{a} = \bar{a}a$ and is in the center, we have: $(A,A,B) = [-(a_1,b_2,\bar{a}_2) - (a_1,\bar{a}_1,b_1)+(b_1,a_2,\bar{a}_2)] + e[(\bar{a}_1,b_1,a_2)-(\bar{a}_1,a_1,b_2) + (b_2,\bar{a}_2,a_2)]$. Now using the flexible property and the properties of conjugates and associators, we have the result.//

From this we get another theorem of Schafer's (cf. [28], p. 46) as a corollary.

COROLLARY 4.4. <u>A Cayley-Dickson algebra $\mathcal{B}$ is alternative if and only if the generating algebra $\mathcal{A}$ is associative</u>.

Proof. If $\mathcal{B}$ is alternative, our equation for (A,A,B) reduces to $(a_1,b_2,a_2)-e(a_1,b_1,a_2)$ which is identically zero if and only if the algebra $\mathcal{A}$ is associative.//

Finally, we comment here, that it is easy to see that the center of each Cayley-Dickson algebra is exactly F1, the field times the identity.

V. Exponent Properties.

It is a well known fact that each Cayley-Dickson algebra is "power associative."

Definition 5.1. An algebra is power associative if $x^1 = x$, $x^{i+1} = xx^i$, and $x^i x^j = x^{i+j}$ for all x in the algebra, and i,j positive integers.

This means that $(x^m, x^n, x^p) = 0$ for all positive integers m,n,p and x any element of a Cayley-Dickson algebra. What we wish to do in this section is extend this definition to integer powers (non-positive as well), and show that all Cayley-Dickson algebras still obey this associative property.

LEMMA 5.1. In any Cayley-Dickson algebra, $(\bar{x})^n = \overline{x^n}$.

Proof. We use power associativity and induction on n. It suffices to note that:

$$(\bar{x})^{k+1} = (\bar{x})^k \bar{x} = \overline{(x^k)}\bar{x} = \overline{x(x^k)} = \overline{x^{k+1}}.//$$

Although every element in a Cayley-Dickson algebra has an inverse, it is not true that it is unique since zero divisors exist. For example, if $aa^{-1} = 1$ and if $ab = 0$, then $a(a^{-1}+b) = 1$ also. To avoid this problem, we adopt the convention that a^{-1} will always mean $\bar{a}/N(a)$. Define $a^0 = 1$, for $a \neq 0$.

LEMMA 5.2. In any Cayley-Dickson algebra, $a^{-1}a^n = a^{n-1}$ for all integers n greater than or equal to 1.

Proof. We use induction on n. It is clear that the statement is true for n = 1 by our definitions. For $n \geq 2$, assume $a^{-1}a^k = a^{k-1}$ for $1 < k \leq n-1$. Recall that a satisfies the quadratic equation $a^2 - T(a)a + N(a)1 = 0$. Then $a^k = T(a)a^{k-1} - N(a)a^{k-2}$. Thus $a^{-1}a^n = a^{-1}[T(a)a-N(a)]a^{n-2}$ $= [T(a)a-N(a)]a^{-1}a^{n-2} = [T(a)a-N(a)]a^{n-3} = a^{n-1}$.//

COROLLARY 5.3. <u>The only idempotent elements of</u> A_n <u>are</u> 0 <u>and</u> 1.

<u>Proof</u>. $a^2 = a$ if and only if $a = 0$ or $a = a^{-1}a^2 = a^{-1}a = 1$.//

COROLLARY 5.4. <u>For</u> $n \geq 1$, <u>we have</u>:

i) $a^{-1}a^n = a^na^{-1}$

ii) $\bar{a}a^n = a^n\bar{a} = N(a)a^{n-1}$.

<u>Proof</u>. For i) we note that $(a^{n-1},a,a^{-1}) = -(a^{-1},a,a^{n-1})$ by flexibility. Thus $a^na^{-1}-a^{n-1} = -a^{n-1}+a^{-1}a^n = 0$.

To see ii) note that $[\bar{a}/N(a)]a^n = a^{-1}a^n = a^{n-1} = a^na^{-1}$ $= a^n[\bar{a}/N(a)]$.//

COROLLARY 5.5. <u>There are no nilpotent elements in any Cayley-Dickson algebra</u>, i.e., $a^n = 0$ <u>implies</u> $a = 0$.

<u>Proof</u>. Suppose $a \neq 0$. Let n be the smallest positive integer such that $a^n = 0$. Then $a^{n-1} = a^{-1}a^n = a^{-1} \cdot 0 = 0$, so $a^{n-1} = 0$, a contradiction.//

THEOREM 5.6. $N^p(a) = N(a^p)$, <u>where</u> p <u>is a positive integer and</u> $N^p(a)$ <u>is</u> $[N(a)]^p$.

<u>Proof</u>. First note that $0 = (a^k,a,a^{k-1}) = (a^k,\bar{a},\overline{a^{k-1}})$ $= (a^k,\bar{a},(\bar{a})^{k-1})$. Thus $(a^k\,\bar{a})(\bar{a})^{k-1} = a^k(\bar{a})^k$. Now the proof of the theorem is by induction on p. Assume the statement true for $1 \leq p \leq k-1$. Then $N(a^k) = a^k\overline{a^k} =$

$(a^k\bar{a})(\bar{a})^{k-1} = N(a)a^{k-1}(\bar{a})^{k-1} = N(a)a^{k-1}\overline{a^{k-1}} = N(a)N(a^{k-1}) =$
$N(a)[N(a)]^{k-1} = N^k(a)$.//

COROLLARY 5.7. $(a^m)^{-1} = (a^{-1})^m$.

Proof. $(a^m)^{-1} = \overline{a^m}/N(a^m) = (\bar{a})^m/N^m(a) = [\bar{a}/N(a)]^m =$
$(a^{-1})^m$.//

Notation. Let a^{-m} be defined to be $(a^{-1})^m$.

LEMMA 5.8. $T(a^2) = T^2(a) - 2N(a)$.

Proof. $T(a^2) = a^2+\overline{a^2} = a^2+\bar{a}^2 = a^2+\bar{a}^2+2a\bar{a} - 2a\bar{a} =$
$(a + \bar{a})^2 - 2a\bar{a} = T^2(a) - 2N(a)$.//

Theorem 5.6 is the best result possible for norms of products; for, although $N(ab) = N(a)N(b)$ is valid for Cayley-Dickson algebras of dimension 1,2,4, and 8, it is not valid for higher dimensions, because of zero divisors. For example, if $a,b \neq 0$, but $ab = 0$, then $0 = N(ab) \neq N(a)N(b)$. In fact, as an example in the next chapter shows, both $N(ab) > N(a)N(b)$ and $N(a'b') < N(a')N(b')$ can occur.

Definition 5.2. An algebra is integer power associative if $x^1 = x$, $x^{i+1} = xx^i$, and $x^ix^j = x^{i+j}$ for all x in the algebra and i,j any integers.

To the author's knowledge this idea is not found in the literature. The following leads to the fact that the Cayley-Dickson algebras are integer power associative.

LEMMA 5.9. For m,n positive integers, $(a^m)^{-1}(a^n)^{-1} =$
$(a^ma^n)^{-1}$.

Proof. $(a^m)^{-1}(a^n)^{-1} = \dfrac{\overline{a^m}\;\overline{a^n}}{N(a^m)N(a^n)} = \dfrac{(\bar{a})^m(\bar{a})^n}{N^m(a)N^n(a)} =$
$\dfrac{(\bar{a})^{m+n}}{N^{m+n}(a)} = \dfrac{\overline{a^{m+n}}}{N(a^{m+n})} = (a^{m+n})^{-1} = (a^ma^n)^{-1}$.//

LEMMA 5.10. *For* m,n *positive integers*, $a^m(\bar{a})^n = (\bar{a})^n a^m = N^n(a)a^{m-n}$.

Proof. Note that $0 = (a^m, a, a^{n-1}) = (a^m, \bar{a}, \overline{a^{n-1}}) = (a^m, \bar{a}, (\bar{a})^{n-1}) = (a^m\bar{a})(\bar{a})^{n-1} - a^m[\bar{a}(\bar{a})^{n-1}]$. Hence $a^m(\bar{a})^n = N(a)a^{m-1}(\bar{a})^{n-1}$. By induction we have $a^m(\bar{a})^n = N^j(a)a^{m-j}(\bar{a})^{n-j}$ for all j, $1 \leq j \leq m,n$.

Now if $n = m$, then $a^m(\bar{a})^n = N^n(a)$.

If $n < m$, then $a^m(\bar{a})^n = N^n(a)a^{m-n}$ for $j = n$.

If $n > m$, then $a^m(\bar{a})^n = N^m(a)(\bar{a})^{n-m} = N^m(a)(\bar{a})^{n-m}\left(\frac{N^{n-m}(a)}{N^{n-m}(a)}\right) = N^{m+n-m}(a)[(\bar{a})^{n-m}/N^{n-m}(a)] = N^n(a)[\bar{a}/N(a)]^{n-m} = N^n(a)(a^{-1})^{n-m} = N^n(a)a^{m-n}$.

The proof of the other part is similar.//

THEOREM 5.11. $a^m a^n = a^{m+n}$, *for all integers* m,n.

Proof. Case 1: m or n = 0; this follows from the definition of a^0.

Case 2: m,n positive; this follows from power associativity.

Case 3: m,n negative; $a^m a^n = (a^{-m})^{-1}(a^{-n})^{-1} = (a^{-m}a^{-n})^{-1} = (a^{-m-n})^{-1} = a^{m+n}$.

Case 4: n negative, m positive; $a^m a^n = a^m(a^{-n})^{-1} = a^m[\overline{a^{-n}}/N(a^{-n})] = a^m[(\bar{a})^{-n}/N^{-n}(a)] = [N^{-n}(a)a^{m+n}]/N^{-n}(a) = a^{m+n}$.

Case 5: n positive, m negative; $a^m a^n = (a^{-m})^{-1}a^n = [(\overline{a^{-m}})/N(a^{-m})]a^n = [(\bar{a})^{-m}/N^{-m}(a)]a^n = [N^{-m}(a)a^{n-(-m)}]/N^{-m}(a) = a^{m+n}$.//

THEOREM 5.12. $(a^m)^n = a^{mn}$ *for all integers* m *and* n.

Proof. Case 1: m or n = 0; the result is obvious.

Case 2: m,n positive; this follows by power associativity.

Case 3: m,n negative; $(a^m)^n = ([(a^{-m})^{-1}]^{-n})^{-1} = [(a^{-m})^n]^{-1} = (a^{-m})^{-n} = a^{(-m)(-n)} = a^{mn}$.

Case 4: m negative, n positive; $(a^m)^n = [(a^{-m})^{-1}]^n = (a^{-m})^{-n} = [[a^{(-m)}]^n]^{-1} = a^{mn}$.

Case 5: m positive, n negative; $(a^m)^n = [(a^m)^{-n}]^{-1} = [a^{m(-n)}]^{-1} = a^{mn}$.//

We now note that it is not possible easily to extend these results to fractional exponents since even $a^{1/2}$ is ambiguous. For example, each basis element $e_i \neq 1$ satisfies the equation $x^2 = -1$. An interesting side light does occur here, however, in that the square of an associator is in fact a negative real number, as we show below.

LEMMA 5.13. $T(x) = 0$ *implies* $x^2 = -N(x)$.

Proof. Recall, for any x, we have $x^2 - T(x)x + N(x)1 = 0$. So, if $T(x) = 0$, the result follows.//

LEMMA 5.14. $(a,b,c)^2 = -N((a,b,c))$.

Proof. Recall that $(a,b,c) = \overline{(c,b,a)} = -\overline{(a,b,c)}$ so $T((a,b,c)) = 0$. (Theorem 3.1). Thus by Lemma 5.13, the result follows.//

It is now possible to show that flexibility, $(a,b,a) = 0$, and the noncommutative Jordan identity, $(a^2,b,a) = 0$, are special cases of a more general identity involving the associator.

LEMMA 5.15. $(a^n,b,a) = (a,b,a^n) = 0$ *for all non-negative integers* n.

Proof. The statement is clear for $n = 0,1,2$. We use induction and the flexible property to finish the proof. Assume $(a^n,b,a) = 0$ for all $n \leq k$. Then $(a^{k+1},b,a) = (a^k[T(a)-\bar{a}],b,a) = T(a)(a^k,b,a)-N(a)(a^{k-1},b,a) = 0.$//

LEMMA 5.16. $(a^m,b,a^n) = 0$ *for all non-negative integers* m,n.

Proof. If $m = n$, the result follows from the flexible property. If m or $n = 0$, the result is obvious. If $m \neq n$, and if neither is zero, then $(a^m,b,a^n) = (a^{m-1}[T(a)-\bar{a}],b,a^{n-1}[T(a)-\bar{a}]) = T^2(a)(a^{m-1},b,a^{n-1}) - T(a)N(a)(a^{m-1},b,a^{n-2}) - T(a)N(a)(a^{m-2},b,a^{n-1}) - N^2(a)(a^{m-2},b,a^{n-2})$, which clearly vanishes by a finite induction argument. Thus the theorem follows.//

THEOREM 5.17. $(a^m,b,a^n) = 0$ *for all integers* m,n.

Proof. If $m = n$, the result is clear. If m,n are non-negative, the result follows by Lemma 5.16. If m,n are negative, then $(a^m,b,a^n) = (\overline{a^m},b,\overline{a^n}) = ((\bar{a})^m,b,(\bar{a})^n) = (N^m(a)[a^{-1}]^m,b,N^n(a)[a^{-1}]^n) = N^{n+m}(a)(a^{-m},b,a^{-n}) = 0.$

If m is negative, and n is positive, then $(a^m,b,a^n) = -(\overline{a^m},b,a^n) = -((\bar{a})^m,b,a^n) = -(N^m(a)a^{-m},b,a^n) = -N^m(a)(a^{-m},b,a^n) = 0.$

If m is positive, and n is negative, then $(a^m,b,a^n) = -(a^m,b,\overline{a^n}) = -(a^m,b,(\bar{a})^n) = -(a^m,b,N^n(a)a^{-n}) = -N(a)(a^m,b,a^{-n}) = 0.$//

VI. Basis Element Properties.

Some very useful identities which are true in alternative algebras are the Moufang identities:

i) $(xax)y = x[a(xy)]$

ii) $y(xax) = [(yx)a]x$

iii) $(xy)(ax) = x(ya)x.$

Unfortunately, these identities do not hold in Cayley-Dickson algebras of dimension higher than eight. We will now show, however, that the Moufang identities and the alternative property do hold in a restricted, but useful, sense.

Consider the Cayley-Dickson algebra $\mathcal{B}$ of dimension 2^n formed by adjoining $e_{2^{n-1}}$ to the basis of the 2^{n-1} dimensional algebra. *Call the adjoined basis element* e *for notational convenience.*

THEOREM 6.1. $(e,A,B) + (A,e,B) = 0$ *for all* A,B *in* $\mathcal{B}$.

Proof. Let $A = a_1+ea_2$ and $B = b_1+eb_2$. Then, $(e,A,B) + (A,e,B) = (eA)B - e(AB) + (Ae)B - A(eB) = (-a_2+ea_1)(b_1+eb_2) - (0+e1)[(a_1b_1-b_2\bar{a}_2)+e(\bar{a}_1b_2+b_1a_2)] + (-\bar{a}_2+e\bar{a}_1)(b_1+eb_2) - (a_1+ea_2)(-b_2+eb_1) = [(-a_2b_1-b_2\bar{a}_1)+e(-\bar{a}_2b_2+b_1a_1)] + [(\bar{a}_1b_2+b_1a_2)-e(a_1b_1-b_2\bar{a}_2)] + [(-\bar{a}_2b_1-b_2a_1)+e(-a_2b_2+b_1\bar{a}_1)] + [(a_1b_2+b_1\bar{a}_2)+e(-\bar{a}_1b_1+b_2a_2)] =$

$[-a_2b_1-b_2\bar{a}_1+\bar{a}_1b_2+b_1a_2-\bar{a}_2b_1-b_2a_1+a_1b_2+b_1\bar{a}_2] +$

$e[-\bar{a}_2b_2+b_1a_1-a_1b_1+b_2\bar{a}_2-a_2b_2+b_1\bar{a}_1-\bar{a}_1b_1+b_2a_2] =$

$[-T(a_2)b_1 + T(a_1)b_2 + T(a_2)b_1 - T(a_1)b_2] +$

$e[-T(a_2)b_2 + T(a_1)b_1 - T(a_1)b_1 + T(a_2)b_2] = 0.$//

LEMMA 6.2. *The following are true for all* A,B *in* $\mathcal{B}$.

i) (A,B,e) + (A,e,B) = 0

ii) (e,A,A) = (A,A,e) = (e,e,A) = (A,e,e) = 0

iii) (e,A,B) + (e,B,A) = 0

iv) (A,B,e) + (B,A,e) = 0.

Proof. To see i), note that (A,B,e) + (A,e,B) = −(e,B,A) − (B,e,A) = 0, using the flexible property and Theorem 6.1. For the same reasons, 0 = (e,A,A) + (A,e,A) = (e,A,A); 0 = (A,A,e) + (A,e,A) = (A,A,e); 0 = (e,e,A) + (e,A,e) = (e,e,A); and 0 = (A,e,e) + (e,A,e) = (A,e,e) proving ii). To see iii), note that (e,A,B) = −(A,e,B) = (B,e,A) = −(e,B,A). Similarly for iv), (A,B,e) = −(A,e,B) = (B,e,A) = −(e,B,A).//

With this Theorem and Lemma, the restricted Moufang identities follow.

THEOREM 6.3. *Let* e *be the adjoined basis element used to form* $\mathcal{B}$. *For any* A,B *in* $\mathcal{B}$, *the following restricted Moufang identities hold:*

i) (eAe)B = e[A(eB)]

ii) B(eAe) = [(Be)A]e

iii) (eB)(Ae) = e(BA)e.

Proof. i) (eAe)B−e[A(eB)] = (eA,e,B)+(e,A,eB) = −(e,eA,B)−(e,eB,A) = −[e(eA)]B+e[(eA)B]+e[(eB)A]−[e(eB)]A = $-(e^2A)B-(e^2B)A$+e[(eA)B+(eB)A] = AB+BA+e[(eA)B+(eB)A] = e[−e(AB)−e(BA)+(eA)B+(eB)A] = e[(e,A,B)+(e,B,A)] = 0.

The proof of ii) is similar. [(Be)A]e−B(eAe) = (Be,A,e)+(B,e,Ae) = −(B,Ae,e)−(A,Be,e) = −[B(Ae)]e +

$B[(Ae)e]-[A(Be)]e+A[(Be)e] = B[Ae^2]+A[Be^2]-[B(Ae)+A(Be)]e =$ $-BA-AB-[B(Ae)+A(Be)]e = [(BA)e+(AB)e-B(Ae)-A(Be)]e =$ $[(B,A,e)+(A,B,e)]e = 0.$

For iii), consider: $(eB)(Ae)-e(BA)e = (e,B,Ae) +$ $e[B(Ae)-(BA)e] = -(e,Ae,B) - e(B,A,e) = -(eAe)B +$ $e[(Ae)B-(B,A,e)] = -e[A(eB)] + e[(Ae)B-(B,A,e)] =$ $-e[A(eB)-(Ae)B+(B,A,e)] = -e[-(A,e,B)+(B,A,e)] =$ $-e[(B,e,A)+(B,A,e)] = 0.$//

COROLLARY 6.4. $(B,eA,e) = -(B,e,A)e.$

Proof. $(B,eA,e) = [B(eA)]e - B(eAe) = [B(eA)]e -$ $[(Be)A]e = [B(eA)-(Be)A]e = -(B,e,A)e.$//

Considering the Cayley-Dickson multiplication, a question is why $(a_1+ea_2)(b_1+eb_2) = (a_1b_1-b_2\bar{a}_2) +$ $e(\bar{a}_1b_2+b_1a_2)$ is a natural definition of multiplication. It would also be natural to multiply as polynomials to obtain $a_1b_1+(ea_2)(eb_2)+a_1(eb_2)+(ea_2)b_1$. Schafer [28] proves that, for alternative algebras, the Cayley-Dickson multiplication and the polynomial multiplication are the same.

THEOREM 6.5. (cf, [28], p.46). *Let A be an alternative algebra with 1, and with involution, and let the adjoined element e be such that $e^2 = -1$ and $ae = e\bar{a}$ for all a in A. Then the polynomial multiplication in $A + eA$ is the Cayley-Dickson multiplication.*

Proof. Following Schafer's proof, first note that the alternative laws imply that, for all a,b in A, the following hold:

i) $a(eb) = e(\bar{a}b)$

ii) $(ea)b = e(ba)$

iii) $(ea)(eb) = -b\bar{a}$.

The remainder of the proof is easy.//

Hence, up to the octonions, the Cayley-Dickson multiplication and the polynomial multiplication are the same. Actually, the restriction that A be an alternative algebra is stronger than necessary.

THEOREM 6.6. *Let A be any Cayley-Dickson algebra. Let e be the adjoined basis element used to construct $B = A + eA$. Assuming that we have already defined in B a distributive multiplication, let e satisfy:*

i) $e^2 = -1$

ii) $ae = e\bar{a}$ *for all* a *in* A

iii) $(e,a,b) + (a,e,b) = 0$ *for all* a,b *in* A.

Then, for all a,b *in* A, *we have:*

i) $a(eb) = e(\bar{a}b)$

ii) $(ea)b = e(ba)$

iii) $(ea)(eb) = -b\bar{a}$.

Proof. We first notice that assuming $(e,a,b) + (a,e,b) = 0$ for all a,b in A gives us Lemma 6.2, Theorem 6.3, and Corollary 6.4 for a,b in A. Now to see i), consider: $0 = -(e,a,b) - (a,e,b) = (e,\bar{a},b) + (\bar{a},e,b) = (e\bar{a})b - e(\bar{a}b) + (\bar{a}e)b - \bar{a}(eb) = (e\bar{a})b - e(\bar{a}b) + (ea)b - \bar{a}(eb) = [e(T(a)-a)]b + (ea)b - \bar{a}(eb) - e(\bar{a}b) = T(a)(eb) - (ea)b + (ea)b - \bar{a}(eb) - e(\bar{a}b) = [T(a)-\bar{a}](eb) - e(\bar{a}b) = a(eb) - e(\bar{a}b)$.

Similarly, for ii), $0 = (a,b,e) + (a,e,b) = (\bar{a},\bar{b},e) + (\bar{a},e,\bar{b}) = (\bar{a}\bar{b})e - \bar{a}(\bar{b}e) + (\bar{a}e)\bar{b} - \bar{a}(e\bar{b}) = e(ba) + (\bar{a}e)\bar{b} - \bar{a}(\bar{b}e-e\bar{b}) = e(ba) + (\bar{a}e)\bar{b} - \bar{a}(\bar{b}e-be) = e(ba) + (\bar{a}e)\bar{b} - T(b)(\bar{a}e) = e(ba) + (\bar{a}e)(\bar{b}-T(b)) = e(ba) + (\bar{a}e)b = e(ba) + (ea)b$.

To show iii), we need one of the restricted Moufang identities. $(ea)(eb) = (ea)(\bar{b}e) = e(a\bar{b})e = e[(a\bar{b})e] = e[e(b\bar{a})] = -b\bar{a}$.//

COROLLARY 6.7. Under the same hypotheses as given in Theorem 6.6, the Cayley-Dickson multiplication in $\mathcal{B}$ is the same as polynomial multiplication in $\mathcal{B}$.

Proof. $a_1b_1 + (ea_2)(eb_2) + a_1(eb_2) + (ea_2)b_1 = a_1b_1 + (-b_2\bar{a}_2) + e(\bar{a}_1b_2) + e(b_1a_2)$.//

It must not be supposed, however, that $ae = e\bar{a}$ is true for arbitrary elements in the algebra $\mathcal{B}$. In particular we have:

LEMMA 6.8. Let $x = a_1+ea_2$. Then $xe = e\bar{x}$ if and only if $T(a_2) = 0$.

Proof. $(a_1+ea_2)e = a_1e+ea_2e = -\bar{a}_2+e\bar{a}_1$ whereas $e(\bar{a}_1-ea_2) = e\bar{a}_1-e(ea_2) = a_2+e\bar{a}_1$. These are equal if and only if $a_2 = -\bar{a}_2$.//

This leads one to ask under what conditions the conclusions of Theorem 6.6 are true for arbitrary a and b in $\mathcal{B}$, assuming the Cayley-Dickson multiplication in $\mathcal{B}$.

LEMMA 6.9. Let A, B be elements of $\mathcal{B}$, where $A = a_1+ea_2$ and $B = b_1+eb_2$. Then,

i) $A(eB) = e(\bar{A}B)$ if and only if $T(a_2) = 0$

ii) $(eA)B = e(BA)$ *if and only if* $T(AB) = T(b_2) = 0$

iii) $(eA)(eB) = -B\bar{A}$ *if and only if* $T(A\bar{B}) = T(b_2) = 0$.

Proof. Since Theorem 6.1, Lemma 6.2, and Theorem 6.3 are true for A,B in $\mathcal{B}$, we need only note that in the proof of Theorem 6.6 for i) we must have $\bar{A}e = eA$ to get $A(eB) = e(\bar{A}B)$ which by Lemma 6.8 means $T(a_2) = 0$. Similarly, to show $(eA)B = e(BA)$, we examine the proof of ii) in Theorem 6.6, and see we need $(\bar{A}\bar{B})e = e(BA)$ and $e\bar{B} = Be$. Thus again the result follows by Lemma 6.8. For $(eA)(eB) = -B\bar{A}$, the proof of iii) in Theorem 6.6 requires $eB = \bar{B}e$ and $(A\bar{B})e = e(B\bar{A})$.//

We now wish to examine the basis elements themselves. As stated before, every element in the Cayley-Dickson algebra A_n can be written in the form $\sum_i r_i e_i$, $i = 0,\ldots,2^n-1$, where r_i is real, $e_0 = 1$, and $e_i^2 = -1$ otherwise. The basis $\{e_i \mid i = 0,\ldots,2^n-1\}$ arises from the basis of A_{n-1} by retaining the old basis and adjoining one new element $e_{2^{n-1}}$ and all multiples of it by the old basis elements. So, if $\{e_0,\ldots,e_{2^{n-1}-1}\}$ is the basis for A_{n-1} and the adjoined basis element is $e_{2^{n-1}}$, then the basis for A_n is $\{e_0,e_1,\ldots, e_{2^{n-1}-1}, e_{2^{n-1}},\ldots,e_j e_{2^{n-1}},\ldots,e_{2^n-1}\}$ where $0 < j \leq 2^{n-1}-1$. The Cayley-Dickson multiplication demands that we call $e_j e_{2^{n-1}} = e_{2^{n-1}+j}$. We are now able to observe several interesting properties that are possessed by these basis elements. In the following, e_i is a basis element of A_n.

LEMMA 6.10. Basis elements obey:

i) $e_ie_j = e_je_i$, if i or $j = 0$, or if $i = j$

ii) $e_ie_j = -e_je_i$, if $i,j \neq 0$ and $i \neq j$.

Proof. If i or $j = 0$, or if $i = j$, the result is obvious. Otherwise e_i and e_j may be each of two types, $(e_k+e\cdot 0)$ or $(0-ee_k)$, for $0 \leq k < 2^{n-1}$ where $e = e_{2^{n-1}}$. This gives rise to four cases. Since the result is clearly true for quaternions, it will then follow by induction.

Case 1: $i,j < 2^{n-1}$; follows from the induction hypothesis.

Case 2: $i < 2^{n-1}$, $j \geq 2^{n-1}$; here $e_i = (e_i+e\cdot 0)$ and $e_j = (0-ee_k)$, for some $k < 2^{n-1}$. Now $e_ie_j = (e_i+0)(0-ee_k)$ $= -e(\bar{e}_ie_k) = e(e_ie_k)$, and $e_je_i = (0-ee_k)(e_i+0) = -e(e_ie_k)$.

Case 3: $i \geq 2^{n-1}$, $j < 2^{n-1}$; this proof is similar to Case 2.

Case 4: $i,j \geq 2^{n-1}$; here $e_i = (0-ee_k)$ and $e_j = (0-ee_m)$ for some $k,m < 2^{n-1}$. Then $e_ie_j = (0-ee_k)(0-ee_m) = -\bar{e}_me_k =$ e_me_k, and $e_je_i = (0-ee_m)(0-ee_k) = -e_k\bar{e}_m = e_ke_m = e_me_k$.//

Although not all Cayley-Dickson algebras are alternative, their basis elements do satisfy the alternative property, as is shown by the following theorem proved by Schafer [26].

THEOREM 6.11. $e_i(e_ie_j) = -e_j = (e_je_i)e_i$ for $i \neq 0$.

Proof. The proof is essentially the same as Schafer gives, and is given here for the reader's convenience. By the flexible property $(e_i,e_i,e_j) = -(e_j,e_i,e_i)$, so it suffices to show $e_i(e_ie_j) = -e_j$. The result is true for the octonions, since they are alternative. Thus the proof is by induction on the dimension of A_n. Assume the statement true for A_{n-1}. As in the previous Lemma, there are four cases.

Case 1: $i,j < 2^{n-1}$; the result holds by the induction hypothesis.

Case 2: $i < 2^{n-1}$, $j \geq 2^{n-1}$; here $e_i = (e_i+0)$ and $e_j = (0-ee_k)$, for some $k < 2^{n-1}$. Now $e_i(e_ie_j) =$ $(e_i+0)[(e_i+0)(0-ee_k)] = (e_i+0)[0+e(e_ie_k)] = e[\bar{e}_i(e_ie_k)] =$ $-e[e_i(e_ie_k)] = -e(-e_k) = ee_k = -e_j$.

Case 3: $i \geq 2^{n-1}$, $j < 2^{n-1}$; here $e_i = (0-ee_k)$ and $e_j = (e_j+0)$ for some $e_k < 2^{n-1}$. Then $e_i(e_ie_j) =$ $(0-ee_k)[(0-ee_k)(e_j+0)] = (0-ee_k)(0-e[e_je_k]) = -(e_je_k)\bar{e}_k =$ $(e_je_k)e_k = e_je_k{}^2 = -e_j$.

Case 4: $i,j \geq 2^{n-1}$; here $e_i = (0-ee_k)$ and $e_j = (0-ee_m)$ for some $k,m < 2^{n-1}$. Then $e_i(e_ie_j) = (0-ee_k)[(0-ee_k)(0-ee_m)]$ $= (0-ee_k)(-e_m\bar{e}_k+0) = e[(e_m\bar{e}_k)e_k] = -e[e_m(e_k{}^2)] = ee_m =$ $-e_j$.//

COROLLARY 6.12. <u>If e_i is any basis element of A_n, and if x is any element of A_n, then</u> $(e_i,e_i,x) = 0$.

<u>Proof</u>. Let $x = \sum_j r_je_j$, $j = 0,\ldots,2^n-1$, r_j real. By the linearity of the associator, $(e_i,e_i,x) = \sum_j r_j(e_i,e_i,e_j)$ $= 0$.//

Notice, however, that Corollary 6.12 does not imply $(e_i,e_j,x) = -(e_j,e_i,x)$, since this would imply that $e_i(e_jx) = -e_j(e_ix)$, whenever $e_ie_j = -e_je_i$. This is false in general, as the following example shows. In any Cayley-Dickson algebra containing A_4, we have $(e_1-e_{15}, e_1-e_{15}, e_4) =$ $(e_1, e_1, e_4) - (e_1, e_{15}, e_4) - (e_{15}, e_1, e_4) + (e_{15}, e_{15}, e_4) =$ $-(e_1, e_{15}, e_4) - (e_{15}, e_1, e_4) = 2e_{10} \neq 0$. The reason this wasn't zero is that the linearization of $(x,x,y) = 0$ to get $(x,y,z) = -(y,x,z)$ required $(x,x,y) = 0$ <u>for all</u> elements of the algebra.

THEOREM 6.13.

$$e_i e_j e_i = \begin{cases} -e_j, & \text{\underline{if} } i,j \neq 0 \text{ \underline{and} } i = j, \\ & \text{\underline{or} \underline{if} } i \neq 0,\ j = 0. \\ e_j, & \text{\underline{if} } i = 0, \text{ \underline{or} \underline{if} } i,j \neq 0 \\ & \text{\underline{and} } i \neq j. \end{cases}$$

<u>Proof</u>. The result is immediate if $i = j$ and $i,j \neq 0$, or if $i \neq 0$ and $j = 0$. For the other situation, the proof is by induction on the dimension of A_n.

Case 1: $i,j < 2^{n-1}$; this follows by the induction hypothesis.

Case 2: $i < 2^{n-1}$, $j \geq 2^{n-1}$; here $e_i = (e_i+0)$ and $e_j(0-ee_k)$, for $k < 2^{n-1}$. Thus $e_i e_j e_i = [e_i(0-ee_k)]e_i =$ $[-e(\bar{e}_i e_k)]e_i = -e[e_i(\bar{e}_i e_k)] = e[e_i(e_i e_k)] = -ee_k = e_j$.

Case 3: $i \geq 2^{n-1}$, $j < 2^{n-1}$; here $e_i = (0-ee_k)$ and $e_j = (e_j+0)$, for $k < 2^{n-1}$. So $e_i e_j e_i = [(0-ee_k)e_j](0-ee_k) =$ $-[e(e_j e_k)](0-ee_k) = -e_k\overline{(e_j e_k)} = -e_k(e_k e_j) = e_j$.

Case 4: $i,j \geq 2^{n-1}$; here $e_i = (0-ee_k)$ and $e_j = (0-ee_m)$, for $k,m < 2^{n-1}$. So $\underline{e_i e_j e_i} = [(0-ee_k)(0-ee_m)](0-ee_k) =$ $(-e_m\bar{e}_k)(0-ee_k) = e[(e_m\bar{e}_k)e_k] = e[(e_k\bar{e}_m)e_k] = -e[(e_k e_m)e_k] =$ $-ee_m = e_j$, since if $k = m$, then $e_i = e_j$, a contradiction.//

We are now able to prove a theorem which may seem obvious, but the proof of which involves several cases.

THEOREM 6.14. $(e_ie_j)e_k = \pm e_i(e_je_k)$.

Proof. The result is clear if i,j,k is 0, or if any two basis elements are the same. Therefore, assume that each is not the identity and no two are the same. The proof is by induction on the dimension of A_n and there are eight cases. In the proof, great use is made of Theorem 6.6, and the observation that the induction hypothesis and Lemma 6.10 allow $(e_ie_j)e_k$ to be rearranged and reassociated in any order, with only perhaps a sign change, as long as i,j,k are less than 2^{n-1}.

Case 1: $i,j,k < 2^{n-1}$; this follows by the induction hypothesis.

Case 2: $j,k < 2^{n-1}$, $i \geq 2^{n-1}$; here $e_i = e_me$, where $m < 2^{n-1}$. Then $(e_ie_j)e_k = [(e_me)e_j]e_k = \pm[(ee_m)e_j]e_k = \pm[e(e_je_m)]e_k = \pm e[e_k(e_je_m)]$. Likewise, $e_i(e_je_k) = (e_me)(e_je_k) = \pm(ee_m)(e_je_k) = \pm e[(e_je_k)e_m]$.

Case 3: $i,k < 2^{n-1}$, $j \geq 2^{n-1}$; here the proof is similar to Case 2.

Case 4: $i,j < 2^{n-1}$, $k \geq 2^{n-1}$; this case is similar to Case 2.

Case 5: $i,j \geq 2^{n-1}$, $k < 2^{n-1}$; here $e_i = e_me$ and $e_j = e_pe$, for $m,p < 2^{n-1}$. Therefore $(e_ie_j)e_k = [(e_me)(e_pe)]e_k = \pm(e_pe_m)e_k$, while $e_i(e_je_k) = (e_me)[(e_pe)e_k] = \pm(e_me)[e(e_pe_k)] = \pm(e_pe_k)e_m$.

Case 6: $i,k \geq 2^{n-1}$, $j < 2^{n-1}$; this case is similar to Case 5.

Case 7: $j,k \geq 2^{n-1}$, $i < 2^{n-1}$; this case is similar to Case 5.

Case 8: $i,j,k \geq 2^{n-1}$; here $e_i = e_m e$, $e_j = e_p e$, and $e_k = e_q e$, for some $m,p,q < 2^{n-1}$. Then $(e_i e_j)e_k =$ $[(e_m e)(e_p e)](e_q e) = \pm(e_p e_m)(e_q e) = \pm(e_p e_m)e_q$, and $e_i(e_j e_k) =$ $(e_m e)[(e_p e)(e_q e)] = \pm(e_m e)(e_q e_p) = \pm e_m(e_q e_p)$.//

LEMMA 6.15. <u>For basis elements of the octonions</u>:

$$(e_i e_j)e_k = \begin{cases} e_i(e_j e_k), & \underline{\text{if}}\ i, j, \underline{\text{or}}\ k = 0, \\ & \underline{\text{or}}\ \underline{\text{if}}\ i = j,\ j = k,\ \underline{\text{or}}\ i = k, \\ & \underline{\text{or}}\ \underline{\text{if}}\ e_i e_j = \pm e_k. \\ -e_i(e_j e_k), & \underline{\text{otherwise}}. \end{cases}$$

<u>Proof</u>. The proof for $(e_i e_j)e_k = e_i(e_j e_k)$ under the hypotheses is easy. It should be noted, however, that $e_i e_j = \pm e_k$ is equivalent, because of the alternative property, to the statement that the product of any two is plus or minus the third. The proof of the last situation involves carefully reviewing the proof of the previous theorem with $e = e_4$.//

For one application of this material, we consider the following definition.

<u>Definition</u> 6.1. The <u>anticommutator</u> $]x,y[$ is defined to be $]x,y[= xy + yx$.

For the following, let $A = \sum_i a_i e_i$ and $B = \sum_i b_i e_i$ for $i = 0,\ldots,n$, where a_i and b_i are real.

LEMMA 6.16. $AB + BA = 2[a_0b_0 - \sum_i a_ib_i]e_0 + 2\sum_i[(a_0b_i+a_ib_0)e_i]$, $i = 1,\ldots,n$.

Proof. Recall that $e_ie_j = -e_je_i$ if $i,j \neq 0$ and $i \neq j$, and $e_ie_j = e_je_i$ otherwise. Consider the multiplication chart for AB given below.

	b_0e_0	b_1e_1	b_2e_2	$\cdots$	b_ne_n
a_0e_0	$a_0b_0e_0$	$+a_0b_1e_1$	$+a_0b_2e_2$	$\cdots$	$+a_0b_ne_n$
a_1e_1	$a_1b_0e_1$	$+a_1b_1e_1^2$	$+a_1b_2e_1e_2$	$\cdots$	$+a_1b_ne_1e_n$
a_2e_2	$a_2b_0e_2$	$+a_2b_1e_2e_1$	$+a_2b_2e_2^2$	$\cdots$	$+a_2b_ne_2e_n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
a_ne_n	$a_nb_0e_n$	$+a_nb_1e_ne_1$	$+a_nb_2e_ne_2$	$\cdots$	$+a_nb_ne_n^2$

A multiplication chart for BA would be similar, but with the entries in reverse order. Thus, for each summand $a_ib_je_ie_j$ in AB, there is a summand $b_ja_ie_je_i$ in BA. Hence, all terms for AB + BA cancel, except those of the first row, the first column, and the diagonal. From the AB table, we obtain $\sum_i(a_0b_0 - a_ib_i)e_0$, $\sum_i a_0b_ie_i$, and $\sum_i a_ib_0e_i$ from the diagonal, the first row, and the first column. We obtain similar sums from the BA table, except the results of the first row and the first column are interchanged. Thus, adding all those terms which do not cancel, the result of the lemma is obtained.//

THEOREM 6.17. $]A,B[= 0$ _if and only if_ $T(A) = T(B) = 0$ _and_ $\sum_i a_ib_i = 0$, $i = 1,\ldots,n$, _where_ $A,B \neq 0$.

Proof. From Lemma 6.16, $AB + BA = 0$ implies

$(a_0b_0 - \sum_i a_ib_i) = 0$ and $(a_0b_i + a_ib_0) = 0$ for $i = 1,\ldots,n$. If $a_0 = 0$, then $a_ib_0 = 0$ for all i implies $a_i = 0$ for all i or else $b_0 = 0$. But $a_i = 0$ for all i contradicts $A \neq 0$. Thus $T(B) = 0$. On the other hand, if $a_0 \neq 0$, then $b_i = [-a_ib_0]/a_0$ so that $a_0b_0 + \sum_i a_i[(a_ib_0)/a_0] = 0$. Then $b_0(a_0^2 + \ldots + a_n^2) = 0$, so $b_0 = 0$ or $A = 0$. Thus $T(B) = 0$. But $AB + BA = 0$ if and only if $BA + AB = 0$ so $T(A) = 0$ also. Now by Lemma 6.16, if $T(A) = T(B) = 0$, then $AB + BA = -2[\sum_i a_ib_i]e_0$. Hence $AB + BA = 0$ implies $T(A) = T(B) = 0$ and $\sum_i a_ib_i = 0$.

Conversely, if $T(A) = T(B) = 0$ and $\sum_i a_ib_i = 0$, then Lemma 6.16 implies $AB + BA = 0$.//

VII. Zero Divisors.

In this section, we consider zero divisors in the Cayley-Dickson algebras. First we consider norms, and then use the properties of norms to examine zero divisors.

A. A. Albert [2] defines an absolute-valued algebra to be an algebra over the reals with a function $f: A \to R$, such that $f(0) = 0$, $f(a) > 0$ if $a \neq 0$, and $f(ab) = f(a)f(b)$. He proves that the octonions are the only absolute-valued nonassociative algebra. He also defines a normed algebra in a similar way, but requires only $f(ab) \leq f(a)f(b)$. Albert then shows that every real algebra is a normed algebra under $f(a) = \sum_i |r_i|$, where $a = \sum_i r_i e_i$, with e_i a basis element of the algebra, $i = 1,\ldots,n$.

In any Cayley-Dickson algebra of dimension higher than eight, for the norm $N(a) = a\bar{a}$, we have $N(a) = 0$ if and only if $a = 0$, $N(a) > 0$ for $a \neq 0$; but, in general, $N(ab) \neq N(a)N(b)$, since if a and b are mutual zero divisors, $N(a)N(b) \neq 0$ and $N(ab) = 0$. What is striking, though, is that under this norm, these algebras are not even normed algebras. Consider the following example from A_4:

Let $A = e_1+e_{10}$ and $B = e_0+e_1+e_4-e_{15}$. Then $AB = -e_0+e_1+e_{10}+e_{11}$. Hence, $N(A) = 2$, $N(B) = 4$; but $N(AB) = 4$. i.e., $N(A)N(B) > N(AB)$.

Now, let $A = e_1-e_{10}$ and B be as before. Here, $AB = -e_0+e_1+2e_5-e_{10}-e_{11}+2e_{14}$. Thus, $N(A) = 2$, $N(B) = 4$; but $N(AB) = 12$. i.e., $N(A)N(B) < N(AB)$.

The question then arises as to what exactly is the relationship between N(AB) and N(A)N(B).

THEOREM 7.1. *In any Cayley-Dickson algebra* $N(A)N(B) - N(AB) = [N(a_1)N(b_1)-N(a_1b_1)] + [N(a_1)N(b_2)-N(a_1b_2)] + [N(a_2)N(b_1)-N(a_2b_1)] + [N(a_2)N(b_2)-N(a_2b_2)] + T[(a_1b_1)(a_2\bar{b}_2)-(\bar{a}_1b_2)(\bar{a}_2\bar{b}_1)]$, *where* $A = a_1+ea_2$ *and* $B = b_1+eb_2$.

Proof. Recall that $N(A) = N(a_1) + N(a_2)$. Therefore $N(A)N(B) = [N(a_1)+N(a_2)][N(b_1)+N(b_2)] = N(a_1)N(b_1) + N(a_2)N(b_1) + N(a_1)N(b_2) + N(a_2)N(b_2)$.

On the other hand, $AB = (a_1b_1-b_2\bar{a}_2) + e(\bar{a}_1b_2+b_1a_2)$, so $N(AB) = N(a_1b_1) + N(b_2\bar{a}_2) - T[(a_1b_1)\overline{(b_2\bar{a}_2)}] + N(\bar{a}_1b_2) + N(b_1a_2) + T[(\bar{a}_1b_2)(\overline{b_1a_2})] = N(a_1b_1) + N(\bar{a}_1b_2) + N(b_2\bar{a}_2) + N(b_1a_2) + T[(a_1b_1)(a_2\bar{b}_2)-(\bar{a}_1b_2)(\bar{a}_2\bar{b}_1)]$. The result now follows after recalling the facts that: $N(ab) = N(ba) = N(\bar{a}b) = N(a\bar{b})$, $T(ab) = T(ba)$, $T([ab]c) = T(a[bc])$, and $T(a\bar{b}) = T(\bar{a}b)$.//

COROLLARY 7.2. *In* A_4, $N(A)N(B) - N(AB) = T[(a_1b_1)(a_2\bar{b}_2)-(\bar{a}_1b_2)(\bar{a}_2\bar{b}_1)]$.

Proof. Here, a_i, b_j are octonions and hence, $N(a_ib_j) = N(a_i)N(b_j)$.//

Note that now an alternate proof of $N^2(A) = N(A^2)$ in A_4 is obtained by observing that $T[N(a_2)a_1^2-(a_1a_2)(\bar{a}_2\bar{a}_1)] = 0$, because of the Moufang identities in A_3.

A major difference between A_4 and the octonions is the existence of zero divisors in A_4. We now wish to examine zero divisors in detail. In the following, assume that A and B are mutual zero divisors, A and B are elements of A_n, and that $A = a_1+ea_2$, $B = b_1+eb_2$.

THEOREM 7.3. <u>The following are equivalent</u>:

i) $AB = 0$

ii) $BA = 0$

iii) $\bar{A}B = 0$

iv) $A^{-1}B = 0$

v) $(eA)(eB) = 0$

vi) $(eA)B = 0$

vii) $A(eB) = 0$.

<u>In addition, the above imply</u>:

viii) $T(A) = 0$

ix) $T(a_1) = 0$

x) $T(a_2) = 0$

xi) A <u>and</u> B <u>are linearly independent</u>.

<u>Proof</u>. i) $\leftrightarrow$ ii) This follows from the facts that $N(A) = 0$ if and only if $A = 0$, and $N(AB) = N(BA)$.

i) $\rightarrow$ xi) Recall that there are no nilpotent elements in A_n. Let $mA + nB = 0$, for m,n real. Then, $mA^2 + nAB = 0$ implies $mA^2 = 0$. This implies $m = 0$. Similarly for n.

i) $\leftrightarrow$ iii) In any flexible ring with involution $N(AB) = N(\bar{A}B)$. Thus the following are all equivalent: $AB = 0$, $N(AB) = 0$, $N(\bar{A}B) = 0$, and $\bar{A}B = 0$.

iii) $\leftrightarrow$ iv) This follows from the identity $A^{-1}B = [\bar{A}B]/N(A)$.

i) $\rightarrow$ viii) $\bar{A}B = [T(A)-A]B - AB$, so $AB = 0$ implies $\bar{A}B = 0$ and $T(A)B = 0$. Thus $T(A) = 0$.

viii) $\rightarrow$ ix) This follows from the fact that for $A = a_1 + ea_2$, $T(A) = T(a_1)$.

iii) $\leftrightarrow$ v) By the restricted Moufang identities, $(eA)(eB) = 0$ if and only if $0 = (eA)(\overline{eB}) = (eA)(\bar{B}e) = e(A\bar{B})e$ if and only if $A\bar{B} = 0$.

v) $\rightarrow$ x) $eA = -a_2 + ea_1$. Hence $T(eA) = T(a_2) = 0$.

ii) $\leftrightarrow$ vi) Since $T(AB) = T(\bar{A}B) = T(a_1) = T(a_2) = T(b_1) = T(b_2) = 0$, the result follows from Lemma 6.9, i.e., $(eA)B = e(BA)$.

iii) $\leftrightarrow$ vii) By the same reasoning as above, Lemma 6.9 gives $A(eB) = e(\bar{A}B)$.//

LEMMA 7.4. <u>The following are equivalent</u>:

i) $T(A) = 0$

ii) $\bar{A} = -A$

iii) $A^{-1} = -A/N(A)$

iv) $A^2 = -N(A)$.

<u>Proof</u>. $A = r_0e_0 + \sum_i r_ie_i$ and $\bar{A} = r_0e_0 - \sum_i r_ie_i$, $i = 1,\ldots,n$. Moreover, $T(A) = 0$ means $r_0 = 0$. Hence, i), ii), and iii) are equivalent. To see iv), recall that $A^2 - T(A)A + N(A)1 = 0$.//

We now find it desirable to define an isomorphism, *, from A_n to A_n in the following way.

<u>Definition</u> 7.1. <u>Let</u> $A^* = a_1 - ea_2$ <u>if</u> $A = a_1 + ea_2$.

The * isomorphism gives some added information about zero divisors, but first we need to consider some properties of the * operator.

THEOREM 7.5. <u>The following properties hold for the</u> * <u>operator</u>.

i) $A + A^* = 2a_1$

ii) $A^* = \bar{A}$ <u>if and only if</u> a_1 <u>is real</u>.

iii) $A^* = A$ <u>if and only if</u> $a_2 = 0$.

iv) $T(A^*) = T(A)$

v) $N(A^*) = N(A)$

vi) $(A+B)^* = A^*+B^*$

vii) $(AB)^* = A^*B^*$

viii) $(rA)^* = rA^*$, r <u>real</u>

ix) $(eA)^* = -eA^*$

x) $[A,B]^* = [A^*,B^*]$

xi) $(A,B,C)^* = (A^*,B^*,C^*)$

xii) $(\bar{A})^* = \overline{A^*}$

xiii) $A^{**} = A$.

<u>Proof</u>. i) $A + A^* = (a_1+ea_2)+(a_1-ea_2) = 2a_1$.

ii) $A^* = a_1-ea_2 = \bar{a}_1-ea_2 = \bar{A}$ if and only if $a_1 = \bar{a}_1$, i.e., if and only if a_1 is real.

iii) $A^* = a_1-ea_2 = a_1+ea_2 = A$ if and only if $a_2 = 0$.

iv) $T(A^*) = A^* + \overline{A^*} = (a_1-ea_2)+(\bar{a}_1+ea_2) = a_1+\bar{a}_1 = T(A)$.

v) $N(A^*) = A^*\overline{A^*} = (a_1-ea_2)(\bar{a}_1+ea_2) = a_1\bar{a}_1+a_2\bar{a}_2 = N(A)$.

vi) $(A+B)^* = [(a_1+b_1)+e(a_2+b_2)]^* = (a_1+b_1)-e(a_2+b_2) = (a_1-ea_2)+(b_1-eb_2) = A^*+B^*$.

vii) $(AB)^* = [(a_1b_1-b_2\bar{a}_2)+e(\bar{a}_1b_2+b_1a_2)]^* = (a_1b_1-b_2\bar{a}_2) -e(\bar{a}_1b_2+b_1a_2) = (a_1-ea_2)(b_1-eb_2) = A^*B^*$.

viii) $(rA)^* = r^*A^* = rA^*$.

ix) $(eA)^* = e^*A^* = -eA^*$.

x) $[A,B]^* = (AB-BA)^* = A^*B^*-B^*A^* = [A^*,B^*]$.

xi) $(A,B,C)^* = [(AB)C-A(BC)]^* = (A^*B^*)C^*-A^*(B^*C^*) = (A^*,B^*,C^*)$.

xii) $(\bar{A})^* = (\bar{a}_1-ea_2)^* = \bar{a}_1+ea_2 = \overline{(a_1-ea_2)} = \overline{A^*}$.

xiii) $A^{**} = (a_1-ea_2)^* = a_1+ea_2 = A$.//

We are now able to consider * operators and zero divisors.

LEMMA 7.6. $AB = 0$ *if and only if* $A^*B^* = 0$.

Proof. Since from the definition of A^*, it is clear that $A^* = 0$ if and only if $A = 0$, we have $0 = AB = (AB)^* = A^*B^*$.//

Definition 7.2. Let $\langle A,B\rangle$ be the vector subspace generated by elements A and B.

Now, we consider the structure of the subspace $\langle A,B\rangle$ generated by mutual zero divisors A and B. We already know, by Theorem 7.3, that A and B are linearly independent.

THEOREM 7.7. *If A is a zero divisor, then*

$$A^n = \begin{cases} [-N(A)]^{n/2}, & \text{if } n \text{ is an even integer.} \\ ([-N(A)]^{(n-1)/2})A, & \text{if } n \text{ is an odd integer.} \end{cases}$$

Proof. First we observe that the result is true for n an even or odd positive integer. This follows from the fact that $A^2 = -N(A)$. Thus if n is even, $A^n = (A^2)^{n/2} = [-N(A)]^{n/2}$. If n is odd, $A^n = (A^{n-1})A = ([-N(A)]^{(n-1)/2})A$.

Now if n is zero, the result is obvious.

If n is negative, the $A^n = (A^{-1})^{-n} = [-A/N(A)]^{-n} = ([-1/N(A)]^{-n})A^{-n}$. Hence, if n is even, $A^n =$

$[-1/N(A)]^{-n}[-N(A)]^{-n/2} = [-N(A)]^{(-n/2)+n}$. If n is odd, $A^n =$ $[-1/N(A)]^{-n}[-N(A)]^{(-n-1)/2} = ([-N(A)]^{[(-n-1)/2]+n})A$.//

Note that, since AB = 0 if and only if BA = 0, the results of Theorem 7.7 hold true for B as well.

THEOREM 7.8. *If* A *and* B *are mutual zero divisors, then:*

$$A^nB^m = B^mA^n = \begin{cases} [-N(A)]^{n/2}[-N(B)]^{m/2}, & n,m \text{ even integers.} \\ 0, & n,m \text{ odd integers.} \\ ([-N(A)]^{n/2}[-N(B)]^{(m-1)/2})B, & n \text{ even, } m \text{ odd.} \\ ([-N(A)]^{(n-1)/2}[-N(B)]^{m/2})A, & n \text{ odd, } m \text{ even.} \end{cases}$$

Proof. Simply multiply A^n by B^m using the results of Theorem 7.7.//

THEOREM 7.9. *Let* A *and* B *be mutual zero divisors, then:*

i) $\langle A,B\rangle = r_1+r_2A+r_3B$, r_i *real*,

ii) $\langle A,B\rangle$ *is a commutative Jordan subalgebra of* A_n,

iii) $\langle A,B\rangle$ *is not alternative*.

Proof. i) This is clear by Theorem 7.8.

ii) This follows since the basis elements A and B are commutative, and since all elements of A_n satisfy the Jordan identity.

iii) This follows since, $A^2B = -N(A)B \neq A(AB) = 0$.//

Thus, the vector subspace generated by any two mutual zero divisors is a three dimensional, non-alternative, commutative Jordan algebra with zero divisors.

Now, let us go back to the nature of the zero divisors themselves. Our objective is to determine just when an element A in A_n will be a zero divisor.

LEMMA 7.10. *AB = 0 is equivalent to any pairing of an equation i) or ii) with an equation iii) or iv), where:*

i) $a_1b_1+b_2a_2 = 0$ iii) $b_1a_2-a_1b_2 = 0$

ii) $b_1a_1+a_2b_2 = 0$ iv) $a_2b_1-b_2a_1 = 0.$

Proof. Recall $T(a) = 0$ implies $\bar{a} = -a$. Then AB = 0 if and only if equations i) and iii) hold; B(eA) = 0 if and only if equations i) and iv) hold; BA = 0 if and only if equations ii) and iii) hold; and (eA)B = 0 if and only if equations ii) and iv) hold.//

Definition 7.3. The *antiassociator*)A,B,C(is defined by)A,B,C(= (AB)C + A(BC).

LEMMA 7.11. *The antiassociator is linear in each argument.*

Proof. The proof is similar to that of showing the associator is linear in each argument.//

We may now prove the major theorem of this section.

THEOREM 7.12. *Let* $A = a_1+ea_2$ *and* $B = b_1+eb_2$, *where* $A,B \neq 0$. *Let* $a_1 \neq 0$ *and* $a_2 \neq 0$ *have no zero divisors. Then* AB = 0 *if and only if* i) *and* ii) *hold, where:*

i) $)a_1,b_1,a_2(= -(a_1,a_1,b_2)+(a_2,a_2,b_2)+[N(a_2)-N(a_1)]b_2$

ii) $(a_1,b_1,a_2) = (a_1,a_1,b_2)+(a_2,a_2,b_2)+[N(a_2)+N(a_1)]b_2.$

Proof. If AB = 0, then $(a_1b_1+b_2a_2) = 0$ and $(a_1b_2-b_1a_2) = 0$ by Lemma 7.10. Thus $a_1b_1 = -b_2a_2$ and $a_1b_2 = b_1a_2$. Thus $(a_1b_1)a_2 = -(b_2a_2)a_2$ and $a_1(a_1b_2) = a_1(b_1a_2)$. Adding gives $(a_1b_1)a_2 + a_1(b_1a_2) = a_1(a_1b_2) - (b_2a_2)a_2 = -(a_1,a_1,b_2) + a_1{}^2b_2 - (b_2,a_2,a_2) - b_2a_2{}^2 =$

$-(a_1,a_1,b_2)+(a_2,a_2,b_2) + (a_1{}^2-a_2{}^2)b_2$, using the flexible property. Now, recall $T(a) = 0$ implies $N(a) = -a^2$. Thus, $)a_1,b_1,a_2(= -(a_1,a_1,b_2)+(a_2,a_2,b_2)+[N(a_2)-N(a_1)]b_2$. Similarly, $(a_1b_1)a_2-a_1(b_1a_2) = -(b_2a_2)a_2 - a_1(a_1b_2) = -(b_2,a_2,a_2)-b_1a_2{}^2+(a_1,a_1,b_2)-a_1{}^2b_2$. Thus, $(a_1,b_1,a_2) = (a_1,a_1,b_2)+(a_2,a_2,b_2)+[N(a_2)+N(a_1)]b_2$.

Conversely, it is easy to see that equation i) implies $(a_1b_1)a_2+a_1(b_1a_2) = a_1(a_1b_2)-(b_2a_2)a_2$ and that equation ii) implies $(a_1b_1)a_2-a_1(b_1a_2) = -(b_2a_2)a_2-a_1(a_1b_2)$. Now, if we add these two equations, and then subtract them, we get $2(a_1b_1)a_2 = -2(b_2a_2)a_2$ and $2a_1(b_1a_2) = 2a_1(a_1b_2)$. i.e., $(a_1b_1+b_2a_2)a_2 = 0$ and $a_1(b_1a_2-a_1b_2) = 0$. Now if a_1 and a_2 have no zero divisors, then $a_1b_1+b_2a_2 = 0$ and $a_1b_2-b_1a_2 = 0$. Hence, by Lemma 7.10, $AB = 0$.//

THEOREM 7.13. _Let_ A _and_ B _be_ _mutual_ _zero_ _divisors_. _Then_:

i) $T([a_1a_2]b_1) = T([a_1a_2]b_2) = 0$

ii) $a_1 = \pm a_2$ _implies_ $a_1b_1 = a_1b_2 = 0$

iii) a_1 and a_2 _are_ _not_ _nonzero_ _real_ _numbers_

iv) $a_1 = 0$ _implies_ $a_2b_1 = a_2b_2 = 0$

v) $a_2 = 0$ _implies_ $a_1b_1 = a_1b_2 = 0$.

Proof. i) From $a_1b_1 = -b_2a_2$ and $b_1a_1 = -a_2b_2$, we see that $(a_1b_1)a_1 = -(b_2a_2)a_1$ and $-a_1(b_1a_1) = a_1(a_2b_2)$. Adding, and using the flexible property, we obtain $0 = a_1(a_2b_2)-(b_2a_2)a_1 = a_1(a_2b_2) - \overline{\bar{a}_1(\bar{a}_2\bar{b}_2)} = a_1(a_2b_2)+\overline{a_1(a_2b_2)} = T(a_1[a_2b_2]) = T([a_1a_2]b_2)$. In a similar manner, we also have $(a_1b_2)a_1 = (b_1a_2)a_1$ and $a_1(a_2b_1) = a_1(b_2a_1)$.

Subtracting and using the flexible property gives 0 = $a_1(a_2b_1)-(b_1a_2)a_1$, and the proof follows as before.

ii) If $a_1 = a_2$, then $a_1b_1+b_2a_1 = 0$ and $a_1b_1-b_2a_1 = 0$. Thus, since $T(a_1) = T(b_2) = 0$, and $\overline{b_2a_1} = a_1b_2$, we obtain $a_1b_1 = a_1b_2 = 0$. If $a_1 = -a_2$, then $a_1b_1-b_2a_1 = 0$ and $a_1b_1+b_2a_1 = 0$; and the result follows as before.

iii) This follows directly from $T(a_1) = T(a_2) = 0$.

iv) and v) follow directly from $AB = 0$ if and only if $a_1b_1+b_2a_2 = 0$ and $a_1b_2-b_1a_2 = 0$ if and only if $b_1a_1+a_2b_2 = 0$ and $b_2a_1-a_2b_1 = 0$.//

We now consider how zero divisors behave in A_4.

THEOREM 7.14. _Let_ A _be a zero divisor in_ A_4. _Then:_

i) $a_1 \neq \pm a_2$

ii) a_1 _and_ a_2 _are not real numbers_

iii) $N(a_1) = N(a_2)$.

Proof. i) This follows from Theorem 7.13, ii), and the fact that a_i, b_i are octonions and have no zero divisors.

ii) This follows from Theorem 7.13, iii) and the fact that if a_i were zero, Theorem 7.13, iv) and v), implies A or B = 0.

iii) Recall that the octonions are an absolute-valued algebra, i.e., $N(ab) = N(a)N(b)$. Also, in any Cayley-Dickson algebra, $N(\bar{a}) = N(-a) = N(a) \geq 0$. Thus, $AB = 0$ if and only if $a_1b_1+b_2a_2 = 0$ and $a_1b_2-b_1a_2 = 0$. Hence $N(a_1)N(b_1) = N(b_2)N(a_2)$ and $N(a_1)N(b_2) = N(b_1)N(a_2)$. Dividing, we obtain $[N(a_1)]/N(a_2) = [N(a_2)]/N(a_1)$. Thus, $N^2(a_1) = N^2(a_2)$ and the result follows.//

We are now able to prove our most useful result for finding zero divisors in A_4.

THEOREM 7.15. <u>A and B are mutual zero divisors in</u> A_4 <u>if and only if the following three conditions hold:</u>

i) $N(a_1) = N(a_2)$

ii) $b_2 = [(a_1b_1)a_2]/N(a_1)$

iii) $)a_1,b_1,a_2(= 0,$

<u>where</u> $A = a_1+ea_2$ <u>and</u> $B = b_1+eb_2$.

<u>Proof</u>. If AB = 0, then by Theorem 7.12 and the fact that octonions are alternative, we obtain $)a_1,b_1,a_2(= [N(a_2)-N(a_1)]b_2$ and $(a_1,b_1,a_2) = [N(a_2)+N(a_1)]b_2$. Now by Theorem 7.14, iii) $N(a_2) = N(a_1)$. Thus, $)a_1,b_1,a_2(= 0$ and $(a_1,b_1,a_2) = 2N(a_1)b_2$. Now $2(a_1b_1)a_2 = (a_1,b_1,a_2) +)a_1,b_1,a_2(= 2N(a_1)b_2$. Thus, the result follows.

Conversely, if $N(a_1) = N(a_2)$, then $b_2 = [(a_1b_1)a_2]/N(a_1)$ implies $[N(a_2)+N(a_1)]b_2 = 2(a_1b_1)a_2 = (a_1b_1)a_2-a_1(b_1a_2)+(a_1b_1)a_2+a_1(b_1a_2) = (a_1,b_1,a_2) +)a_1,b_1,a_2(= (a_1,b_1,a_2)$ by iii). Finally, $N(a_1) = N(a_2)$ and $)a_1,b_1,a_2(= 0$ imply $)a_1,b_1,a_2(= [N(a_2)-N(a_1)]b_2$. Now since by the alternative property, $(a_1,a_1,b_2) = (a_2,a_2,b_2) = 0$, the result follows by Theorem 7.12.//

COROLLARY 7.16. <u>If A and B are mutual zero divisors, then the following holds in</u> A_4:

i) <u>No three of</u> $a_1,a_2,b_1,$ <u>and</u> b_2 <u>can be quaternions</u>.

ii) <u>Each of the following antiassociators is zero</u>:

$)a_1,b_1,a_2(;\ )b_1,a_1,b_2(;\ )a_1,b_2,a_2(;\ )b_1,a_2,b_2(;$
$)a_2,b_1,a_1(;\ )b_2,a_1,b_1(;\ )a_2,b_2,a_1(;\ )b_2,a_2,b_1(.$

Proof. We first prove ii) by noting that the following are all equivalent: $AB = 0$, $BA = 0$, $(eA)B = 0$, $B(eA) = 0$, $(eA)(eB) = 0$, $(eB)(eA) = 0$, $(eB)A = 0$, and $A(eB) = 0$. Thus the order of the a_i in a_1+ea_2 and the b_i in b_1+eb_2 may be permuted.

i) If any three were quaternions, they would be associative, contradicting part ii).//

Considering the above, one would hope to be able to look at a particular $A \neq 0$, at least in A_4, and determine whether or not it is a divisor of zero by inspection. In fact, we do know that if $A = a_1+ea_2$ is in A_4, and if $N(a_1) \neq N(a_2)$, $a_1 = \pm a_2$, $T(a_1)$ or $T(a_2) \neq 0$, or either a_1 or a_2 real, then A is not a zero divisor by Theorem 7.14, and Theorem 7.3. Each of the necessary and sufficient conditions for zero divisors, however, involved considering B as well as A. The following example illustrates that putting all the restrictions on A that we have encountered as necessary (without considering B) still doesn't guarantee that A will be a divisor of zero.

Let $A = (e_5+e_6+e_7)+e_8(e_5-e_6+e_7)$. Then $N(a_1) = N(a_2)$, $T(a_1) = T(a_2) = 0$, and $a_1 \neq \pm a_2$. Yet letting $b_1 = \sum_i r_i e_i$, $i = 0,\ldots,7$, and computing $)a_1,b_1,a_2(= 0$ yields $b_1 = 0$. Hence $B = 0$ and A is not a zero divisor.

Thus, it appears that there is no way to look at a specific A and be sure if it is a zero divisor, unless you actually solve for B such that $AB = 0$. There are two obvious ways to do this. The first would be to set

$B = \sum_i b_i e_i$, $i = 0,\ldots,2^n-1$. Then after calculating $AB = \sum_i c_i e_i$, $i = 0,\ldots,2^n-1$, one could set each $c_i = 0$ and solve. Unfortunately, the multiplication could be extremely long, and solving $c_i = 0$ could result in solving 2^n-1 equations in 2^n-1 unknowns. The author used a PL-1 computer program to aid in working several examples in this way. This program, which multiplies arbitrary Cayley-Dickson elements in A_4, is included in Appendix C. Even a computer is little help in solving the system of equations when its solutions are not rational, however.

The other way suggested by our previous work is to set $b_1 = \sum_i r_i e_i$, $i = 0,\ldots,2^{n-1}-1$, and solve equations i) and ii) of Theorem 7.12. In A_4, this reduces the problem to, at worst, seven equations in seven unknowns. The aid of a computer in performing the multiplication is useful, even here. Solving the equations remains tedious, however. It seems apparent that what is needed is further investigation into the antiassociator. From here, we concentrate on zero divisors in A_4, where the elements a_i, b_i acted on by the antiassociator will be octonions.

LEMMA 7.17. _If_ a,b,c _are_ _octonions_, _then_ $)a,b,c(= 0$ _implies_ $T(a) = T(b) = T(c) = 0$.

Proof. By Theorem 7.15 $(ab)c+a(bc) = 0$ implies $(a/\sqrt{N(a)} + e_8[c/\sqrt{N(c)}])$ is a zero divisor. Hence, $T(a/\sqrt{N(a)}) = 0$. Thus $T(a) = 0$. The other parts are similar.//

We now consider for which basis elements of the octonions the antiassociator vanishes.

THEOREM 7.18. *For basis elements of the octonions,* $)e_i,e_j,e_k(= 0$ *if and only if*

i) *None of* $i,j,$ *or* k *is* 0,

ii) *No two of* i,j,k *are the same*,

iii) $e_ie_j \neq \pm e_k$.

Otherwise, $)e_i,e_j,e_k(= 2(e_ie_j)e_k$.

Proof. Recall Lemma 6.15.//

THEOREM 7.19. *For basis elements of the octonions, if* $i,j,k \neq 0$, *then*:

i) $)e_i,e_i,e_j(=)e_j,e_i,e_i(= -2e_j$,

ii) $$)e_i,e_j,e_i(= \begin{cases} -2e_j, & \text{if } i = j, \\ 2e_j, & \text{if } i \neq j, \end{cases}$$

iii) $)e_i,e_j,e_k(= -\overline{)e_k,e_j,e_i(}$.

Proof. i) Since basis elements are alternative, we have $)e_i,e_i,e_j(= e_i^2e_j + e_i(e_ie_j) = 2e_i^2e_j = -2e_j$.

ii) Recall Theorem 6.13.

iii) If $i,j,k \neq 0$, then $-\overline{)e_k,e_j,e_i(} = -\overline{(e_ke_j)e_i}-\overline{e_k(e_je_i)}$ $= -\bar{e}_i(\bar{e}_j\bar{e}_k)-(\bar{e}_i\bar{e}_j)\bar{e}_k = e_i(e_je_k)+(e_ie_j)e_k =)e_i,e_j,e_k($.//

Since we are interested in antiassociators of elements each of which has trace 0, the condition that $i,j,k \neq 0$ in Theorem 7.19 is no restriction to us.

THEOREM 7.20. *Let* e_i,e_j,e_k *be basis elements of any Cayley-Dickson algebra. Then* $)e_i,\sum_j r_je_j,e_k(= 0$ *implies* $)e_i,e_j,e_k(= 0$ *for all* $j = 0,\ldots,n$.

<u>Proof</u>. Notice first, that since $(e_ie_j)e_k = \pm e_i(e_je_k)$ by Theorem 6.14, $)e_1,e_j,e_k(=$ either 0 or $2(e_ie_j)e_k$. Now by linearity we may write $)e_i,\sum_j r_je_j,e_k($ as $\sum_j r_j)e_i,e_j,e_k($. Suppose not all these terms are zero, say $r_m)e_i,e_m,e_k(\neq 0$, $m = 1,\ldots,h$. Then $0 = \sum_m r_m)e_i,e_m,e_k(= \sum_m r_m\cdot 2(e_ie_m)e_k = 2[e_i(\sum_m r_me_m)]e_k$.

But Corollary 6.12 says $(A,e_k,e_k) = 0$, hence if $Ae_k = 0$, then $0 = (Ae_k)e_k = A(e_k^{\,2}) = \pm A$. Likewise, if $e_iB = 0$, $B = 0$. Thus $\sum_m r_me_m = 0$ which is a contradiction of the linear independence of the e_m's.//

With the apparatus above, we often may calculate all the zero divisors of a given number in A_4 quickly since many, if not most, antiassociators will vanish. In fact, by Theorem 7.18, since we assume $i,j,k \neq 0$, $)e_i,e_j,e_k(= 0$ unless $i = j$, $j = k$, or $i = k$, or $e_ie_j = \pm e_k$. The only time we even need to consult the basis multiplication table is for that last case, $e_ie_j = \pm e_k$.

Consider the following example. Let $A = e_1+e_8e_2$. We seek B such that $AB = 0$. Let $b_1 = \sum_i r_ie_i$, $i = 0,\ldots,7$. Since $T(b_1) = 0$, $r_0 = 0$. Since $T([a_1a_2]b_1) = 0$, $T([e_1e_2]b_1) = T(e_3b_1) = 0$, implying $r_3 = 0$. Now expanding $0 =)e_1,\sum_i r_ie_i,e_2($ and discarding any antiassociator which is zero, we obtain $0 = r_1)e_1,e_1,e_2(+r_2)e_1,e_2,e_2(= -2r_1e_2-2r_2e_1$. Thus $r_1 = r_2 = 0$. Hence $b_1 = r_4e_4+r_5e_5+r_6e_6+r_7e_7$. Now letting $b_2 = [(a_1b_1)a_2/N(a_1) = [e_1(r_4e_4+r_5e_5+r_6e_6+r_7e_7)]e_2$ we get $b_2 = r_7e_4-r_6e_5+r_5e_6-r_4e_7$.

Hence, $B = (r_4e_4+r_5e_5+r_6e_6+r_7e_7)+e_8(r_7e_4-r_6e_5+r_5e_6-r_4e_7) = r_4(e_4+e_{15})+r_5(e_5-e_{14})+r_6(e_6+e_{13})+r_7(e_7-e_{12})$, for all real r_i, $i = 4,\ldots,7$.

Thus the properties of the antiassociator allow us to perform otherwise long tedious multiplications with a minimum of steps.

The last thing we do in this section is examine the equation $AX = B$ and its solutions.

LEMMA 7.21. *In an alternative algebra with involution,* $AX = B$, $A \neq 0$, *always has* $A^{-1}B$ *as a solution*.

Proof. Since $A(A^{-1}B) = -(A,A^{-1},B)+B = -(A,\bar{A}/N(A),B) + B = (A,A,B)/N(A) + B = B$, we conclude that $A^{-1}B$ is always a solution.//

This seemingly obvious choice of a solution need not work in A_4, however.

THEOREM 7.22. *In* A_4, *we have the following*: $A(A^{-1}B) = B + [(a_1,b_2,a_2)-e(a_1,b_1,a_2)]/N(A)$, *where* $A = a_1+ea_2$ *and* $B = b_1+eb_2$.

Proof. $A(A^{-1}B) = B + (A,A,B)/N(A)$, and by Theorem 4.3, $(A,A,B) = (a_1,b_2,a_2)-e(a_1,b_1,a_2)$ for A,B in A_4.//

It is clear that only certain equations will have the solution $A^{-1}B$. However, even then, the solution may not be unique. For example, if A has a zero divisor C, so that $AC = 0$, then $A(X+C) = B$ if $AX = B$. The situation can be worse, however, because for some equations of this type, there are no solutions. Consider the following example. Let $A = e_5+e_{15}$ and $X = \sum_i r_ie_i$, $i = 0,\ldots,15$. Then a little calculation shows $AX = (-r_5-r_{15})e_0 + (r_5-r_{15})e_{10} - (r_4+r_{14})(e_1+e_{11}) + (r_7+r_{13})e_2 + (r_7-r_{13})e_8 + (r_{12}-r_6)(e_3+e_9) + (r_1-r_{11})(e_4+e_{14}) + (r_0-r_{10})e_5 + (r_0+r_{10})e_{15} + (r_3+r_9)(e_6-e_{12}) - (r_2+r_8)e_7 + (r_8-r_2)e_{13}$. Notice the pairing of e_1 and e_{11}, e_3 and e_9, e_4 and e_{14}, and e_6 and e_{12}. It is clear that no choice of r_i will let AX be any single element of any of these pairs. In particular, $(e_5+e_{15})X = e_1$ has no solution in A_4.

An unproven conjecture is that if AX = B has a solution, then $X = A^{-1}B$ is a solution. In particular, all zero divisors for (e_5+e_{15}) must be of the form:
$m(e_1+e_{11}) + n(e_3-e_9) + p(e_6+e_{12}) + q(e_4-e_{14})$, m,n,p,q real. Hence, another unproven conjecture is that if A has no zero divisor, then AX = B has the unique solution $X = A^{-1}B$.

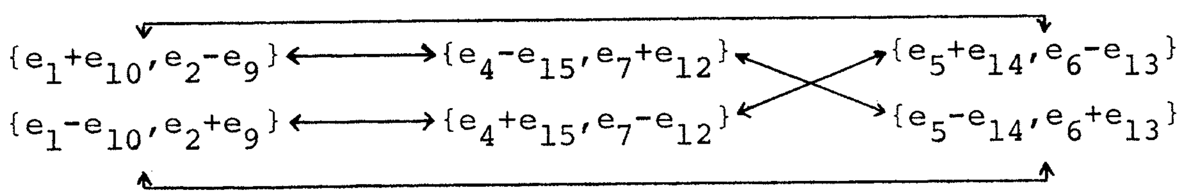
VIII. Examples and Counter Examples.

In the first part of this section, we wish to find the set of zero divisors of a given nonzero A in A_4. Then we wish to find all zero divisors of that set. In general, this is extremely complicated; but if A is of the form $A = e_i + e_8 e_j$, $1 \leq i, j \leq 7$, the situation results in seven disjoint systems of interrelated zero divisors.

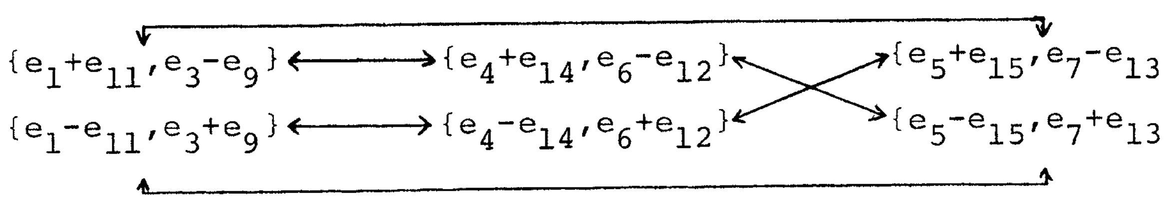
In the following, let {A,B} be the set of all linear combinations of A and B. Also, let $\{X\} \leftrightarrow \{Y\}$ indicate that $xy = 0$ for all x in X and for all y in Y.

THEOREM 8.1. *Each A of the form* $e_i + e_8 e_j$ *which has a zero divisor is in one and only one system below, and all of its zero divisors are in the same system.*

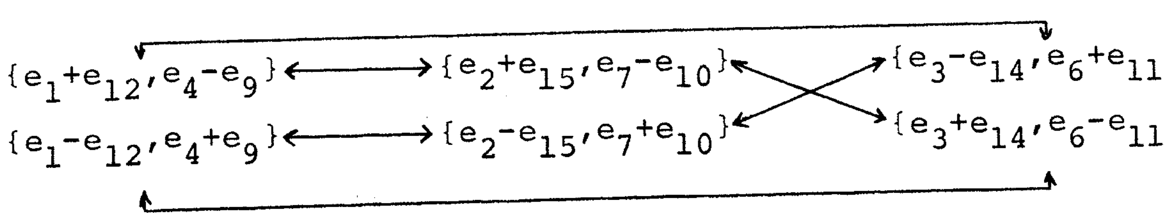
System 1:



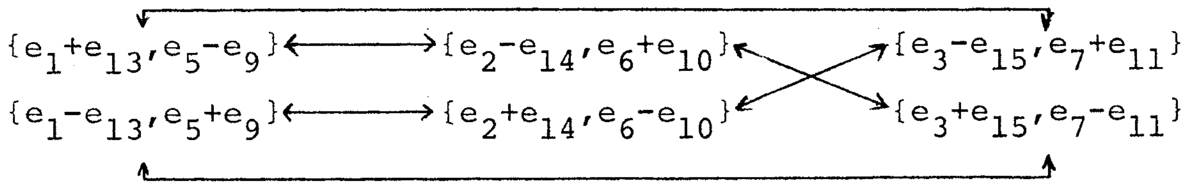
System 2:



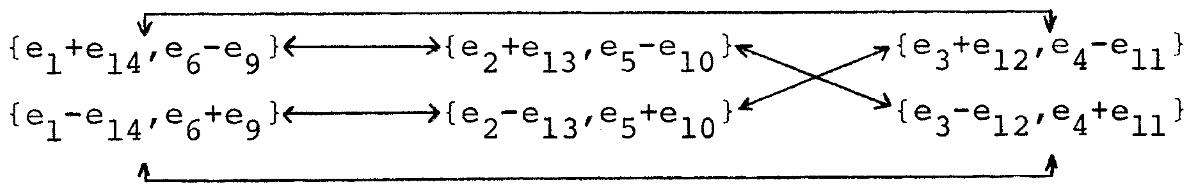
System 3:



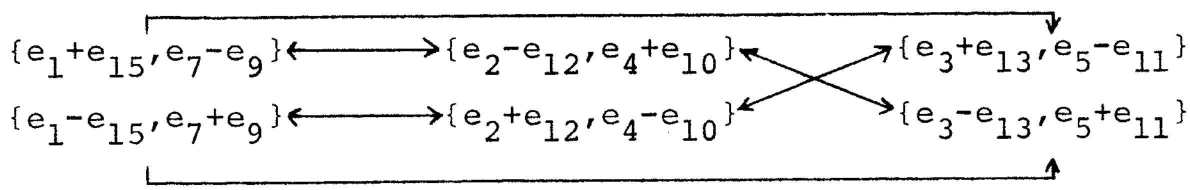
System 4:



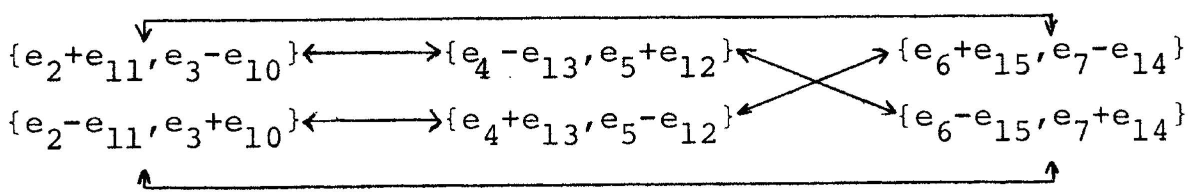
System 5:



System 6:

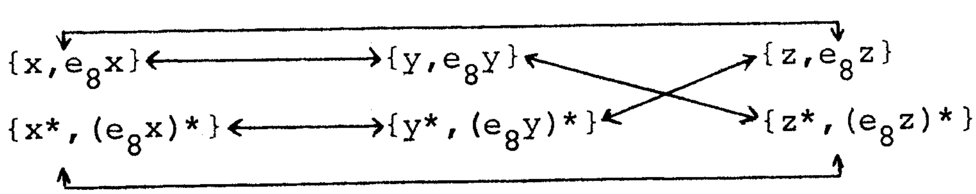


System 7:



<u>Proof</u>. To see that this theorem is true, one must compute all zero divisors of elements of the form $e_i+e_8e_j$. Although this is quite lengthy, it is easy to do using the methods of the last chapter.//

Recall that $A^* = a_1-ea_2$ if $A = a_1+ea_2$. Then notice that each system above has the following form:



Thus we may find all zero divisors of the element e_1+e_{10}, for example, by looking at system 1, and noting that all zero divisors of $m_1(e_1+e_{10}) + m_2(e_2-e_9)$ are of the form $n_1(e_4-e_{15}) + n_2(e_7+e_{12}) + n_3(e_5+e_{14}) + n_4(e_6-e_{13})$, where n_i and m_i are arbitrary real numbers.

Consideration of the examples above suggests the following.

THEOREM 8.2. <u>Let</u> A <u>be</u> <u>a</u> <u>fixed</u> <u>nonzero</u> <u>element</u> <u>in</u> $\mathcal{A}_4$. <u>Let</u> $\mathcal{B} = \{B \mid AB = 0\}$ <u>and</u> $e_8\mathcal{B} = \{e_8B \mid B$ <u>is</u> <u>in</u> $\mathcal{B}\}$. <u>Then</u> $\mathcal{B}$ <u>is</u> <u>an</u> <u>additive</u> <u>group</u> <u>closed</u> <u>under</u> <u>multiplication</u> <u>by</u> e_8.

<u>Proof</u>. First $e_8\mathcal{B}$ is contained in $\mathcal{B}$ since $(e_8B)A = 0$ if and only if $AB = 0$. Similarly, $\mathcal{B}$ is contained in $e_8\mathcal{B}$ since $B = -e_8(e_8B)$. Thus $\mathcal{B}$ is closed under multiplication by e_8.

To see that $\mathcal{B}$ is an additive group, consider that $AB_1 = 0$ and $AB_2 = 0$ implies $A(B_1+B_2) = 0$ and the fact that $AB = 0$ implies $A(-B) = 0$.//

It is not true, in general, that B_iB_j is in $\mathcal{B}$ when B_i and B_j are in $\mathcal{B}$. For example: Let $A = e_1+e_{10}$, $B_1 = e_4-e_{14}$, and $B_2 = e_5+e_{14}$. Then $AB_1 = AB_2 = 0$. But $B_1B_2 = 2(e_1-e_{10})$ so that $A(B_1B_2) = 4e_{11} \neq 0$.

An unanswered question remains as to whether or not there exists A,B,C in $\mathcal{A}_4$ such that $AB = AC = BC = 0$. From the systems of zero divisors above, it is clear that such A,B,C do not exist of the form $e_i+e_8e_j$. If such A,B,C do exist in any Cayley-Dickson algebra, then the subspace generated by A, B, and C would be a subalgebra.

Another unanswered question also deals with mutual zero divisors. Let $\mathcal{B} = \{B \mid AB = 0\}$. Then we have observed that $(e_8A)\mathcal{B} = 0$. Now does $X\mathcal{B} = 0$ imply $X = 0$, $X = A$, or $X = e_8A$? If $A = e_i+e_8e_j$, the answer is yes, in view of the systems above. In fact, for every example the author has considered, the answer is in the affirmative.

Earlier, we observed that $AB = 0$ implies that $\langle A,B\rangle$ is a commutative Jordan algebra with zero divisors. We now consider an example of the nature of $\langle A,\mathcal{B}\rangle$ where $\mathcal{B} = \{B \mid AB = 0\}$. Let $A = e_4-e_{15}$. Then considering system 1, we see that $\mathcal{B} = \{r_1B_1+r_2B_2+r_3B_3+r_4B_4\}$, where $B_1 = e_1+e_{10}$, $B_2 = e_2-e_9$, $B_3 = e_5-e_{14}$, and $B_4 = e_6+e_{13}$, where r_i is an arbitrary real number. Note that $\mathcal{B}$ is a four dimensional subset of A_4. In every example of dozens worked out, the same thing occured. For $A = e_i+e_8e_j$, the systems above show that $\mathcal{B}$ is always four dimensional.

Now consider the set generated by A and $\mathcal{B}$. It is easy to verify that $-A^2/2 = e_0$, $(A-[B_1B_3]/2)/2 = e_4$, $(A+[B_1B_3]/2)/2 = e_{15}$, $B_1B_2/2 = e_8$, $e_4e_8 = e_{12}$, $e_{15}e_4 = e_{11}$, $e_8e_{15} = e_7$, and $e_4e_7 = e_3$. Moreover, the following table shows closure under multiplication for these basis elements.

	e_0	e_3	e_4	e_7	e_8	e_{11}	e_{12}	e_{15}
e_0	e_0	e_3	e_4	e_7	e_8	e_{11}	e_{12}	e_{15}
e_3	e_3	$-e_0$	e_7	$-e_4$	e_{11}	$-e_8$	$-e_{15}$	e_{12}
e_4	e_4	$-e_7$	$-e_0$	e_3	e_{12}	e_{15}	$-e_8$	$-e_{11}$
e_7	e_7	e_4	$-e_3$	$-e_0$	e_{15}	$-e_{12}$	e_{11}	$-e_8$
e_8	e_8	$-e_{11}$	$-e_{12}$	$-e_{15}$	$-e_0$	e_3	e_4	e_7
e_{11}	e_{11}	e_8	$-e_{15}$	e_{12}	$-e_3$	$-e_0$	$-e_7$	e_4
e_{12}	e_{12}	e_{15}	e_8	$-e_{11}$	$-e_4$	e_7	$-e_0$	$-e_3$
e_{15}	e_{15}	$-e_{12}$	e_{11}	e_8	$-e_7$	$-e_4$	e_3	$-e_0$

Thus, $\langle A, B\rangle$ is an eight dimensional subalgebra of A_4. Note that $\langle A, B\rangle$ is clearly not isomorphic to the octonions since it contains zero divisors. The following result is now apparent.

LEMMA 8.3. <u>Not all eight dimensional subalgebras of A_4 are isomorphic to the octonions</u>.

One of the original questions which prompted this paper, was how A_5 differed from A_4. Some answers to this can now be considered.

First of all, recall from Theorem 7.5, i), that if $A = a_1+ea_2$ is in A_n, then $A + A^*$ is in A_{n-1}.

THEOREM 8.4. $(A+A^*, A+A^*, B+B^*) = 0$ <u>for all</u> A, B <u>in</u> A_4, <u>but</u> $(A+A^*, A+A^*, B+B^*) \neq 0$ <u>for all</u> A, B <u>in</u> A_5.

<u>Proof</u>. A_3 is alternative, A_4 is not.//

As was seen in the last chapter, a major difference between A_4 and A_5 is in the nature of their zero divisors. Let us recall here these differences. If A and B are mutual zero divisors, then the following hold for A,B in A_4, but do not necessarily hold for A,B in A_5:

i) $a_1 \neq \pm a_2$,

ii) a_1 and a_2 must be nonzero,

iii) $N(a_1) = N(a_2)$.

The fact that each of these holds for zero divisors in A_4 was shown in Theorem 7.14. The fact that they do not hold in A_5 is easy to see. For, let a be an element of A_4 such that for b_1 and b_2 in A_4 we have $ab_1 = ab_2 = 0$ (as in the example immediately following Theorem 8.2). Now, letting $A = (a+e_{16}a)$ and $B = (b_1+e_{16}b_2)$, we see that A and B are elements of A_5; but $AB = 0$, contradicting i). Similarly let $A = (0+e_{16}a)$ and B be as before. Then $AB = 0$, contradicting ii) and iii).

Finally, it was mentioned in the beginning of this paper that many results have been obtained in non-associative algebras by considering "defining identities". All algebras satisfying a particular identity, or group of identities, are classed together and then studied. Examples of this which we have seen are the flexible algebras, the alternative algebras, and the noncommutative Jordan algebras. Several such classes of algebras are well known, and others appear only recently in the literature. Each is an attempt to restrict a broad class of algebras such as the noncommutative Jordan algebras to a smaller, more fruitful, class of algebras. In order to proceed, we need the following definitions.

Definition 6.1. A finite-dimensional algebra is said to be a standard algebra if it satisfies the identities:

i) $(x,y,z) + (z,x,y) - (x,z,y) = 0$,

ii) $(w,x,yz) - (wy,x,z) - (wz,x,y) = 0$,

for all w,x,y,z in the algebra.

This definition was first given by A. A. Albert [3].

Definition 6.2. A nonassociative algebra over a field F of characteristic $\neq 2$ is a generalized standard algebra A if:

i) A is flexible,

ii) $H(x,y,z)x = H(x,y,xz)$, where $H(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y)$,

iii) $(x,y,wz) + (w,y,xz) + (z,y,xw) = [x,(w,z,y)] + (x,w,[y,z])$,

iv) if F has characteristic 3, then $(x,y,x^2) = 0$,

v) $y[x(wz)]-x[y(wz)]+(x,wz,y)+[(wz)x]y-[(wz)y]x = [y(xw)-x(yw)+(x,w,y)+(wx)y-(wy)x]z + w[y(wz)-x(yz)+(x,z,y)+(zx)y-(zy)x]$,

for all w,x,y,z in A.

Schafer first gave this definition in 1968 [29].

Definition 6.3. An accessible algebra is defined by the identities:

i) $(x,y,z) + (z,x,y) - (x,z,y) = 0$,

ii) $([w,x],y,z) = 0$.

These were first given by E. Kleinfeld [18].

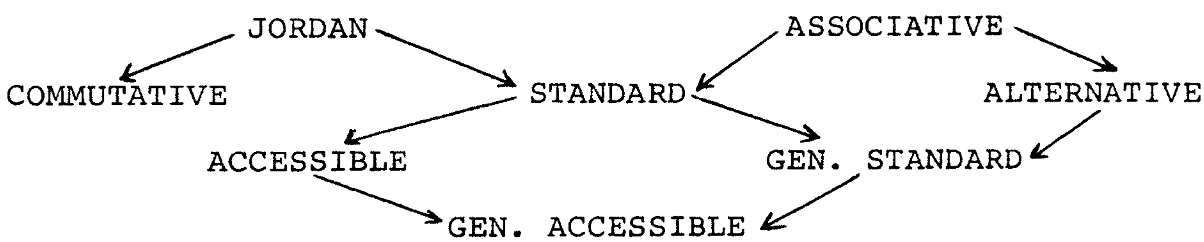
Definition 6.4. A generalized accessible algebra satisfies the identities:

i) $[x,(z,y,y)] = 0$,

ii) $3(x,y,[w,z]) = -[w,(x,y,z)]-2[x,(y,z,w)]+$
$2[y,(z,w,x)]+[z,(w,x,y)]$.

These were first defined by E. Kleinfeld, M. H. Kleinfeld, and J. F. Kosier [69].

The best account of the interrelation between these ideas is given in the last mentioned paper, and is condensed in the following diagram, where $\longrightarrow$ stands for implication:



We now wish to show, by means of counter examples, that not only the Cayley-Dickson algebras are not generalized accessible; but also that they do not satisfy any of the defining identities given above, except $(x,y,x) = 0$ and $(x,y,x^2) = 0$, i.e., the flexible property and the noncommutative Jordan identity, respectively.

Proceeding through the identities in the order given, the standard identities are not satisfied.

i) If $x = e_5$, $y = e_8$, $z = e_{10}$, then $(x,y,z) + (z,x,y) - (x,z,y) = 6e_7 \neq 0$.

ii) If $x = e_1$, $y = e_1+e_{10}$, $z = e_4-e_{15}$, $w = e_7+e_{12}$, so that $yz = yw = 0$, then $(w,x,yz) - (wy,x,z) - (wz,x,y) = 0 - 0 - (2e_8,e_1,e_1+e_{10}) = -4e_3 \neq 0$.

The generalized standard identities ii), iii), and v) do not hold.

ii) Let $x = e_1$, $y = e_{15}$, $z = e_4$. Then $H(x,y,z)x = -2e_{11}$, but $H(x,y,xz) = 2e_{11}$.

iii) Let $x = y = e_1$, $w = e_{15}$, $z = e_4$. Then all terms in $(x,y,wz)+(w,y,xz)+(z,y,xw) = [x,(w,z,y)] + (x,w,[y,z])$ are zero except $(x,w,[y,z])$ which is $4e_{11}$.

v) Let $x = e_5-e_{14}$, $y = e_5+e_{14}$, $z = e_4-e_{15}$, $w = e_1+e_{10}$. Then $y[x(wz)]-x[y(wz)]+(x,wz,y)+[(wz)x]y-[(wz)y]x = 0$. On the other hand: $[y(xw)-x(yw)+(x,w,y)+(wx)y-(wy)x]z +w[y(xz)-x(yz)+(x,z,y)+(zx)y-(zy)x] = -2e_4 - 4e_5 + 4e_{14} + 2e_{15} \neq 0$.

The accessible identities do not hold.

i) $(x,y,z) + (z,x,y) - (x,z,y) = 0$ was one of the standard identities.

ii) $([w,x],y,z) = 0$ doesn't hold, for let $w = e_1$, $x = e_2$, $y = e_5$, and $z = e_8$. Then $([w,x],y,z) = (wx,y,z) - (xw,y,z) = (e_3,e_5,e_8)-(-e_3,e_5,e_8) = 2(e_3,e_5,e_8) = -4e_{14} \neq 0$.

The generalized accessible identities fail to hold.

i) Let $x = e_3$, $y = e_1-e_{15}$, $z = e_4$. Then $[x,(z,y,y)] = [e_3,-2e_{10}] = -4e_9 \neq 0$.

ii) Let $x = z$ and $y = w$. Then $3(x,y,[w,z]) = -[w,(x,y,z)]-2[x,(y,z,w)]+2[y,(z,w,x)]+[x,(w,x,y)]$ becomes $3(x,y,[y,z]) = 0$ because of the flexible property. Letting $xy = -yx$ yields $(x,y,[y,x]) = -2(xy)^2+2x[y(xy)]$, so letting $x = e_1-e_{15}$ and $y = e_7+e_{13}$, we obtain $(x,y,[y,x]) = 4(e_4+e_{10}) \neq 0$.

Richard Block [7], in 1969, further generalized the generalized standard algebras by considering noncommutative

Jordan algebras that also satisfy what he calls alternativity conditions. He gives the following definitions.

Definition 6.5. An algebra A is completely alternative if $(x,[y,z],w) + (x,w,[y,z]) = 0$, for all x,y,z,w in A.

Definition 6.6. An algebra A is semicompletely alternative if $([x,y],z,z) = 0$, for all x,y,z in A.

Definition 6.7. An algebra A is strongly alternative if $([v,w],[x,y],z) + ([v,w],z,[x,y]) = 0$, for all x,y,z,v,w in A.

We now show that the Cayley-Dickson algebras do not satisfy any of these identities.

The Cayley-Dickson algebras are not completely alternative. If $w = x$ and $y = x$, then $(x,[y,z],z) + (x,x,[x,z]) = (x,x,[x,z])$ by the flexible property. Now, let $z = x = e_1-e_{15}$, $y = e_4$. Then $(x,x,[x,z]) = 4(e_5-e_{11}) \neq 0$.

The semicompletely alternative identity fails by letting $z = x = e_1-e_{15}$, and $y = e_4$. For $([x,y],z,z) = -4(e_5-e_{11}) \neq 0$.

Finally, the strongly alternative identity fails. For, if $v = e_9/2$, $w = e_{13}$, $x = e_1/2$, $y = e_{14}$, and $z = e_1$, we have $([v,w],[x,y],z)+([v,w],z,[x,y]) = (e_4,e_{15},e_1) + (e_4,e_1,e_{15}) = 2e_{10} \neq 0$.

Thus, not only are the Cayley-Dickson algebras not members of any of the algebras defined above, not one of the defining identities holds in the sixteen dimensional Cayley-Dickson algebra.

IX. Conclusions.

In this paper we have examined the structure of zero divisors in real Cayley-Dickson algebras. A necessary and sufficient condition for zero divisors in a real Cayley-Dickson algebra of any dimension was found. Antiassociators played a major role in the description of zero divisors.

It was found that the use of an IBM 360 computer was very valuable in finding examples and in checking hypotheses. It appears that computers hold much promise in the further study of higher dimensional Cayley-Dickson algebras.

Finally, we have observed that Cayley-Dickson algebras are a class of noncommutative Jordan algebras which are integer power associative. The fact that A_4 satisfies none of the defining identities of the preceding chapter is significant. It means that there remains much work to be done in classifying noncommutative Jordan algebras. It appears no one has considered as a class noncommutative Jordan algebras which are integer power associative. The author feels that favorable results could be obtained from such a study.

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VITA

Harmon Caril Brown, the son of Reuben R. and Lela M. Brown, was born in Grand Blanc, Michigan on June 19, 1941. After attending elementary and secondary school at Grand Blanc, Michigan, he attended Harding College, Searcy, Arkansas where he received the B.S. degree in mathematics in June, 1963. While at Harding, he was active in intercollegiate debating and was president of the student body. There he married the former Sarah Ellen Colvin of Vivian, Louisiana. They now have three children: Mark, Sarah, and David. After a summer at Louisiana Polytech, Ruston, Louisiana, he taught 7th and 8th grade mathematics in Bernie, Missouri the school year of 1963-1964. Then he attended Michigan State University, receiving the M.A. degree in mathematics in March, 1966. From September 1966 to 1972, he was instructor of mathematics at the University of Missouri at Rolla, where he received the Ph.D. degree in mathematics in May, 1972.

Appendix A

BASIS MULTIPLICATION TABLE FOR OCTONIONS

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Appendix B

BASIS MULTIPLICATION TABLE FOR A_4

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6	e_9	$-e_8$	$-e_{11}$
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$	e_{10}	e_{11}	$-e_8$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$	e_{11}	$-e_{10}$	e_9
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3	e_{12}	e_{13}	e_{14}
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2	e_{13}	$-e_{12}$	e_{15}
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$	e_{14}	$-e_{15}$	$-e_{12}$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$	e_{15}	e_{14}	$-e_{13}$
e_8	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_0$	e_1	e_2
e_9	e_9	e_8	$-e_{11}$	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$	$-e_1$	$-e_0$	$-e_3$
e_{10}	e_{10}	e_{11}	e_8	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}	$-e_2$	e_3	$-e_0$
e_{11}	e_{11}	$-e_{10}$	e_9	e_8	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}	$-e_3$	$-e_2$	e_1
e_{12}	e_{12}	e_{13}	e_{14}	e_{15}	e_8	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	e_5	e_6
e_{13}	e_{13}	$-e_{12}$	e_{15}	$-e_{14}$	e_9	e_8	e_{11}	$-e_{10}$	$-e_5$	$-e_4$	e_7
e_{14}	e_{14}	$-e_{15}$	$-e_{12}$	e_{13}	e_{10}	$-e_{11}$	e_8	e_9	$-e_6$	$-e_7$	$-e_4$
e_{15}	e_{15}	e_{14}	$-e_{13}$	$-e_{12}$	e_{11}	e_{10}	$-e_9$	e_8	$-e_7$	e_6	$-e_5$

Appendix B continued

BASIS MULTIPLICATION TABLE FOR A_4

	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_0	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
e_1	e_{10}	$-e_{13}$	e_{12}	e_{15}	$-e_{14}$
e_2	$-e_9$	$-e_{14}$	$-e_{15}$	e_{12}	e_{13}
e_3	$-e_8$	$-e_{15}$	e_{14}	$-e_{13}$	e_{12}
e_4	e_{15}	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
e_5	$-e_{14}$	e_9	$-e_8$	e_{11}	$-e_{10}$
e_6	e_{13}	e_{10}	$-e_{11}$	$-e_8$	e_9
e_7	$-e_{12}$	e_{11}	e_{10}	$-e_9$	$-e_8$
e_8	e_3	e_4	e_5	e_6	e_7
e_9	e_2	$-e_5$	e_4	e_7	$-e_6$
e_{10}	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_{11}	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
e_{12}	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_{13}	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
e_{14}	e_5	e_2	$-e_3$	$-e_0$	e_1
e_{15}	$-e_4$	e_3	e_2	$-e_1$	$-e_0$

Appendix C

PL-1 PROGRAM FOR CHARACTER STRING MULTIPLICATION CAYLEY-DICKSON ALGEBRA ELEMENTS

```
SKC: PROCEDURE OPTIONS (MAIN) ;
   DCL
      ST1(*) CHAR(100) VAR CONTROLLED ,
      TEM  FIXED ,
      H FIXED ,
      ST2(*) CHAR(100) VAR CONTROLLED ,
                B FIXED ,
                ZPQ CHAR(2) ,
                ST3(*) CHAR(64) VAR CONTROLLED ,
                CHZQ CHAR(15) VAR ,
                ZQZ FIXED ,
                LU(0:15,0:15) FIXED ,
      XYZ CHAR(100) VARYING ,
      PROD(*,*) CHAR(100) VAR CONTROLLED ,
                NU1(*) PICTURE 'SSSS9' CONTROLLED ,
                NU2(*) PICTURE 'SSSS9' CONTROLLED ,
      F FIXED ,
                NU3(*) PICTURE 'SSSS9' CONTROLLED ,
                NUK(*) PICTURE 'SSSS9' CONTROLLED ,
                (M,O,I,K,J,L,MM,MY,IX,Q,IH,OO,IK,IQ) FIXED ,
                OU CHAR(80) VAR ,
                TN CHAR(1) ,
                CAT CHAR(64) VAR ,
                S CHAR(1) ,
                NUE1(*) FIXED CONTROLLED ,
                NUE2(*) FIXED CONTROLLED ,
                NUE3(*) FIXED CONTROLLED ,
                NUM(*,*) PICTURE  'SSSS9' CONTROLLED ,
                SAB PICTURE 'SSSS9' ,
                SAV CHAR(64) VAR ,
                NEE(*,*) FIXED  CONTROLLED ,
                (Z,Y,X,W,V,U,T,SS,R, P,A) FIXED ,
                C(*) CHAR(32) VAR CONTROLLED ,
      TEMP CHAR(100) VAR ,
                OUT CHAR(200) VAR ,
                TEMF FIXED ,
                TEML FIXED ;
 /*
ALL ELEMENT SHOULD BE PUT ON A NEW DATA CARD AND EACH PART
    SEPARATED BY COMMAS
IN FRONT OF EACH DATA SECTION A DATA CARD SHOULD BE PLACED
    WHICH CONTAINS THE NUMBER OF PART AND THEN THE NUMBER OF
    ELEMENT IN EACH PART. THESE NUMBERS SHOULD BE RIGHT
    JUSTIFIED IN A 5 COLUMN FIELD. ALL REMAINING 5 COLUMN
    FIELDS SHOULD BE FILL BY 0.
BEFORE RUNNING THIS PROGRAM ONE MUST FIND OUT THE LARGEST
    LENGTH OF ANY OUTPUT ELEMENT AND THEN CHANGE THE FIELD
               PROD(*,*) CHAR(---) VAR CONTROLLED ,
    LOCATED IN THE DECLARE STATEMETS TO FIT THIS ELEMENT
I IS THE NUMBER OF POLYNOMIAL
K IS THE NUMBER OF ELEMENTS IN THE FIRST POLYNOMIAL
J IS THE NUMBER OF ELEMENTS IN THE SECOND POLYNOMIAL
Z IS THE NUMBER OF ELEMENTS IN THE THIRD POLYNOMIAL
Y IS THE NUMBER OF ELEMENTS IN THE FOURTH POLYNOMIAL
X IS THE NUMBER OF ELEMENTS IN THE FIFTH POLYNOMIAL
W IS THE NUMBER OF ELEMENTS IN THE SIXTH POLYNOMIAL
```

```
V IS THE NUMBER OF ELEMENTS IN THE SEVENTH POLYNOMIAL
U IS THE NUMBER OF ELEMENTS IN THE EIGHTH POLYNOMIAL
T IS THE NUMBER OF ELEMENTS IN THE NINTH POLYNOMIAL
SS IS THE NUMBER OF ELEMENTS IN THE TENTH POLYNOMIAL
R IS THE NUMBER OF ELEMENTS IN THE ELEVENTH POLYNOMIAL
H IS THE NUMBER OF ELEMENTS IN THE TWELFTH POLYNOMIAL
B IS THE NUMBER OF ELEMENTS IN THE THIRTEENTH POLYNOMIAL
A IS THE NUMBER OF ELEMENTS IN THE FOURTHTEENTH POLYNOMIAL
L IS THE CHECK DIGIT FOR BACKWARDS MULT.
      NV1(*)   ST1(*) NUE1(*)      NU2(*)   ST2(*)   NUE2(*)
                NUM(**)    PROD(*,*)  NEE(*,*)
 */
      ON ENDFILE(SYSIN) GO TO OUTT;
 /* GET TABLE */
      DO M = 0 BY 1 TO 15 ;
      GET EDIT ((LU(M,O) DO O = 0 BY 1 TO 15),DU)
      (16(F(3),X(1)),A(16));
      PUT EDIT ((LU(M,O) DO O = 0 BY 1 TO 15))
      (SKIP(1),16(X(2),F(3)));
      END ;
 /* GET THE NUMBER OF ELEMENTS */
STA: GET EDIT (I,K,J,Z,Y,X,W,V,U,T,SS,R,H,B,A,L)
      (16(F(5.0))) ;
      PUT PAGE ;
      PUT SKIP DATA (I,K,J,Z,Y,X,W,V,U,T,SS,R,H,B,A,L) ;
      ZQZ=1 ;
      ALLOCATE ST1(K) ;       I=I-2        ;
      ALLOCATE ST2(J) ;    ALLOCATE NUE1(K) ;
      ALLOCATE NU1(K) ;    ALLOCATE NUE2(J) ;
      ALLOCATE NU2(J) ;
      ALLOCATE NUM(K,J) ;
      ALLOCATE PROD(K,J) ;
      ALLOCATE NEE(K,J) ;  PUT PAGE LIST ('POLYNOMIAL 1') ;
      DO MZ = 1 BY 1 TO K ;
      O=0 ;
      GET EDIT (ST1(MZ),DU) (A(64),A(16)) ;
      NU1(MZ)=0 ;      PUT SKIP DATA (ST1(MZ)) ;
      IF  SUBSTR(ST1(MZ),1,1)='1' |
          SUBSTR(ST1(MZ),1,1)='2' |
          SUBSTR(ST1(MZ),1,1)='-' |
          SUBSTR(ST1(MZ),1,1)='3' |
          SUBSTR(ST1(MZ),1,1)='4' |
          SUBSTR(ST1(MZ),1,1)='5' |
          SUBSTR(ST1(MZ),1,1)='6' |
          SUBSTR(ST1(MZ),1,1)='7' |
          SUBSTR(ST1(MZ),1,1)='8' |
          SUBSTR(ST1(MZ),1,1)='9' |
          SUBSTR(ST1(MZ),1,1)='0' THEN DO ;
      DO O = 1 BY 1 TO 64 WHILE(SUBSTR(ST1(MZ),O,1)¬=',') ;
      END ;
      CAT=SUBSTR(ST1(MZ),1,O-1) ;
      IH=-1 ;
      DO IX = LENGTH(CAT) BY -1 TO 1 ;
      IH=IH+1 ;
      IF IX=1 & SUBSTR(CAT,1,1)='-' THEN DO ;
      NU1(MZ)=-1*NU1(MZ) ;
      GO TO XXX ;
      END ;
      TN=SUBSTR(CAT,IX,1) ;
      DO Q = 0 BY 1 TO 9 ;
      PUT STRING(S) EDIT (Q) (F(1.0)) ;
      IF TN=S THEN DO ;
            NU1(MZ)=NU1(MZ)+Q*10**IH ;
            GO TO JUMP ;
            END ;
      END ;
 JUMP: END ;
XXX:  END ;
      ELSE NU1(MZ)=1 ;
      DO OO =1 BY 1 TO 64 WHILE (SUBSTR(ST1(MZ),OO,1)¬=' ') ;
      END ;
```

```
      SAV=SUBSTR(ST1(MZ),OO-2,2) ;
      TN=SUBSTR(SAV,2,1) ;
      DO Q = 0 BY 1 TO 9 ;
      PUT STRING(S) EDIT (Q) (F(1.0)) ;
      IF TN=S THEN DO ;
          NUE1(MZ)=Q ;
          GO TO JQ ;
          END ;
      END ;
 JQ:   TN=SUBSTR(SAV,1,1) ;
      DO Q = 0 BY 1 TO 9 ;
      PUT STRING(S) EDIT (Q) (F(1.0)) ;
      IF TN=S THEN DO ;
          NUE1(MZ)=NUE1(MZ)+Q*10 ;
          GO TO JQQ ;
          END ;
      END ;
 JQQ: IF OO-5-O=0 THEN ST1(MZ)='' ;
      ELSE ST1(MZ)=SUBSTR(ST1(MZ),O+1,OO-5-O) ;
      END ;
RE:  ZQZ=ZQZ+1 ;
      PUT STRING(ZPQ) EDIT (ZQZ) (F(2,0)) ;
      CHZQ='POLYNOMIAL ' || ZPQ ;
      PUT PAGE LIST (CHZQ) ;
      DO MZ = 1 BY 1 TO J ;
      GET EDIT (ST2(MZ),DU) (A(64),A(16)) ;
      O=0 ;
      NU2(MZ)=0 ;
      PUT SKIP DATA (ST2(MZ)) ;
      IF  SUBSTR(ST2(MZ),1,1)='1' |
          SUBSTR(ST2(MZ),1,1)='2' |
          SUBSTR(ST2(MZ),1,1)='-' |
          SUBSTR(ST2(MZ),1,1)='3' |
          SUBSTR(ST2(MZ),1,1)='4' |
          SUBSTR(ST2(MZ),1,1)='5' |
          SUBSTR(ST2(MZ),1,1)='6' |
          SUBSTR(ST2(MZ),1,1)='7' |
          SUBSTR(ST2(MZ),1,1)='8' |
          SUBSTR(ST2(MZ),1,1)='9' |
          SUBSTR(ST2(MZ),1,1)='0' THEN DO ;
      DO O = 1 BY 1 TO 64 WHILE(SUBSTR(ST2(MZ),O,1)¬=',') ;
      END ;
      CAT=SUBSTR(ST2(MZ),1,O-1) ;
      IH=-1 ;
      DO IX = LENGTH(CAT) BY -1 TO 1 ;
      IH=IH+1 ;
      IF IX=1 & SUBSTR(CAT,1,1)='-' THEN DO ;
      NU2(MZ)=-1*NU2(MZ) ;
      GO TO QQQ ;
      END ;
      TN=SUBSTR(CAT,IX,1) ;
      DO Q = 0 BY 1 TO 9 ;
      PUT STRING(S) EDIT (Q) (F(1.0)) ;
      IF TN=S THEN DO ;
          NU2(MZ)=NU2(MZ)+Q*10**IH ;
          GO TO JUMP2 ;
          END ;
      END ;
 JUMP2: END ;
QQQ: END ;
      ELSE NU2(MZ)=1 ;
      DO OO = 1 BY 1 TO 64 WHILE(SUBSTR(ST2(MZ),OO,1)¬=' ') ;
      END ;
      SAV=SUBSTR(ST2(MZ),OO-2,2) ;
      TN=SUBSTR(SAV,2,1) ;
      DO Q = 0 BY 1 TO 9 ;
      PUT STRING(S) EDIT (Q) (F(1.0)) ;
      IF TN=S THEN DO ;
          NUE2(MZ)=Q ;
          GO TO JQ2;
          END ;
```

```
         END ;
JQ2:  TN=SUBSTR(SAV,1,1) ;
      DO Q = 0 BY 1 TO 9 ;
      PUT STRING(S) EDIT (Q) (F(1.0)) ;
      IF TN=S THEN DO ;
           NUE2(MZ)=NUE2(MZ)+Q*10 ;
           GO TO JQQ7 ;
           END ;
      END ;
JQQ7: IF QQ-5-Q=0 THEN ST2(MZ)='' ;
      ELSE ST2(MZ)=SUBSTR(ST2(MZ),Q+1,QQ-5-Q) ;
      END ;
/*    MULTIPLY */
      DO MZ = 1 BY 1 TO K ;
      DO MY = 1 BY 1 TO J ;
      IF L=1 THEN SAB=LU(NUE2(MY),NUE1(MZ)) ;
      ELSE SAB=LU(NUE1(MZ),NUE2(MY)) ;
      IF SAB<0  THEN DO ;
                 NUM(MZ,MY)=-1*(NU1(MZ)*NU2(MY)) ;
                 NEE(MZ,MY)=-1*SAB ;
                 END ;
      ELSE IF SAB=100 THEN DO ;
                 NUM(MZ,MY)=-1*(NU1(MZ)*NU2(MY)) ;
                 NEE(MZ,MY)=0;
                 END ;
      ELSE DO ;
                 NUM(MZ,MY)=NU1(MZ)*NU2(MY) ;
                 NEE(MZ,MY)=SAB ;
                 END ;
      PROD(MZ,MY)=ST1(MZ)||ST2(MY) ;
      END ;
      END ;
      IF I>0 THEN DO ;
           FREE ST1 ;
           FREE NU1 ;
           FREE NUE1 ;
           FREE ST2 ;
           FREE NU2 ;
           FREE NUE2 ;
           M=K*J ;
           ALLOCATE ST1(M) ;
           ALLOCATE NU1(M) ;
           ALLOCATE NUE1(M) ;
           IK=0 ;
      DO MZ=1 BY 1 TO K ;
           DO MY = 1 BY 1 TO J ;
           IK=IK+1 ;
      ST1(IK)=PROD(MZ,MY) ;
           NU1(IK)=NUM(MZ,MY) ;
           NUE1(IK)=NEE(MZ,MY) ;
           END ;
           END ;
           K=M ;
           IF I=1 THEN J=Z ;
           ELSE IF I=2 THEN  J=Y ;
           ELSE IF I=3 THEN  J=X ;
           ELSE IF I=4 THEN J=W ;
           ELSE IF I=5 THEN J=V ;
           ELSE IF I=6 THEN J=U ;
           ELSE IF I=7 THEN J=T ;
           ELSE IF I=8 THEN J=SS ;
           ELSE IF I=9 THEN J=R ;
           ELSE IF I=10 THEN J=H ;
           ELSE IF I=11 THEN J=B ;
           ELSE IF I=12 THEN J=A ;
      I=I-1 ;
           ALLOCATE ST2(J) ;
           ALLOCATE NU2(J) ;
           ALLOCATE NUE2(J);
           FREE PROD ;
           FREE NUM ;
```

```
            FREE NEE ;
            ALLOCATE PROD(K,J) ;
            ALLOCATE NUM(K,J) ;
            ALLOCATE NEE(K,J) ;
        GO TO RE ;
        END ;
        FREE ST1 ;
        FREE ST2 ;
        FREE NU1 ;
        FREE NU2 ;
        FREE NUE1 ;
        FREE NUE2 ;
        M=K*J ;
        ALLOCATE ST1(M) ;
        ALLOCATE NU1(M) ;
        ALLOCATE NUE1(M) ;    IK=0 ;
        DO MZ = 1 BY 1 TO K ;
        DO MY = 1 BY 1 TO J ;
        IK=IK+1 ;
        ST1(IK)=PROD(MZ,MY) ;
        NU1(IK)=NUM(MZ,MY) ;
        NUE1(IK)=NEE(MZ,MY) ;
        END ;
        END ;
        MM=M*32 ;
PUT SKIP DATA(M,MM);
        ALLOCATE C(MM) ;
        DO MZ=1 BY 1 TO M ;
        IQ=1 ;
        IK=1 ;
        DO O = 1 BY 1 TO LENGTH(ST1(MZ)) ;
        IF SUBSTR(ST1(MZ),O,1)=',' THEN DO ;
            C(IK)=SUBSTR(ST1(MZ),IQ,O-IQ) ;
            IK=IK+1 ;
            IQ=O+1 ;
            END ;
        END ;
        IK=IK-1 ;
        IF IK>1 THEN DO ;
 ST:    DO P=2 BY 1 TO IK ;
        IF C(P-1)>C(P) THEN GO TO DO ;
        GO TO EN ;
 DO:    TEMP=C(P) ;
        C(P)=C(P-1) ;
        C(P-1)=TEMP ;
        GO TO ST ;
 EN:    END ;
        END ;
        ST1(MZ)='' ;
        DO P=1 BY 1 TO IK ;
        ST1(MZ)=ST1(MZ)||C(P)||',' ;
        END ;
        END ;
        FREE C ;
 /*     CHECK FOR THE SAME ELEMENTS */
        DO P=1 BY 1 TO M-1 ;
        DO O=P+1 BY 1 TO M ;
        IF ST1(P)=ST1(O) & NUE1(P)=NUE1(O) THEN DO ;
        NU1(P)=NU1(P)+NU1(O) ;
        NU1(O)=0 ;
        END ;
        END ;
        END ;
        DO P=1 BY 1 TO M-1 ;
        IF NU1(P)=0 THEN GO TO ZAZ ;
        XYZ='('||NU1(P)||ST1(P) ;
        DO O=P+1 BY 1 TO M ;
        IF NU1(O)=0 THEN GO TO QE ;
        IF NUE1(O)=99 THEN GO TO QE ;
        IF NUE1(P)=NUE1(O) THEN DO ;
        XYZ=XYZ||NU1(O)||ST1(O) ;
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      NUE1(O)=99 ;
      END ;
 QE:  END ;
      ST1(P)=XYZ||')'||'E'||NUE1(P) ;
ZAZ:  END ;
      IF NU1(M)=0 THEN GO TO SZ ;
      IF NUE1(M)=99 THEN GO TO SZ ;
      ST1(M)='('||NU1(M)||ST1(M)||')E'||NUE1(M) ;
 SZ:  DO P=2 BY 1 TO M ;
      IF NUE1(P-1)>NUE1(P) THEN GO TO DZ ;
      GO TO  EZ ;
 DZ:  TEMP=ST1(P) ; TEMF=NU1(P) ; TEML=NUE1(P)              ;
      ST1(P)=ST1(P-1) ; NU1(P)=NU1(P-1) ; NUE1(P)=NUE1(P-1) ;
      ST1(P-1)=TEMP ;  NU1(P-1)=TEMF ;    NUE1(P-1)=TEML ;
      GO TO SZ ;
 EZ:  END ;      PUT PAGE LIST ('ANSWER =') ; PUT SKIP(2) ;
 /*   LIST FINAL ELEMENT */
      IX=0 ;
      DO O=1 BY 1 TO M ;
      IF NUE1(O)=99 THEN GO TO EA ;
      IF NU1(O)=0 THEN GO TO EA ;
      IF ST1(O)='' THEN GO TO EA ;
      F=INDEX(ST1(O),' ') ;
      DO WHILE (F>0) ;
      ST1(O)=SUBSTR(ST1(O),1,F-1)||
      SUBSTR(ST1(O),F+1,LENGTH(ST1(O))-F) ;
      F=INDEX(ST1(O),' ') ;
      END ;
      F=INDEX(ST1(O),',)') ;
      IF F>0 THEN
      ST1(O)=SUBSTR(ST1(O),1,F-1)||
      SUBSTR(ST1(O),F+1,LENGTH(ST1(O))-F) ;
      IX=IX+1 ;
      IF IX>1 THEN ST1(O)='+'||ST1(O) ;
      PUT SKIP LIST (ST1(O)) ;
 EA:  END ;
      IF IX=0 THEN PUT SKIP LIST ('0') ;
      FREE ST1 ;
      FREE NU1 ;
      FREE NUE1 ;
      GO TO STA ;
 OUTT: END ;
```