Vietnam Journal of Science and Technology ***55*** *(5) (2017) 598-611*
DOI: 10.15625/2525-2518/55/5/9140

THE TRANSFER MATRIX METHOD FOR MODAL ANALYSIS OF CRACKED MULTISTEP BEAM

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Received: 9 January 2017; Accepted for publication: 28 May 2017

ABSTRACT

The present study addresses the modal analysis of multistep beam with arbitrary number of cracks by using the transfer matrix method and modal testing technique. First, there is conducted general solution of free vibration problem for uniform beam element with arbitrary number of cracks that allows one to simplify the transfer matrix for cracked multistep beam. The transferring beam state needs to undertake only at the steps of beam but not through crack positions. Such simplified the transfer matrix method makes straightforward to investigate effect of cracks mutually with cross-section step in beam on natural frequencies. It is revealed that step-down and step-up in beam could modify notably sensitivity of natural frequencies to crack so that the analysis provides useful indication for crack detection in multistep beam. The proposed theory was validated by an experimental case study.

Keywords: stepped beam; cracked beam; modal analysis; transfer matrix method.

1. INTRODUCTION

Stepped beam structures have found widespread application in engineering fields such as bridges, rotating machines, robotics and aerospace structures. In the engineering application, vibration of the structures is the problem of a great importance and it is studied in the enormous literature. Sato [1] studied an interesting problem that proposed to calculate natural frequency of beam with a groove in dependence on size of the groove. Using a model of stepped beam and Transfer Matrix Method (TMM) combined with Finite Element Method (FEM) the author demonstrated that (a) fundamental frequency of the structure increases with growing thickness and reducing length of the mid-step; (b) the mid-step could be modeled by a beam element, therefore, the TMM is reliably applicable for the stepped beam if ratio of its length to the beam thickness ($r = L_2/h$) is equal or greater than 4.0. Comparing with experimental results the author concluded that error of the TMM may be up to 20 % if the ratio is less than 0.2. Latter, Jang and Bert [2, 3] used the conventional technique for calculating natural frequencies of two-step beam and shown that natural frequencies of the structure are dependent not only on the change in cross-section but also on the beam boundary conditions. Namely, stepping up (increasing height) loads to increasing natural frequencies for any boundary conditions except the clamped ends beam and stepping down (decreasing height) reduces the frequencies except the cantilevered beam. The findings are important to show the dynamic property of stepped beam and which method could be useful for vibration analysis of the beam. Other methods such as Adomian Decomposition Method (ADM) and Differential Quadrature Element Method (DEM) have been developed in [4] and [5], respectively, for free vibration analysis of multi-step beams. Cunha and Junior [6] investigated effect of elastic boundary supports on natural frequencies and mode shapes of multiple stepped beam. Kukla and Zamojska [7] studied effect of axial force on natural frequencies and longitudinal or torsional vibrations of stepped one-dimensional structures such as bars or shafts were studied in [8] by using Distributed Transfer Function Method (DTFM). Jaworski and Dowell [9] have compared different theoretical methods and beam theories used for free vibration analysis of multistep cantilever beam with experimental results. It was shown by the authors that there is disaggreement between theoretical and experimental results. Wattanasakulpong and Charoensuk [10] studied one-step beam made of functionally graded material.

Vibration of stepped beam structures with cracks have been also intensively examined due to that cracks are potential to reduce the serviceability of a structure and in consequence may lead to a serious accidence if it is not early detected. To detect cracks in a structure its vibration analysis is crucially important. Nandwana and Maiti [11] have established frequency equation of an n-step Euler-Bernoulli beam with single crack in a form of $4(n+1)$ order determinant and used for crack detection by natural frequencies. Using TMM, Tsai and Wang [12] obtained frequency equation for cracked multistep Timoshenko beam in much simplified form of 4x4-dimention determinant that simplifies significantly computation of the beam's natural frequencies. Maghsoodi et al. [13] have obtained an explicit expression of natural frequencies through the crack magnitudes for multistep Euler-Bernoulli beam that provide a system of linear algebraic equations for crack detection from natural frequencies. Li [14] was able to conduct a recurrent relationship between vibration modes of adjacent steps that is straightforward to obtain an explicit expression of frequency equation for multiple cracked and stepped beam. The TMM is completely developed and used for solving both the forward and inverse problems for multistep Euler-Bernoulli beam with arbitrary number of cracks by Attar in [15]. However, in the latter publication the transfer matrix is very complicated because it should be assembled not only at the steps of beam but also over the crack sections.

This paper presents the TMM developed for modal analysis of cracked multistep beam based on an explicit expression for mode shape of multiple cracked uniform beam element. This enables to much simplify the transfer matrix of multiple cracked multistep beam compared to that was developed in [15] and it is validated by an experimental study.

2. GENERAL SHAPE FUNCTION FOR MULTIPLE CRACKED BEAM ELEMENT

Consider a uniform beam element of length L; material density (ρ); elasticity (E) and shear (G) modulus; section area $A = b \times h$ and moment of inertia $I = bh^3/12$. Assume furthermore that the beam is cracked at the positions $0 < e_1 < ... < e_n < L$ and the cracks of depth $a_1,...,a_n$ are modeled by equivalent springs of stiffness $K_1,...,K_n$. The springs stiffness is calculated from the crack depths using formulas given in Appendix. For the beam element, free vibration is governed by equation

$$d^4\varphi(x)/dx_4 - \lambda^4\varphi(x) = 0,\ \lambda^4 = \rho A\omega^2/EI, \tag{2.1}$$

that is solved under the conditions at the crack positions

$$\phi(e_j+0) = \phi(e_j-0)\ ;\phi''(e_j+0) = \phi''(e_j-0) = \phi''(e_j);\phi'''(e_j+0) = \phi'''(e_j-0)\ ;$$
$$\varphi'(e_j+0) = \varphi'(e_j-0) + \gamma_j\varphi''(e_j), \tag{2.2}$$

where $\gamma_j = EI/K_j,\ j = 1,2,...,n$ called hereby magnitudes of the cracks. Introducing Krylov's functions

$$\begin{aligned} L_{01}(x) &= (\cosh\lambda x + \cos\lambda x)/2; L_{02}(x) = (\sinh\lambda x - \sin\lambda x)/2;\\ L_{03}(x) &= (\cosh\lambda x - \cos\lambda x)/2; L_{04}(x) = (\sinh\lambda x + \sin\lambda x)/2, \end{aligned} \tag{2.3}$$

that are all continuous particular solutions of Eq. (2.1), we can prove that the functions

$$L_k(x) = L_{0k}(x) + \sum_{j=1}^{n}\mu_{kj}K(x-e_j), k = 1,2,3,4, \tag{2.4}$$

where

$$K^{(p)}(x) = \begin{cases} S^{(p)}(x):\ x \geq 0;\\ 0: \quad\quad x < 0; \end{cases}\ S(x) = \frac{\sinh\lambda x + \sin\lambda x}{2\lambda}, p = 0,1,2,3, \tag{2.5}$$

p is derivative order and so-called damage parameters $\mu_{kj},\ j = 1,2,...,n; k = 1,2,3,4$ are defined as

$$\mu_{kj} = \gamma_j[L''_{0k}(e_j) + \sum_{i=1}^{j-1}\mu_{ki}S''(e_j - e_i), k = 1,2,3,4, \tag{2.6}$$

are solutions of Eq. (2.1) satisfying also conditions (2.2).

Since functions (2.3) and function $S(x)$ defined in (2.5) are continuous solutions of Eq. (2.1), the functions (2.4) would satisfy also the equation except crack positions where they need to satisfy conditions (2.2). Indeed, since

$$S(0) = S''(0) = S'''(0) = 0; S'(0) = 1, \tag{2.7}$$

one has got

$$L_k(e_j+0) = L_{0k}(e_j+0) + \sum_{i=1}^{j}\mu_{ki}S(e_j-e_i) = L_{0k}(e_j-0) + \sum_{i=1}^{j-1}\mu_{ki}S(e_j-e_i) = L_k(e_j-0)\ ;$$

$$L''_k(e_j+0) = L''_{0k}(e_j+0) + \sum_{i=1}^{j}\mu_{ki}S''(e_j-e_i) = L''_{0k}(e_j-0) + \sum_{i=1}^{j-1}\mu_{ki}S''(e_j-e_i) = L''_k(e_j-0)\ ;$$

$$L'''_k(e_j+0) = L'''_{0k}(e_j+0) + \sum_{i=1}^{j}\mu_{ki}S'''(e_j-e_i) = L'''_{0k}(e_j-0) + \sum_{i=1}^{j-1}\mu_{ki}S'''(e_j-e_i) = L'''_k(e_j-0)\ ;$$

$$L'_k(e_j+0) = L'_{0k}(e_j+0) + \sum_{i=1}^{j}\mu_{ki}S'(e_j-e_i) = L'_{0k}(e_j-0) + \sum_{i=1}^{j-1}\mu_{ki}S'(e_j-e_i) + \mu_{kj} =$$

$$= L'_k(e_j-0) + \gamma_j[L''_{0k}(e_j) + \sum_{i=1}^{j-1}\mu_{ki}S''(e_j-e_i)] = L'_k(e_j-0) + \gamma_j L''_k(e_j-0).$$

Thus, general solution of Eq. (2.1) satisfying conditions (2.2) can be found in the form

$$\phi(x) = C_1L_1(x) + C_2L_2(x) + C_3L_3(x) + C_4L_4(x) \tag{2.8}$$

where functions $L_k(x), k = 1,2,3,4$ are determined in (2.4)-(2.6) and C_1, C_2, C_3, C_4 are arbitrary constants would be found using boundary conditions for the beam. Using the expression (2.8) one is able to calculate displacement, slope, moment and shear force respectively as follows

$$W(x) \equiv \phi(x) = L_1(x)C_1 + L_2(x)C_2 + L_3(x)C_3 + L_4(x)C_4;$$
$$\Theta(x) \equiv \phi'(x) = L_1'(x)C_1 + L_2'(x)C_2 + L_3'(x)C_3 + L_4'(x)C_4;$$
$$M(x) \equiv EI\varphi''(x) = EIL_1''(x)C_1 + EIL_2''(x)C_2 + EIL_3''(x)C_3 + EIL_4''(x)C_4; \tag{2.9}$$
$$Q(x) \equiv EI\phi'''(x) = EIL_1'''(x)C_1 + EIL_{21}'''(x)C_2 + EIL_3'''(x)C_3 + EIL_4'''(x)C_4.$$

that can be rewritten in the matrix form

$$\{\mathbf{V}(x)\} = [\mathbf{H}(x)] \cdot \{\mathbf{C}\}, \tag{2.10}$$

where vectors $\{\mathbf{V}(x)\} = \{W(x), \Theta(x), M(x), Q(x)\}^T, \{\mathbf{C}\} = \{C_1, C_2, C_3, C_4\}^T$ and matrix

$$[\mathbf{H}(x)] = \begin{bmatrix} L_1(x) & L_2(x) & L_3(x) & L_4(x) \\ L_1'(x) & L_2'(x) & L_3'(x) & L_4'(x) \\ EIL_1''(x) & EIL_2''(x) & EIL_3''(x) & EIL_4''(x) \\ EIL_1'''(x) & EIL_2'''(x) & EIL_3'''(x) & EIL_4'''(x) \end{bmatrix}. \tag{2.11}$$

This representation (2.10) of beam state will be employed below to develop the transfer matrix method for multiple cracked stepped beam.

3. TRANSFER MATRIX FOR STEPPED BEAM WITH MULTIPLE CRACKS

Let's consider now a stepped beam composed of *m* uniform beam segments designated with subscript *j*, *j*=1,2,…,*m*. Namely, material and geometry constants of *j*-th beam segment are denoted by $E_j, \rho_j, b_j, h_j, L_j$. Suppose that each of the beam segments contains a number (n_j) of cracks represented by its position $e_{ji}, i = 1,...,n_j$ and magnitude $\gamma_{ji} = E_j I_j / K_{ji}$ and a crack of magnitude $\alpha_j = E_j I_j / K_j$ occurs at joint of (*j*+1)-th and *j*-th segments.

Introduce state vector for *j*-th step as $\mathbf{V}_j(x) = \{W_j(x), \Theta_j(x), M_j(x), Q_j(x)\}^T$ defined in (2.9) symbolized with subscript *j*. Therefore, continuity conditions at step joints are

$$W_{j+1}(0) = W_j(L_j); \Theta_{j+1}(0) = \Theta_j(L_j) + \bar{\alpha}_j M_j(L_j); M_{j+1}(0) = M_j(L_j); Q_{j+1}(0) = Q_j(L_j)$$

or

$$\mathbf{V}_{j+1}(0) = \Gamma(\bar{\alpha}_j)\mathbf{V}_j(L_j),\ j = 1, 2, ..., m, \tag{3.1}$$

where $\bar{\alpha}_j = \gamma_j / E_j I_j = 1/K_j$ and

$$\Gamma(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.2}$$

On the other hand, using representation (2.10) the introduced state vector $\mathbf{V}_j(x)$ could be now rewritten as

$$\mathbf{V}_j(x) = [\mathbf{H}_j(x)]\mathbf{C}_j, \tag{3.3}$$

where

$$[\mathbf{H}_j(x)] = \begin{bmatrix} L_{j1}(x) & L_{j2}(x) & L_{j3}(x) & L_{j4}(x) \\ L'_{j1}(x) & L'_{j2}(x) & L'_{j3}(x) & L'_{j4}(x) \\ E_jI_jL''_{j1}(x) & E_jI_jL''_{j2}(x) & E_jI_jL''_{j3}(x) & E_jI_jL''_{j4}(x) \\ E_jI_jL'''_{j1}(x) & E_jI_jL'''_{j2}(x) & E_jI_jL'''_{j3}(x) & E_jI_jL'''_{j4}(x) \end{bmatrix}. \tag{3.4}$$

The shape functions $L_{jk}(x)$ defined in (2.4) where beam constants are replaced by those with subscript j. Specifically, the frequency parameter λ defined in (2.1) now is

$$\lambda_j = (\omega^2 \rho_j b_j h_j / E_j I_j)^{1/4}.$$

It is easily to show that expression (3.2) yields

$$\mathbf{V}_j(L_j) = \mathbf{T}(j)\mathbf{V}_j(0)\,;\mathbf{T}(j) = \mathbf{H}_j(L_j)\mathbf{H}_j^{-1}(0). \tag{3.5}$$

So, combining the relationship (3.5) with (3.1) for j =1, 2, …, m one obtains finally

$$\mathbf{V}_m(L_m) = [\mathbf{T}]\{\mathbf{V}_1(0)\}; \tag{3.6}$$

$$[\mathbf{T}] = [\mathbf{T}(m)\Gamma(\bar{\alpha}_{m-1})\mathbf{T}(m-1)...\Gamma(\bar{\alpha}_1)\mathbf{T}(1)]. \tag{3.7}$$

Usually, conventional boundary conditions are expressed by

$$\mathbf{B}_0\{\mathbf{V}_1(0)\} = 0; \mathbf{B}_L\{\mathbf{V}_m(L_m)\} = 0, \tag{3.8}$$

where B_0, B_L are matrices of 2×4 dimension. For instance, if both ends of the beam are clamped the boundary matrices get the form

$$\mathbf{B}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \bar{\alpha}_0 & 0 \end{bmatrix}; \mathbf{B}_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \bar{\alpha}_L & 0 \end{bmatrix}.$$

with $\bar{\alpha}_0, \bar{\alpha}_L$ being magnitudes of possible cracks at the end clamps. Consequently,

$$[\mathbf{B}(\omega)]\mathbf{V}_1(0) = 0, \tag{3.9}$$

where

$$\mathbf{B}(\omega) = \begin{bmatrix} \mathbf{B}_0 \\ \mathbf{B}_L\mathbf{T} \end{bmatrix}. \tag{3.10}$$

Eq. (3.5) would have nontrivial solution with respect to $\mathbf{V}_1(0)$ under the condition

$$D(\omega) \equiv \det[\mathbf{B}(\omega)]) = 0, \tag{3.11}$$

that is frequency equation desired for the stepped beam with cracks.

For instance, if the left end of beam is clamped and the other one is free, i. e. the beam is cantilevered, the boundary conditions are

$$W_1(0) = \Theta_1(0) = M_m(L_m) = Q_m(L_m) = 0.$$

Therefore, the frequency equation (3.10) gets to be

$$D_{CF}(\omega) \equiv \det\begin{bmatrix} T_{33} & T_{34} \\ T_{43} & T_{44} \end{bmatrix} = T_{33}T_{44} - T_{43}T_{34} = 0, \tag{3.12}$$

where $T_{ik}, i,k = 1,2,3,4$ are elements of the total transfer matrix [**T**] defined in (3.4). Similarly, frequency equation of stepped FGM beam can be obtained as determinant of a 2x2 matrix for other cases of boundary conditions such as simple supports or clamped ends. Namely, for simply supported beam with

$$W_1(0) = M_1(0) = W_m(L_m) = M_m(L_m) = 0,$$

frequency equation is

$$D_{SS}(\omega) \equiv \det\begin{bmatrix} T_{12} & T_{14} \\ T_{32} & T_{34} \end{bmatrix} = T_{12}T_{34} - T_{32}T_{14} = 0 \ . \tag{3.13}$$

For beam with clamped ends where

$$W_1(0) = \Theta_1(0) = W_m(L_m) = \Theta_m(L_m) = 0,$$

one has got

$$D_{CC}(\omega) \equiv \det\begin{bmatrix} T_{13} & T_{14} \\ T_{23} & T_{24} \end{bmatrix} = T_{13}T_{24} - T_{23}T_{14} = 0. \tag{3.14}$$

Solving the frequency equations gives rise natural frequencies $\omega_k, k = 1,2,3,...$ of the beam that in turn allow one to find corresponding solution of Eq. (3.8) as $\mathbf{V}_1(0) = D_k \overline{\mathbf{V}}_1$ with an arbitrary constant D_k and normalized solution $\overline{\mathbf{V}}_1$. Afterward, mode shape corresponding to natural frequency ω_k is determined for every beam step as follows

$$\mathbf{\Phi}_{jk}(x) = W_{jk}(x) = D_k\{L_{j1}(x)\overline{C}_{j1} + L_{j2}(x)\overline{C}_{j2} + L_{j3}(x)\overline{C}_{j3} + L_{j4}(x)\overline{C}_{j4}\}_{\omega=\omega_k};$$

$$\overline{\mathbf{C}}_j = [\mathbf{H}_j(0)]^{-1}[\mathbf{T}(j-1)\mathbf{T}(j-2)...\mathbf{T}(1)]\{\overline{\mathbf{V}}_1\},\ j = 1,2,...,m \ . \tag{3.15}$$

The arbitrary constant D_k is determined by a chosen normalized condition, for example,

$$\max_{(x,j)} \mathbf{\Phi}_{jk}(x) = 1.$$

Thus, the free vibration problem for stepped beam with multiple cracks is completely solved by the simplified transfer matrix method.

4. EXPERIMENTAL SETUP AND MODAL TESTING TECHNIQUE

In this section experimental modal analysis is accomplished for the stepped beam with clamped ends as shown in Fig. 1. Geometry and material parameters of the beam models are given in Table 1. $E = 210MPa; \rho = 7855kg/m^3; \nu = 0.3$.

Crack is produced by saw cut with very small wide and different depth 0 %, 10 %, 20 %, 30 % and 40 % beam thickness at fixed positions on beam. Therefore, the saw-cut can be treated as an approximate model of open transverse crack described in [16]. Three scenarios of cracked beam are investigated: single crack at position 450 mm; double cracks at the positions 200 mm; 450 mm from the left end and triple cracks at positions 200 mm, 450 mm, 800 mm from the left end. In the first scenario, single crack of various depth (10 – 40 %) is examined. The second scenario is tested with various depth of crack at the first span and the crack at intermediate span has fixed depth of 40 %. The last crack scenario is carried out for the beam with three cracks of equal depth 40 %.

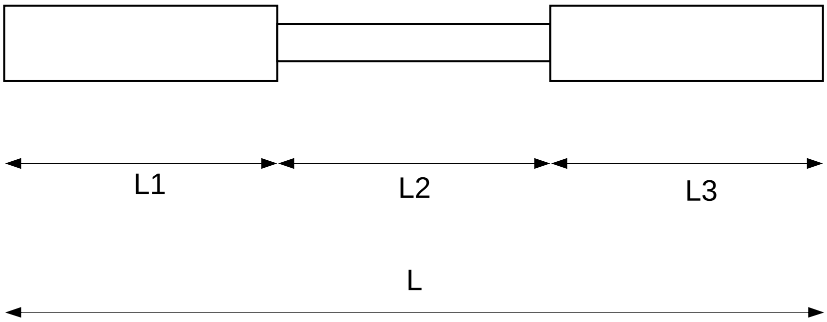

Figure 1. Model of stepped beam used in experimentation.

Table 1. Geometrical dimensions and material properties of two-step (three span) beam.

Geometrical parameters (mm)	Beam spans		
	1st span	2nd span	3rd span
Wide, *b*	20	20	20
Height, h	15.4	7.5	15.4
Length, *L*	315	400	315
Total length	1230		
Material properties $E = 210MPa; \rho = 7855kg/m^3; \nu = 0.3$			

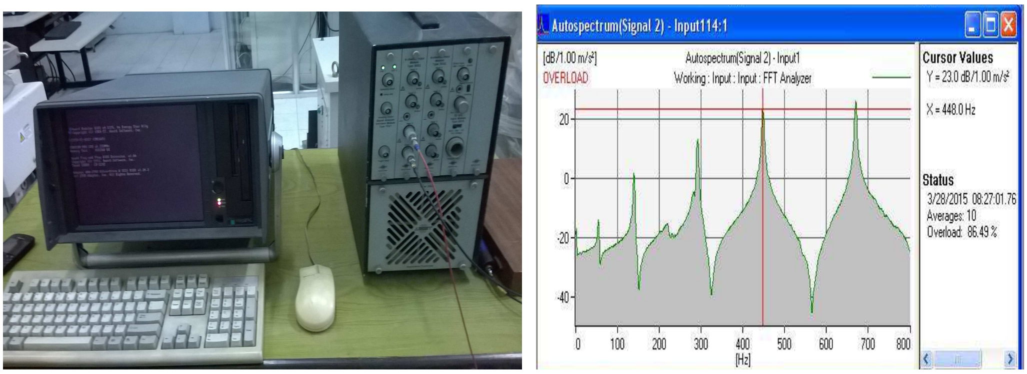

Figure 2. Measurement system PULSE 360.

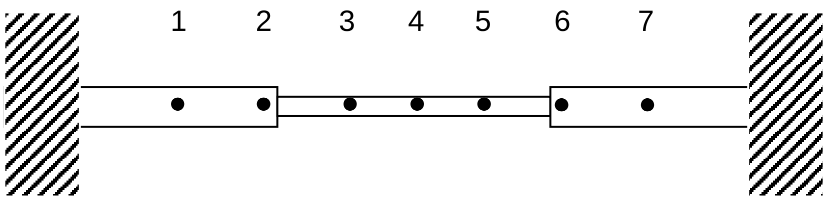

Figure 3. Experimental model with measurement points.

The PULSE B&K360 system, Fig. 2a, is employed for gathering and processing measured data. An impact hammer (Fig. 3) is used for generating an excitation at position A denoted by $X(\omega)$ in the frequency domain and an accelerometer has been employed for measuring response ($Y(\omega)$) at the position B on the beam. Hence, the signal processor installed in the measurement system provides Frequency Response Function (FRF) between the positions A, B calculated as

$$H_{AB}(\omega) = \frac{S_{XY}(\omega)}{S_{XX}(\omega)}, \tag{4.1}$$

where $S_{XX}(\omega), S_{XY}(\omega)$ are auto- and cross correlation functions respectively of the signals *X* and *Y*. Magnitude of the function (4.1) is shown for instance in Fig. 2b. Multiple measurement of FRF is performed for varying positions of excitation and response and all the measured data gathered should give rise the same modal parameters of testing structure.

In the theory of structural vibration, it was shown that the FRF (4.1) can be expressed in term of natural vibration modes as

$$H_{AB}(\omega) = \sum_{\ell=1}^{n} \frac{\bar{\phi}_\ell(A)\bar{\phi}_\ell(B)}{[\omega_\ell^2 - \omega^2 + i\omega\zeta_\ell]}. \tag{4.2}$$

where ω_ℓ, ζ_ℓ are natural frequency and damping ratio respectively of mode ℓ and $\bar{\phi}_\ell(A)$ - normalized $\ell-$th mode shape measured at position A. Moreover, analysis of the function (4.2) in the frequency domain exhibited that in the case of small damping and sparse distribution of natural frequencies the frequency response function reaches its local maximums at resonant frequencies

$$\hat{\omega}_\ell = \sqrt{\omega_\ell^2 - \zeta_\ell^2 / 2}, \ell = 1,2,3... \tag{4.3}$$

The damping ratio is represented by sharpness of the resonant peak that determined by

$$\hat{\zeta}_\ell = (\bar{\omega}_{\ell 2} - \bar{\omega}_{\ell 1}) / 2\hat{\omega}_\ell, \tag{4.4}$$

where $\bar{\omega}_{\ell 2}, \bar{\omega}_{\ell 1}$ are two frequencies in both sides of $\hat{\omega}_\ell$ defined by

$$H_{AB}(\bar{\omega}_{\ell 1}) = H_{AB}(\bar{\omega}_{\ell 1}) = H_{AB}(\hat{\omega}_\ell) / \sqrt{2}\ . \tag{4.5}$$

So that natural frequencies are determined from the measured data as

$$\omega_\ell^* = \sqrt{\hat{\omega}_\ell^2 + \hat{\zeta}_\ell^2 / 2}, \ell = 1,2,3... \tag{4.6}$$

and results are given in Table 2 in comparison with the numerically computed ones.

5. RESULTS AND DISCUSSION

5.1. Theoretical validation

Note that in the case of uncracked beam the transfer matrix method proposed in section 3 leads to its classical version. This can be validated first by using the method for computing five lowest eigenvalues ($\lambda_k = \sqrt[4]{\rho A \omega_k^2 / EI}, k = 1, 2, 3, 4, 5$) of a uniform beam with clamped ends. Results of the computation compared to those obtained by the classical analytical method (see Table 2) show that the transfer matrix method is really an exact method equivalent to the analytical one. Moreover, natural frequencies of an intact stepped beam calculated by the classical transfer matrix method are given in the first row of Table 3. The results compared to those obtained by FEM and measured demonstrate the fact that measured natural frequencies are more closed (almost identical for three lower frequencies) to the analytical ones than FEM results. This validates reliability of measured data.

Table 2. Eigenvalues of uniform beam calculated by the TMM compared to analytical method [17].

Eigenvalues	λ_1	λ_2	λ_3	λ_4	λ_5
TMM	4.7300	7.8532	10.9956	14.1372	17.2788
Analytical	4.7300	7.8532	10.9956	14.1371	17.2787

5.2. Experimental validation

Table 3. Comparison of calculated and measured natural frequencies of three-span stepped beam with clamped ends.

Crack scenarios		Natural frequencies (Hz)				
		1	2	3	4	5
Intact beam	TMM	73.2781	144.5188	301.1640	529.0126	726.2999
	Experiment	**73.38**	**144.30**	**301.10**	**526.50**	**723.81**
	FEM	74.8296	146.6345	304.9121	530.7197	729.8934
Single crack	TMM	72.4574	143.6455	294.4666	519.7713	721.3785
	Experiment	**72.31**	**143.70**	**294.40**	**517.56**	**714.88**
Double cracks	TMM	71.4113	143.0440	287.3652	491.1691	680.994
	Experiment	**71.63**	**143.70**	**290.90**	**493.06**	**689.81**
Triple cracks	TMM	70.9058	142.6157	285.9632	482.6887	671.1254
	Experiment	**71.06**	**142.90**	**287.50**	**480.06**	**674.69**
Single crack at 450 mm; Double cracks at 200; 450 mm; Triple cracks at 200; 450; 800 mm from the left end of beam and all the cracks are of equal depth 40 %.						

Measurements of natural frequencies have been performed at 7 points (see Fig. 7) and measured data are processed accordingly to that procedure presented in section 4.

In Table 3 there are depicted five lowest natural frequencies calculated and measured for three crack scenarios described in the last row of the table. The results show that discrepancy between calculated and measured natural frequencies is within 2 %. So the theoretical development proposed above in the sections 2 and 3 is thus experimentally validated and it can be surely used for analysis of crack effect on natural frequencies accomplished below.

5.3. Effect of crack position and depth

For analysis of crack effect on natural frequencies of stepped beam two types of the beam are investigated. The beam of first type called down-stepped (B1S) is shown in Fig. 3 that was examined also in the experiment. The other one has intermediate span of thickness greater than that of the end spans and this type is called up-stepped beam (B2S). Both the types of stepped beam investigated below are clamped at the ends and have the following configurations:

$$\text{B1S}: L_1 = L_2 = L_3 = 1m; b_1 = b_2 = b_3 = 0.1m; h_1 = h_3 = 0.15m; h_2 = 0.10m;$$

$$\text{B2S}: L_1 = L_2 = L_3 = 1m; b_1 = b_2 = b_3 = 0.1m; h_1 = h_3 = 0.10m; h_2 = 0.15m.$$

First, ratios of three lowest natural frequencies (cracked to intact) are computed for the beams with single crack of different depth from 10 % to 40 % and position running from the left to the right ends through the steps. Results are shown in Fig. 4 where the frequency ratios of beam B1 on the left and those of beam B2 on the right. Observing graphs presented in Fig. 4 allows one to make the following notations: (1) Likely to the uniform beam, there exist positions on the stepped beam crack occurred at which does not change a certain natural frequency. Such positions are called frequency node and they are given in Table 4 for uniform and stepped beams. Obviously, natural frequency nodes are located symmetrically about the beam middle for symmetric boundary conditions; (2) The frequency ratios undergo a jump when crack passing beam steps (this means discontinuity of frequency variation due to crack at beam steps). Expanse of the jumps is different for various modes and it is certainly dependent on height of the steps; (3) Natural frequencies, as well known, are monotonically decreasing with growing crack depth.

Table 4. Frequency nodes of five lowest modes for uniform and stepped beams.

Mode No	Stepped beam B_1	Uniform beam B_0	Stepped beam B_2
1	0.85-2.15	0.67-2.33	0.56-2.44
2	0.46-**<u>1.5</u>** -2.54	0.4 - **<u>1.5</u>** - 2.6	0.38 - **<u>1.5</u>** - 2.62
3	0.3-1.16-1.84-2.7	0.28-1.07-1.93-2.72	0.27 - 0.94 -2.06 - 2.73
4	0.24-0.94-**<u>1.5</u>**-2.06-2.76	0.22-0.83-**<u>1.5</u>**-2.17-2.76	0.21- 0.76 - **<u>1.5</u>** - 2.24-2.79
5	0.2-0.75-1.26-1.74-2.25-2.8	0.18-0.68-1.23-1.77-2.32-2.82	0.17-0.65-1.16-1.84-2.35-2.83
Total beam length L = 3 *m*; Span length $L_1 = L_2 = L_3 = 1$ *m*; Steps at 1 *m* and 2 *m*			

5.1. Effect of beam steps and crack position

In this subsection, aimed to study effect of steps (abrupt change in beam height), two uniform beams and two stepped beams (with changed both sizes of cross section) of the following geometry are investigated in addition to the stepped beams considered in the previous subsection.

$$\text{BU1}: L_1 = L_2 = L_3 = 1m; b_1 = b_2 = b_3 = 0.1m; h_1 = h_2 = h_3 = 0.15m;$$

$$\text{BU2}: L_1 = L_2 = L_3 = 1m; b_1 = b_2 = b_3 = 0.1m; h_1 = h_2 = h_3 = 0.10m;$$

$$\text{B3S}: L_1 = L_2 = L_3 = 1m; b_1 = 0.15; b_2 = 0.1; b_3 = 0.15m; h_1 = 0.15; h_2 = 0.1; h_3 = 0.15m;$$

$$\text{B4S}: L_1 = L_2 = L_3 = 1m; b_1 = 0.1; b_2 = 0.15; b_3 = 0.1m; h_1 = 0.1; h_2 = 0.15; h_3 = 0.10m$$

As the stepped beams B1S and B2S have uniform width, the beams B3S, B4S are stepped in both sizes of cross section (width and height). The first three frequency ratios computed in dependence on the crack position for the uniform and stepped beams are presented in Fig. 5.

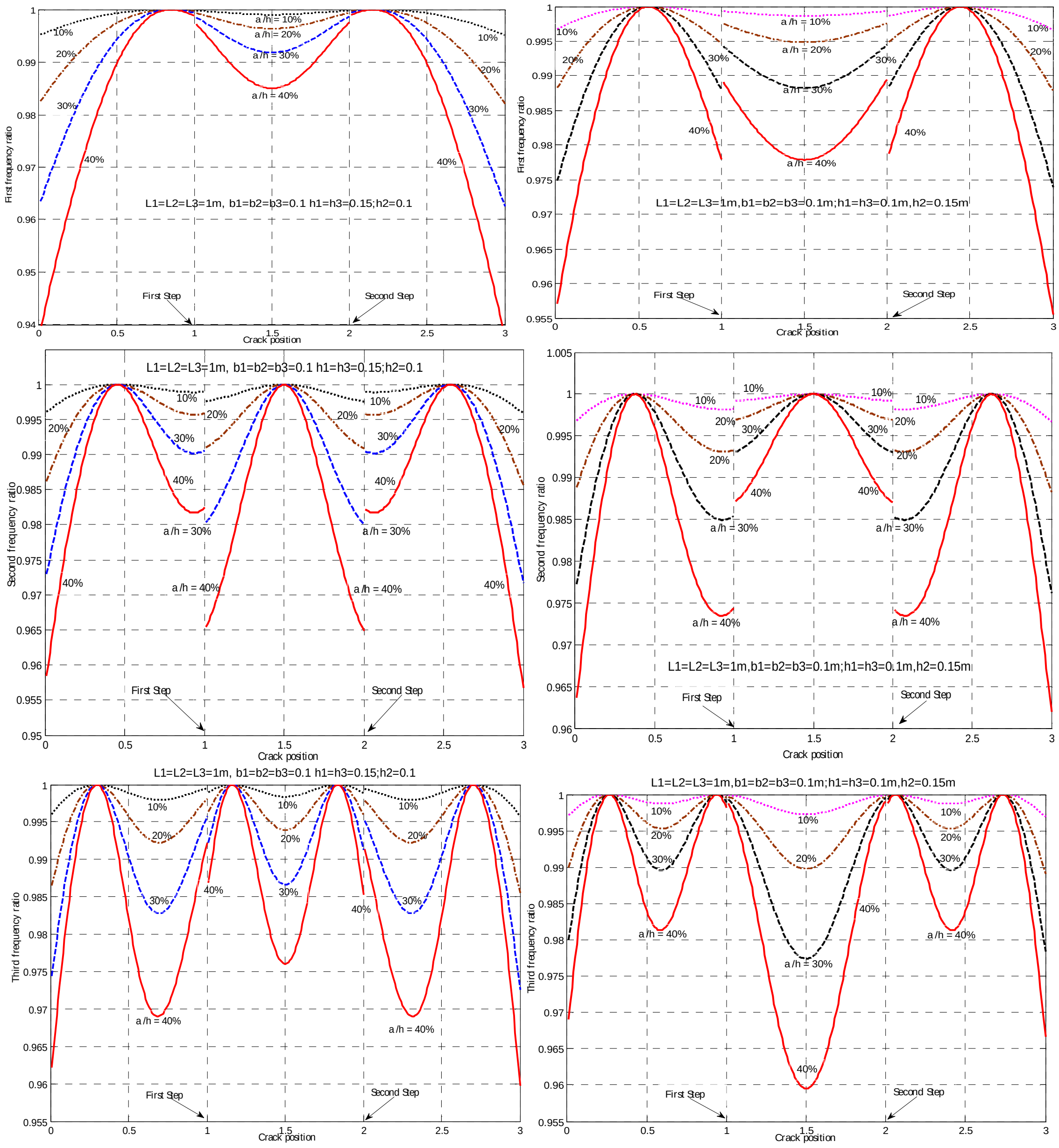

Figure 4. Effect of position and depth of crack on natural frequencies of beams B1 (left) and B2 (right).

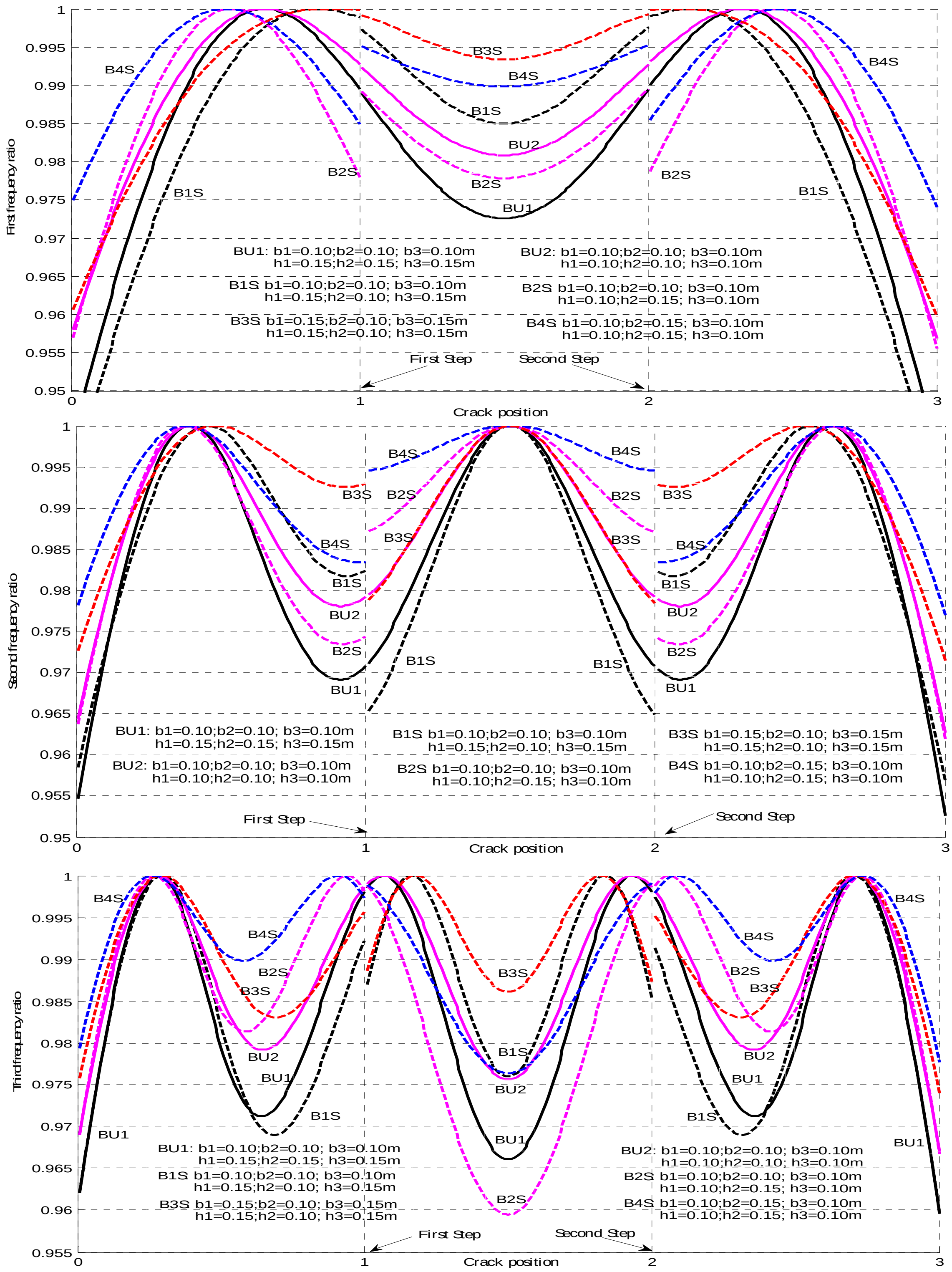

Figure 5. Effect of beam thickness variaton (steps) and crack position on natural frequencies.

Comparing the ratios computed for beam BU1 and BU2 allows one to find that increasing thickness of uniform beam makes all its natural frequencies more sensitive to crack. This highlights the well-known fact that more stiff beam is more sensitive to crack. However, the increasing or decreasing thickness of only mid-span in stepped beam leads to diminish or magnify the second frequency sensitivity to crack occurred at the span. So, steps in beam thickness may increase or decrease natural frequency sensitivity to crack in dependence on where crack is located and which frequency is considered. Graphs given in Fig.5 show also that frequency nodes of stepped-down beam (B1) are thrusted to the beam middle and the nodes are pulled away from the middle for stepped-up beam (B2). Nevertheless, the steps do not shift the node located at the beam middle and uniformly increasing thickness of uniform beam does not change the frequency nodes.

6. CONCLUSION

In the present paper a simplified version of the transfer matrix method has been developed for modal analysis of multiple cracked stepped beam based on an explicit expression of mode shape of multiple cracked uniform beam element. The simplification consists of that the beam state needs to be transferred only through steps of beam but not over the cracks as done in the earlier publications.

An experimental modal analysis of cracked multistep beam has been carried out and comparison of computed and measured natural frequencies demonstrated a good agreement of the theory with experiment.

Using the simplified TMM it was found that likely to the uniform beam there exist on beam positions crack appeared at which does not change a certain natural frequency. Such critical points on beam are called herein frequency nodes and it was shown that step-down shifts the nodes to the beam middle and step-up pulls them to the beam ends.

Finally, the performed modal analysis shows significant influence of steps on the natural frequency sensitivity to cracks and this is a useful indication for crack detection in stepped beam by measurement of natural frequencies.

Acknowledgement. This work was completed with financial support from NAFOSTED of Vietnam under Grant of number 107.01-2015.20.

APPENDIX

CALCULATION OF CRACK MAGNITUDE

The so-called crack magnitude introduced above is calculated as [16]

$$\gamma_0 = E_0 I / K_0 = 6\pi(1-\nu_0^2) h f_0(a/h); \quad \text{(A.1)}$$

$$f_0(z) = z^2(0.6272 - 1.04533z + 4.5948z^2 - 9.9736z^3 + 20.2948z^4 - 33.0351z^5 + \\ + 47.1063z^6 - 40.7556z^7 + 19.6z^8).$$

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