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Analysis and Design of One- and Two-Sided Cusum Charts with Known and Estimated Parameters

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ANALYSIS AND DESIGN OF ONE- AND TWO-SIDED CUSUM CHARTS WITH KNOWN AND ESTIMATED PARAMETERS

by

Martin Xavier Dunbar

(Under the Direction of Charles W. Champ)

ABSTRACT

The integral equation and Markov chain methods for analyzing the performances of one- and two-sided CUSUM $\overline{X}$ charts with known and estimated in-control process parameters are studied. Some new integral equations for analyzing the two-sided CUSUM $\overline{X}$ are derived. These methods provide us with ways to approximate the run length distribution of the chart. Since parameters of the run length distribution are commonly used measures of the performance of a control chart, it is important to choose an accurate approximation method. We develop some new Markov chain approximations using methods similar to the methods for approximating a solution to integral equations that describe the run length distribution.

INDEX WORDS: ARL, CUSUM, Integral Equations, Markov chain, Run Length

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by

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Table of Contents

LIST OF TABLES

LIST OF FIGURES

Page

Chapter 1

INTRODUCTION

The Shewhart $\overline{X}$ (with runs rules), cumulative sum (CUSUM), and exponentially weighted moving average (EWMA) charts are well know methods used by practitioners as aids in monitoring for a change in the mean of the distribution of a quality measurement. There are three methods that are commonly used to evaluate the run length performance of these charts: simulation, Markov chain, and integral equation approaches. Champ (1986) and Champ and Woodall (1987) showed how the Shewhart $\overline{X}$ chart with supplementary runs rules can be represented as a Markov chain. Page (1954) introduced one- and two-sided CUSUM $\overline{X}$ charts in which he used an integral equation approach to study the average run length (ARL) of the the one-sided charts. Brook and Evans (1972) gave a method for obtaining a Markov chain approximation to the CUSUM chart. The EWMA chart was introduced by Roberts (1959). A Markov chain approximation was used by Lucas and Saccucci (1990) to analyze the performance of this chart. Crowder (1987) gave some integral equations that are useful for evaluating the properties of the run length distribution of the EWMA chart using integral equations. Champ, Rigdon, and Scharnagl (2001) show how various integral equations are derived that are useful in control chart performance analysis. It is shown by Champ and Rigdon (1991) under what conditions the Markov chain approximation of Brook and Evans (1972) and integral equation methods are equivalent.

We will study the one- and two-sided CUSUM $\overline{X}$ charts of Page (1954) and the two-sided CUSUM $\overline{X}$ chart of Crosier (1986). Using the method presented in Champ, Rigdon, and Scharnagl (2001), we derive integral equations useful in analyzing the run length distribution of the two-sided CUSUM $\overline{X}$ chart of Crosier (1986). It is shown that if a certain method is used to approximate the solutions to these integral equations gives the same results as the Markov chain method used by Crosier (1986). The Markov chain approximation of the two-sided CUSUM $\overline{X}$ chart of Page (1954) is derived using the method presented in Woodall (1984) and implement this method in FORTRAN. A parameters estimated version of the two-sided CUSUM $\overline{X}$ chart of Crosier (1986) is presented.

Chapter 2

MODELING AND SAMPLING PROCEDURE

2.1 Introduction

Typically, statistical methods are designed for a given data model and sampling procedure. Control charts are no exception. The independent normal model is the most commonly assumed model with samples being collected periodically the most commonly assumed sampling procedure. Using a model for the data does not imply that the data follows this model nor is this of concern. The importance is how well the procedure performs for the practitioner. There are two ways to examine a statistical method's performance. One way is to design the method under one model and then study the robustness of the method under other models. Selecting the model under which the method is designed and the models to examine the robustness is part of the difficulty in studying the performance of the method. The second way to study the performance of a procedure is in actual practice. The acceptance of the Shewhart control chart is due in large part to how it has been perceived to perform in practice. When the quality measurement is a continuous measurement, the independent normal model for the design of the charts has produced charts that work exceptionally well in practice. This also holds for other control charts.

In this chapter, we will examine the independent normal model. It is not our intent to study the robustness of control charts. The sampling method under which we study our methods assumes the practitioner obtains information about the process in the form of the quality meausurements on independent random samples of size n taken periodically from the output of the process. A question often asked by the practitioner is what should be the value of n and in some cases how many samples should be taken? These questions will not be studied in this thesis, but we will provided results that will be useful in answering these questions.

Other sampling methods, include among others; those that vary the sample size, those that vary the sampling intervals, and those that may only look at part of the sample in order to make a decision about the process. Charts that used these sampling methods are referred to as variable sampling size (VSS), variable sampling interval (VSI), and multiple sampling control charts, respectively. For each of these charts, sample data is used to decide on the sample size, the interval, and how much of the sample to measure. In general, when data is used to determine what chart will be used at the next sampling stage to aid the practitioner in making a decision about the quality of the process, these charts are referred to as adaptive control charts.

2.2 Meaning of In- and Out-of-Control

In the first phase (Phase I) of the process, it is of interest to bring the process into a state of statistical in-control. Shewhart (1931) describes two causes of variability, "natural" and "assignable." Natural causes of variability are inherent in the process and cannot be removed. A process that is operating with only natural causes of variability can only be improved by redesigning the process. Assignable causes of variability can be removed and when removed, the quality of the outputted items is improved. A process that is operating with only natural causes of variability is said to be in a state of statistical in-control and we will simply say the process is in-control. When an assignable cause(s) is present, we will classify the process as being out-of-control. For the normal model, we assume that the process is in-control when $\mu = \mu_0$ and $\sigma = \sigma_0$ for fixed values μ_0 and σ_0. It will be convenient to consider the parameters $\delta = (\mu - \mu_0)/\sigma_0$ and $\lambda = \sigma/\sigma_0$. Writing $\mu = \mu_0 + \delta\sigma_0$, we see that δ is the number of in-control standard deviation σ_0 that μ has shifted from μ_0. Observing that $\sigma = \lambda\sigma_0$, we see that λ is the proportion of σ_0 that the standard deviation σ has shifted from σ_0. It is easy to see that the process is in-control if $\delta = 0$ and $\lambda = 1$. It is our interest to study CUSUM charting procedures that are used in the second phase (Phase II) of the process. In this phase, it is assumed the process is in-control and it is desirable to monitor the process to detect a change from in-control to out-of-control.

2.3 Sampling Method

The random sample is the most commonly used sampling method. Under the assumption that the process is producing items in which the quality measurements are independent, if one further assumes that these quality measurements are identically distributed, then the quality measurements on any n items are independent and identically distributed (iid). To obtain information about the quality of the process, we will periodically select n items from the output of the process and take the quality measurement X on each. The measurements on the n individuals in the tth sample will be denoted by $X_{t,1}, \ldots, X_{t,n}$. Periodic here may indicate the number of items produced or the time between samples is constant.

Other sampling methods discussed in the literature vary the sample size, the number of items or time between samples, or the rule used by the chart to signal. Champ (1986) stated "It is generally the case that one set of rules is selected and only this set is used. The quality of the product may be more appropriately monitored if periodically the runs rules are changed. This can be done by applying less stringent rules if, after a fixed number of observations, the process appears to be in control. It may also be the case that, if the process reaches a point where it appears to be going out-of-control more often between process corrections, more stringent rules may need to be applied."

2.4 Independent Normal Model

The most widely assumed process model is the independent normal model. Under this model, the distribution of the quality measurement X is a normal distribution with mean μ and standard deviation σ. Further, the quality measurements on any two items of output are assumed to be independent. We can express the probability density function (pdf) $f_X(x)$ and the cumulative distribution function (cdf) $F_X(x)$

as

$$f_X(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) \text{ and } F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

where

$$\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2} \text{ and } \Phi(z) = \int_{-\infty}^{z}\phi(u)\,du.$$

The functions $\phi(z)$ and $\Phi(z)$ are, respectively, the pdf and cdf of a random variable Z that has a (standard) normal distribution with mean 0 and variance 1.

2.5 Estimating the Process Mean

We assume Phase I data will be available that is believed to be from an in-control process in the form of m independent random samples each of size n. The measuresments on these sampled items will be represented by $\{X_{i,1},\ldots,X_{i,n}\}$, $i=1,\ldots,m$ and used to estimate the in-control mean μ_0 and in-control standard deviation σ_0. When $n=1$, we will express $X_{i,j}$ as X_i. The most common used estimator for μ_0 is

$$\widehat{\mu}_0 = \overline{\overline{X}} = \frac{1}{mn}\sum\nolimits_{i=1}^{m}\sum\nolimits_{j=1}^{n}X_{i,j} = \frac{1}{m}\sum\nolimits_{i=1}^{m}\overline{X}_i.$$

For the case in which $n=1$, we have that $\overline{X}_i = \overline{\overline{X}}$. The random variable $\overline{\overline{X}}$ is sometimes referred to as the grand mean. Under the independent normal model, one can show that $\overline{\overline{X}}$ has a normal distribution with mean μ_0 and standard deviation $\sigma_0/\sqrt{mn}$. It will be convenient to denote the standardized value of $\overline{\overline{X}}$ by

$$Z_0 = \frac{\overline{\overline{X}}-\mu_0}{\sigma_0/\sqrt{mn}}.$$

Under the independent normal model, the random variable Z_0 has a standard normal distribution. The statistic $\overline{\overline{X}}$ will be used in this thesis to estimate μ_0.

Various linear combinations of the sample data or their order statistics have been suggested as estimators for the mean (see David (1981)). The sample mean is one example. Others include the sample median, the trimmed means, and average of the smallest and largest of the sample. In general, an estimator $\widehat{\mu}_0$ for μ_0 is expressed as

$$\widehat{\mu}_0 = \frac{1}{m}\sum\nolimits_{i=1}^{m}\widehat{\mu}_{i,0}, \text{ with } \widehat{\mu}_{i,0} = \sum\nolimits_{j=1}^{n}a_{j:n}X_{i,j:n}$$

where $X_{i,1:n},\ldots,X_{i,n:n}$ represent the order statistics of the sample data $X_{i,1},\ldots,X_{i,n}$. For the sample mean, $a_{j:n}=1/n$ for all values of i. To obtain the sample median, one sets $a_{n/2:n}=a_{n/2+1:n}=1/2$ and $a_{i:n}$ is zero otherwise for an even sample size. For n odd, one choses $a_{(n+1)/2:n}=1$ and $a_{j:n}=0$ otherwise. An example of a (weighted) trimmed mean is

$$\widehat{\mu}_{i,0}=\overline{X}_{i,Trimmed}=\sum\nolimits_{j=2}^{n-1}a_{j:n}X_{i,j:n}.$$

The average of the smallest and largest sets $a_{1:n}=a_{n:n}=1/2$ with $a_{j:n}=0$ for $j=2,\ldots,n-1$.

The criteria used to select the estimator $\widehat{\mu}_0$ is often based on the estimator having certain distributional proporties, as well as, nonstatistical reasons such as easy of use and interpretation. Distributional properties that are typically considered are (1) unbiasedness, (2) precision, and (3) robustness. For example, choosing the $a_{j:n}$'s such that

$$\sum\nolimits_{j=1}^{n}a_{j:n}=1\text{ and }\sum\nolimits_{j=1}^{n}a_{j:n}E\left(Z_{i,j:n}\right)=0$$

yields unbiased estimators $\widehat{\mu}_{i,0}$ and $\widehat{\mu}_0$ of μ_0, where $Z_{i,j:n}$ is the standardized form of $X_{i,j:n}$.One choice is to set $a_{j:n}=1/n$ for $j=1,\ldots,n$.

2.6 Estimating the Process Standard Deviation

Several estimators for σ_0 have been discussed in the literature. In the quality control, literature, the most commonly used estimators for σ_0 are functions of the sample range R and the sample standard deviation S. The sample range is defined by

$$R_i=X_{i,n:n}-X_{i,1:n}.$$

We see that

$$R_i=\left(\mu+\sigma Z_{i,n:n}\right)-\left(\mu+\sigma Z_{i,1:n}\right)=\sigma\left(Z_{i,n:n}-Z_{i,1:n}\right).$$

The statisics $Z_{i,n:n}-Z_{i,1:n}$ is called the standardized range. The mean and standard deviation of its distribution are expressed by $d_2=d_2\left(n\right)$ and $d_3=d_3\left(n\right)$, respectively, which are functions of the sample size n. Under the independent normal model, the mean μ_R and standard deviation σ_R of the distribution of the sample range R are

$$\mu_R = d_2\sigma \text{ and } \sigma_R = d_3\sigma.$$

The values of $d_2 = d_2(n)$ and $d_3 = d_3(n)$ are tabled in Harter (1961). Dividing R_i by d_2 yields an unbiased estimator of σ_0. The standard deviation of R_i/d_2 is $(d_3/d_2)\,\sigma_0$. One commonly used pooled unbiased estimator of σ_0 based on the sample ranges is $\overline{R}/d_2$, where

$$\overline{R} = \frac{1}{m}\sum\nolimits_{i=1}^{m} R_i.$$

The standard deviation of $\overline{R}/d_2$ is given by

$$\sigma_{\overline{R}/d_2} = \frac{1}{\sqrt{m}}\frac{d_3}{d_2}\sigma_0.$$

The sample standard deviation S_i is defined by

$$S_i = \sqrt{\frac{1}{n-1}\sum\nolimits_{j=1}^{n}\left(X_{i,j} - \overline{X}_i\right)^2}.$$

The mean and standard deviation of the distribution of the sample standard deviation are

$$\mu_S = c_4\sigma_0 \text{ and } \sigma_S = \sqrt{1-c_4^2}\sigma_0, \text{ with } c_4 = c_4(n) = \frac{\sqrt{2}\Gamma\left(\frac{n}{2}\right)}{\sqrt{n-1}\Gamma\left(\frac{n-1}{2}\right)}.$$

It is easy to see that S_i/c_4 is an unbiased estimator of σ_0 having standard deviation

$$\sigma_{S_i/c_4} = \sqrt{\frac{1-c_4^2}{c_4^2}}\sigma_0.$$

A pooled unbiased estimator of σ_0 based on the sample standard deviations is $\overline{S}/c_4$, where

$$\overline{S} = \frac{1}{m}\sum\nolimits_{i=1}^{m} S_i.$$

The standard deviation of this estimator is

$$\sigma_{\overline{S}/c_4} = \frac{1}{\sqrt{m}}\sqrt{\frac{1-c_4^2}{c_4^2}}\sigma_0.$$

A pooled unbiased estimator of σ_0 based on the sample variances is $S_p/c_{4,m}$, where

$$S_p^2 = \frac{1}{m}\sum_{i=1}^{m} S_i^2 \text{ and } c_{4,m} = c_4\left(m\left(n-1\right)\right) = \frac{\sqrt{2}\Gamma\left(\frac{m(n-1)+1}{2}\right)}{\sqrt{m\left(n-1\right)}\Gamma\left(\frac{m(n-1)}{2}\right)}.$$

Under the independent normal model, the random variable $m\left(n-1\right)S_p^2/\sigma_0^2$ has a chi square distribution with $m\left(n-1\right)$ degrees of freedom. It then follows that the variance of this estimator is given by

$$\sigma_{V_0/c_{4,m}} = \sqrt{\frac{1-c_{4,m}^2}{c_{4,m}^2}}\sigma_0.$$

It can be demonstrated that

$$\sigma^2_{S_p^2/c_{4,m}} \leq \sigma^2_{\overline{S}/c_4} \leq \sigma^2_{\overline{R}/d_2}.$$

Further, each of the estimators $\overline{R}$, $\overline{S}$, and V are stochastically independent of the sample mean $\overline{X}$ as well as the grand mean $\overline{\overline{X}}$. The estimator $S_p/c_{4,m}$ will be used in this thesis to provide unbiased estimates of σ_0.

Linear estimators of σ_0 have the form

$$\widehat{\sigma}_{i,0} = \sum_{j=1}^{n} b_{j:n}X_{i,j:n}$$

with pooled estimator

$$\widehat{\sigma}_0 = \frac{1}{m}\sum_{i=1}^{m} \widehat{\sigma}_{i,0}.$$

One can obtain unbiased estimators $\widehat{\sigma}_{i,0}$ and $\widehat{\sigma}_0$ by selecting the $b_{j:n}$'s such that

$$\sum_{j=1}^{n} b_{j:n} = 0 \text{ and } \sum_{j=1}^{n} b_{j:n}E\left(Z_{i,j:n}\right) = 1.$$

One choice is to set

$$b_{j:n} = \frac{E\left(Z_{i,j:n}\right)}{\sum_{j=1}^{n}\left[E\left(Z_{i,j:n}\right)\right]^2}.$$

Little information is available in the literature about the distributions of these estimators of σ_0.

2.7 Conclusion

The model and sampling method for the process data were discussed. The meanings of in- and out-of-control processes were defined in terms of the model and a reparameterization of the model. Various methods from the literature were discussed for estimating the parameters of the model. In this thesis, the estimators $\overline{\overline{X}}$ and $S_p/c_{4,m}$ will be used as estimators for μ_0 and σ_0, respectively. The distributions of these estimators for the in-control mean and standard deviation were given. These results will be useful in analyzing the run length distribution of the CUSUM $\overline{X}$ charts with estimated parameters.

Chapter 3

CUSUM CHARTS FOR MONITORING THE PROCESS MEAN

3.1 Introduction

Various CUSUM charts have been introduced in the literature beginning with Page (1954). In this chapter, we will discuss the one- and two-sided CUSUM $\overline{X}$ charts of Page (1954) and the two-sided CUSUM $\overline{X}$ chart of Crosier (1986). Each of these are special cases of a family of cumulative sum type charts described by Champ, Woodall, and Mohsen (1991). We also discuss the method of Healy (1987) for deriving the one-sided CUSUM $\overline{X}$ charts of Page (1954). An example is given illustrating how a practitioner would used these charts.

3.2 CUSUM $\overline{X}$ Charts of Page (1954)

In his seminal paper, Page (1954) introduced the CUSUM quality control chart for monitoring the process mean. The tabular form of the CUSUM chart was developed by Ewan and Kemp (1960). Jones, Champ, and Rigdon (2004) expressed in tabular form the statistic for an upper one-sided CUSUM $\overline{X}$ chart with estimated parameters for detecting an increase in the process mean as

$$C_t^+ = \max\left\{0, C_{t-1}^+ + \frac{\overline{X}_t - \widehat{\mu}_0}{\widehat{\sigma}_0/\sqrt{n}} - k^+\right\}$$

with $C_0^+ = 0$. Here, the statistics $\widehat{\mu}_0$ and $\widehat{\sigma}_0$ are estimators of μ_0 and σ_0, respectively. The chart is a plot of the points $\left(t, C_t^+\right)$ for $t = 1, 2, \ldots$. The chart signals a potential out-of-control process at sampling stage t if $C_t^+ > h^+$, where $h^+ \geq 0$. The associated lower sided CUSUM $\overline{X}$ chart for detecting decreases in the mean plots C_t^- versus t, where

$$C_t^- = \min\left\{0, C_{t-1}^- + \frac{\overline{X}_t - \widehat{\mu}_0}{\widehat{\sigma}_0/\sqrt{n}} + k^-\right\}$$

with $C_0^- = 0$. The lower one-sided CUSUM $\overline{X}$ chart signals at sampling stage t if $C_t^- < h^-$ with $h^- \leq 0$. The combined plot of the points $\left(t, C_t^-\right)$ and $\left(t, C_t^+\right)$ for $t = 1, 2, \ldots$ will be referred to as a two-sided CUSUM $\overline{X}$ chart. The chart parameters k^-

and k^+ are often referred to as "reference" values and the chart parameters h^- and h^+ are called the control limits of the charts. One can see that the CUSUM $\overline{X}$ chart (with estimated parameters) reduces to the Shewhart $\overline{X}$ chart (with estimated parameters) if both h^- and h^+ are set to zero. Selecting $0 < C_0^+ < h^+$ (or $h^- < C_0^- < 0$) gives the charts that Lucas and Crosier (1982) referred to as a "head-start." A head-start increases the charts ability to detect a process that is initially out-of-control.

3.3 Healy's Derivation

Healy (1987) developed the CUSUM chart for monitoring a process mean under the assumption that μ_0 and σ_0 are known. For the upper one-sided CUSUM $\overline{X}$ chart, let $\mu_1^+ > \mu_0$ be a value of the process mean when the process is out-of-control for which the practitioner is interested in detecting quickly. According to Healy (1987), the upper one-sided CUSUM $\overline{X}$ chart can be expressed as

$$C_t^{+(*)} = \max\left\{0, C_{t-1}^{+(*)} + \log\left(\frac{f_{\overline{X}}\left(\overline{X}_t \,|\mu = \mu_1^+, \sigma_0\right)}{f_{\overline{X}}\left(\overline{X}_t \,|\mu = \mu_0, \sigma_0\right)}\right)\right\}.$$

The chart signals at sampling stage t if $C_t^{+(*)} > h^{+(*)}$. Note that we can write

$$\log\left(\frac{f_{\overline{X}}\left(\overline{X}_t \,|\mu = \mu_1^+, \sigma_0\right)}{f_{\overline{X}}\left(\overline{X}_t \,|\mu = \mu_0, \sigma_0\right)}\right) = \frac{\mu_1^+ - \mu_0}{\sigma_0/\sqrt{n}}\left(\frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} - k^+\right)$$

where $k^+ = 0.5\,(\mu_1 - \mu_0)\,/\,(\sigma_0/\sqrt{n})$. Defining

$$C_t^+ = \frac{\sigma_0/\sqrt{n}}{\mu_1^+ - \mu_0} C_t^{+(*)} \text{ and } h^+ = \frac{\sigma_0/\sqrt{n}}{\mu_1^+ - \mu_0} h^{+(*)}$$

then

$$C_t^+ = \max\left\{0, C_{t-1}^+ + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} - k^+\right\}.$$

In a similar way, the lower one-sided CUSUM $\overline{X}$ chart with known parameters can be derived with plotted statistic C_t^- given by

$$C_t^- = \min\left\{0, C_{t-1}^+ + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} + k^-\right\},$$

where $k^- = 0.5\left(\mu_0 - \mu_1^-\right)/\left(\sigma_0/\sqrt{n}\right)$ with $\mu_1^- < \mu_0$.

3.4 Crosier's Two-Sided CUSUM $\overline{X}$ Chart

Crosier (1986) introduced a CUSUM chart for monitoring the mean of a quality measurement. First, he discusses the one-sided CUSUM $\overline{X}$ charts of Page (1954). He expresses the plotted statistics in a way that shows how the statistics are shrunk toward zero by multiplying by factors between zero and one. In particular, the plotted statistic for the upper one-sided CUSUM $\overline{X}$ of Page (1954) is expressed as the the statistic $C_t^{+(*)}$, where $C_t^{+(*)}$ is determined by first calculating

$$A_t^{+(*)} = C_{t-1}^{+(*)} + \overline{X}_t - \mu_0$$

and then determining $C_t^{+(*)}$ by

$$C_t^{+(*)} = \begin{cases} 0, & A_t^{+(*)} \leq k^+\sigma_0/\sqrt{n}; \\ \left(A_t^{+(*)}\right)\left(1 - k^+\left(\sigma_0/\sqrt{n}\right)/A_t^{+(*)}\right), & A_t^{+(*)} > k^+\sigma_0/\sqrt{n}. \end{cases}$$

with $C_0^{+(*)} = 0$ and $k^+ > 0$. The chart signals at the first sampling stage t such that $C_t^{+(*)} > h^+\sigma_0/\sqrt{n}$, where $h^+ \geq 0$. For an equivalent form of this chart, we define

$$A_t^+ = \left(\sqrt{n}/\sigma_0\right) A_t^{+(*)} \text{ and } C_t^+ = \left(\sqrt{n}/\sigma_0\right) C_t^{+(*)}.$$

It follows that

$$\begin{aligned} A_t^+ &= C_{t-1}^+ + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}}; \\ C_t^+ &= \begin{cases} 0, & A_t^+ \leq k^+; \\ \left(A_t^+\right)\left(1 - k^+/A_t^+\right), & A_t^+ > k^+. \end{cases} \end{aligned}$$

with $C_0^+ = 0$ and $k^+ > 0$. A signal is given if $C_t^+ > h^+$.

It is not difficult to see that the upper one-sided CUSUM chart of Crosier (1986) is equivalent to the one-sided tabular form of the CUSUM $\overline{X}$ chart of Page (1954) as given by Ewan and Kemp (1960). For the upper one-sided chart $C_t^+ = 0$ provided $A_t^+ \leq k^+$ or equivalently if

$$C_{t-1}^+ + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} - k^+ \leq 0.$$

If $A_t^+ > k^+$ or equivalently if

$$C_{t-1}^+ + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} - k^+ > 0,$$

then

$$C_t^+ = \left(A_t^+\right)\left(1 - k^+/A_t^+\right) = A_t^+ - k^+ = C_{t-1}^+ + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} - k^+.$$

This is equivalent to defining

$$C_t^+ = \max\left\{0, C_{t-1}^+ + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} - k^+\right\}$$

which is the statistic of the one-sided CUSUM $\overline{X}$ described by Ewan and Kemp (1960).

Although Crosier (1986) did not give a lower one-sided version of his upper one-sided chart, we give a version here. We define the lower one-sided CUSUM chart statistic $C_t^{-(*)}$ by first defining

$$A_t^{-(*)} = C_{t-1}^{-(*)} + \overline{X}_t - \mu_0$$

and then defining

$$C_t^{-(*)} = \begin{cases} 0, & A_t^{-(*)} \geq k^-\left(\sigma_0/\sqrt{n}\right); \\ \left(A_t^{-(*)}\right)\left(1 - k^-\left(\sigma_0/\sqrt{n}\right)/A_t^{-(*)}\right), & A_t^{+(*)} < k^-\left(\sigma_0/\sqrt{n}\right). \end{cases}$$

with $C_0^{-(*)} = 0$. The chart signals a potential out-of-control process at sampling stage t if $C_t^{-(*)} < h^-\left(\sigma_0/\sqrt{n}\right)$, where $h^- \leq 0$. An equivalent form sets

$$A_t^- = \left(\sqrt{n}/\sigma_0\right) A_t^{-(*)} \text{ and } C_t^- = \left(\sqrt{n}/\sigma_0\right) C_t^{-(*)}.$$

One can show that this chart is equivalent to the lower one-sided CUSUM $\overline{X}$ described by Ewan and Kemp (1960).

Crosier (1986) however defined a two-sided CUSUM chart so that a single statistic is ploted versus the sample number t. His two-sided CUSUM chart is equivalent to the CUSUM $\overline{X}$ chart that plots the statistic C_t^* versus t that is obtained by first computing

$$
\begin{aligned}
A_t^* &= \left|C_{t-1}^* + \overline{X}_t - \mu_0\right| \text{ followed by} \\
C_t^* &= \begin{cases} 0, & A_t^* \leq k\left(\sigma_0/\sqrt{n}\right); \\ \left(C_{t-1}^* + \overline{X}_t - \mu_0\right)\left(1 - k\left(\sigma_0/\sqrt{n}\right)/A_t^*\right), & A_t^* > k\left(\sigma_0/\sqrt{n}\right). \end{cases}
\end{aligned}
$$

with $C_0^* = 0$ and $k > 0$. A more general version of the CUSUM chart of Crosier (1986) signals at sampling stage t if $C_t^* < -h\left(\sigma_0/\sqrt{n}\right)$ or $C_t^* > h\left(\sigma_0/\sqrt{n}\right)$ with $h \geq 0$. An equivalent form of this chart is a plot of the points (t, C_t), where one first computes

$$
\begin{aligned}
A_t &= \left|C_{t-1} + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}}\right| \text{ followed by} \\
C_t &= \begin{cases} 0, & A_t \leq k; \\ \left(C_{t-1} + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}}\right)\left(1 - k/A_t\right), & A_t > k. \end{cases}
\end{aligned}
$$

with $C_0^* = 0$ and $k > 0$. The chart signals at sampling stage t if $C_t < h^-$ or $C_t > h^+$. This chart can also be expressed as

$$
C_t = \begin{cases} 0, & \text{if } \left|C_{t-1} + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}}\right| \leq k; \\ C_{t-1} + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} + k, & \text{if } C_{t-1} + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} < -k; \\ C_{t-1} + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} - k, & \text{if } C_{t-1} + \frac{\overline{X}_t - \mu_0}{\sigma_0/\sqrt{n}} > k. \end{cases}
$$

The estimated parameters versions of these charts replace the in-control values μ_0 and σ_0 with their respective estimates $\widehat{\mu}_0$ and $\widehat{\sigma}_0$.

In order to analyze the performance of a chart, we observe that we can express

$$
Y_t = \frac{\overline{X}_t - \widehat{\mu}_0}{\widehat{\sigma}_0/\sqrt{n}} = V_0^{-1}\left(\lambda Z_t + \sqrt{n}\delta - Z_0/\sqrt{m}\right),
$$

where

$$
\delta = \frac{\mu - \mu_0}{\sigma_0},\ \lambda = \frac{\sigma}{\sigma_0},\ Z_t = \frac{\overline{X}_t - \mu}{\sigma/\sqrt{n}},\ Z_0 = \frac{\widehat{\mu}_0 - \mu_0}{\sigma_0/\sqrt{mn}}, \text{ and } V_0 = \frac{\widehat{\sigma}_0}{\sigma_0}.
$$

If μ_0 is known and used in place of $\widehat{\mu}_0$, this is equivalent to setting $Z_0 = 0$. Similarly, if σ_0 is known and used in place of $\widehat{\sigma}_0$, this is equivalent to setting $V_0 = 1$. Under the normal model, the random variable Z_t has a standard normal distribution. Further, if the average of the sample means and the average of the sample variances of m independent random samples each of size n are used respectively to estimate μ_0 and σ_0^2, then Z_0 has a standard normal distribution and $m\,(n-1)\,V_0^2$ has a chi square distribution with $m\,(n-1)$ degrees of freedom.

We observe that the joint distribution of $Y_1, \ldots, Y_t$ depends on the process parameters δ and λ, the number of preliminary samples m, the sample size n, and the statistics Z_0 and V_0. It then follows that the joint distributions of

$$C_1^-, \ldots, C_t^-, C_1^+, \ldots, C_t^+,\ C_1, \ldots, C_t$$

depend on these values as well as the chart parameters k^-, k^+, h^-, and h^+. The process determines the values of δ and λ and the variability in Z_0 and V_0. The practitioner selects the values m, n, k^-, h^-, k^+, and h^+. One often discussed method for selecting the chart parameters m, n, k^-, h^-, k^+, and h^+ is based on the distribution of the run length.

3.5 Example

Montgomery (1997) gives an example of a forging process that is producing piston rings for automotive engines. One quality measurement of interest is the inside diameter X of a piston ring. He gives $m = 25$ independent samples each of size $n = 5$ that was collected initially from the output of the production process. The Phase I $\overline{X}$ and R charts were used as aids by the practitioner to infer that these data were measurements on output from an in-control process. The data is then used to estimate the in-control mean μ_0 and in-control standard deviation σ_0 as

$$\widehat{\mu}_0 = \overline{\overline{x}} = \frac{1}{25}\sum_{i=1}^{25}\overline{x}_i = \frac{1850.028}{25} = 74.001256, \text{ and}$$

$$\widehat{\sigma}_0 = \frac{s_p}{c_{4,25}} = \sqrt{\frac{1}{25}\sum\nolimits_{i=1}^{25} s_i^2}/c_{4,25} = 0.009912678176.$$

The unbiasing constant $c_{4,25}$ is

$$c_{4,25} = \frac{\sqrt{2}\Gamma\left(\frac{25(5-1)+1}{2}\right)}{\sqrt{25\,(5-1)}\Gamma\left(\frac{25(5-1)}{2}\right)} = 0.9900524688.$$

Fifteen samples of size $n = 5$ are given by Montgomery (1997) taken from the process in Phase II (the monitoring phase). The statistics C_t^- and C_t^+, respectively, of the lower and upper CUSUM $\overline{X}$ chart of Page (1954) and the statistic C_t of the two-sided CUSUM $\overline{X}$ chart of Crosier (1986) are given in Table 3.1 along with the sample mean $\overline{X}_t$, for $t = 26, \ldots, 40$.

Table 1: CUSUM Statistics

t	$\overline{X}_t$	C_t^-	C_t^+	C_t
26	74.0086	0.0000	0.7409	0.7409
27	74.0022	0.0000	0.8361	0.8361
28	73.9922	−0.9136	0.0000	−0.0775
29	74.0036	−0.6771	0.2365	0.1590
30	73.7974	−21.2423	0.0000	−20.4062
31	74.2072	−0.4665	20.7758	0.3696
32	73.8056	−20.2044	1.0379	−19.3683
33	73.7978	−40.7293	0.0000	−39.8932
34	74.0112	−39.7261	1.0032	−38.8900
35	73.8126	−58.7579	0.0000	−57.9218
36	73.8040	−78.6572	0.0000	−77.8211
37	73.2166	−157.8141	0.0000	−156.9780
38	73.8196	−176.1397	0.0000	−175.3036
39	73.8234	−194.0820	0.0000	−193.2459
40	74.0128	−192.9174	1.1646	−192.0813

Figure 3.1 is a graph of the points $\left(t, \overline{X}_t\right)$ for $t = 26, \ldots, 40$ along with the charts lower (LCD) and upper (UCL) control limits given by

$$\begin{aligned} LCL &= 74.001256 - 2.7641\frac{0.9900524688}{\sqrt{5}} = 72.777 \text{ and} \\ UCL &= 74.001256 + 2.7641\frac{0.9900524688}{\sqrt{5}} = 75.225. \end{aligned}$$

The control limits are chosen using simulation so the chart has an in-control ARL of 200.

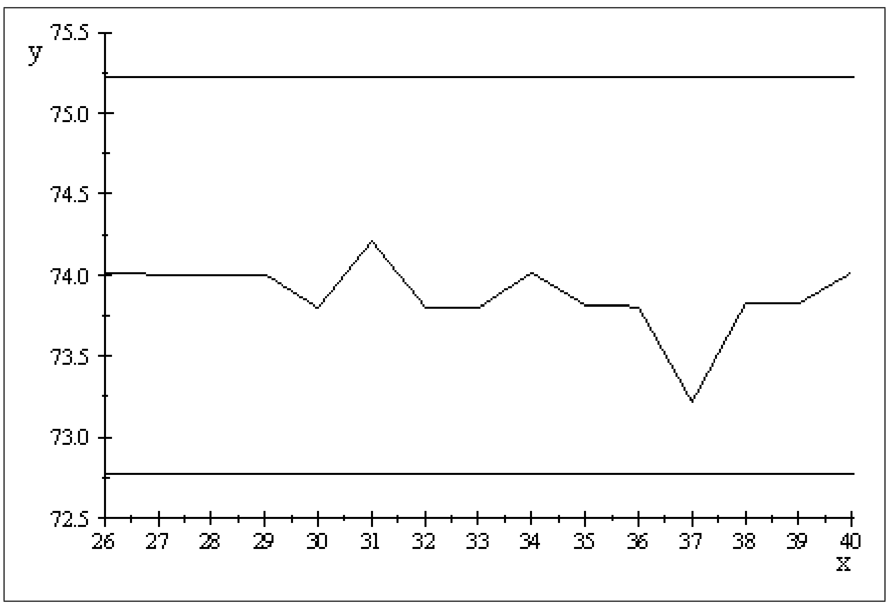

Figure 3.1: Phase II $\overline{X}$ Chart

The plot of the points $\left(t, C_t^-\right)$ and $\left(t, C_t^+\right)$ is illustrate in Figure 3.2 along with the charts lower (LCD) and upper (UCL) control limits, using random-number generation are given by

$$LCL = -4.3687 \text{ and } UCL = 4.3687.$$

up to the time the chart signals. These limits are chosen using simulation so the chart has an in-control ARL of 200. The chart first signal at time $t = 30$.

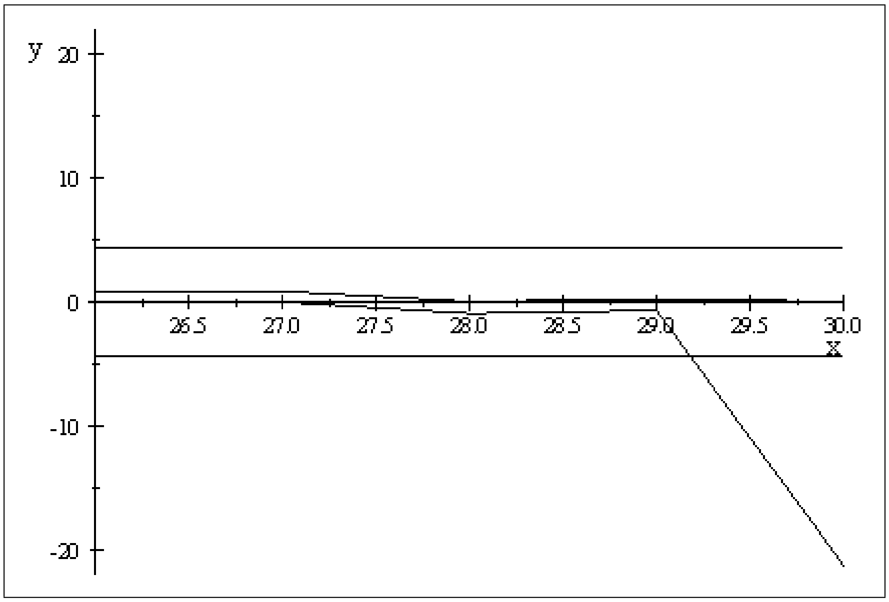

Figure 3.2. Two-Side CUSUM $\overline{X}$ Chart of Page (1954)

The plot of the statistic C_t versus t is given in Figure 3.3 up to the time the chart signals at time $t = 30$ with lower (LCL) and upper (UCL) control limits given by

$$LCL = -4.1445 \text{ and } UCL = 4.1445.$$

These limits are also plotted on the chart.

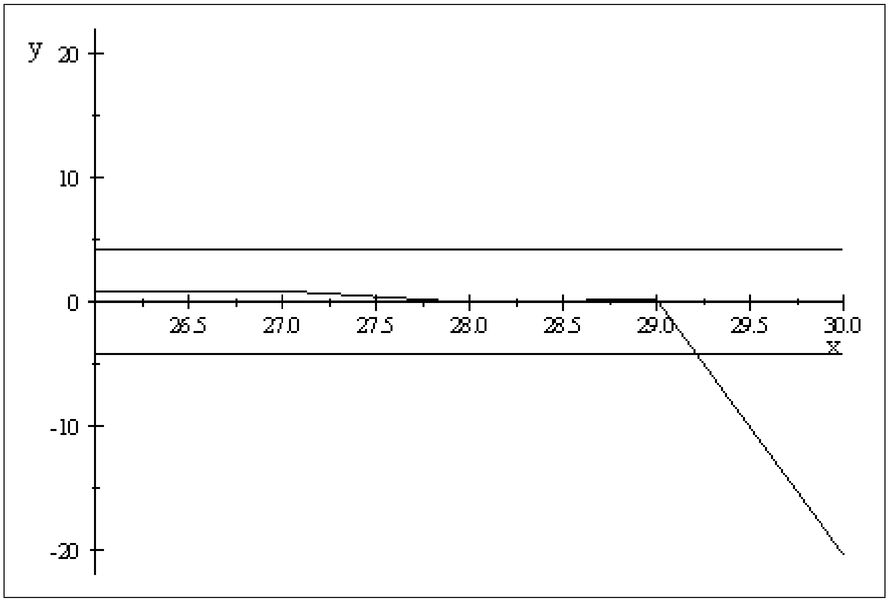

Figure 3.3. Two-Sided CUSUM $\overline{X}$ Chart of Crosier (1986)

The control limits for this chart were chosen using simulation so the chart would have an ARL of 200. The Shewhart $\overline{X}$ chart with 3-sigma limits does not indicate the mean has shifted whereas both the two-sided CUSUM $\overline{X}$ charts of Page (1954) and Crosier (1986) do.

3.6 Analysis of the Run Length Distribution

The run length of the chart is the number of samples taken in Phase II when the chart first signals a potential out-of-control process. Let T^- and T^+ be the run lengths of the lower and upper one-sided CUSUM $\overline{X}$ charts of Page (1954), respectively. Further, let T be the run length of the two-sided CUSUM $\overline{X}$ chart of Page (1954). It is not difficult to show that

$$T = \min\left\{T^-, T^+\right\}.$$

We propose to select the chart parameters k^-, h^-, k^+, and h^+ given m and n such that the run length distribution of the chart has the following properties: (1) The mean of the run length distribution, known as the average run length (ARL), will have a given value ARL_0 when the process is in-control. (2) For specified values of δ and λ, δ_1 and λ_1, the out-of-control ARL will be a minimum. Consequently, it will be one of our interest to determine the distributions of T^-, T^+, and T.

We will examine three methods for determining the run length distribution of a CUSUM chart. One of these methods is the Markov chain approach. This method is discussed in Brook and Evans (1972) for the one-sided CUSUM chart. The authors discuss a Markov chain as an exact representation of a CUSUM chart based on attribute data and as an approximation when the quality measurement is a continuous random variable. Woodall (1984) extended the method by Brook and Evans (1972) by giving a Markov chain representation of a two-sided CUSUM based on an attribute quality measurement. We extend the method in Woodall (1984) for CUSUM charts based on a continuous quality measurement.

Page (1954) gives an integral equation whose solution is the ARL of a one-sided CUSUM chart when the support of the quality measurement is the set of real numbers. Other integral equations useful in determining various properties of the run length distribution are discussed in van Dobben de Bruyn (1968). Champ, Rigdon, and Scharnagl (2000) give a method for deriving the integral equations useful in studying the run length distribution of various control charts. Champ and Ridgon (1991) showed that if a particular method is used to approximate the integral equations that describe the run length distribution of a CUSUM $\overline{X}$ the results are equivalent to the Markov chain approximation method of Brook and Evans (1972). We will show that any approximation method to the integral equations used to describe the run length distribution of one-sided CUSUM $\overline{X}$ charts will also provide an approximate Markov chain representation of the chart. The integral equation method is used to obtain integral equations useful in determining run length the run length distribution of two-sided CUSUM $\overline{X}$ chart of Crosier (1986). We will compare the Markov chain approximation presented by Crosier (1986) for his two-sided chart with the integral equation method.

3.7 Conclusion

Descriptions of the one- and two-sided CUSUM $\overline{X}$ of Page (1954) as expressed in tabular form by Ewan and Kemp (1960) and the two-sided CUSUM $\overline{X}$ of Crosier (1986) were discussed when parameters are known. These charts were also discussed when the in-control parameters are estimated. The method of Healy (1987) for deriving the one-sided CUSUM $\overline{X}$ was discussed. Equivalent forms of these charts are derived that are useful in analyzing the run length performance of the charts. These forms will be used in the following chapters in the derivation of methods for analyzing the run length distribution of the charts. For more discussion of CUSUM charts, see Hawkins and Olwell (1998).

Chapter 4

INTEGRAL EQUATION METHOD

4.1 Introduction

As previously stated, Page (1954) and van Dobben de Bruyn (1968) give various integral equations useful in determining the run length distribution of one-sided CUSUM $\overline{X}$ charts. Champ, Rigdon, and Scharnagl (2001) give a method for deriving these integral equations. We will use their method to derive various integral equations that will be useful in describing the run length distribution the two-sided CUSUM chart of Crosier (1986). An iterative method is used to describe the probability mass function of the run length distribution. We show how the tail probabilities of the run length distribution can be approximated using the results of Woodall (1983). Other integral equations are given whose exact solution are parameters of the run length distribution such as the ARL and standard deviation run length ($SDRL$). While the solution of these integral equations cannot be determined exactly, they can be approximated accurately. The approximation method using Gaussian quadrature is discussed.

4.2 One-Sided CUSUM Charts

For the upper one-sided CUSUM $\overline{X}$ chart of Page (1954), we will represent the probability that the run length equals t given that $C_0^+ = u^+$ by $pr\left(t\,|u^+\right)$. As will be seen, the probability $pr^+\left(t\,|u^+\right)$ is also a function of the process parameters δ and λ, the number of preliminary samples m, the sample size n, and the chart parameters k^+ and h^+. Using the method of Champ, Rigdon, and Scharnagl (2001), it can be show that

$$pr^+\left(t\,\middle|u^+\right) = \begin{cases} 1 - \Phi\left(\frac{v_0\left(h^+ - u^+ + k^+\right) - \zeta}{\lambda}\right), & t = 1 \\ pr^+\left(t-1\,|0\right)\Phi\left(\frac{v_0\left(0 - u^+ + k^+\right) - \zeta}{\lambda}\right) & \\ + \int_0^{h^+} pr^+\left(t-1\,\middle|c_1^+\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^+ - u^+ + k^+\right) - \zeta}{\lambda}\right)dc_1^+, & t \geq 2. \end{cases}$$

where $\zeta = \sqrt{n}\delta - z_0/\sqrt{m}$. Similar results hold for the lower one-sided CUSUM chart. In particular, the probability $pr^-(t\,|u^-)$ the lower one-sided CUSUM $\overline{X}$ of Page (1954) signals at time t given that $C_0^- = u^-$ is determined by

$$pr^-\left(t\left|u^-\right.\right) = \begin{cases} \Phi\left(\frac{v_0\left(h^- - u^- - k^-\right) - \zeta}{\lambda}\right), & t = 1 \\ pr^-\left(t-1\,|0\right)\Phi\left(\frac{v_0\left(0 - u^- - k^-\right) - \zeta}{\lambda}\right) & \\ + \int_{h^-}^{0} pr^-\left(t-1\left|c_1^-\right.\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^- - u^- - k^-\right) - \zeta}{\lambda}\right)dc_1^-, & t \geq 2. \end{cases}$$

It is convenient to express the average run length (ARL^+) of the upper one-sided CUSUM $\overline{X}$ chart of Page (1954) with $C_0^+ = u^+$ by $M^+(u^+)$. That is,

$$M^+\left(u^+\right) = \sum_{t=1}^{\infty} t pr^+\left(t\left|u^+\right.\right).$$

It can be shown using the method presented in Champ, Rigdon, and Scharnagl (2001) that the function $M^+(u^+)$ is the exact solution to the integral equation

$$\begin{aligned} M^+\left(u^+\right) &= 1 + M^+(0)\,\Phi\left(\frac{v_0\left(0 - u^+ + k^+\right) - \zeta}{\lambda}\right) \\ &\quad + \int_0^{h^+} M^+\left(c_1^+\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^+ - u^+ + k^+\right) - \zeta}{\lambda}\right)dc_1^+. \end{aligned}$$

This is similar to the integral equation for average run length given by Page (1954). Defining the function $M_2^+(u^+)$ by

$$M_2^+\left(u^+\right) = \sum_{t=1}^{\infty} t^2 pr^+\left(t\left|u^+\right.\right),$$

it can be shown using the method presented by Champ, Rigdon, and Scharnagl (2001) that the function $M_2^+(u^+)$ is a solution to the integral equation

$$\begin{aligned} M_2^+\left(u^+\right) &= 1 + 2M^+(0)\,\Phi\left(\frac{v_0\left(0 - u^+ + k^+\right) - \zeta}{\lambda}\right) \\ &\quad + 2\int_0^{h^+} M^+\left(c_1^+\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^+ - u^+ + k^+\right) - \zeta}{\lambda}\right)dc_1^+ \\ &\quad + M_2^+(0)\,\Phi\left(\frac{v_0\left(0 - u^+ + k^+\right) - \zeta}{\lambda}\right) \\ &\quad + \int_0^{h^+} M_2^+\left(c_1^+\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^+ - u^+ + k^+\right) - \zeta}{\lambda}\right)dc_1^+. \end{aligned}$$

Similarly for the lower one-sided CUSUM $\overline{X}$ chart of Page (1954) the average run length $M^{-}\left(u^{-}\right)$ and the expected value of the square of the run length $M_2^{-}\left(u^{-}\right)$ can be showed to be, respectively, the solutions of the integral equations

$$\begin{aligned} M^{-}\left(u^{-}\right) &= 1+M^{-}(0)\left[1-\Phi\left(\frac{v_0\left(0-u^{-}-k^{-}\right)-c}{\lambda}\right)\right] \\ &\quad +\int_{h^{-}}^{0} M^{-}\left(c_1^{-}\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^{-}-u^{-}-k^{-}\right)-\zeta}{\lambda}\right)dc_1^{-}. \end{aligned}$$

and

$$\begin{aligned} M_2^{-}\left(u^{-}\right) &= 1+2M^{-}(0)\left[1-\Phi\left(\frac{v_0\left(0-u^{-}-k^{-}\right)-\zeta}{\lambda}\right)\right] \\ &\quad +2\int_{h^{-}}^{0} M^{-}\left(c_1^{-}\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^{-}-u^{-}-k^{-}\right)-\zeta}{\lambda}\right)dc_1^{-} \\ &\quad +M_2^{-}(0)\,\Phi\left(\frac{v_0\left(0-u^{-}-k^{-}\right)-\zeta}{\lambda}\right) \\ &\quad +\int_{h^{-}}^{0} M_2^{-}\left(c_1^{-}\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^{-}-u^{-}-k^{-}\right)-\zeta}{\lambda}\right)dc_1^{-}. \end{aligned}$$

For integral equations whose solutions are other parameters of the run length distribution see van Dobben de Bruyn (1968) as well as Champ, Rigdon, and Scharnagl (2001). We note that the functions $M^{-}\left(u^{-}\right)$, $M_2^{-}\left(u^{-}\right)$, $M^{+}\left(u^{+}\right)$, and $M_2^{+}\left(u^{+}\right)$ are also functions of the process parameters δ and λ, the number of preliminary samples m, the sample size n, and the chart parameters k^{-}, h^{-}, k^{+}, and h^{+}.

4.3 Two-Sided CUSUM Charts

The integral equations describing the distribution of the run length $T = \min\left\{T^{-}, T^{+}\right\}$ of the two-sided CUSUM $\overline{X}$ chart of Page (1954) as described by Ewan and Kemp (1960) are not available. However, Kemp (1961) showed that

$$\frac{1}{ARL}=\frac{1}{ARL^{-}}+\frac{1}{ARL^{+}},$$

given that $C_0^{-}=0$ and $C_0^{+}=0$, where $ARL^{-}=M^{-}(0)$, $ARL^{+}=M^{+}(0)$, and ARL are the means of the distribution of T^{-}, T^{+}, and T respectively.

Let T_c represent the run length of the two-sided CUSUM $\overline{X}$ chart of Crosier (1986) with starting value $C_0 = u \in [-h, h]$. We define

$$pr_c(t\,|u) = P(T_c = t\,|C_0 = u).$$

As we will see, the function $pr_c(t\,|u)$ is also a function of the distributional parameters δ and λ, the number of preliminary samples m, sample size n, and the chart parameters k, and h. For $t = 1$, we have

$$\begin{aligned}
pr_c(1\,|u) &= P(T_c = 1\,|C_0 = u) \\
&= P(C_1 < -h\,|C_0 = u) + P(C_1 > h\,|C_0 = u) \\
&= P\left(u + \frac{\overline{X}_1 - \widehat{\mu}_0}{\widehat{\sigma}_0/\sqrt{n}} + k < -h\right) + P\left(u + \frac{\overline{X}_1 - \widehat{\mu}_0}{\widehat{\sigma}_0/\sqrt{n}} - k > h\right) \\
&= P\left(u + v_0^{-1}\left(\lambda Z_1 + \sqrt{n}\delta - z_0/\sqrt{m}\right) + k < -h\right) \\
&\quad + P\left(u + v_0^{-1}\left(\lambda Z_1 + \sqrt{n}\delta - z_0/\sqrt{m}\right) - k > h\right) \\
&= \Phi\left(\frac{v_0(-h - u - k) - \zeta}{\lambda}\right) \\
&\quad + 1 - \Phi\left(\frac{v_0(h - u + k) - \zeta}{\lambda}\right).
\end{aligned}$$

For $t > 1$, we see that

$$\begin{aligned}
pr_c(t\,|u) &= P(T_c = t\,|C_0 = u) = P(T_c - 1 = t - 1, -h \le C_1 < 0\,|C_0 = u) \\
&\quad + P(T_c - 1 = t - 1, C_1 = 0\,|C_0 = u) \\
&\quad + P(T_c - 1 = t - 1, 0 < C_1 \le h\,|C_0 = u) \\
&= P(T_c - 1 = t - 1\,|C_0 = u, -h \le C_1 < 0)\,P(-h \le C_1 < 0\,|C_0 = u) \\
&\quad + P(T_c - 1 = t - 1\,|C_0 = u, C_1 = 0)\,P(C_1 = 0\,|C_0 = u) \\
&\quad + P(T_c - 1 = t - 1\,|C_0 = u, 0 < C_1 \le h) + P(0 < C_1 \le h\,|C_0 = u) \\
&= \int_{-h}^{0} pr_c(t - 1\,|c_1)\,\frac{v_0}{\lambda}\phi\left(\frac{v_0(c_1 - u - k) - \zeta}{\lambda}\right) dc_1 \\
&\quad + pr_c(t - 1\,|0)\left[\Phi\left(\frac{v_0(0 - u - k) - \zeta}{\lambda}\right) - \Phi\left(\frac{v_0(0 - u + k) - \zeta}{\lambda}\right)\right] \\
&\quad + \int_{0}^{h} pr_c(t - 1\,|c_1)\,\frac{v_0}{\lambda}\phi\left(\frac{v_0(c_1 - u + k) - \zeta}{\lambda}\right) dc_1.
\end{aligned}$$

The expected value of the run length (ARL) $M_c(u)$ and the expected value of the square of the run length $M_{c,2}(u)$ of this chart are given by

$$M_c(u) = \sum\nolimits_{t=1}^{\infty} tpr_c(t\,|u) \text{ and } M_{c,2}(u) = \sum\nolimits_{t=1}^{\infty} t^2 pr_c(t\,|u).$$

Applying the method of Champ, Rigdon, and Scharnagl (2001), we obtain integral equation whose exact solution are the functions $M_c(u)$ and $M_{c,2}(u)$. We see that

$$\begin{aligned} M_c(u) &= \sum\nolimits_{t=1}^{\infty} tpr_c(t\,|u) = pr_c(1\,|u) + \sum\nolimits_{t=2}^{\infty} tpr_c(t\,|u) \\ &= pr_c(1\,|u) + \sum\nolimits_{t=1}^{\infty} (1+t)\,pr_c(1+t\,|u) \\ &= pr_c(1\,|u) + \sum\nolimits_{t=1}^{\infty} pr_c(1+t\,|u) + \sum\nolimits_{t=1}^{\infty} tpr_c(1+t\,|u) \\ &= 1 + \sum\nolimits_{t=1}^{\infty} tP(T_c - 1 = t\,|C_0 = u). \end{aligned}$$

Note that

$$pr_c(1\,|u) + \sum\nolimits_{t=1}^{\infty} pr_c(1+t\,|u) = pr_c(1\,|u) + \sum\nolimits_{t=2}^{\infty} pr_c(t\,|u) = 1.$$

Next we write

$$\begin{aligned} pr_c(1+t\,|u) &= P(T_c - 1 = t\,|C_0 = u) \\ &= P(T_c - 1 = t, C_1 < -h\,|C_0 = u) \\ &\quad + P(T_c - 1 = t, -h \le C_1 < 0\,|C_0 = u) \\ &\quad + P(T_c - 1 = t, C_1 = 0\,|C_0 = u) \\ &\quad + P(T_c - 1 = t, 0 < C_1 \le h\,|C_0 = u) \\ &\quad + P(T_c - 1 = t, C_1 > h\,|C_0 = u). \end{aligned}$$

Observe that

$$P(T_c - 1 = t, C_1 < -h\,|C_0 = u) = P(T_c - 1 = t, C_1 > h\,|C_0 = u) = 0.$$

This follows because for $t \geq 1$ the events $\{T_c - 1 = t\}$ and $\{C_1 < -h\}$ are mutually exclusive as are the events $\{T_c - 1 = t\}$ and $\{C_1 > h\}$. Thus,

$$\begin{aligned}
pr_c\left(1+t\,|u\right) &= P\left(T_c - 1 = t, -h \leq C_1 < 0\,|C_0 = u\right) \\
&\quad + P\left(T_c - 1 = t, C_1 = 0\,|C_0 = u\right) \\
&\quad + P\left(T_c - 1 = t, 0 < C_1 \leq h\,|C_0 = u\right) \\
&= P\left(T_c - 1 = t\,|C_0 = u, -h \leq C_1 < 0\right) P\left(-h \leq C_1 < 0\,|C_0 = u\right) \\
&\quad + P\left(T_c - 1 = t\,|C_0 = u, C_1 = 0\right) P\left(C_1 = 0\,|C_0 = u\right) \\
&\quad + P\left(T_c - 1 = t\,|C_0 = u, 0 < C_1 \leq h\right) P\left(0 < C_1 \leq h\,|C_0 = u\right) \\
&= \int_{-h}^{0} pr_c\left(t\,|c_1\right) f_{C_1|C_0}\left(c_1\,|u\right) dc_1 \\
&\quad + pr_c\left(t\,|0\right) F_{C_1|C_0}\left(0\,|u\right) \\
&\quad + \int_{0}^{h} pr_c\left(t\,|c_1\right) f_{C_1|C_0}\left(c_1\,|u\right) dc_1.
\end{aligned}$$

Using these results, we write

$$\begin{aligned}
M_c\left(u\right) &= 1 + \sum_{t=1}^{\infty} t \int_{-h}^{0} pr_c\left(t\,|c_1\right) f_{C_1|C_0}\left(c_1\,|u\right) dc_1 \\
&\quad + \sum_{t=1}^{\infty} t pr_c\left(t\,|0\right) F_{C_1|C_0}\left(0\,|u\right) \\
&\quad + \sum_{t=1}^{\infty} t \int_{0}^{h} pr_c\left(t\,|c_1\right) f_{C_1|C_0}\left(c_1\,|u\right) dc_1 \\
&= 1 + \int_{-h}^{0} \left(\sum_{t=1}^{\infty} t pr_c\left(t\,|c_1\right)\right) f_{C_1|C_0}\left(c_1\,|u\right) dc_1 \\
&\quad + \left(\sum_{t=1}^{\infty} t pr_c\left(t\,|0\right)\right) F_{C_1|C_0}\left(0\,|u\right) \\
&\quad + \int_{0}^{h} \left(\sum_{t=1}^{\infty} t pr_c\left(t\,|c_1\right)\right) f_{C_1|C_0}\left(c_1\,|u\right) dc_1 \\
&= 1 + \int_{-h}^{0} M_c\left(c_1\right) f_{C_1|C_0}\left(c_1\,|u\right) dc_1 \\
&\quad + M_c\left(0\right) F_{C_1|C_0}\left(0\,|u\right) \\
&\quad + \int_{0}^{h} M_c\left(c_1\right) f_{C_1|C_0}\left(c_1\,|u\right) dc_1
\end{aligned}$$

Observing that

$$
\begin{aligned}
f_{C_1|C_0}\left(c_1\left|u\right.\right) &= \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u-k\right)-c}{\lambda}\right)I_{(-\infty,0)}\left(c_1\right)+F_{C_1|C_0}\left(0\left|u\right.\right)I_{\{0\}}\left(c_1\right) \\
&\quad +\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u+k\right)-c}{\lambda}\right)I_{(0,\infty)}\left(c_1\right), \text{ with} \\
F_{C_1|C_0}\left(0\left|u\right.\right) &= \Phi\left(\frac{v_0\left(0-u-k\right)-c}{\lambda}\right)-\Phi\left(\frac{v_0\left(0-u+k\right)-c}{\lambda}\right),
\end{aligned}
$$

we can write

$$
\begin{aligned}
M_c\left(u\right) &= 1+\textstyle\int_{-h}^{0}M_c\left(c_1\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right)dc_1 \\
&\quad +M_c\left(0\right)\left[\Phi\left(\frac{v_0\left(0-u+k\right)-\zeta}{\lambda}\right)-\Phi\left(\frac{v_0\left(0-u-k\right)-\zeta}{\lambda}\right)\right] \\
&\quad +\textstyle\int_{0}^{h}M_c\left(c_1\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right)dc_1.
\end{aligned}
$$

It is clear that the conditional distribution of C_1 given $C_0=u$ is a mix of a discrete and continuous part. In a similar way, it is shown in the Appendix that

$$
\begin{aligned}
M_{c,2}\left(u\right) &= 1+2\textstyle\int_{-h}^{0}M_2\left(c_1\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right)dc_1 \\
&\quad +2M_2\left(0\right)\left[\Phi\left(\frac{v_0\left(0-u+k\right)-\zeta}{\lambda}\right)-\Phi\left(\frac{v_0\left(0-u-k\right)-\zeta}{\lambda}\right)\right] \\
&\quad +2\textstyle\int_{0}^{h}M_2\left(c_1\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right)dc_1 \\
&\quad +\textstyle\int_{-h}^{0}M_{c,2}\left(c_1\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right)dc_1 \\
&\quad +M_{c,2}\left(0\right)\left[\Phi\left(\frac{v_0\left(0-u+k\right)-\zeta}{\lambda}\right)-\Phi\left(\frac{v_0\left(0-u-k\right)-\zeta}{\lambda}\right)\right] \\
&\quad +\textstyle\int_{0}^{h}M_{c,2}\left(c_1\right)\frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right)dc_1.
\end{aligned}
$$

The 100γth percentage point of the run length distribution is the integer value t_γ that satisfies the inequalities

$$
F_T\left(t_\gamma\right)=\sum\nolimits_{t=1}^{t_\gamma}pr\left(t\left|u\right.\right)\geq\gamma \text{ with } F_T\left(t_\gamma-1\right)=\sum\nolimits_{t=1}^{t_\gamma-1}pr\left(t\left|u\right.\right)<\gamma,
$$

where $pr\left(t\left|u\right.\right)$ is the probability mass function of the run length distribution. For example, if the run length distribution is a geometric distribution with parameter p, then the 100γth percentage point t_γ is determined by

$$t_\gamma = \left\lceil \frac{\ln(1-\gamma)}{\ln(1-p)} \right\rceil .$$

4.4 Approximation Methods

While the probability mass function of the run length distributions for $t > 1$ for both the two one-sided CUSUM $\overline{X}$ charts of Page (1954) and the two-sided CUSUM $\overline{X}$ chart of Crosier (1986) are exact solutions of their respective integral equations, these probabilities cannot be computed exactly. In this section, we will examine methods that give good approximations for these probabilities as well as other parameters of the run length distribution. Approximations for the run length distribution of the two-sided CUSUM $\overline{X}$ chart of Crosier (1986) will be derived in this section. The approximations for the two one-sided CUSUM $\overline{X}$ charts of Page (1954) have been derived by Champ, Rigdon, and Scharnagl (2001) and will only be stated here for completeness.

For $t > 1$, we can use Gaussian quadrature to approximate the probability mass function $pr_c(t\,|u)$ of Crosier's (1986) two-sided CUSUM $\overline{X}$ chart as

$$\begin{aligned} pr_c(t\,|u_i) &= \textstyle\sum_{j=1}^{\eta-1} pr_c(t-1\,|u_j)\,\frac{v_0}{\lambda}\phi\left(\frac{v_0(u_j-u_i-k)-\zeta}{\lambda}\right)w_j + pr_c(t-1\,|u_\eta) \\ &\quad \times\left[\Phi\left(\frac{v_0(u_\eta-u_i-k)-\zeta}{\lambda}\right)-\Phi\left(\frac{v_0(u_\eta-u_i+k)-\zeta}{\lambda}\right)\right] \\ &\quad +\textstyle\sum_{j=\eta+1}^{2\eta-1} pr_c(t-1\,|u_j)\,\frac{v_0}{\lambda}\phi\left(\frac{v_0(u_j-u_i+k)-\zeta}{\lambda}\right)w_j, \end{aligned}$$

where

$$u_j = \begin{cases} \frac{h}{2}\left(u_j^*-1\right), & \text{for } j=1,\ldots,\eta-1; \\ 0, & \text{for } j=\eta;\ \text{and} \\ \frac{h}{2}\left(u_{j-\eta}^*+1\right), & \text{for } j=\eta+1,\ldots,2\eta-1; \end{cases}$$

and

$$w_j = \begin{cases} \frac{h}{2}w_j^*, & \text{for } j=1,\ldots,\eta-1; \\ \frac{h}{2}w_{j-\eta}^*, & \text{for } j=\eta+1,\ldots,2\eta-1. \end{cases}$$

Here, the u_j^*'s are the $\eta - 1$ Gaussian quadrature points and the w_i^*'s the associated weights based on Legendre polynomials. The Gaussian quadrature points u_j^* and weights w_j^* can be found in such reference texts as Abramowitz and Stegun (1972). This yields a sequence of equations that gives approximate solutions for the probability mass function $pr_c\left(t\,|u\right)$ at the values $u_1, \ldots, u_{2\eta-1}$ for $t > 1$ with exact values for $t = 1$. This can be seen by defining the $(2\eta - 1) \times 1$ vector $\mathbf{p}_t$ by

$$\mathbf{p}_t = \left[pr_c\left(t-1\,|u_1\right), \ldots, pr_c\left(t-1\,|u_{2\eta-1}\right)\right]^{\mathbf{T}}$$

and the $(2\eta - 1) \times (2\eta - 1)$ matric $\mathbf{Q}_c$ having (i, j)th component

$$\begin{array}{ll} \frac{v_0}{\lambda}\phi\left(\frac{v_0(u_j-u_i-k)-\zeta}{\lambda}\right)w_j, & \text{for } j = 1, \ldots, \eta - 1; \\ \Phi\left(\frac{v_0(u_0-u_i+k)-\zeta}{\lambda}\right) - \Phi\left(\frac{v_0(u_0-u_i-k)-c}{\lambda}\right), & \text{for } j = \eta; \text{ and} \\ \frac{v_0}{\lambda}\phi\left(\frac{v_0(u_j-u_i+k)-\zeta}{\lambda}\right)w_{j-\eta}, & \text{for } j = \eta + 1, \ldots, 2\eta - 1, \end{array}$$

for $i = 1, \ldots, 2\eta - 1$. It follows that

$$\mathbf{p}_t = \mathbf{Q}\mathbf{p}_{t-1} = \mathbf{Q}^{t-1}\mathbf{p}_1$$

for $t > 1$.

This method may give poor approximations for $\mathbf{p}_t$ if t is large. Woodall (1983) showed that the "tail" probabilities of the one- and two-sided CUSUM $\overline{X}$ chart of Page (1954) can be quite well approximated by a geometric distribution. In particular, he shows that for the upper one-sided CUSUM $\overline{X}$ chart there exist a value $0 < \theta_{u^+}^+ < 1$ and a value $t^{+(*)}$ such that

$$pr^+\left(t\,|u^+\right) \approx \left(\theta_{u^+}^+\right)^{t-t^{+(*)}} pr\left(t^{+(*)}\,|u^+\right)$$

for $t > t^{+(*)}$. Note that $\theta_{u^+}^+$ is a function of the starting value u^+. Further, he gives a method for approximating $\theta_{u^+}^+$. This allows good approximations for the various parameters of the run length distribution. This approximation method also applies to the CUSUM $\overline{X}$ chart of Crosier (1986).

For Crosier's (1986) CUSUM $\overline{X}$ chart, there exist a value $0 < \theta_u < 1$ and a value t^* such that

$$pr_c\left(t\left|u\right.\right) \approx \theta_u^{t-t^*} pr\left(t^*\left|u^+\right.\right)$$

for $t > t^*$. Using these results, we can obtain good approximations for the ARL $(M_c(u))$, standard deviation of the run length ($SDRL$), and various percentage points of the run length distribution. The approximation for the ARL is obtained by observing that

$$\begin{aligned} M_c(u) &= \sum_{t=1}^{\infty} tpr_c\left(t\left|u\right.\right) = \sum_{t=1}^{t^*} tpr_c\left(t\left|u\right.\right) + \sum_{t=t^*+1}^{\infty} tpr_c\left(t\left|u\right.\right) \\ &\approx \sum_{t=1}^{t^*} tpr_c\left(t\left|u\right.\right) + \left(\sum_{t=t^*+1}^{\infty} t\theta_u^{t-t^*}\right) pr\left(t^*\left|u^+\right.\right) \\ &= \sum_{t=1}^{t^*} tpr_c\left(t\left|u\right.\right) + \left(\sum_{t=1}^{\infty} \left(t^*+t\right)\theta_u^t\right) pr\left(t^*\left|u^+\right.\right) \\ & \sum_{t=1}^{t^*} tpr\left(t\left|u\right.\right) + \theta_u pr\left(t^*\left|u\right.\right)\left(\frac{t^*}{1-\theta_u} + \frac{1}{\left(1-\theta_u\right)^2}\right) \end{aligned}$$

To obtain the approximation for the $SDRL$, we first obtain the expectation $M_{c,2}(u)$ of the square of the run length distribution. We have

$$\begin{aligned} M_{c,2}(u) &= \sum_{t=1}^{\infty} t^2 pr_c\left(t\left|u\right.\right) = \sum_{t=1}^{t^*} t^2 pr_c\left(t\left|u\right.\right) + \sum_{t=1}^{\infty} \left(t^*+t\right)^2 pr_c\left(t^*+t\left|u\right.\right) \\ &= \sum_{t=1}^{t^*} t^2 pr_c\left(t\left|u\right.\right) + \left(t^*\right)^2 \sum_{t=1}^{\infty} pr_c\left(t^*+t\left|u\right.\right) \\ & +2t^* \sum_{t=1}^{\infty} pr_c\left(t^*+t\left|u\right.\right) + \sum_{t=1}^{\infty} t^2 pr_c\left(t^*+t\left|u\right.\right) \end{aligned}$$

Using Woodall's (1983) approximation $\theta_u^t pr_c\left(t^*\left|u\right.\right)$ for $pr_c\left(t^*+t\left|u\right.\right)$, we have

$$\begin{aligned} M_{c,2}(u) &\approx \sum_{t=1}^{t^*} t^2 pr_c\left(t\left|u\right.\right) + \left(t^*\right)^2 \left(\sum_{t=1}^{\infty} \theta_u^t\right) pr_c\left(t^*\left|u\right.\right) \\ & +2t^* \left(\sum_{t=1}^{\infty} \theta_u^t\right) pr_c\left(t^*\left|u\right.\right) + \left(\sum_{t=1}^{\infty} t^2\theta_u^t\right) pr_c\left(t^*\left|u\right.\right) \end{aligned}$$

It is easy to show that

$$\sum_{t=1}^{\infty} \theta_u^t = \frac{\theta_u}{1-\theta_u};\ \sum_{t=1}^{\infty} t\theta_u^t = \frac{\theta_u}{\left(1-\theta_u\right)^2};\text{ and } \sum_{t=1}^{\infty} t^2\theta_u^t = \frac{\theta_u\left(1+\theta_u\right)}{\left(1-\theta_u\right)^3}.$$

Thus,

$$M_{c,2}(u) \approx \sum_{t=1}^{t^*} t^2 pr_c\left(t\left|u\right.\right) + \theta_u pr_c\left(t^*\left|u\right.\right)\left(\frac{\left(t^*\right)^2}{1-\theta_u} + \frac{2t^*}{\left(1-\theta_u\right)^2} + \frac{1+\theta_u}{\left(1-\theta_u\right)^3}\right).$$

We can now obtain the $SDRL$ as

$$\sqrt{M_{c,2}\left(u\right)-\left(M_c\left(u\right)\right)^2}.$$

As previously stated the 100γth percentile of the run length distribution given $C_0 = u$ is the integer t_γ such that

$$F_{T_c}\left(t_\gamma\left|u\right.\right)\geq\gamma \text{ and } F_{T_c}\left(t_\gamma-1\left|u\right.\right)<\gamma,$$

where

$$F_{T_c}\left(t\left|u\right.\right)=\sum\nolimits_{\tau=1}^{t}pr_c\left(\tau\left|u\right.\right).$$

If $F_{T_c}\left(t^*\left|u\right.\right) \geq \gamma$, then $t_\gamma \leq t^*$. In this case, the value of t_γ can be determined using a search method. For the case in which $F_{T_c}\left(t^*\left|u\right.\right) < \gamma$, then $t_\gamma > t^*$ and can be approximated using Woodall's (1983) method. For this case, we have that t_γ approximately satisfies the following inequalities.

$$\begin{aligned}
F_{T_c}\left(t^*\left|u\right.\right)+\left(\sum\nolimits_{t=1}^{t_\gamma-t^*}\theta_u^t\right)pr\left(t^*\left|u^+\right.\right) &\geq \gamma \text{ and} \\
F_{T_c}\left(t^*\left|u\right.\right)+\left(\sum\nolimits_{t=1}^{t_\gamma-1-t^*}\theta_u^t\right)pr\left(t^*\left|u^+\right.\right) &< \gamma.
\end{aligned}$$

Observing that

$$\sum\nolimits_{t=1}^{t_\gamma-t^*}\theta_u^t=\frac{\theta_u\left(1-\theta_u^{t_\gamma-t^*}\right)}{1-\theta_u} \text{ and } \sum\nolimits_{t=1}^{t_\gamma-1-t^*}\theta_u^t=\frac{\theta_u\left(1-\theta_u^{t_\gamma-1-t^*}\right)}{1-\theta_u},$$

then t_γ (approximately) satisfies the compound inequality

$$t^*+\frac{\ln\left(1-\frac{1-\theta_u}{\theta_u}\frac{\gamma-F_{T_c}\left(t^*\left|u\right.\right)}{pr\left(t^*\left|u^+\right.\right)}\right)}{\ln\left(\theta_u\right)}\leq t_\gamma<t^*+1+\frac{\ln\left(1-\frac{1-\theta_u}{\theta_u}\frac{\gamma-F_{T_c}\left(t^*\left|u\right.\right)}{pr\left(t^*\left|u^+\right.\right)}\right)}{\ln\left(\theta_u\right)}.$$

We then see that the integer t_γ is determined by

$$t_\gamma=\left\lceil t^*+\frac{\ln\left(1-\frac{1-\theta_u}{\theta_u}\frac{\gamma-F_{T_c}\left(t^*\left|u\right.\right)}{pr\left(t^*\left|u^+\right.\right)}\right)}{\ln\left(\theta_u\right)}\right\rceil.$$

Similar results to these holds for the lower and upper one-sided CUSUM $\overline{X}$ chart of Page (1954). One only needs to replace the appropriate values of the run length distribution in place of those for the run length distribution of the two-sided CUSUM $\overline{X}$ of Crosier (1986).

As in the case with the probability mass function for the two-sided CUSUM $\overline{X}$ chart of Crosier (1986), we can obtain an approximation based on Guassian quadrature of the integral equation whose exact solution with the average run length $M_c(u)$ of the chart. Using our previous results, we have the following approximation

$$\begin{aligned} M_c(u_i) &= 1 + \textstyle\sum_{j=1}^{\eta-1} M_c(u_j)\, \frac{v_0}{\lambda}\phi\left(\frac{v_0(u_j - u_j - k) - \zeta}{\lambda}\right) w_j \\ &\quad + M_c(u_\eta)\left[\Phi\left(\frac{v_0(u_\eta - u_i - k) - \zeta}{\lambda}\right) - \Phi\left(\frac{v_0(u_\eta - u_i + k) - \zeta}{\lambda}\right)\right] \\ &\quad + \textstyle\sum_{j=\eta+1}^{2\eta-1} M_c(u_j)\, \frac{v_0}{\lambda}\phi\left(\frac{v_0(u_j - u_i - k) - \zeta}{\lambda}\right) w_{j-\eta}. \end{aligned}$$

This is a system of $2\eta - 1$ equations in the $2\eta - 1$ unknows $M_c(u_1), \ldots, M_c(u_{2\eta-1})$. We can express this system in matrix form as

$$\mathbf{M}_c = \mathbf{Q}_c\mathbf{M}_c + \mathbf{1} \text{ or } (\mathbf{I} - \mathbf{Q}_c)\,\mathbf{M}_c = \mathbf{1}$$

with the $(2\eta - 1) \times (2\eta - 1)$ matrix $\mathbf{Q}_c$ as previously defined in this section, the $(2\eta - 1) \times 1$ vector $\mathbf{M}_c$ defined by

$$\mathbf{M}_c = [M_c(u_1), \ldots, M_c(u_{2\eta-1})]^{\mathbf{T}},$$

and $\mathbf{1}$ a $(2\eta - 1) \times 1$ vector of ones. It follows that

$$\mathbf{M}_c = (\mathbf{I} - \mathbf{Q}_c)^{-1}\,\mathbf{1}.$$

If the practitioner chooses to set $C_0 = u_i$, then the ηth component $M_c(u_i)$ of $\mathbf{M}_c$ is the approximation to the ARL of the two-sided CUSUM $\overline{X}$ chart. In particular, $M_c(u_\eta) = M_c(0)$ is the ARL when $C_0 = 0$.

4.5 Conclusion

In this chapter, various integral equations were studied that described the distribution of the run length. Integral equations useful for describing the run length distribution of both the lower and upper one-sided CUSUM $\overline{X}$ charts of Page (1954) can be found in the literature but not for the two-sided version of the chart. Crosier (1986) used a Markov chain method for approximating the average run length of his two-sided CUSUM $\overline{X}$ chart. We derived integral equations that can be used to obtain more accurate results for this chart. A Gaussian quadrature method was used to obtain approximate solutions to the integral equations presented. It was shown how Woodall's (1983) approximation method can be used to obtain an approximation for the tail probabilities useful in obtaining various run length distribution parameters.

Chapter 5

MARKOV CHAIN METHOD

5.1 Introduction

In this chapter, we will examine Markov chain approximations of the Markov processes $\{C_t\}$, $\left\{C_t^-\right\}$, and $\left\{C_t^+\right\}$. We begin by examining the Markov chain representations for the lower and upper one-sided CUSUM $\overline{X}$ charts of Page (1954). It was shown by Champ and Rigdon (1991) that if the integral equation for the upper on-sided CUSUM $\overline{X}$ of Page (1954) is approximated a certain way it would give the same approximation as the Markov chain method. We will show that this also holds for the run length distribution.

The method of Woodall (1984) is extended to a Markov chain representation of the two-sided CUSUM $\overline{X}$ charts of Page (1954). Crosier (1986) presents the Markov chain approximation of his two-sided CUSUM $\overline{X}$ chart. It is presented here for completeness. We show that if the integral equation that has its exact solution the ARL of the two-sided CUSUM $\overline{X}$ chart of Crosier (1986) is approximated a certain way gives equivalent results to the Markov chain method. We also show that this holds for the run length distribution of this chart.

5.2 One-Sided CUSUM $\overline{X}$ Charts of Page (1954)

Brook and Evans (1972) develop a Markov chain representation of the one-sided CUSUM $\overline{X}$ charts of Page (1954). The representation is exact for attribute and approximate for variables data. The η^+ nonabsorbing states of the Markov chain approximation of the upper one-sided CUSUM $\overline{X}$ are the points in the interval $[0, h^+]$ that have the values i^+w^+ for $i^+ = 0, 1, \ldots, \eta^+ - 1$, where $w^+ = 2h^+/\left(2\eta^+ - 1\right)$. The probability that the chain transitions from the nonabsorbing state i^+w^+ to nonabsorbing state j^+w^+ is given by

$$\begin{array}{l} \Phi\left(\frac{v_0\left(\left(j^+ + 0.5 - i^+\right)w^+ + k^+\right) - \zeta}{\lambda}\right), \\ \text{if } j^+ = 0; \\ \Phi\left(\frac{v_0\left(\left(j^+ + 0.5 - i^+\right)w^+ + k^+\right) - \zeta}{\lambda}\right) - \Phi\left(\frac{v_0\left(\left(j^+ - 0.5 - i^+\right)w^+ + k^+\right) - \zeta}{\lambda}\right), \\ \text{if } j^+ = 1, \ldots, \eta^+ - 1. \end{array}$$

Define the $\eta^+ \times \eta^+$ matrix $\mathbf{R}^+$ such the (i^+, j^+) component is the probability the Markov chain approximation to the upper one-sided CUSUM $\overline{X}$ chart of Page (1954) transitions from the nonabsorbing state i^+w^+ to the nonabsorbing state j^+w^+ for $i^+, j^+ = 0, \ldots, \eta^+ - 1$. This is the matrix that results from removing from the transition matrix the row and column associated with the absorbing state.

It can be shown for $t > 1$ that

$$\mathbf{p}_t^+ \approx \mathbf{R}^+ \mathbf{p}_{t-1}^+ \text{ with } \mathbf{p}_1^+ \approx \left(\mathbf{I} - \mathbf{R}^+\right)\mathbf{1},$$

where $\mathbf{I}$ is the $\eta^+ \times \eta^+$ identity matrix and $\mathbf{1}$ is a $\eta^+ \times 1$ vector of ones. Here, $\mathbf{p}_t$ is defined by

$$\mathbf{p}_t^+ = \left[pr^+\left(t\left|0w^+\right.\right), \ldots, pr^+\left(t\left|\left(\eta^+ - 1\right)w^+\right.\right)\right]^{\mathbf{T}}.$$

It follows that the vector $\mathbf{M}^+$ of ARLs can be determined approximately by

$$\mathbf{M}^+ \approx \left(\mathbf{I} - \mathbf{R}^+\right)^{-1}\mathbf{1},$$

where

$$\mathbf{M}^+ = \left[M^+\left(0w^+\right), \ldots, M^+\left(\left(\eta^+ - 1\right)w^+\right)\right]^{\mathbf{T}}.$$

Similar results hold for the lower one-sided CUSUM $\overline{X}$ chart of Page (1954). The η^- nonabsorbing states of the Markov chain approximation have the form $-i^-w^-$ for $i^- = 0, \ldots, \eta^- - 1$, where $w^- = 2\left|h^-\right| / \left(2\eta^- - 1\right)$. The probability that the chain transitions from the nonabsorbing state $-i^-w^-$ to nonabsorbing state $-j^-w^-$ is given by

$$1-\Phi\left(\frac{v_0\left(\left(-j^- - 0.5 + i^-\right)w^- - k^-\right)-\zeta}{\lambda}\right),$$

if $j^- = 0$;

$$\Phi\left(\frac{v_0\left(\left(-j^- + 0.5 + i^-\right)w^- - k^-\right)-\zeta}{\lambda}\right) - \Phi\left(\frac{v_0\left(\left(-j^- - 0.5 + i^-\right)w^- - k^-\right)-\zeta}{\lambda}\right),$$

if $j^- = 1, \ldots, \eta^- - 1$.

These are the transition probabilities of the $\eta^- \times \eta^-$ matrix $\mathbf{R}^-$ determined from the transition by removing the row and column associated with the absorbing state.

5.3 Two-Sided CUSUM $\overline{X}$ Charts of Page (1954)

The Markov chain representation of the two-sided CUSUM $\overline{X}$ chart of Page (1954) has nonabsorbing states of the form $(-i^- w^-, i^+ w^+)$, where $-i^- w^-$ and $i^+ w^+$ are nonabsorbing states of the Markov chain approximation of the lower and upper one-sided CUSUM $\overline{X}$ charts of Page (1954). The probability the Markov chain transitions from the nonabsorbing state $(-i^- w^-, i^+ w^+)$ to the nonabsorbing state $(-j^- w^-, j^+ w^+)$ is given by

$$\Phi\left(\frac{v_0\left(\left(j^+ + 0.5 - i^+\right)w^+ + k^+\right)-\zeta}{\lambda}\right) - \Phi\left(\frac{v_0\left(\left(-j^- - 0.5 + i^-\right)w^- - k^-\right)-\zeta}{\lambda}\right),$$

if $j^- = 0,\ j^+ = 0$;

$$\Phi\left(\frac{v_0\left(\left(j^+ + 0.5 - i^+\right)w^+ + k^+\right)-\zeta}{\lambda}\right)$$
$$-\Phi\left(\max\left\{\frac{v_0\left(\left(-0.5 + i^-\right)w^- - k^-\right)-\zeta}{\lambda}, \frac{v_0\left(\left(j^+ - 0.5 - i^+\right)w^+ + k^+\right)-\zeta}{\lambda}\right\}\right),$$

if $j^- = 0,\ j^+ = 1, \ldots, \eta^+ - 1$;

$$\Phi\left(\min\left\{\frac{v_0\left(\left(-j^- + 0.5 + i^-\right)w^- - k^-\right)-\zeta}{\lambda}, \frac{v_0\left(\left(j^+ + 0.5 - i^+\right)w^+ + k^+\right)-\zeta}{\lambda}\right\}\right)$$
$$-\Phi\left(\frac{v_0\left(\left(-j^- - 0.5 + i^-\right)w^- - k^-\right)-\zeta}{\lambda}\right),$$

if $j^- = 1, \ldots, \eta^- - 1,\ j^+ = 0$;

$$\Phi\left(\min\left\{\frac{v_0\left(\left(-j^- + 0.5 + i^-\right)w^- - k^-\right)-\zeta}{\lambda}, \frac{v_0\left(\left(j^+ + 0.5 - i^+\right)w^+ + k^+\right)-\zeta}{\lambda}\right\}\right)$$
$$-\Phi\left(\max\left\{\frac{v_0\left(\left(-0.5 + i^-\right)w^- - k^-\right)-\zeta}{\lambda}, \frac{v_0\left(\left(j^+ - 0.5 - i^+\right)w^+ + k^+\right)-\zeta}{\lambda}\right\}\right),$$

if $j^- = 1, \ldots, \eta^- - 1,\ j^+ = 1, \ldots, \eta^+ - 1$.

There are $\eta^{-}\eta^{+}$ nonabsorbing states. For convenience, we number the states by the nonabsorbing state $(-i^{-}w^{-}, i^{+}w^{+})$ being numbered $i^{-}\eta^{+} + i^{+}$.

Letting $\mathbf{R}$ be the $\eta^{-}\eta^{+} \times \eta^{-}\eta^{+}$ matrix obtained from the transition matrix by removing the row and column assoicated with the absorbing state, we can express the run length distribution in the form

$$\mathbf{p}_t \approx \mathbf{R}\mathbf{p}_{t-1} \text{ with } \mathbf{p}_1 \approx (\mathbf{I} - \mathbf{R})\,\mathbf{1}.$$

The vector $\mathbf{M}$ of ARLs of the chart can then be determined approximately by

$$\mathbf{M} \approx (\mathbf{I} - \mathbf{R})^{-1}\,\mathbf{1},$$

where $\mathbf{1}$ is an $\eta^{-}\eta^{+} \times 1$ vector of ones. A FORTRAN program is given in the Appendix for determining the ARL of the chart when the in-control parameters are known. The transitions probabilities for this case are obtained from the transition probabilities when parameters are estimated by setting $z_0 = 0$ and $v_0 = 1$.

5.4 Two-Sided CUSUM $\overline{X}$ Chart of Crosier (1986)

Crosier (1986) gives a Markov chain approximation to his two sided CUSUM $\overline{X}$ chart. The states are the values

$$-(\eta - 1)\,w, \ldots, -w, 0, w, \ldots, (\eta - 1)\,w.$$

There is one absorbing state. In general, we will express the ith nonabsorbing state by $(i - \eta)\,w$ with i ranging from 1 to $2\eta - 1$. The probabilities of making a transition from nonaborbing state $(i - \eta)\,w$ to nonabsorbing state $(j - \eta)\,w$ is

$$\Phi\left(\tfrac{v_0((j+0.5-i)w-k)-\zeta}{\lambda}\right) - \Phi\left(\tfrac{v_0((j-0.5-i)w-k)-\zeta}{\lambda}\right),$$

if $j = 1, \ldots, \eta - 1$;

$$\Phi\left(\tfrac{v_0((j+0.5-i)w+k)-\zeta}{\lambda}\right) + 1 - \Phi\left(\tfrac{v_0((j-0.5-i)w+k)-\zeta}{\lambda}\right),$$

if $j = \eta$;

$$\Phi\left(\tfrac{v_0((j+0.5-i)w+k)-\zeta}{\lambda}\right) - \Phi\left(\tfrac{v_0((j-0.5-i)w+k)-\zeta}{\lambda}\right),$$

if $j = \eta + 1, \ldots, 2\eta - 1$.

for $i = 1, \ldots, 2\eta - 1$. Define the $(2\eta - 1) \times (2\eta - 1)$ matrix $\mathbf{R}$ with (i, j)th component the probability the Markov chain makes a transition from nonabsorbing state $(i - \eta)\, w$ to nonabsorbing state $(j - \eta)\, w$. The matrix $\mathbf{R}$ is obtained from the transition matrix by removing the row and column associated with the absorbing state. As with the one-sided charts, the run length distribution $\mathbf{p}_{c,t}$ can be approximated by

$$\mathbf{p}_{c,t} \approx \mathbf{R}_c \mathbf{p}_{c,t-1} \text{ with } \mathbf{p}_{c,1} \approx (\mathbf{I} - \mathbf{R}_c)\, \mathbf{1}.$$

The vector $\mathbf{M}_c$ of ARLs of the chart can then be determined approximately by

$$\mathbf{M}_c \approx (\mathbf{I} - \mathbf{R}_c)^{-1}\, \mathbf{1},$$

where $\mathbf{I}$ is a $(2\eta - 1) \times (2\eta - 1)$ identity matrix and $\mathbf{1}$ is a $(2\eta - 1) \times 1$ vector of ones.

5.5 Equivalence of the Markov Chain and Integral Equation Approaches

As previously stated, Champ and Rigdon (1991) expressed the integral equation whose exact solution is the function $M^+(u^+)$ by

$$\begin{aligned} M^+\left(u^+\right) &= 1 + M^+(0)\, \Phi\left(\frac{v_0\left(0 - u^+ + k^+\right) - \zeta}{\lambda}\right) \\ &\quad + \int_0^{0.5w^+} M^+\left(c_1^+\right) \frac{v_0}{\lambda} \phi\left(\frac{v_0\left(c_1^+ - u^+ + k^+\right) - \zeta}{\lambda}\right) dc_1^+ \\ &\quad + \textstyle\sum_{j^+=1}^{\eta^+-1} \displaystyle\int_{(j^+-0.5)w^+}^{(j^++0.5)w^+} M^+\left(c_1^+\right) \frac{v_0}{\lambda} \phi\left(\frac{v_0\left(c_1^+ - u^+ + k^+\right) - \zeta}{\lambda}\right) dc_1^+. \end{aligned}$$

By the mean value theorem for integrals, there exists values

$$\xi_0^+, \xi_1^+, \ldots, \xi_{\eta^+-1}^+$$

with $(j^+ - 0.5)\, w^+ < \xi_{j^+}^+ \leq (j^+ + 0.5)\, w^+$ such that

$$\begin{aligned} M^+\left(u^+\right) &= 1 + M^+(0)\, \Phi\left(\frac{v_0\left(0 - u^+ + k^+\right) - \zeta}{\lambda}\right) \\ &\quad + M^+\left(\xi_0^+\right) \int_0^{0.5w^+} \frac{v_0}{\lambda} \phi\left(\frac{v_0\left(c_1^+ - u^+ + k^+\right) - \zeta}{\lambda}\right) dc_1^+ \\ &\quad + \textstyle\sum_{j^+=1}^{\eta^+-1} M^+\left(\xi_{j^+}^+\right) \displaystyle\int_{(j^+-0.5)w^+}^{(j^++0.5)w^+} \frac{v_0}{\lambda} \phi\left(\frac{v_0\left(c_1^+ - u^+ + k^+\right) - \zeta}{\lambda}\right) dc_1^+. \end{aligned}$$

for $j^+ = 1, \ldots, \eta^+ - 1$. Replacing $\xi^+_{j^+}$ with j^+w^+, it can be stated approximately that

$$\begin{aligned}
M^+\left(i^+w^+\right) &= 1 + M^+(0)\,\Phi\left(\frac{v_0\left(0 - i^+w^+ + k^+\right) - \zeta}{\lambda}\right) \\
&\quad + M^+(0)\int_0^{0.5w^+} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^+ - i^+w^+ + k^+\right) - \zeta}{\lambda}\right) dc_1^+ \\
&\quad + \textstyle\sum_{j^+=1}^{\eta^+-1} M^+\left(j^+w^+\right) \\
&\quad \times \int_{\left(j^+-0.5\right)w^+}^{\left(j^++0.5\right)w^+} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1^+ - i^+w^+ + k^+\right) - \zeta}{\lambda}\right) dc_1^+ \\
&= 1 + M^+(0)\,\Phi\left(\frac{v_0\left(\left(0.5 - i^+\right)w^+ + k^+\right) - \zeta}{\lambda}\right) \\
&\quad + \textstyle\sum_{j^+=1}^{\eta^+-1} M^+\left(j^+w^+\right) \times [\Phi\left(\frac{v_0\left(\left(j^+ + 0.5 - i^+\right)w^+ + k^+\right) - \zeta}{\lambda}\right) \\
&\quad - \Phi\left(\frac{v_0\left(\left(j^+ - 0.5 - i^+\right)w^+ + k^+\right) - \zeta}{\lambda}\right)]
\end{aligned}$$

In this case, the $\eta^+ \times \eta^+$ matrix $\mathbf{Q}^+$ such that $\mathbf{M}^+ \approx \left(\mathbf{I} - \mathbf{Q}^+\right)^{-1}\mathbf{1}$ has (i^+, j^+)th component

$$\begin{array}{l}
\Phi\left(\frac{v_0\left(\left(j^++0.5-i^+\right)w^++k^+\right)-\zeta}{\lambda}\right), \\
i^+ = 0, \ldots, \eta^+ - 1,\ j^+ = 0; \\
\Phi\left(\frac{v_0\left(\left(j^++0.5-i^+\right)w^++k^+\right)-\zeta}{\lambda}\right) - \Phi\left(\frac{v_0\left(\left(j^+-0.5-i^+\right)w^++k^+\right)-\zeta}{\lambda}\right), \\
i^+ = 0, \ldots, \eta^+ - 1,\ j^+ = 1, \ldots, \eta^+ - 1.
\end{array}$$

By inspection, we see that $\mathbf{R}^+ = \mathbf{Q}^+$. Thus, the Markov chain approximation and this approximation to the integral equation for the mean give the same results.

It would then follow that

$$\mathbf{p}_t^+ \approx \mathbf{R}^+\mathbf{p}_{t-1}^+$$

for $t > 1$. For the integral equation method, the i^+ component of $\mathbf{p}_1^+$ is given exactly by

$$\Phi\left(\frac{v_0\left(-h - i^+w^+ - k\right) - \zeta}{\lambda}\right) + 1 - \Phi\left(\frac{v_0\left(h - i^+w^+ + k\right) - \zeta}{\lambda}\right).$$

Adding the components in the i^{+} row of $\mathbf{Q}^{+}$, we have

$$
\begin{aligned}
& \Phi\left(\frac{v_0\left(\left(j^{+}+0.5-i^{+}\right) w^{+}+k^{+}\right)-\zeta}{\lambda}\right) \\
& +\sum_{j^{+}=1}^{\eta^{+}-1}\left[\Phi\left(\frac{v_0\left(\left(j^{+}+0.5-i^{+}\right) w^{+}+k^{+}\right)-\zeta}{\lambda}\right)\right. \\
& \left.-\Phi\left(\frac{v_0\left(\left(j^{+}-0.5-i^{+}\right) w^{+}+k^{+}\right)-\zeta}{\lambda}\right)\right] \\
= \; & \Phi\left(\frac{v_0\left(h-i^{+} w^{+}+k\right)-\zeta}{\lambda}\right)-\Phi\left(\frac{v_0\left(-h-i^{+} w^{+}-k\right)-\zeta}{\lambda}\right) .
\end{aligned}
$$

This is the probability the chart does not signal at sampling stage $t=1$ for a chart beginning in state $i^{+}w^{+}$. It would then follow that one minus this probability which is

$$
\Phi\left(\frac{v_0\left(-h-i^{+} w^{+}-k\right)-\zeta}{\lambda}\right)+1-\Phi\left(\frac{v_0\left(h-i^{+} w^{+}+k\right)-\zeta}{\lambda}\right) .
$$

is the probability that the chart signals at sampling stage $t=1$. Hence, we have

$$
\mathbf{p}_1^{+}=\left(\mathbf{I}-\mathbf{R}^{+}\right) \mathbf{1}=\left(\mathbf{I}-\mathbf{Q}^{+}\right) \mathbf{1}
$$

is exact. Similar results hold for the lower one-sided CUSUM $\overline{X}$ chart of Page (1954).

For the CUSUM $\overline{X}$ chart of Crosier (1986), we can write the integral equation in which the exact solution is the function $M_c(u)$as

$$
\begin{aligned}
M_c(u) = \; & 1+\sum_{j=1}^{\eta-1} \int_{(j-\eta-0.5) w}^{(j-\eta+0.5) w} M_c\left(c_1\right) \frac{v_0}{\lambda} \phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right) d c_1 \\
& +\int_{-0.5 w}^0 M_c\left(c_1\right) \frac{v_0}{\lambda} \phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right) d c_1 \\
& +M_c(0)\left[\Phi\left(\frac{v_0(0-u+k)-\zeta}{\lambda}\right)-\Phi\left(\frac{v_0(0-u-k)-\zeta}{\lambda}\right)\right] \\
& +\int_0^{0.5 w} M_c\left(c_1\right) \frac{v_0}{\lambda} \phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right) d c_1 \\
& +\sum_{j=\eta+1}^{2 \eta-1} \int_{(j-\eta-0.5) w}^{(j-\eta+0.5) w} \int M_c\left(c_1\right) \frac{v_0}{\lambda} \phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right) d c_1
\end{aligned}
$$

By the mean value theorem for integrals, there exists values

$$
\xi_{-(\eta-1)}, \ldots,, \xi_{-1}, \xi_{-0}, \xi_0, \xi_1, \ldots, \xi_{\eta-1}
$$

such that

$$\begin{aligned}
M_c(u) &= 1 + \sum_{j=1}^{\eta-1} M_c\left(\xi_{j-\eta}\right) \int_{(j-\eta-0.5)w}^{(j-\eta+0.5)w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right) dc_1 \\
&\quad + M_c\left(\xi_{-0}\right) \int_{-0.5w}^{0} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right) dc_1 \\
&\quad + M_c(0)\left[\Phi\left(\frac{v_0\left(0-u+k\right)-\zeta}{\lambda}\right) - \Phi\left(\frac{v_0\left(0-u-k\right)-\zeta}{\lambda}\right)\right] \\
&\quad + M_c\left(\xi_0\right) \int_{0}^{0.5w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right) dc_1 \\
&\quad + \sum_{j=\eta+1}^{2\eta-1} M_c\left(\xi_{j-\eta}\right) \int_{(j-\eta-0.5)w}^{(j-\eta+0.5)w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right) dc_1
\end{aligned}$$

with

$$(j-0.5)\,w \le \xi_{j-\eta} < (j+0.5)\,, \text{ for } j = 1,\ldots,\eta-1;$$

$$-0.5w \le \xi_{-0} \le 0;\ 0 \le \xi_0 \le 0.5w;$$

$$(j-0.5) < w\xi_{j-\eta} \le (j+0.5)\,, \text{ for } j = \eta+1,\ldots,2\eta-1.$$

The symbols ξ_{-0} and ξ_0 are being used to represent distinct values. We then have

$$\begin{aligned}
M_c(u) &= 1 + \sum_{j=1}^{\eta-1} M_c\left(\xi_{j-\eta}\right) \int_{(j-\eta-0.5)w}^{(j-\eta+0.5)w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right) dc_1 \\
&\quad + M_c\left(\xi_{-0}\right) \int_{-0.5w}^{0} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right) dc_1 \\
&\quad + M_c(0)\left[\Phi\left(\frac{v_0\left(0-u+k\right)-\zeta}{\lambda}\right) - \Phi\left(\frac{v_0\left(0-u-k\right)-\zeta}{\lambda}\right)\right] \\
&\quad + M_c\left(\xi_0\right) \int_{0}^{0.5w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right) dc_1 \\
&\quad + \sum_{j=\eta+1}^{2\eta-1} M_c\left(\xi_{j-\eta}\right) \int_{(j-\eta-0.5)w}^{(j-\eta+0.5)w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right) dc_1
\end{aligned}$$

Approximating $\xi_{i-\eta} = (i-\eta)\,w$ for $i = 1,\ldots,\eta-1,\eta+1,\ldots,2\eta-1$ and $\xi_{-0} = \xi_0 = 0$, then we can obtain approximate values of $M_c\left((i-\eta)\,w\right)$ from the system of $2\eta-1$

equations in the $2\eta - 1$ unknowns $M_c\left((i-\eta)\,w\right)$ given by

$$\begin{aligned}
M_c\left((i-\eta)\,w\right) &= 1+\sum\nolimits_{j=1}^{\eta-1} M_c\left((j-\eta)\,w\right) \\
&\times \int_{(j-\eta-0.5)w}^{(j-\eta+0.5)w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-(i-\eta)\,w-k\right)-\zeta}{\lambda}\right)dc_1 \\
&+M_c\left(0\right)\int_{-0.5w}^{0} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-(i-\eta)\,w-k\right)-\zeta}{\lambda}\right)dc_1 \\
&+M_c\left(0\right)[\Phi\left(\frac{v_0\left(-(i-\eta)\,w+k\right)-\zeta}{\lambda}\right) \\
&-\Phi\left(\frac{v_0\left(-(i-\eta)\,w-k\right)-\zeta}{\lambda}\right)] \\
&+M_c\left(0\right)\int_{0}^{0.5w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-(i-\eta)\,w+k\right)-\zeta}{\lambda}\right)dc_1 \\
&+\sum\nolimits_{j=\eta+1}^{2\eta-1} M_c\left((j-\eta)\,w\right) \\
&\times \int_{(j-\eta-0.5)w}^{(j-\eta+0.5)w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-(i-\eta)\,w+k\right)-\zeta}{\lambda}\right)dc_1
\end{aligned}$$

Making the substitutions

$$\begin{aligned}
\Phi\left(\frac{v_0\left(0-u-k\right)-\zeta}{\lambda}\right) &= -\int_0^\infty \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u-k\right)-\zeta}{\lambda}\right)dc_1 \text{ and} \\
\Phi\left(\frac{v_0\left(0-u+k\right)-\zeta}{\lambda}\right) &= \int_{-\infty}^0 \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-u+k\right)-\zeta}{\lambda}\right)dc_1,
\end{aligned}$$

we have

$$\begin{aligned}
M_c\left((i-\eta)\,w\right) &= 1+\sum\nolimits_{j=1}^{\eta-1} M_c\left((j-\eta)\,w\right) \\
&\times \int_{(j-\eta-0.5)w}^{(j-\eta+0.5)w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-(i-\eta)\,w-k\right)-\zeta}{\lambda}\right)dc_1 \\
&+M_c\left(0\right)[\int_{-0.5w}^{\infty} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-(i-\eta)\,w-k\right)-\zeta}{\lambda}\right)dc_1 \\
&+\int_{-\infty}^{0.5w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-(i-\eta)\,w+k\right)-\zeta}{\lambda}\right)dc_1] \\
&+\sum\nolimits_{j=\eta+1}^{2\eta-1} M_c\left((j-\eta)\,w\right) \\
&\times \int_{(j-\eta-0.5)w}^{(j-\eta+0.5)w} \frac{v_0}{\lambda}\phi\left(\frac{v_0\left(c_1-(i-\eta)\,w+k\right)-\zeta}{\lambda}\right)dc_1
\end{aligned}$$

We can also write

$$\begin{aligned}
M_c\left((i-\eta)\,w\right) &= 1+\sum\nolimits_{j=1}^{\eta-1} M_c\left((j-\eta)\,w\right)[\Phi\left(\frac{v_0\left((j+0.5-i)\,w-k\right)-\zeta}{\lambda}\right) \\
&\quad -\Phi\left(\frac{v_0\left((j-0.5-i)\,w\right)-k-\zeta}{\lambda}\right)] \\
&\quad +M_c\left(0\right)[\Phi\left(\frac{v_0\left((0.5-i)\,w+k\right)-\zeta}{\lambda}\right) \\
&\quad +1-\Phi\left(\frac{v_0\left((-0.5-i)\,w-k\right)-\zeta}{\lambda}\right)] \\
&\quad +\sum\nolimits_{j=\eta+1}^{2\eta-1} M_c\left((j-\eta)\,w\right) \\
&\quad \times[\Phi\left(\frac{v_0\left((j-\eta+0.5)\,w-(i-\eta)\,w+k\right)-\zeta}{\lambda}\right) \\
&\quad -\Phi\left(\frac{v_0\left((j-\eta-0.5)\,w-(i-\eta)\,w+k\right)-\zeta}{\lambda}\right)]
\end{aligned}$$

This system of equations can be expressed in the form

$$\mathbf{M}=\mathbf{1}+\mathbf{QM},$$

where

$$\mathbf{M}=\left[M_c\left(-\left(\eta-1\right)w\right),\ldots,M_c\left(\left(\eta-1\right)w\right)\right].$$

By inspection, we see that $\mathbf{Q}=\mathbf{R}$, where $\mathbf{R}$ is the $(2\eta-1)\times(2\eta-1)$ matrix obtained from the transition matrix of Markov chain representation of the chart after the row and column associated with the absorbing state is removed.

5.6 Conclusion

In this chapter, we discussed the Markov chain approximations of the one- and two-sided CUSUM $\overline{X}$ charts. The Markov chain approximation of the two-sided CUSUM $\overline{X}$ chart was implemented in FORTRAN. It was shown that approximating the integral equations describing the run length distribution in a certain way gives the same approximations as the Markov chain approach. In this sense, the two methods were shown to be equivalent.

Chapter 6

CONCLUSION

6.1 General Conclusions

The integral equation and Markov chain methods were presented that are useful in analyzing the run length distributions of the one- and two- sided CUSUM $\overline{X}$ charts of Page (1954) and the two- sided CUSUM $\overline{X}$ chart of Crosier (1986). The two-sided CUSUM $\overline{X}$ chart of Page (1954) was implemented in FORTRAN. For the two- sided CUSUM $\overline{X}$ chart of Crosier (1986), we presented an estimated parameters version, derived integral equations useful in studying the run length performance of the chart, and showed under what conditions the integral equation method and the Markov chain method are equivalent.

6.2 Areas for Further Research

It is our interest to develop a FORTRAN program to determine the run length distribution of the two-sided CUSUM $\overline{X}$ chart of Page (1954) when parameters are estimated using a Markov chain approach. Also, we plan to develop a FORTRAN program to determine the run length distribution of the two- sided CUSUM $\overline{X}$ chart of Crosier (1986). Little work has been done with CUSUM charts that are useful in monitoring for a change in the standard deviation of the quality measurement of interest. We would like to extend our work to the CUSUM R (sample range) and S (sample standard deviation) charts. Integral equations that would be useful in describing the run length distribution of the two-sided CUSUM $\overline{X}$ of Page (1954) do not presently exist. We plan to look into developing this method for the two-sided CUSUM $\overline{X}$ of Page (1954).

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Derivation of $M_{c,2}(u)$

The expected value $M_{c,2}(u)$ of the square of the run length for Crosier's (1986) two-sided CUSUM $\overline{X}$ chart is determined as follows.

$$\begin{aligned} M_{c,2}(u) &= \sum_{t=1}^{\infty} t^2 pr_c(t|u) = pr_c(1|u) + \sum_{t=2}^{\infty} t^2 pr_c(t|u) \\ &= pr_c(1|u) + \sum_{t=1}^{\infty} (1+t)^2 pr_c(1+t|u) \\ &= pr_c(1|u) + \sum_{t=2}^{\infty} pr_c(t|u) + 2\sum_{t=1}^{\infty} t pr_c(1+t|u) \\ &\quad + \sum_{t=1}^{\infty} t^2 pr_c(1+t|u). \end{aligned}$$

Since

$$pr_c(1|u) + \sum_{t=1}^{\infty} pr_c(1+t|u) = pr_c(1|u) + \sum_{t=2}^{\infty} pr_c(t|u) = 1,$$

then

$$M_{c,2}(u) = 1 + 2\sum_{t=1}^{\infty} t pr_c(1+t|u) + \sum_{t=1}^{\infty} t^2 pr_c(1+t|u).$$

Writing

$$\begin{aligned} pr_c(1+t|u) &= \int_{-h}^{0} pr_c(t|c_1) f_{C_1|C_0}(c_1|u)\,dc_1 + pr_c(t|0) F_{C_1|C_0}(0|u) \\ &\quad + \int_{0}^{h} pr_c(t|c_1) f_{C_1|C_0}(c_1|u)\,dc_1, \end{aligned}$$

we have

$$\begin{aligned} M_{c,2}(u) &= 1 + 2\sum_{t=1}^{\infty} t \int_{-h}^{0} pr_c(t|c_1) f_{C_1|C_0}(c_1|u)\,dc_1 \\ &\quad +2\sum_{t=1}^{\infty} t pr_c(t|0) F_{C_1|C_0}(0|u) \\ &\quad +2\sum_{t=1}^{\infty} t \int_{0}^{h} pr_c(t|c_1) f_{C_1|C_0}(c_1|u)\,dc_1 \\ &\quad +\sum_{t=1}^{\infty} t^2 \int_{-h}^{0} pr_c(t|c_1) f_{C_1|C_0}(c_1|u)\,dc_1 \\ &\quad +\sum_{t=1}^{\infty} t^2 pr_c(t|0) F_{C_1|C_0}(0|u) \\ &\quad +\sum_{t=1}^{\infty} t^2 \int_{0}^{h} pr_c(t|c_1) f_{C_1|C_0}(c_1|u)\,dc_1. \end{aligned}$$

After regrouping, we have

$$
\begin{aligned}
M_{c,2}(u) &= 1 + 2\int_{-h}^{0}\left(\sum_{t=1}^{\infty} tpr_c(t\,|c_1)\right) f_{C_1|C_0}(c_1\,|u)\,dc_1 \\
&+2\left(\sum_{t=1}^{\infty} tpr_c(t\,|0)\right) F_{C_1|C_0}(0\,|u) \\
&+2\int_{0}^{h}\left(\sum_{t=1}^{\infty} tpr_c(t\,|c_1)\right) f_{C_1|C_0}(c_1\,|u)\,dc_1 \\
&+\int_{-h}^{0}\left(\sum_{t=1}^{\infty} t^2pr_c(t\,|c_1)\right) f_{C_1|C_0}(c_1\,|u)\,dc_1 \\
&+\left(\sum_{t=1}^{\infty} t^2pr_c(t\,|0)\right) F_{C_1|C_0}(0\,|u) \\
&+\int_{0}^{h}\left(\sum_{t=1}^{\infty} t^2pr_c(t\,|c_1)\right) f_{C_1|C_0}(c_1\,|u)\,dc_1.
\end{aligned}
$$

Observing that

$$
M(c_1) = \sum_{t=1}^{\infty} tpr_c(t\,|c_1)\,; M_2(c_1) = \sum_{t=1}^{\infty} t^2pr_c(t\,|c_1)\,;
$$

and making these substitutions, then

$$
\begin{aligned}
M_{c,2}(u) &= 1 + 2\int_{-h}^{0} M_c(c_1)\frac{v_0}{\lambda}\phi\left(\frac{v_0(c_1-u-k)-c}{\lambda}\right)dc_1 + 2M_c(0) \\
&\times\left[\Phi\left(\frac{v_0(0-u+k)-c}{\lambda}\right) - \Phi\left(\frac{v_0(0-u-k)-c}{\lambda}\right)\right] \\
&+2\int_{0}^{h} M_c(c_1)\frac{v_0}{\lambda}\phi\left(\frac{v_0(c_1-u+k)-c}{\lambda}\right)dc_1 \\
&+\int_{-h}^{0} M_{c,2}(c_1)\frac{v_0}{\lambda}\phi\left(\frac{v_0(c_1-u-k)-c}{\lambda}\right)dc_1 + M_{c,2}(0) \\
&\times\left[\Phi\left(\frac{v_0(0-u+k)-c}{\lambda}\right) - \Phi\left(\frac{v_0(0-u-k)-c}{\lambda}\right)\right] \\
&+\int_{0}^{h} M_{c,2}(c_1)\frac{v_0}{\lambda}\phi\left(\frac{v_0(c_1-u+k)-c}{\lambda}\right)dc_1
\end{aligned}
$$

FORTRAN PROGRAM

```
c*---------------------------------------------------*

   c* The ARL and SDRL of the two-sided CUSUM chart of *

   c* Page (1954) are computed using a Markov chain *

   c* method.   *
```

```
c*-----------------------------------------------------*
c* Variables List *
c* Integer Variables *
c* etan - number of states lower sided chart *
c* etap - number of states upper sided chart *
c* eta=etn*etp *
c* etamax -- mazimum of eta *
c* m - number of preliminary samples *
c* n - sample size *
c* npmf -- size of the pmf vector *
c* ts -- run length after which run length *
c* approximated by geometric distribution *
c* Double Precision Variables *
c* arl - estimated average run length of chart *
c* delta -- standardized shift in the mean *
c* hn -- lower control limit *
c* hp -- upper control limit *
c* kn -- k of lower sided chart *
c* kp -- k of upper sided chart *
c* lambda -- sigma divided by sigma_0 *
c* Pmf(200) -- probability mass function to 200 *
c* q(1000,1000) -- Q matrix, max 1000x1000 *
c* sdrl -- standard deviation of the run length *
c* v0 -- unbiased estimator of sigma_0 divided by *
c* sigma_0 *
c* z0 -- standardized value of estimate of mu_0 *
c*-----------------------------------------------------*
```

```
c*
 integer eta,etamax,etan,etap,m,n,npmf,ts
 double precision arl,delta,hn,hp,kn,kp,lp,
 & lambda,pmf(200),q(1000,1000),sdrl,v0,z0
c*
 etamax=1000
 npmf=200
c*
c*-----------------------------------------------------*
c* Various variables are initialized *
c*-----------------------------------------------------*
c*
 etan=30
 etap=30
 delta=0.0d0
 lambda=1.0d0
 m=10
 n=5
 z0=0.0d0
 v0=1.0d0
 kn=0.5d0
 kp=0.5d0
 hn=-4.292d0
 hp=-hn
c*
c*-----------------------------------------------------*
c* Matrix Q is determined *
```

```
c*-----------------------------------------------------*
c*
 call qcusum2(q,eta,etamax,delta,lambda,
 & m,n,z0,v0,kn,kp,hn,hp,etan,etap)
c*
c*-----------------------------------------------------*
c* Run length distribution is determined with tail *
c* approximated by a geometric distribution *
c*-----------------------------------------------------*
c*
 call rldistr(q,eta,etamax,pmf,npmf,ts,lp)
c*
c*-----------------------------------------------------*
c* ARL and SDRL are determined *
c*-----------------------------------------------------*
c*
 call rlpar(pmf,npmf,ts,lp,arl,sdrl)
c*
c*-----------------------------------------------------*
c* ARL and SDRL reported *
c*-----------------------------------------------------*
c*
 write(*,60) arl,sdrl
60 format(2(1x,f8.3))
c*
 stop
 end
```

```
c*
c*-----------------------------------------------------*
c* The Q matrix is calculated *
c*-----------------------------------------------------*
c*
 subroutine qcusum2(q,eta,etamax,delta,lambda,
 & m,n,z0,v0,kn,kp,hn,hp,etan,etap)
c*
c*-----------------------------------------------------*
c* Variables List *
c* Defined in Main Routine *
c*-----------------------------------------------------*
c*
 integer eta,etamax,etan,etap,i0,i1,j0,j1,m,n
 double precision c,delta,DNML,hn,hp,kn,kp,
 & lambda,q(etamax,etamax),tp,tpmax,tpmin,wn,
 & wp,v0,z0
c*
c*-----------------------------------------------------*
c* Initializing some variables *
c*-----------------------------------------------------*
c*
 dsqrm=dsqrt(1.0d0*m)
 dsqrn=dsqrt(1.0d0*n)
 c=(dsqrn*delta)-(z0/dsqrm)
c*
c*-----------------------------------------------------*
```

```
c* i0=the initial state in the negative direction *
c* j0=the initial state in the positive direction *
c* i1=the next state in the negative direction *
c* j1=the next state in the positive direction *
c*-----------------------------------------------------*
c*
 eta=etan*etap
 wn=2.0d0*dabs(hn)/(2.0d0*etan-1)
 wp=2.0d0*dabs(hp)/(2.0d0*etap-1)
c
 do 2 i=1,eta
 do 1 j=1,eta
c
 j0=(i-1)/etan
 i0=(i-1)-etan*j0
 j1=(j-1)/etan
 i1=(j-1)-etan*j1
c
 if ((i1.eq.0).and.(j1.eq.0)) then
 tpmax=(dsqrt(v0)*((i0-0.5d0)*wn-kn)-c)/lambda
 tpmin=(dsqrt(v0)*((0.5d0-j0)*wp+kp)-c)/lambda
 endif
c
 if ((i1.ne.0).and.(j1.eq.0)) then
 tpmax=(dsqrt(v0)*((i0-i1-0.5d0)*wn-kn)-c)/lambda
 tpmin=(dsqrt(v0)*((i0-i1+0.5d0)*wn-kn)-c)/lambda
 tp=(dsqrt(v0)*((0.5d0-j0)*wp+kp)-c)/lambda
```

```
      if (tp.lt.tpmin) tpmin=tp
      endif
c
      if ((i1.eq.0).and.(j1.ne.0)) then
      tpmax=(dsqrt(v0)*((i0-0.5d0)*wn-kn)-c)/lambda
      tp=(dsqrt(v0)*((j1-j0-0.5d0)*wp+kp)-c)/lambda
      if (tp.gt.tpmax) tpmax=tp
      tpmin=(dsqrt(v0)*((j1-j0+0.5d0)*wp+kp)-c)/lambda
      endif
c
      if ((i1.ne.0).and.(j1.ne.0)) then
      tpmax=(dsqrt(v0)*((i0-i1-0.5d0)*wn-kn)-c)/lambda
      tp=(dsqrt(v0)*((j1-j0-0.5d0)*wp+kp)-c)/lambda
      if (tp.gt.tpmax) tpmax=tp
      tpmin=(dsqrt(v0)*((i0-i1+0.5d0)*wn-kn)-c)/lambda
      tp=(dsqrt(v0)*((j1-j0+0.5d0)*wp+kp)-c)/lambda
      if (tp.lt.tpmin) tpmin=tp
      endif
      q(i,j)=0.0d0
      if (tpmax.lt.tpmin)
     & q(i,j)=DNML(tpmin)-DNML(tpmax)
c
1     continue
2     continue
c
      return
      end
```

```
c*
c*------------------------------------------------------*
c* The probability mass function (pmf) for the run *
c* distribution is determined for a given Q matrix *
c* describing the (approximate) Markov chain *
c* representation of the chart.  *
c*------------------------------------------------------*
c*
 subroutine rldistr(q,eta,etamax,pmf,npmf,ts,lp)
c*
c*------------------------------------------------------*
c* Variables List *
c* See list in Main Routine *
c* lhat -- one estimate of parameter in geometric *
c* approximation of the tail probabilities *
c* lp -- another estimate of parameter in geometric *
c* approximation of the tail probabilities *
c* pr(1000) -- vector of run length probabilities *
c* associated with the nonabsorbing states *
c* pr1(1000) -- previous value of pr(1000) *
c*------------------------------------------------------*
c*
    integer ck,eta,etamax,i,istate,j,npmf,t,ts
    double precision cdf,lhat,lp,q(etamax,etamax),
 & pmf(npmf),pr(1000),pr1(1000)
c*
c*------------------------------------------------------*
```

```
c* Probability mass function of the run length *
c* distribution is calculated iteratively by *
c* pr(eta) = Q*pr1(eta) *
c*---------------------------------------------------*
c*
 ck=0
 istate=1
c*
 t=1
 do 2 i=1,eta
 pr(i)=0.0d0
 do 1 j=1,eta
 pr(i)=pr(i)+q(i,j)
1 continue
 pr(i)=1.0d0-pr(i)
2 continue
 pmf(1)=pr(istate)
 cdf=pmf(1)
c*
3 t=t+1
 do 4 i=1,eta
 pr1(i)=pr(i)
4 continue
c*
 do 6 i=1,eta
 pr(i)=0.0d0
 do 5 j=1,eta
```

```
 pr(i)=pr(i)+q(i,j)*pr1(j)
5 continue
6 continue
 pmf(t)=pr(istate)
 cdf=cdf+pmf(t)
 if (t.le.20) goto 3
 if (ck.eq.1) goto 9
c*
 lhat=pmf(t)/pmf(t-1)
 lp=1.0d0-cdf+pmf(t)
 lp=(1.0d0-cdf)/lp
 epsilon=dabs(lhat-lp)
 if (epsilon.gt.0.000001d0) goto3
c*
 ts=t
 ck=1
 goto 3
c*
9 return
 end
c*
c*----------------------------------------------------*
c* ARL and SDRL are determined
c*----------------------------------------------------*
c*
 subroutine rlpar(pmf,npmf,ts,lp,arl,sdrl)
c*
```

```
c*-----------------------------------------------------*
c* Variables list *
c* See main routine *
c* i -- indexing variable *
c* t -- value of the run length *
c* arl -- average run length *
c* sdrl -- standard deviation of the run length *
c* tmp -- temporary variable *
c*-----------------------------------------------------*
c*
 integer i,npmf,t,ts
 double precision arl,cdf,lp,pmf(npmf),
 & sdrl,tmp
c*
c*----------------------------------------------------*
c* ARL and mean of the square of the run length are *
c* calculated exactly up to an including t* (ts) *
c* with the tail part of each approximated using *
c* a geometric distribution to approximate the tail *
c* probabilities.  *
c*----------------------------------------------------*
 i=1
 arl=0.0d0
 sdrl=0.0d0
 cdf=0.0d0
 do 1 t=1,ts
 arl=arl+t*pmf(t)
```

```
 sdrl=sdrl+t*t*pmf(t)
 cdf=cdf+pmf(t)
1 continue
 tmp=pmf(ts+1)*(ts/(1.0d0-lp)
 & +1/((1.0d0-lp)*(1.0d0-lp)))
 arl=arl+tmp
 tmp=pmf(ts+1)*(ts*ts/(1.0d0-lp)
 & +(2.0d0*ts-1.0d0)/((1.0d0-lp)*(1.0d0-lp))
 & +2.0d0/((1.0d0-lp)*(1.0d0-lp)*(1.0d0-lp)))
 sdrl=sdrl+tmp
 sdrl=sdrl-arl*arl
 sdrl=dsqrt(sdrl)
c*
 return
 end
C
 DOUBLE PRECISION FUNCTION DNML(X)
C
C COMPUTES THE CUMULATIVE DISTRIBUTION FUNCTION
C P(Y<=X) OF A RANDOM VARIABLE Y HAVING A
C STANDARD NORMAL DISTRIBUTION.
C
 DOUBLE PRECISION X,Y,S,RN,ZERO,ONE,ERF,SQRT2,PI
 DATA SQRT2,ONE/1.414213562373095,1.D0/
 DATA PI,ZERO/3.141592653589793,0.D0/
 Y=X/SQRT2
 IF (X.LT.ZERO) Y=-Y
```

```
      S=ZERO
      DO 1 N=1,37
      RN=N
      S=S+DEXP(-RN*RN/25)/N*DSIN(2*N*Y/5)
    1 CONTINUE
      S=S+Y/5
      ERF=2*S/PI
      DNML=(ONE+ERF)/2
      IF (X.LT.ZERO) DNML=(ONE-ERF)/2
      IF (X.LT.-8.3D0) DNML=ZERO
      IF (X.GT.8.3D0) DNML=ONE
      RETURN
      END
```