



MODAL ANALYSIS OF MULTISTEP TIMOSHENKO BEAM WITH A NUMBER OF CRACKS

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Received: 2 May 2018; Accepted for publication: 15 October 2018

ABSTRACT

Modal analysis of cracked multistep Timoshenko beam is accomplished by the Transfer Matrix Method (TMM) based on a closed-form solution for Timoshenko uniform beam element. Using the solution allows significantly simplifying application of the conventional TMM for multistep beam with multiple cracks. Such simplified transfer matrix method is employed for investigating effect of beam slenderness and stepped change in cross section on sensitivity of natural frequencies to cracks. It is demonstrated that the transfer matrix method based on the Timoshenko beam theory is usefully applicable for beam of arbitrary slenderness while the Euler-Bernoulli beam theory is appropriate only for slender one. Moreover, stepwise change in cross-section leads to a jump in natural frequency variation due to crack at the steps. Both the theoretical development and numerical computation accomplished for the cracked multistep beam have been validated by an experimental study.

Keywords: Timoshenko beam theory; multi-stepped beam; multi-cracked beam; natural frequencies; transfer matrix method.

Classification numbers: 5.4.2; 5.4.3 và 5.4.6

1. INTRODUCTION

Beam-like structures with stepwise changes in cross-section called stepped beams are widely used in the practice of construction and machinery engineering and can be used also as a proper approximation of nonuniform beams. Therefore, dynamics of stepped beams is a problem of great importance. A lot of publications was devoted to study vibration of stepped beams and main results obtained in the earlier studies can be summarized as follow: (1) It was discovered that an abrupt change in cross-section leads to typical variation of the dynamic properties such as natural frequencies [1-3], mode shapes [4-6] or frequency response functions [4, 7] of beams; (2) The variation is strongly dependent on location of the discontinuity [8] and boundary conditions of beam [6, 9, 10]; (3) Shear deformation and rotary inertia make also a remarkable effect on the change in dynamic properties caused by the varying cross-section [8, 11]; (4) The correlation

between the dynamic properties and geometrical discontinuity provides a beneficial effect for design of a stepped beam [12]. Also, numerous methods have been developed to study vibration of the beams such as Transfer Matrix Method (TMM) [1-3, 9]; Adomian decomposition method [5] or differential quadrature element method [10]; Green's function method [11]; Galerkin's or Rayleigh-Ritz method [13,14]. The short outline enables to make the following notices: firstly, since a segment in a stepped beam is rarely a slender or long beam element, the Timoshenko beam theory should be more appropriately used for analysis of multistep beams; secondly, among the proposed methods the TMM shows to be most convenient technique that is efficiently applicable also for investigating the stepped beams with other discontinuities such as cracks.

Vibration of cracked structures is a problem of significant interest during the last decades and a lot of methods have been proposed for analysis and identification of stepped beams with cracks [14-21]. From the studies it is worthy to highlight two important results: (a) Li established in his work [20] a recurrent connection of free vibration shapes of segments in a multistep beam that enables to easily conduct explicit frequency equation of the beam with multiple cracks; (b) Attar [21] has completely developed the TMM for not only free vibration analysis but also crack identification problem of multistep Euler-Bernoulli beams with a number of transverse cracks. Nevertheless, the achievements have been accomplished for Euler-Bernoulli beams only, therefore, expanding the obtained results for Timoshenko multistep beams with multiple cracks is essential. Actually, Timoshenko beams with cracks were studied by numerous authors for instance in Refs. [22-27] that allow one to make the following remarks: (a) The Timoshenko beam theory gives rise results more close to experimental ones and those obtained by FEM than the Euler-Bernoulli theory; discrepancy between the beam theories increases with decreasing slenderness ratio (L/h) and increasing crack depth; (b) Reduction of beam slenderness ratio leads dynamic characteristics of beam to be more sensitive to crack; (c) Among the studies on cracked Timoshenko beams there is very few publications on cracked multistep Timoshenko beams, except the Ref. [27] where a stepped shaft with single crack was investigated by using the TMM and Timoshenko beam theory.

The present paper addresses the problem for free vibration of multistep Timoshenko beams with arbitrary number of cracks, continuing the work accomplished in [28], where the problem was studied on the base of Euler-Bernoulli beam theory. First, the obtained general solution of uniform Timoshenko beam is employed to develop the TMM for modal analysis of multistep Timoshenko beam with multiple cracks. Then, effect of beam slenderness and stepped change in cross section on sensitivity of natural frequencies to cracks is thoroughly examined.

2. GENERAL SOLUTION FOR FREE VIBRATION OF CRACKED TIMOSHENKO UNIFORM BEAM

Consider a uniform beam element of length L ; material density (ρ); elasticity (E) and shear (G) modulus; area $A = b \times h$ and moment of inertia $I = bh^3 / 12$ of cross section. Assuming first order shear deformation (Timoshenko) theory of beam, the displacement field in cross-section at x is

$$u(x, z, t) = u_0(x, t) - z\theta(x, t); w(x, z, t) = w_0(x, t), \quad (2.1)$$

with $u_0(x, t)$, $w_0(x, t)$, $\theta(x, t)$ being respectively the displacements and slope at central axis. Therefore, constituting equations get the form

$$\varepsilon_x = \partial u_0 / \partial x - z \partial \theta / \partial x; \gamma_{xz} = \partial w_0 / \partial x - \theta; \sigma_x = E \varepsilon_x; \tau_{xz} = \kappa G \gamma_{xz}. \quad (2.2)$$

Using Hamilton principle equations for free vibration of the beam element can be established in the form

$$\rho A \ddot{w} - \kappa G A (w'' - \theta') = 0; \rho I \ddot{\theta} - EI \theta'' - \kappa G A (w' - \theta) = 0. \quad (2.3)$$

Seeking solution of (2.3) in the form

$$w(x, t) = W(x)e^{i\omega t}, \theta(x, t) = \Theta(x)e^{i\omega t}, \quad (2.4)$$

one gets

$$\omega^2 \rho W(x) + \kappa G (W'' - \Theta') = 0; \omega^2 \rho I \Theta(x) + EI \Theta''(x) + \kappa G A (W' - \Theta) = 0. \quad (2.5)$$

Furthermore, it is assumed that the beam has been cracked at positions $e_j, j = 1, \dots, n$ and the cracks are modeled by equivalent rotational springs of stiffness K_j calculated from crack depth (see Appendix). Therefore, conditions that must be satisfied at the crack section are

$$W(e_j + 0) = W(e_j - 0); \Theta(e_j + 0) = \Theta(e_j - 0) + M(e_j) / K_j; \\ Q(e_j + 0) = Q(e_j - 0) = Q(e_j); M(e_j + 0) = M(e_j - 0) = M(e_j), \quad (2.6)$$

where N, Q, M are respectively internal axial, shear forces and bending moment at a section x

$$M = EI \Theta'_x; Q = \kappa G A (W'_x - \Theta). \quad (2.7)$$

Substituting (2.7) into (2.6) one can rewrite the latter conditions as

$$W(e_j + 0) = W(e_j - 0) = W(e_j); \Theta'_x(e_j + 0) = \Theta'_x(e_j - 0) = \Theta'_x(e_j); \\ \Theta(e_j + 0) = \Theta(e_j - 0) + \gamma_j \Theta'_x(e_j); W'_x(e_j + 0) = W'_x(e_j - 0) + \gamma_j \Theta'_x(e_j); \gamma_j = EI / K_j, \quad (2.8)$$

where [29]

$$\gamma_i = EI / K_i = 6\pi(1 - \nu^2)hf(a_i / h); \\ f_0(z) = z^2(0.6272 - 1.04533z + 4.5948z^2 - 9.9736z^3 + 20.2948z^4 - 33.0351z^5 + \\ + 47.1063z^6 - 40.7556z^7 + 19.6z^8).$$

Seeking solution of Eq. (2.5) in the form $W_0(x) = C_w e^{\lambda x}, \Theta_0(x) = C_\theta e^{\lambda x}$ one is able to obtain so-called characteristic equation

$$\lambda^4 + b\lambda^2 - c = 0; \quad (2.9)$$

$$b = \alpha(1 + \beta); c = \alpha(\tau - \alpha\beta); \alpha = \rho\omega^2 / E; \beta = E / \kappa G; \tau = A / I. \quad (2.10)$$

This is a cubic algebraic equation with respect to $\eta = \lambda^2$ that can be elementarily solved to give roots

$$\eta_1 = (-b + \sqrt{b^2 + 4c}) / 2; \eta_2 = -(b + \sqrt{b^2 + 4c}) / 2. \quad (2.11)$$

Note that in the case if $c = 0$ the Eq. (2.9) has the roots

$$\lambda_{1,2} = \pm i\sqrt{b} = \pm i\omega\sqrt{\rho(1 + \beta) / E}; \lambda_{3,4} = 0. \quad (2.12)$$

This occurs when $\omega = \omega_c = \sqrt{12\kappa G / \rho h^2}$ that is acknowledged as cut-off frequency. Otherwise, the Eq. (2.9) has the roots

$$\lambda_{1,2} = \pm k_1; \lambda_{3,4} = \pm ik_2; k_1 = \sqrt{(\sqrt{b^2 + 4c} - b) / 2}, k_2 = \sqrt{(\sqrt{b^2 + 4c} + b) / 2} \quad (2.13)$$

for frequency less than cut-off one, $\omega < \omega_c = \sqrt{\kappa GA / \rho I}$. Since the cut-off frequency is very high, vibration of the beam is often investigated in the lower frequency range $(0, \omega_c)$. Thus, in the frequency range, general continuous solution of Eq. (2.5) can be represented as

$$W_0(x) = C_1 \cosh k_1 x + C_2 \sinh k_1 x + C_3 \cos k_2 x + C_4 \sin k_2 x; \quad (2.14)$$

$$\Theta_0(x) = r_1 C_1 \sinh k_1 x + r_1 C_2 \cosh k_1 x + r_2 C_3 \sin k_2 x - r_2 C_4 \cos k_2 x, \quad (2.15)$$

$$r_1 = (\rho \omega^2 + \kappa G k_1^2) / \kappa G k_1; r_2 = (\rho \omega^2 - \kappa G k_2^2) / \kappa G k_2. \quad (2.16)$$

Particularly, solution (2.14) and (2.15) satisfying the conditions $W_0(0) = 0; W_0'(0) = 1; \Theta_0(0) = 1; \Theta_0'(0) = 0$ is

$$S_w(x) = S_1 \sinh k_1 x + S_2 \sin k_2 x; S_\theta(x) = r_1 S_1 \cosh k_1 x - r_2 S_2 \cos k_2 x; \quad (2.17)$$

$$S_1 = (r_2 + k_2) / (r_1 k_2 + r_2 k_1); S_2 = (r_1 - k_1) / (r_1 k_2 + r_2 k_1). \quad (2.18)$$

Using obtained above particular solution, general solution of Eq. (2.5) satisfying conditions (2.8) at cracks is represented by [30]

$$W(x, \omega) = C_1 W_1(k_1, x) + C_2 W_2(k_1, x) + C_3 W_3(k_2, x) + C_4 W_4(k_2, x); \quad (2.19)$$

$$\Theta(x, \omega) = C_1 \Theta_1(k_1, x) + C_2 \Theta_2(k_1, x) + C_3 \Theta_3(k_2, x) + C_4 \Theta_4(k_2, x), \quad (2.20)$$

where

$$\begin{aligned} \{W_1(x), W_2(x), W_3(x), W_4(x)\}^T &= \\ &= \{\cosh k_1 x, \sinh k_1 x, \cos k_2 x, \sin k_2 x\}^T + \sum_{j=1}^n \{\mu_{1j}, \mu_{2j}, \mu_{3j}, \mu_{4j}\}^T K_w(x - e_j); \end{aligned} \quad (2.21)$$

$$\begin{aligned} \{\Theta_1(x), \Theta_2(x), \Theta_3(x), \Theta_4(x)\}^T &= \\ &= \{r_1 \sinh k_1 x, r_1 \cosh k_1 x, r_2 \sin k_2 x, -r_2 \cos k_2 x\}^T + \sum_{j=1}^n \{\mu_{1j}, \mu_{2j}, \mu_{3j}, \mu_{4j}\}^T K_\theta(x - e_j); \\ K_w(x) &= \{0 : x < 0; S_w(x) : x \geq 0\}; K_w'(x) = \{0 : x < 0; S_w'(x) : x \geq 0\}; \\ K_\theta(x) &= \{0 : x < 0; S_\theta(x) : x \geq 0\}; K_\theta'(x) = \{0 : x < 0; S_\theta'(x) : x \geq 0\}; \\ \mu_{kj} &= \gamma_j L_k(e_j) + \sum_{i=1}^{j-1} \mu_{ki} S_\theta'(e_j - e_i); k = 1, 2, 3, 4; j = 1, 2, \dots, n. \end{aligned} \quad (2.22)$$

$$L_1(x) = k_1 r_1 \cosh k_1 e; L_2(x) = k_1 r_1 \sinh k_1 e; L_3(x) = k_2 r_2 \cos k_2 e; L_4(x) = k_2 r_2 \sin k_2 e.$$

Therefore, shear force and bending moment defined in (2.7) can be represented as

$$M(x, \omega) = C_1 M_1(k_1, x) + C_2 M_2(k_1, x) + C_3 M_3(k_2, x) + C_4 M_4(k_2, x); \quad (2.23)$$

$$Q(x, \omega) = C_1 Q_1(k_1, x) + C_2 Q_2(k_1, x) + C_3 Q_3(k_2, x) + C_4 Q_4(k_2, x), \quad (2.24)$$

where $M_i(k, x) = EI \Theta_i'(k, x); Q_i(k, x) = \kappa GA [W_i'(k, x) - \Theta_i(k, x)]; i = 1, 2, 3, 4$.

Finally, Eqs. (2.19) - (2.22) can be rewritten in the matrix form

$$\begin{Bmatrix} W(x, \omega) \\ \Theta(x, \omega) \\ M(x, \omega) \\ Q(x, \omega) \end{Bmatrix} = \begin{bmatrix} W_1(k_1, x) & W_2(k_1, x) & W_3(k_2, x) & W_4(k_2, x) \\ \Theta_1(k_1, x) & \Theta_2(k_1, x) & \Theta_3(k_2, x) & \Theta_4(k_2, x) \\ M_1(k_1, x) & M_2(k_1, x) & M_3(k_2, x) & M_4(k_2, x) \\ Q_1(k_1, x) & Q_2(k_1, x) & Q_3(k_2, x) & Q_4(k_2, x) \end{bmatrix} \cdot \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}, \quad (2.25)$$

that is fundamental to develop the TMM for multistep Timoshenko beam with multiple cracks in subsequent section.

3. THE TRANSFER MATRIX METHOD FOR MULTIPLE CRACKED AND STEPPED TIMOSHENKO BEAM

Let's consider now a stepped beam composed of m uniform beam segments with size $b_j \times h_j \times L_j$ denoted by subscript j , $j = 1, 2, \dots, m$, shown in Fig. 1. Suppose that each of the beam steps contains a number of crack represented by its position $e_{jk}, k = 1, \dots, n_j$ and magnitude $\gamma_{jk} = EI_{jk} / K_{jk}$.

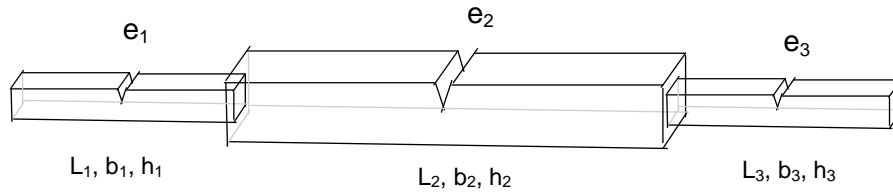


Figure 1. Model of cracked multistep beam.

Introduce state vector for j -th step as $\mathbf{V}_j(x) = \left\{ \begin{matrix} W_j(x) \\ \Theta_j(x) \\ M_j(x) \\ Q_j(x) \end{matrix} \right\}$ with bending moment $M_j(x)$ and shear force $Q_j(x)$ are defined by Eq. (2.7). So, continuity conditions at step joints are

$$\mathbf{V}_{j+1}(0) = \mathbf{V}_j(L_j), j = 1, 2, \dots, m, \tag{3.1}$$

On the other hand, using (2.23) the introduced state vector $\mathbf{V}_j(x)$ can be represented as

$$\mathbf{V}_j(x) = [\mathbf{H}_j(x)]\mathbf{C}_j, \tag{3.2}$$

where

$$[\mathbf{H}_j(x)] = \begin{bmatrix} W_1(k_{j1}, x) & W_2(k_{j1}, x) & W_3(k_{j2}, x) & W_4(k_{j2}, x) \\ \Theta_1(k_{j1}, x) & \Theta_2(k_{j1}, x) & \Theta_3(k_{j2}, x) & \Theta_4(k_{j2}, x) \\ M_1(k_{j1}, x) & M_2(k_{j1}, x) & M_3(k_{j2}, x) & M_4(k_{j2}, x) \\ Q_1(k_{j1}, x) & Q_2(k_{j1}, x) & Q_3(k_{j2}, x) & Q_4(k_{j2}, x) \end{bmatrix}. \tag{3.3}$$

So that the state vector is transferred from the left to right ends of the beam span by

$$\mathbf{V}_j(L_j) = [\mathbf{H}_j(L_j)]\mathbf{H}_j^{-1}(0)\mathbf{V}_j(0) = \mathbf{T}(j)\mathbf{V}_j(0). \tag{3.4}$$

Subsequently combining (3.5) with (3.1) for $j = 1, 2, \dots, m$ one obtains

$$\mathbf{V}_m(L_m) = [\mathbf{T}(m)\mathbf{T}(m-1)\dots\mathbf{T}(1)]\{\mathbf{V}_1(0)\} = [\mathbf{T}]\{\mathbf{V}_1(0)\}. \tag{3.5}$$

Usually, conventional boundary conditions are expressed by

$$\mathbf{B}_0\{\mathbf{V}_1(0)\} = 0; \mathbf{B}_L\{\mathbf{V}_m(L_m)\} = 0. \tag{3.6}$$

Consequently,

$$[\mathbf{B}(\omega)]\mathbf{V}_1(0) = 0, \tag{3.7}$$

where

$$\mathbf{B}(\omega) = \begin{bmatrix} \mathbf{B}_0 \\ \mathbf{B}_L \mathbf{T} \end{bmatrix}. \tag{3.8}$$

Equation (3.7) would have nontrivial solution with respect to $\mathbf{V}_1(0)$ under the condition

$$D(\omega) \equiv \det[\mathbf{B}(\omega)] = 0, \tag{3.9}$$

that is frequency equation desired for the stepped beam with cracks.

For instance, if the left end of beam is clamped and the other one is free, i. e. the beam is cantilevered, the boundary conditions are $W_1(0) = \Theta_1(0) = M_m(L_m) = Q_m(L_m) = 0$ that allows one to have got frequency equation as

$$D_{CF}(\omega) = T_{11}T_{22} - T_{21}T_{12} = 0, \quad (3.10)$$

where $T_{ik}, i, k = 1, 2, 3, 4$ are elements of the total transfer matrix $[\mathbf{T}]$ defined in (3.6). Similarly, frequency equation of stepped FGM beam can be obtained as determinant of a 2x2 matrix for other cases of boundary conditions such as simple supports or clamped ends. Namely, for simply supported beam with $W_1(0) = M_1(0) = W_m(L_m) = M_m(L_m) = 0$, frequency equation is

$$D_{SS}(\omega) = T_{12}T_{34} - T_{32}T_{14} = 0. \quad (3.11)$$

For beam with clamped ends where $W_1(0) = \Theta_1(0) = W_m(L_m) = \Theta_m(L_m) = 0$ one has got

$$D_{CC}(\omega) = T_{13}T_{24} - T_{23}T_{14} = 0. \quad (3.12)$$

Solving the frequency equations gives rise natural frequencies $\omega_k, k = 1, 2, 3, \dots$ of the beam that in turn allow one to find corresponding solution of Eq. (3.8) as $\mathbf{V}_1(0) = D_k \bar{\mathbf{V}}_1$ with an arbitrary constant D_k and normalized solution $\bar{\mathbf{V}}_1$. Subsequently, mode shape corresponding to natural frequency ω_k is determined for every beam step as follows

$$\Phi_{jk}(x) = \{W_{jk}(x), \Theta_{jk}(x)\}^T = D_k [\mathbf{G}_c(x, \omega_k)] \bar{\mathbf{C}}_j, \quad (3.13)$$

$$\bar{\mathbf{C}}_j = [\mathbf{H}_j(0)]^{-1} [\mathbf{T}(j-1)\mathbf{T}(j-2)\dots\mathbf{T}(1)] \{\bar{\mathbf{V}}_1\}, j = 1, 2, \dots, m. \quad (3.14)$$

The arbitrary constant D_k is determined by a chosen normalized condition, for example

$$\max_{(x,j)} \Phi_{jk}(x) = 1. \quad (3.15)$$

3. NUMERICAL AND EXPERIMENTAL VALIDATION

To validate the theoretical development of the transfer matrix method proposed above for cracked multistep beam, first three natural frequencies of the beam model (see Table 1) with crack scenarios given in Table 2 are computed by using both Euler-Bernoulli and Timoshenko beam theories and then compared to the measured results (see Fig. 2). The graphs presented in the Figure demonstrate a good agreement of the beam theories applied for cracked multistep beam with experiment. The closeness of natural frequencies computed by the beam theories is explained by the fact that slenderness ratios of the test beam segments are all greater than 20. Nevertheless, it can be observed that Euler-Bernoulli beam theory gives natural frequencies all overestimated in comparison with Timoshenko beam theory and measured frequencies are lower the computed ones. It is because the stiffness of theoretical models is generally higher than that of testing beam. Moreover, natural frequencies computed by different methods (analytical method [31]; Galerkin's method [23] and transfer matrix method) for uniform beam are compared in Tables 3-4. Table 3 shows that the transfer matrix method is really one of the exact methods for computing natural frequencies of beam-like structures. The Galerkin's method gives natural frequencies almost identical to those obtained by TMM in application for uniform beam with different slenderness ratio. However, disagreement of the methods is apparent when they are applied for cracked beam and miscalculation of Galerkin's method can be noticeable from that results in reduction of second and fourth frequencies as the crack appeared at the middle of

beam whereas the frequencies should be unchanged due to crack. Finally, it can be seen from Table 4 that Timoshenko beam model is more useful to apply for calculating natural frequencies of cracked beam.

Table 1. Geometry and material properties of beam with $E = 2GPa$; $\rho = 7855kg / m^3$; $\nu = 0.3$.

Geometrical parameters (mm)	Spans		
	1 st (Left)	2 nd (Middle)	3 rd (Right)
Wide, b	20	20	20
Height, h	15.4	7.8	15.4
Length, L	318	405	318
Total length	1131		

Table 2. Crack scenarios in experimental study of three-step cracked beam.

Crack scenarios	Description	Number of cracks	Positions	Relative depths
1	Intact beam	No crack	-	0 % - 0 % - 0 %
2				0 % - 10 % - 0 %
3	Single crack at midspan	1	0.403	0 % - 20 % - 0 %
4				0 % - 30 % - 0 %
5				0 % - 40 % - 0 %
6				10 % - 40 % - 0 %
7	Two cracks at the left and mid-span	2	0.218 ; 0.403	20 % - 40 % - 0 %
8				30 % - 40 % - 0 %
9				40 % - 40 % - 0 %
10				40 % - 40 % - 10 %
11				40 % - 40 % - 20 %
12	One crack at all three spans	3	0.218; 0.403; 0.823	40 % - 40 % - 30 %
13				40 % - 40 % - 40 %
14				40 % - 50 % - 40 %
15				50 % - 50 % - 40 %
16				50 % - 50 % - 50 %

Table 3. Comparison of frequency parameter ($\lambda_k = [\omega_k^2 \rho A / EI]^{1/4}$) computed by using different beam theories and methods for simply supported uniform intact beam.

Eigenvalue No	1	2	3	4	5
Euler-Bernoulli –Analytical [31]	π	2π	3π	4π	5π
Euler-Bernoulli –TMM (present)	3.1416	6.2832	9.4248	12.5664	15.7080
Timoshenko – Analytical [31]	3.1155	6.0867	8.8180	11.2766	13.4740
Timoshenko – TMM (Present)	3.1157	6.0907	8.8405	11.3431	13.6132
Beam parameters	$E = 2e11; \rho = 7855; \nu = 0.3; \kappa = 5/6; L = 1.0; b = 0.1; h = 0.1$ (m)				

Table 4. Comparison of natural frequencies computed by using different beam theories and methods for simply supported uniform beam with various slenderness (L/h).

Frequency No	1		2		3		4	
	ω_0	ω_c / ω_0	ω_0	ω_c / ω_0	ω_0	ω_c / ω_0	ω_0	ω_c / ω_0
$L/h=15$								
EB – GM [23]	103.64	1.8836	214.56	1.9801	1732.77	0.9185	4808.26	0.9673
EB – TMM (present)	103.64	1.8383	213.10	1.0000	1732.80	0.8740	4851.50	1.0000
TB – GM [23]	101.34	1.8844	179.28	1.9806	1565.03	0.9234	4366.67	0.9707
TB – TMM (Present)	101.30	1.8397	179.30	1.0000	1565.00	0.8827	4367.70	1.0000
$L/h=10$								
EB – GM [23]	155.46	1.8268	821.85	1.9588	1099.15	0.8906	7287.39	0.9397
EB – TMM (present)	155.46	1.7319	819.70	1.0000	1099.20	0.8430	7277.30	1.0000
TB – GM [23]	147.84	1.8293	710.02	1.9613	1599.00	0.9030	5918.77	0.9509
TB – TMM (Present)	147.80	1.7857	710.00	1.0000	1599.00	0.8628	5918.80	1.0000
$L/h=5$								
EB – GM [23]	110.92	1.6855	1643.72	1.8721	1198.31	0.8245	14574.77	0.8545
EB – TMM (present)	110.92	1.6631	1639.30	1.0000	1198.30	0.7922	14555.00	1.0000
TB – GM [23]	155.01	1.6985	1959.38	1.8936	1643.70	0.8686	8551.50	0.9069
TB – TMM (Present)	155.00	1.6799	1959.40	1.0000	1643.70	0.8484	8511.50	1.0000
Beam parameters $E = 62.1\text{GPa}$; $G = 23.3\text{GPa}$; $\rho = 2770$; $\nu = 0.3$; $\kappa = 5/6$								
EB – Euler Beam; TB – Timoshenko Beam; GM – Galerkin Method; TMM – Transfer Matrix Method; ω_0 - natural frequency of intact beam; ω_c / ω_0 - ratio of cracked to intact frequencies								

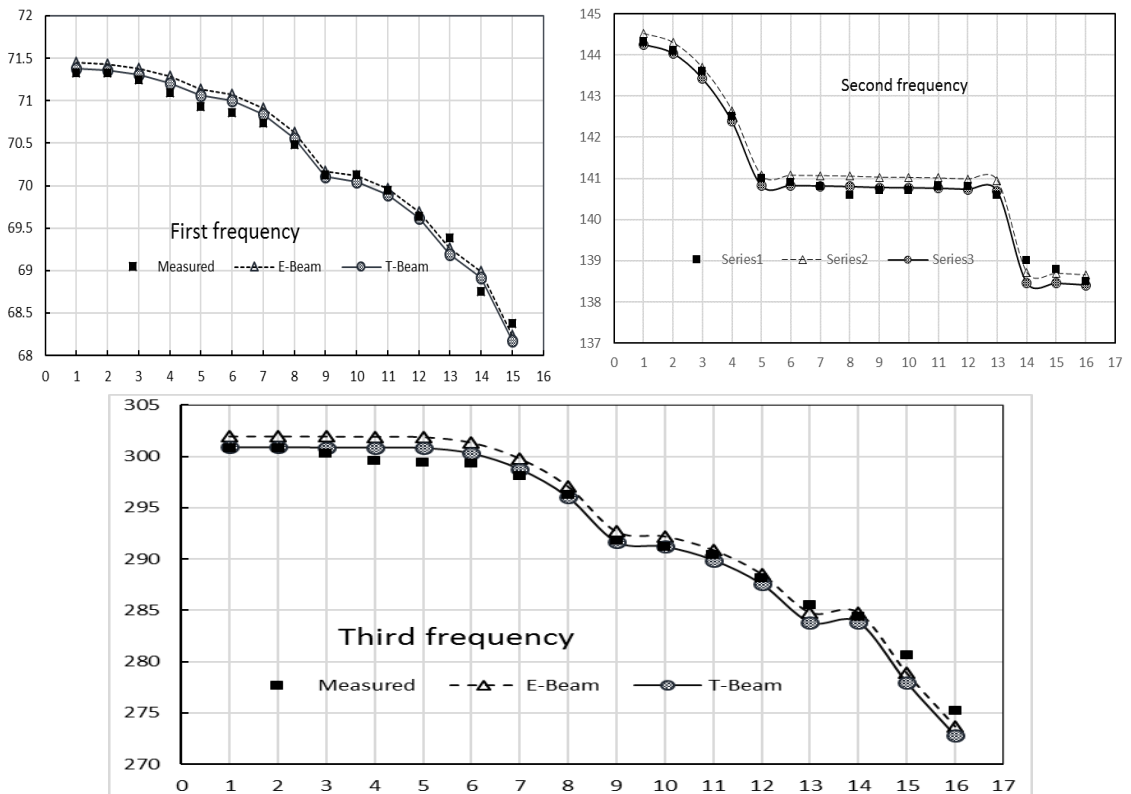


Figure 2. Comparison of natural frequencies computed by the Euler-Bernoulli and Timoshenko beam theories with measured ones for stepped beam in different scenarios of multiple cracks.

4. RESULTS AND DISCUSSION

4.1. Effect of beam slenderness ratio and number of cracks

The aim of present subsection is to discuss on using the beam theories for sensitivity analysis of beam to crack although this question has been addressed by some authors but only in the cases of individual cracks. The sensitivity of natural frequencies to cracks is acknowledged herein as ratio of a frequency of cracked beam to that of intact one and it is computed versus of crack position along beam segments with various scenarios of cracks. Frequency parameters, $\lambda_k = [\omega_k^2 \rho A / EI]^{1/4}$, computed for stepped beam of various slenderness and boundary conditions are tabulated in Table 5. The obtained results allow one to reaffirm the conclusions made on the variation of natural frequencies versus slenderness ratio for stepped beam as follow: (1) Natural frequencies of stepped Euler-Bernoulli beam are always higher than those of stepped Timoshenko beam and their deviation gets to be more significant for decreasing slenderness ratio L/h ; (2) The deviation can be reached to 50 % for $L/h = 5$ and it becomes insignificant for slenderness greater 30; (3) For obtaining reliable natural frequencies of stepped beam in any case of slenderness it is recommended to use the Timoshenko beam theory.

4.2. Effect of step change in beam thickness

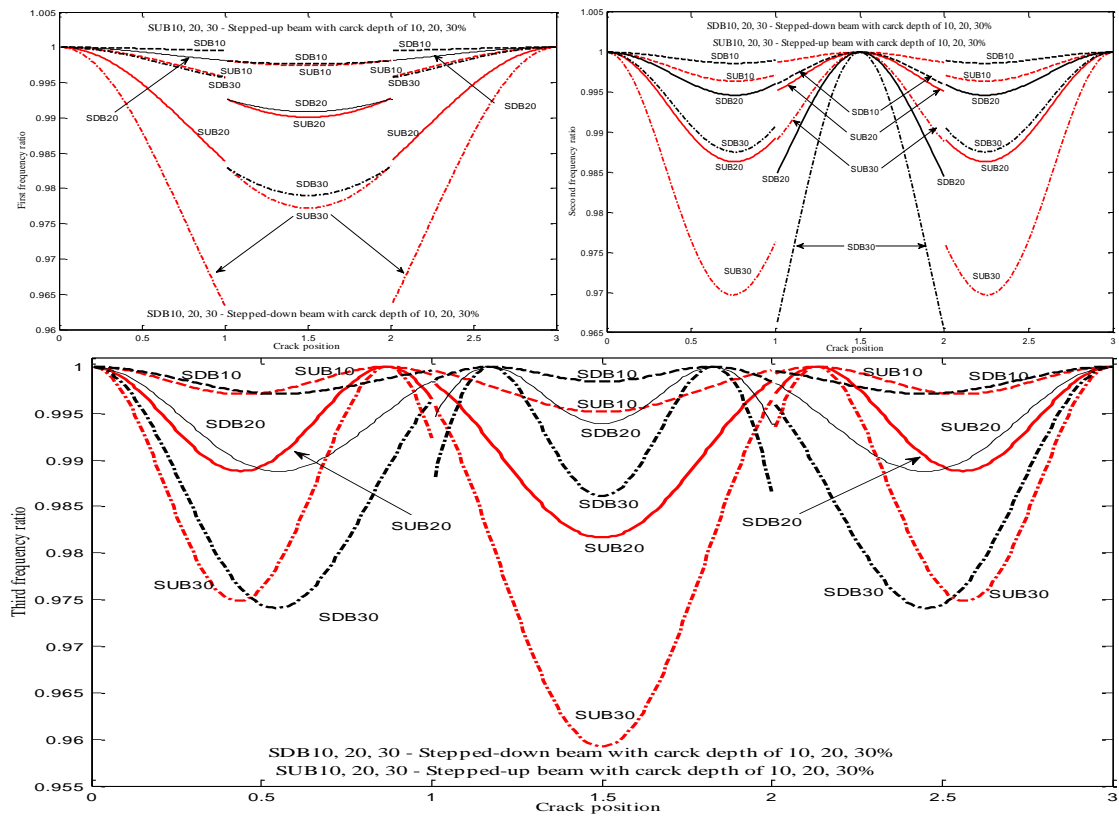


Figure 3. Crack-induced change in natural frequencies computed for step-down (SD) and step-up (SU) simply supported Timoshenko beam.

Two types of stepped beam are investigated in this study that are called step-up beam (SUB) and step-down beam (SDB) and both have three spans (or segments) of equal length. The first type has intermediate segment thicker two other and the other type has thinner intermediate segment. The natural frequency ratios of three lowest frequencies are computed for the SUB and SDB beams with the classical boundary conditions mentioned above as SS-, CC-, CF-beams. The obtained ratios are plotted versus crack position along the beam span for various crack depth (10;20;30 %) and shown in Figures 3 - 5. It is observed jumps in the graphs at the beam steps where thickness of beam undertakes an abrupt change. It can be seen that increase (decrease) of thickness in step-up (step-down) makes natural frequencies less (more) sensitive to crack. Compared to the uniform beam, crack at the central span of SUB makes less change in natural frequencies than that of SDB and it is independent on the boundary conditions of the beam. On the other hand, graphs in the Figures demonstrate that, likely to the uniform beam, there exist positions on stepped beams crack occurred at which does not change a specific natural frequency. Such positions on beam acknowledged herein as frequency nodes can be evidently found in the Figures 3-5. Obviously, step change in thickness of beam shifts the frequency nodes to the left or to the right dependently on whether the thickness variation is step-up or step-down. The shift of frequency nodes is dependent also on the boundary conditions of beam, for instance, the frequency node of second mode in beam with symmetric boundary conditions (SS or CC) is unchanged due to step variation of beam thickness.

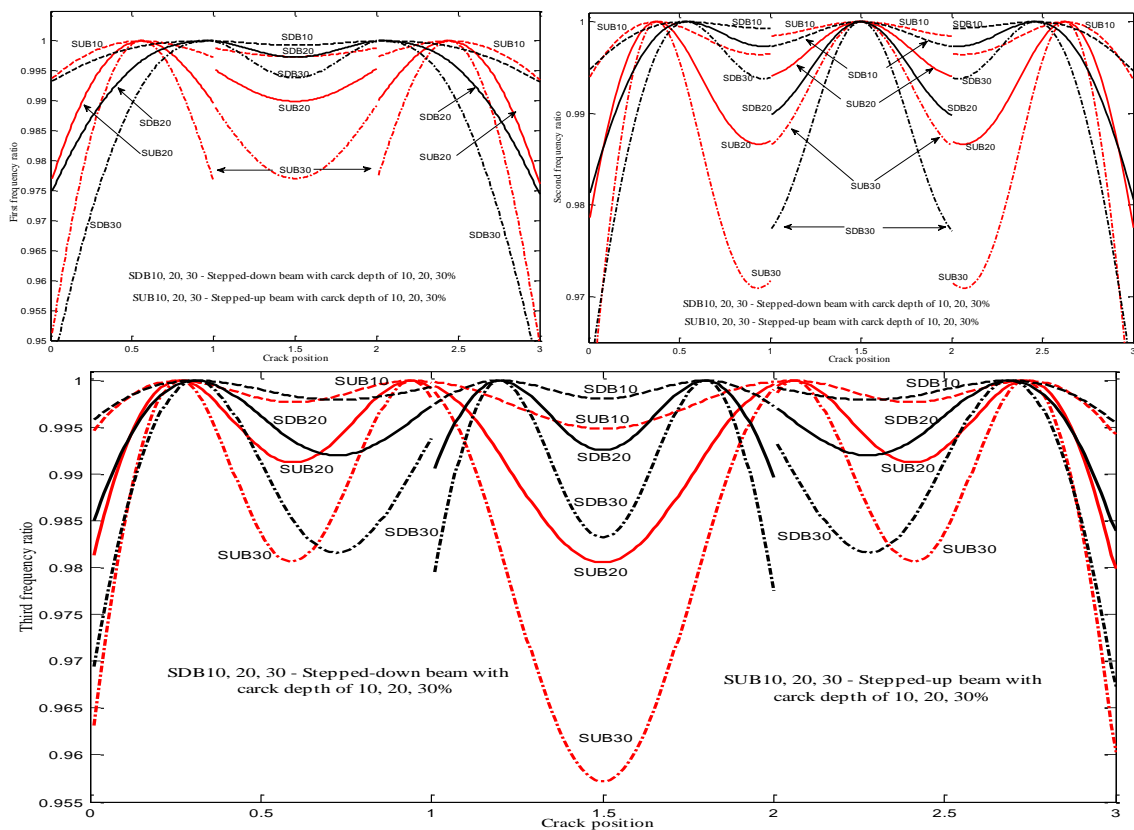


Figure 4. Crack-induced change in natural frequencies computed for stepped-up (SU) and stepped-down (SD) clamped end Timoshenko beam.

To investigate influence of number of cracks on natural frequencies, different scenarios of crack occurrence on the beam are considered. Five frequency ratios of the SUB and SDB with the boundary condition cases are computed in seven crack scenarios: 3 cases of single crack occurred at every segment; 3 cases of double cracks at every pair of the segments and the case when all three segments are cracked. All the cracks are at the middle of beam segments and they have equal depth of 30 %.

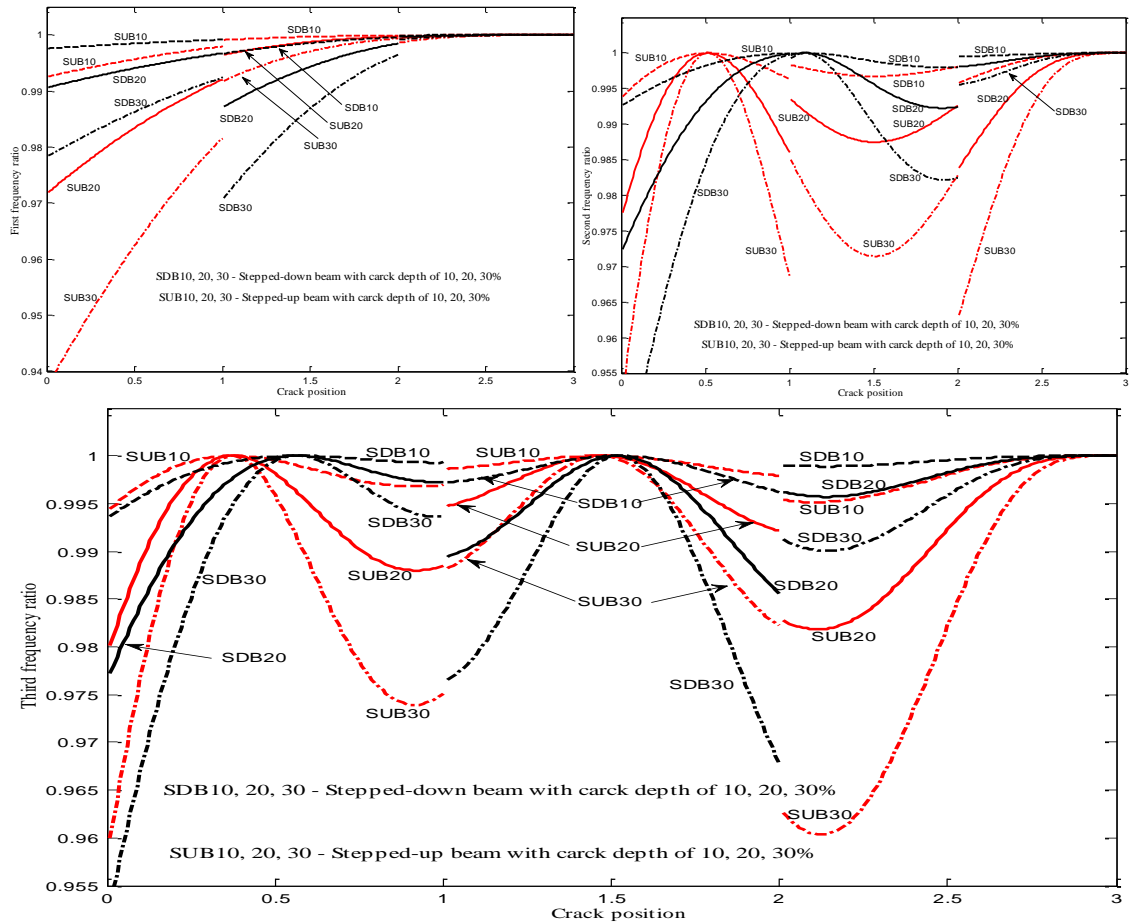


Figure 5. Crack-induced change in natural frequencies of stepped-up (SU) and stepped-down (SD) Timoshenko cantilever beam.

Results of computation by using TBT are given in Table 6 that allows one to make the following notations: (1) Increasing number of cracks in stepped beam leads, in general, to more reduction of natural frequencies, but magnitude of the reduction is dependent much on where the cracks are located; (2) Symmetrical cracks in stepped beam with symmetric variation of thickness and symmetric boundary conditions affect equally on natural frequencies; (3) The midpoints of beam segments that are frequency nodes can be found in Table 6 where the ratio equals to unique (the bold results).

Table 6. Comparison of frequency ratios (cracked/intact) computed for stepped beam with various number of cracks and different boundary conditions.

Crack Scenarios		Single crack at	Single crack at	Single Crack at	Two cracks at 1 st +2 nd spans	Two cracks at 2 nd +3 rd spans	Two cracks at 1 st +3 rd spans	Three cracks at all three spans
BC	Mode	1 st span	2 nd span	3 rd span				
Step-up beam (h1=0.10; h2=0.15; h3=0.10, b1=b2=b3=0.10;L1=L2=L3=1.0)								
SS	1	0.9878	0.9772	0.9878	0.9659	0.9659	0.9762	0.9550
	2	0.9765	1.0000	0.9765	0.9765	0.9765	0.9538	0.9538
	3	0.9762	0.9592	0.9762	0.9367	0.9367	0.9534	0.9150
	4	0.9840	1.0000	0.9840	0.9839	0.9839	0.9674	0.9674
	5	0.9966	0.9622	0.9966	0.9581	0.9581	0.9931	0.9538
CC	1	0.9994	0.9771	0.9994	0.9764	0.9764	0.9989	0.9757
	2	0.9957	1.0000	0.9957	0.9957	0.9957	0.9916	0.9916
	3	0.9837	0.9572	0.9837	0.9426	0.9426	0.9685	0.9289
	4	0.9786	1.0000	0.9786	0.9784	0.9784	0.9577	0.9577
	5	0.9884	0.9664	0.9884	0.9542	0.9542	0.9760	0.9411
CF	1	0.9624	0.9971	0.9999	0.9598	0.9970	0.9623	0.9597
	2	1.0000	0.9714	0.9947	0.9713	0.9667	0.9947	0.9665
	3	0.9959	0.9999	0.9822	0.9957	0.9822	0.9784	0.9784
	4	0.9836	0.9574	0.9729	0.9428	0.9326	0.9581	0.9192
	5	0.9784	1.0000	0.9718	0.9780	0.9714	0.9506	0.9506
Step-down beam (h1=0.15; h2=0.10; h3=0.15, b1=b2=b3=0.10;L1=L2=L3=1.0)								
SS	1	0.9953	0.9627	0.9953	0.9584	0.9584	0.9906	0.9543
	2	0.9783	1.0000	0.9783	0.9783	0.9783	0.9581	0.9581
	3	0.9601	0.9773	0.9601	0.9391	0.9391	0.9241	0.9043
	4	0.9684	1.0000	0.9684	0.9681	0.9681	0.9353	0.9353
	5	0.9882	0.9720	0.9882	0.9597	0.9597	0.9757	0.9464
CC	1	0.9889	0.9841	0.9889	0.9726	0.9726	0.9775	0.9608
	2	0.9995	1.0000	0.9995	0.9995	0.9995	0.9990	0.9990
	3	0.9816	0.9755	0.9816	0.9587	0.9587	0.9652	0.9436
	4	0.9700	1.0000	0.9700	0.9697	0.9697	0.9414	0.9414
	5	0.9795	0.9738	0.9795	0.9534	0.9534	0.9579	0.9314
CF	1	0.9666	0.9829	0.9998	0.9511	0.9827	0.9665	0.9510
	2	0.9853	0.9746	0.9978	0.9592	0.9727	0.9832	0.9573
	3	0.9996	1.0000	0.9884	0.9996	0.9884	0.9881	0.9881
	4	0.9806	0.9753	0.9665	0.9577	.9440	0.9504	0.9291
	5	0.9700	0.9999	0.9571	0.9692	0.9567	0.9285	0.9285

BC – Boundary Conditions:SS – Simply Supports,CC – Clamp-Clamp,CF – Cantilever; $e_1=L_1/2$; $e_2=L_2/2$; $e_3=L_3/2$

5. CONCLUSION

The main results obtained in the present paper can be summarized as follows:

The transfer matrix method was developed for free vibration analysis of multistep Timoshenko beam with arbitrary number of cracks based on a closed form solution for multiple-cracked Timoshenko beam element;

The proposed theoretical development, that was validated by both numerical and experimental results, demonstrate usefulness of transfer matrix method and Timoshenko beam theory for modal analysis of cracked multistep beams;

Change in natural frequencies due to cross-section variation is strongly dependent on the boundary condition and slenderness ratio of beam;

The stepwise variation of beam's thickness leads to abrupt change in sensitivity of natural frequencies to crack that is a good indicator for crack detection in stepped beam by measurement of natural frequencies.

Acknowledgement: The first author is sincerely thankful to University of Transport and Communication for financial support under Grant number T2018-CB-7.

REFERENCES

1. Sato H. - Free vibration of beams with abrupt changes of cross-section. *Journal of Sound and Vibration* **89** (1983) 59-64.
2. Jang S. K. and Bert C. W. - Free vibration of stepped beams: exact and numerical solutions, *Journal of Sound and Vibration* **130** (1989) 342-346.
3. Maurizi M. J. and Belles P. M. - Natural frequencies of one-span beams with stepwise variable cross-section. *Journal of Sound and Vibration* **168** (1) (1993) 184-188.
4. Stanton S. C. and Mann B. P. - On the dynamic response of beams with multiple geometric or material discontinuities, *Mechanical Systems and Signal Processing* **24** (2010) 1409-1419.
5. Mao Q. - Free vibration analysis of multiple-stepped beams by using Adomian decomposition method, *Mechanical and Computer Modelling* **54** (2012) 756-764.
6. Cunha-Vaz J. and Lima-Junior J. J. - Vibration analysis of Euler-Bernoulli beams in multiple steps and different shapes of cross section, *Journal of Vibration and Control* (2014) DOI: 10.1177/1077546314528366.
7. Koplów M.A., Bhattachayya A. and Mann B.P. - Closed form solutions for the dynamic response of Euler-Bernoulli beams with step changes in cross section. *Journal of Sound and Vibration* **295** (2006) 214-225.
8. Farghary S.H. - Vibration and stability analysis of Timoshenko beams with discontinuities in cross-section. *Journal of Sound and Vibration* **174**(5) (1994) 591-605.
9. Maguleswaran S. - Vibration of a Euler-Bernoulli beam on elastic end supports and with up to three step changes in cross-section, *Intern. J. Mech. Sciences* **44** (2002) 2541-2555.
10. Wang X. W. and Wang Y. L. - Free vibration analysis of multiple-stepped beams by the differential quadrature element method, *Applied Mathematics and Computation* **219** (2013) 5802-5810.

11. Sarigul (Aydin) A.S. and Aksu G. - A finite difference method for free vibration analysis of stepped Timoshenko beams and shafts, *Mechanism and Machine Theory* **21** (1986) 1-12.
12. Subramanian G. and Balasubramanian T.S. - Beneficial effects of steps on the free vibration characteristics of beams, *Journal of Sound and Vibration* **118** (3) (1987) 555-560.
13. Kukla S. and Zamojska I. - Frequency analysis of axially loaded stepped beams by Green's function method, *Journal of Sound and Vibration* **300** (2007) 1341-1041.
14. Jaworski J. W. and Dowell E. H. - Free vibration of a cantilevered beam with multiple steps: Comparison of several theoretical methods with experiment, *Journal of Sound and Vibration* **312** (2008) 713-725.
15. Nandwana B. P. and Maiti S. K. - Detection of the location and size of a crack in stepped cantilever beams based on measurements of natural frequencies, *Journal of Sound and Vibration* **203** (3) (1997) 435-446.
16. Kukla S. - Free vibrations and stability of stepped columns with cracks, *Journal of Sound and Vibration* **319** (2009) 1301-1311.
17. Zhang W., Wang Z. and Ma H. - Crack identification in stepped cantilever beam combining wavelet analysis with transform matrix, *Acta Mechanica Solida Sinica* **22** (4) (2009) 360-368.
18. Maghsoodi A., Ghadami A. and Mirdamadi H. R. - Multiple crack damage detection in multi-step beams by a novel local flexibility-based damage index, *Journal of Sound and Vibration* **332** (2013) 294-305.
19. Skrinar M. - Computational analysis of multi-stepped beams and beams with linearly-varying heights implementing closed-form finite element formulation for multi-cracked beam elements. *International Journal of Solids and Structures* **50** (2013) 2527-2541.
20. Li Q.S. - Vibratory characteristics of multi-step beams with an arbitrary number of cracks and concentrated masses, *Applied Acoustics* **62** (2001) 691-706.
21. Attar M. - A transfer matrix method for free vibration analysis and crack identification of stepped beams with multiple edge cracks and different boundary conditions, *International Journal of Mechanical Sciences* **57** (2012) 19-33.
22. Lele S.P. and Maiti S.K. - Modeling of transverse vibration of short beams for crack detection and measurement of crack extension. *Journal of Sound and Vibration* **257** (3) (2002) 559-583.
23. Swamidass A.S.J., Yang X. and Seshadri R. - Identification of cracking in beam structures using Timoshenko and Euler formulations, *ASCE J. Eng. Mech.* **130** (11) (2004) 1297-1338.
24. Loya J. A., Rubio L. and Fernandez-Sàez J. - Natural frequencies for bending vibrations of Timoshenko cracked beams, *Journal of Sound and Vibration* **290** (2006) 640-653.
25. Li Q. S. - Vibratory characteristics of Timoshenko beams with arbitrary number of cracks, *ASCE J. Eng. Mech.* **129** (2009) 1355-1359.
26. Aydin K. - Influence of Crack and Slenderness Ratio on the Eigenfrequencies of Euler-Bernoulli and Timoshenko Beams, *Mechanics of Advanced Materials and Structures* **20** (5) (2012) 339-352.
27. Tsai T. C. and Wang Y. Z. - Vibration analysis and diagnosis of a cracked shaft, *Journal of Sound and Vibration* **192** (3) (1996) 607-620.

28. Vu Thi An Ninh, Luu Quynh Huong, Tran Thanh Hai, Nguyen Tien Khiem - The transfer matrix method for modal analysis of cracked multistep beam, *Vietnam Journal of Science and Technology* **55** (5) (2017) 598-611.
29. Chondros T. G., Dimarogonas A. D. and Yao J. - Longitudinal vibration of a continuous cracked bar, *Engineering Fracture Mechanics* **61** (1998) 593-606.
30. Khiem N.T. and Hung D. T. - A closed form solution for free vibration of multiple cracked Timoshenko beam and application, *Vietnam Journal of Mechanics* **39** (4) (2017) 315-328.
31. Karnovsky I. A. and Lebed O. I. - *Formulas for Structural Dynamics: Tables, Graphs and Solutions.*, McGraw-Hill, Inc. NY 2000, 638p.

Table 5. Comparison of frequency parameter computed by using Euler-Bernoulli and Timoshenko beam theories for stepped beam with various slenderness and different boundary conditions

Slenderness		L/h = 5		L/h = 10		L/h = 15		L/h = 20		L/h = 30		L/h = 50		L/h = 100	
BC	Fr.No	EBT	TBT	EBT	TBT	EBT	TBT	EBT	TBT	EBT	TBT	EBT	TBT	EBT	TBT
Step-up beam (h1=0.10; h2=0.15; h3=0.10)															
SS	1	6.0027	5.7718	3.0013	2.9700	2.0009	1.9914	1.5007	1.4967	1.0004	0.9993	0.6003	0.6000	0.3001	0.3001
	2	11.1842	10.1363	5.5921	5.4293	3.7281	3.6773	2.7961	2.7742	1.8640	1.8575	1.1184	1.1170	0.5592	0.5590
	3	17.6695	14.6182	8.8348	8.2619	5.8898	5.6997	4.4174	4.3334	2.9449	2.9192	1.7670	1.7613	0.8835	0.8828
	4	23.7384	17.9008	11.8692	10.6233	7.9128	7.4758	5.9346	5.7366	3.9564	3.8943	2.3738	2.3600	1.1869	1.1852
	5	28.8451	20.5981	14.4225	12.5082	9.6150	8.9127	7.2113	6.8853	4.8075	4.7031	2.8845	2.8609	1.4423	1.4392
CC	1	8.4934	7.5242	4.2467	4.0945	2.8311	2.7836	2.1234	2.1029	1.4156	1.4094	0.8493	0.8480	0.4247	0.4245
	2	14.0633	11.4675	7.0317	6.5663	4.6878	4.5361	3.5158	3.4495	2.3439	2.3237	1.4063	1.4019	0.7032	0.7026
	3	20.3472	15.5508	10.1736	9.1740	6.7824	6.4366	5.0868	4.9312	3.3912	3.3427	2.0347	2.0240	1.0174	1.0160
	4	26.7908	18.5834	13.3954	11.4513	8.9303	8.2138	6.6977	6.3648	4.4651	4.3584	2.6791	2.6550	1.3395	1.3365
	5	31.9780	21.0203	15.9890	13.1808	10.6593	9.5776	7.9945	7.4792	5.3297	5.1604	3.1978	3.1590	1.5989	1.5939
CF	1	3.2597	3.2143	1.6298	1.6240	1.0866	1.0848	0.8149	0.8142	0.5433	0.5431	0.3260	0.3259	0.4559	0.4558
	2	9.1194	8.2006	4.5597	4.4208	3.0398	2.9969	2.2799	2.2615	1.5199	1.5144	0.9119	0.9107	0.7135	0.7130
	3	14.2693	12.1261	7.1346	6.7562	4.7564	4.6339	3.5673	3.5138	2.3782	2.3620	1.4269	1.4234	1.0158	1.0146
	4	20.3156	15.8870	10.1578	9.2546	6.7719	6.4618	5.0789	4.9399	3.3859	3.3427	2.0316	2.0220	1.3379	1.3351
	5	26.7578	18.9132	13.3789	11.5655	8.9193	8.2580	6.6894	6.3838	4.4596	4.3621	2.6758	2.6538	1.5983	1.5938
Step-down beam (h1=0.15; h2=0.10; h3=0.15)															
SS	1	4.0879	3.9837	2.0439	2.0300	1.3626	1.3585	1.0220	1.0202	0.6813	0.6808	0.4088	0.4087	0.2044	0.2044
	2	9.0613	8.0611	4.5306	4.3711	3.0204	2.9703	2.2653	2.2437	1.5102	1.5037	0.9061	0.9047	0.4531	0.4529
	3	13.4903	11.1542	6.7451	6.3092	4.4968	4.3523	3.3726	3.3089	2.2484	2.2289	1.3490	1.3447	0.6745	0.6740
	4	17.5137	13.4910	8.7569	7.9194	5.8379	5.5468	4.3784	4.2471	2.9190	2.8779	1.7514	1.7423	0.8757	0.8745
	5	22.6179	15.8899	11.3089	9.7146	7.5393	6.9489	5.6545	5.3791	3.7696	3.6810	2.2618	2.2417	1.1309	1.1283
CC	1	6.8201	6.1488	3.4101	3.3069	2.2734	2.2414	1.7050	1.6913	1.1367	1.1326	0.6820	0.6811	0.3410	0.3409
	2	11.2354	9.1052	5.6177	5.2257	3.7451	3.6165	2.8089	2.7524	1.8726	1.8553	1.1235	1.1198	0.5618	0.5613
	3	15.8232	11.6412	7.9116	7.0093	5.2744	4.9582	3.9558	3.8127	2.6372	2.5924	1.5823	1.5723	0.7912	0.7899
	4	19.7170	13.7082	9.8585	8.4479	6.5723	6.0550	4.9292	4.6895	3.2862	3.2095	1.9717	1.9544	0.9858	0.9836
	5	24.7370	16.0118	12.3685	10.0965	8.2457	7.3635	6.1842	5.7624	4.1228	3.9838	2.4737	2.4418	1.2368	1.2327
CF	1	2.6537	2.6096	1.3269	1.3211	0.8846	0.8829	0.6634	0.6627	0.4423	0.4421	0.2654	0.2653	0.1327	0.1327
	2	6.3458	5.7983	3.1729	3.0918	2.1153	2.0903	1.5865	1.5758	1.0576	1.0544	0.6346	0.6339	0.3173	0.3172
	3	11.1737	9.2556	5.5869	5.2473	3.7246	3.6144	2.7934	2.7453	1.8623	1.8477	1.1174	1.1142	0.5587	0.5583
	4	15.8531	12.0638	7.9266	7.1394	5.2844	5.0124	3.9633	3.8410	2.6422	2.6041	1.5853	1.5769	0.7927	0.7916
	5	19.7330	14.2471	9.8665	8.6111	6.5777	6.1219	4.9333	4.7231	3.2888	3.2219	1.9733	1.9583	0.9867	0.9847

BC – Boundary Conditions; SS – Simply Supports; CC – Clamp-Clamp; CF – Cantilever; EBT – Euler-Bernoulli beam theory; TBT – Timoshenko beam theory