



FREE VIBRATION OF TWO-DIRECTIONAL FGM BEAMS USING A HIGHER-ORDER TIMOSHENKO BEAM ELEMENT

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Abstract. Free vibration of two-directional functionally graded material (2-D FGM) beams is studied by the finite element method (FEM). The material properties are assumed to be graded in both the thickness and longitudinal directions by a power-law distribution. Equations of motion based on Timoshenko beam theory are derived from Hamilton's principle. A higher-order beam element using hierarchical functions to interpolate the displacements and rotation is formulated and employed in the analysis. In order to improve the efficiency of the element, the shear strain is constrained to constant. Validation of the derived element is confirmed by comparing the natural frequencies obtained in the present paper with the data available in the literature. Numerical investigations show that the proposed beam element is efficient, and it is capable to give accurate frequencies by a small number of elements. The effects of the material composition and aspect ratio on the vibration characteristics of the beams are examined in detail and highlighted.

Keywords. 2-D FGM, Timoshenko beam, hierarchical functions, free vibration, FEM.

Classification numbers: 2.9.4; 5.4.2; 5.4.3.

1. INTRODUCTION

Functionally graded materials (FGMs), initiated in Japan in 1984 during a space project [1], are increasingly used as structural elements in modern industries such as aerospace structures, turbine blades and rocket engine components. Many researches on vibration behavior of FGM beam structures have been reported in the literature, the papers that are most relevant to the present work are briefly discussed below.

Chakraborty et al. [2] developed an exact first-order shear deformable beam element for studying the static, free vibration and wave propagation problems of FGM beams. Aydogdu and Taskin [3] investigated the free vibration of simply supported FGM beam by considering Young's modulus of the beam being graded in the thickness direction by the power and exponential laws. The authors considered different beam theories and employed Navier type solution method to obtain frequencies. Li [4] proposed a new unified approach for analyzing the static and dynamic behavior of FGM beams with the rotary inertia and shear deformation effects.

Sina et al. [5] developed a new beam theory for studying the free vibration of FGM beams. The resulting system of ordinary differential equations of the free vibration analysis in the work is solved by an analytical method. In [6], Alshorbagy et al. employed the traditional Euler-Bernoulli beam element to calculate the natural frequencies of FGM beams with the material properties to be graded in the thickness or longitudinal direction by a power-law distribution. Shahba et al. [7] derived the stiffness and mass matrices for free vibration and buckling analyses of tapered axially FGM beams with elastic end supports. The solution of the equilibrium equations of a homogeneous Timoshenko was employed by the authors to interpolate the displacement field. The analytical solutions for the bending and free vibration problems of higher-order shear deformable FGM beams were proposed by Thai and Vo [8]. The static and free vibration problem of FGM beams is also considered by Vo et al. in [9] by using a refined shear deformation theory. A first-order shear deformation theory, in which the transverse shear stiffness is derived from the in-plane stress and the shear correction factor is calculated analytically, was presented by Nguyen et al. [10] for studying the static and free vibration of axially loaded FGM beams. Wattanasakulpong et al. [11] used the modified rule of mixture to describe and approximate material properties in a study of linear and nonlinear free vibration of FGM beams with porosities. The differential transformation method is employed by the author to obtain the natural frequencies of the beams with different elastic supports.

In the above cited papers, the beam material properties are considered to vary in one spatial direction only. The development of FGMs with effective material properties varying in two or three directions to withstand severe general loadings is of great importance in practice, especially in development of structural elements for space structures [12, 13]. Studies on the static and dynamic behavior of beams formed from two-directional functionally graded materials (2-D FGMs) have been recently reported by several researchers. In this line of works, Şimşek [14,15] considered the material properties being varied in both the thickness and length directions by an exponent function in the forced vibration and buckling analyses of 2-D FGM Timoshenko beams. The author showed that the vibration and buckling behavior of the 2-D FGM beams is significantly influenced by the material distribution. Based on an analytical method, Wang et al. [16] investigated the free vibration of FGM beams with the material properties vary through the thickness by an exponential function and along the length by a power-law distribution. The numerical investigations by the authors show that the variation of material properties has a strong influence on the natural frequencies, and there is a critical frequency at which the natural frequencies have an abrupt jump when they across the critical frequency. Based on a finite element procedure, Nguyen et al. [13] studied the forced vibration of 2-D FGM Timoshenko beams excited by a moving load. The material properties in [13] were assumed to vary in both the thickness and longitudinal directions by a power-law function. Recently, Shafiei et al. [17] studied the vibration behavior of 2-D FG nano and microbeams formed from two types of porous FGMs. The generalized differential quadrature method has been employed by the authors to solve the governing equations of motion.

In this paper, a higher-order Timoshenko beam element is developed and employed in studying free vibration of 2-D FGM Timoshenko beams. The material properties of the beams are considered to vary in both the thickness and longitudinal directions by a power-law distribution. Based on Timoshenko beam theory, equations of motion are derived from Hamilton's principle and they are solved by a finite element procedure based on the developed beam element. The beam element, using hierarchical functions to interpolate the displacement field, is formulated by constraining the shear strain constant for improving its efficiency. Validation of the derived element is confirmed by comparing the result obtained in the present work with the published data. A parametric study is carried out to highlight the effects of

material composition on the vibration characteristics of the beams. The influence of the aspect ratio on the natural frequencies is also examined and discussed.

2. MATHEMATICAL FORMULATION

Figure 1 shows a 2-D FGM beam with length L , width b and height h in a Cartesian coordinate system (x,z) . The system (x,z) is chosen such that the x -axis is on the mid-plane, and the z -axis is perpendicular to the mid-plane, and it directs upward.

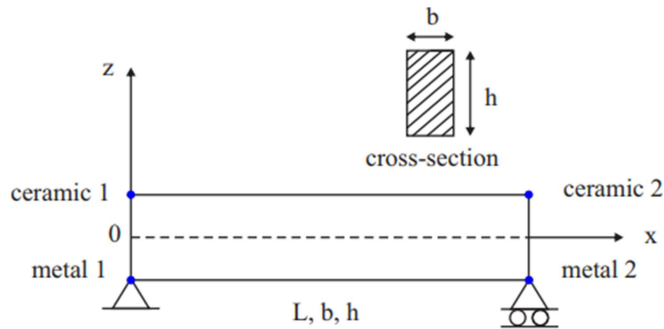


Figure 1. Geometry and coordinates of a 2-D FGM beam.

The beam material is assumed to be formed from two ceramics (referred to as ceramic 1-C1 and ceramic 2-C2) and two metals (referred to as metal 1-M1 and metal 2-M2) whose volume fraction varies in both the thickness and longitudinal directions according to

$$\begin{aligned}
 V_{c1} &= \left(\frac{z}{h} + \frac{1}{2} \right)^{n_z} \left[1 - \left(\frac{x}{L} \right)^{n_x} \right], & V_{c2} &= \left(\frac{z}{h} + \frac{1}{2} \right)^{n_z} \left(\frac{x}{L} \right)^{n_x}, \\
 V_{m1} &= \left[1 - \left(\frac{z}{h} + \frac{1}{2} \right)^{n_z} \right] \left[1 - \left(\frac{x}{L} \right)^{n_x} \right], & V_{m2} &= \left[1 - \left(\frac{z}{h} + \frac{1}{2} \right)^{n_z} \right] \left(\frac{x}{L} \right)^{n_x},
 \end{aligned} \tag{1}$$

where n_z and n_x are the material grading indexes, which dictate the variation of the constituent materials in the thickness and longitudinal directions, respectively. It can be seen from Eq. (1) that the left and right lower corners of the beam contain only M1 and M2, respectively whereas the corresponding upper two corners are, correspondingly, pure C1 and C2. The variation of the volume fraction of C1 and C2 in the z - and x -directions according to Eq. (1) is depicted in Fig. 2 for various values of the grading indexes n_z and n_x .

The effective material properties \mathcal{P} , such as the elastic modulus E and the mass density, are evaluated according to

$$\mathcal{P} = V_{c1} \mathcal{P}_{c1} + V_{c2} \mathcal{P}_{c2} + V_{m1} \mathcal{P}_{m1} + V_{m2} \mathcal{P}_{m2} \tag{2}$$

where \mathcal{P}_{c1} , \mathcal{P}_{c2} , \mathcal{P}_{m1} and \mathcal{P}_{m2} denote the properties of the C1, C2, M1, and M2, respectively.

Substituting Eq. (1) into Eq. (2) leads to

$$\mathcal{P}(x, z) = \left[(\mathcal{P}_{c1} - \mathcal{P}_{m1}) \left(\frac{z}{h} + \frac{1}{2} \right)^{n_z} + \mathcal{P}_{m1} \right] \left[1 - \left(\frac{x}{L} \right)^{n_x} \right] + \left[(\mathcal{P}_{c2} - \mathcal{P}_{m2}) \left(\frac{z}{h} + \frac{1}{2} \right)^{n_z} + \mathcal{P}_{m2} \right] \left(\frac{x}{L} \right)^{n_x} \tag{3}$$

One can verify that if $n_x = 0$, Eq. (3) deduces to the expression for the effective material properties of transversely unidirectional FGM beam composed of C2 and M2 as Eq. (4).

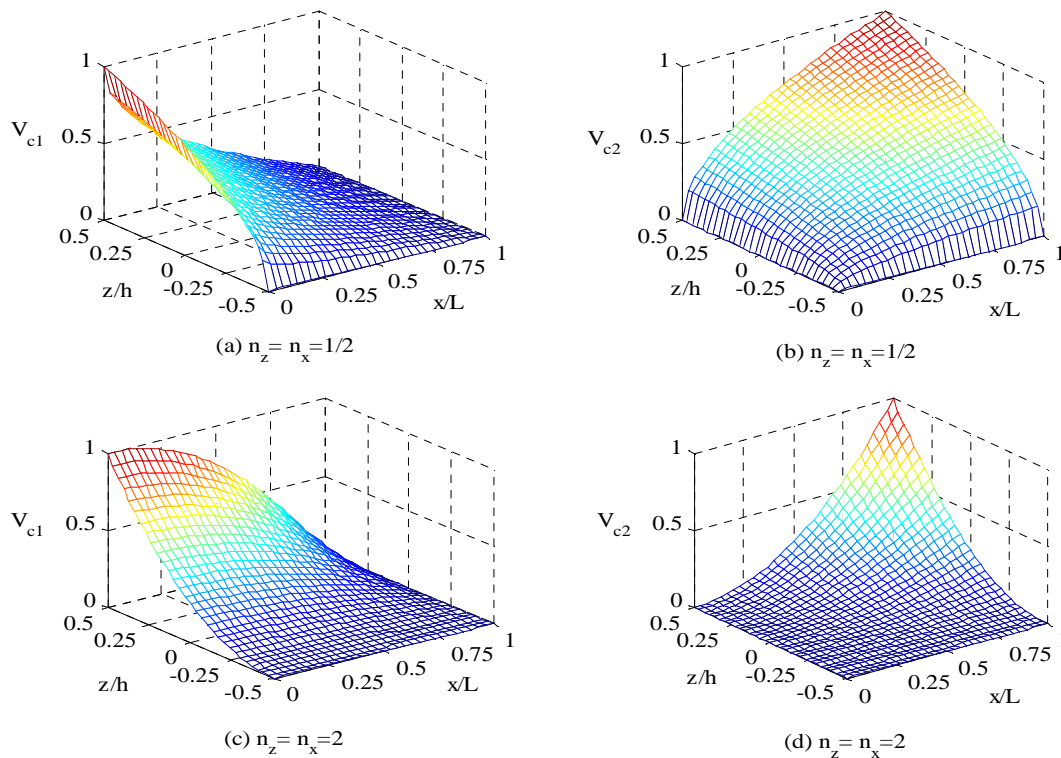


Figure 2. Variation of volume fraction of ceramics in the thickness and longitudinal directions.

$$\mathcal{P}(z) = (\mathcal{P}_{c2} - \mathcal{P}_{m2}) \left(\frac{z}{h} + \frac{1}{2} \right)^{n_z} + \mathcal{P}_{m2} \quad (4)$$

In case $n_z = 0$, Eq. (3) leads to an expression for the effective material properties of an axially FGM beam formed from C1 and C2, namely

$$\mathcal{P}(x) = (\mathcal{P}_{c2} - \mathcal{P}_{c1}) \left(\frac{x}{L} \right)^{n_x} + \mathcal{P}_{c1} \quad (5)$$

Based on Timoshenko beam theory, the displacements in x - and z -directions, $u_1(x, z, t)$ and $u_3(x, z, t)$, respectively, at any point of the beam are given by

$$\begin{aligned} u_1(x, z, t) &= u(x, t) - z\theta(x, t) \\ u_3(x, z, t) &= w(x, t) \end{aligned} \quad (6)$$

where z is the distance from the mid-plane to the considering point; $u(x, t)$ and $w(x, t)$ are, respectively, the axial and transverse displacements of the corresponding point on the mid-plane; $\theta(x, t)$ is the cross-sectional rotation.

The axial strain (ϵ_{xx}) and the shear strain (γ_{xz}) resulted from Eq. (6) are of the forms

$$\varepsilon_{xx} = u_{,x} - z\theta_{,x}; \quad \gamma_{xz} = w_{,x} - \theta, \quad (7)$$

where a subscript comma is used to indicate the derivative of the variable with respect to the spatial coordinate x , that is $(\cdot)_{,x} = \partial(\cdot) / \partial x$.

Based on the Hooke's law, the constitutive relation for the 2-D FGM beam is as follows

$$\sigma_{xx} = E(x, z)\varepsilon_{xx}, \quad \tau_{xz} = \psi G(x, z)\gamma_{xz} \quad (8)$$

where σ_{xx} and τ_{xz} are the axial stress and shear stress, respectively; $E(x, z)$ and $G(x, z)$ are, respectively, the elastic modulus and shear modulus, which are functions of both the coordinates x, z ; ψ is the shear correction factor, equals to 5/6 for the beams with rectangular cross-section considered herein.

The strain energy of the beam (U) resulted from (7) and (8) is as follows

$$U = \frac{1}{2} \int_0^L \int_A (\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz}) dA dx = \frac{1}{2} \int_0^L \left[A_{11}u_{,x}^2 - 2A_{12}u_{,x}\theta_{,x} + A_{22}\theta_{,x}^2 + \psi A_{33}(w_{,x} - \theta)^2 \right] dx \quad (9)$$

and the kinetic energy resulted from Eq. (6) is as follows:

$$\mathcal{T} = \frac{1}{2} \int_0^L \int_A \rho(x, z) (\dot{u}_1^2 + \dot{u}_3^2) dA dx = \frac{1}{2} \int_0^L \left[I_{11}\dot{u}^2 + I_{11}\dot{w}^2 - 2I_{12}\dot{u}\dot{\theta} + I_{22}\dot{\theta}^2 \right] dx \quad (10)$$

In Eqs. (9) and (10), A is the cross-sectional area; A_{11}, A_{12}, A_{22} and A_{33} are, respectively, the extensional, extensional-bending coupling, bending rigidities and shear rigidity, which are defined as follows

$$(A_{11}, A_{12}, A_{22})(x, z) = \int_A E(x, z)(1, z, z^2) dA, \quad A_{33}(x, z) = \int_A G(x, z) dA \quad (11)$$

and I_{11}, I_{12}, I_{22} are the mass moments defined as

$$(I_{11}, I_{12}, I_{22})(x, z) = \int_A \rho(x, z)(1, z, z^2) dA \quad (12)$$

Substituting Eq. (3) into Eq. (11), one can obtain the rigidities as

$$\begin{aligned} A_{11}(x, z) &= A_{11}^{C1M1} - (A_{11}^{C1M1} - A_{11}^{C2M2}) \left(\frac{x}{L} \right)^{n_x}, \quad A_{12}(x, z) = A_{12}^{C1M1} - (A_{12}^{C1M1} - A_{12}^{C2M2}) \left(\frac{x}{L} \right)^{n_x}, \\ A_{22}(x, z) &= A_{22}^{C1M1} - (A_{22}^{C1M1} - A_{22}^{C2M2}) \left(\frac{x}{L} \right)^{n_x}, \quad A_{33}(x, z) = A_{33}^{C1M1} - (A_{33}^{C1M1} - A_{33}^{C2M2}) \left(\frac{x}{L} \right)^{n_x}, \end{aligned} \quad (13)$$

where $A_{11}^{C1M1}, A_{12}^{C1M1}, A_{22}^{C1M1}$ and A_{33}^{C1M1} are the rigidities of the transversely unidirectional FGM beam composed of C1 and M1; $A_{11}^{C2M2}, A_{12}^{C2M2}, A_{22}^{C2M2}$ and A_{33}^{C2M2} are the rigidities of the transverse FGM beam composed of C2 and M2. As can be seen from Eq. (13) that the rigidities of the present 2-D FG beam degenerate to that of the unidirectional FGM beam if $n_x = 0$ or the two ceramics and two metals are identical. Because $A_{ij}^{C1M1}, A_{ij}^{C2M2}$ are functions of z only, the following explicit expressions for the rigidities of the transversely unidirectional FGM beam can be obtained easily

$$\begin{aligned}
 A_{11}^{C1M1} &= \frac{bh(E_{C1} + n_z E_{M1})}{n_z + 1}, \quad A_{12}^{C1M1} = \frac{bh^2 n_z (E_{C1} - E_{M1})}{2(n_z + 1)(n_z + 2)}, \\
 A_{22}^{C1M1} &= \frac{bh^3 (n_z^2 + n_z + 2)(E_{C1} - E_{M1})}{4(n_z + 1)(n_z + 2)(n_z + 3)} + \frac{bh^3}{12} E_{M1}, \quad A_{33}^{C1M1} = \frac{bh(G_{C1} + n_z G_{M1})}{n_z + 1}
 \end{aligned} \tag{14}$$

Similar expressions for A_{ij}^{C2M2} are obtained by replacing Young's modulus of C1 and M1 by that of C2 and M2, respectively. The mass moments can be also written as

$$\begin{aligned}
 I_{11}(x, z) &= I_{11}^{C1M1} - (I_{11}^{C1M1} - I_{11}^{C2M2}) \left(\frac{x}{L} \right)^{n_x}, \quad I_{12}(x, z) = I_{12}^{C1M1} - (I_{12}^{C1M1} - I_{12}^{C2M2}) \left(\frac{x}{L} \right)^{n_x}, \\
 I_{22}(x, z) &= I_{22}^{C1M1} - (I_{22}^{C1M1} - I_{22}^{C2M2}) \left(\frac{x}{L} \right)^{n_x},
 \end{aligned} \tag{15}$$

where I_{ij}^{C1M1} and I_{ij}^{C2M2} are the mass moments of the C1–M1 and C2–M2 beams, respectively. The explicit expressions for I_{ij}^{C1M1} and I_{ij}^{C2M2} have similar forms as in Eq. (14).

Applying Hamilton's principle to Eq. (9) and Eq. (10), one obtains the equations of motion for the 2-D FGM beam as

$$\begin{cases}
 I_{11}\ddot{u} - I_{12}\ddot{\theta} - (A_{11}u_{,x} - A_{12}\theta_{,x})_{,x} = 0 \\
 I_{11}\ddot{w} - \psi [A_{33}(w_{,x} - \theta)]_{,x} = 0 \\
 I_{12}\ddot{u} - I_{22}\ddot{\theta} - (A_{12}u_{,x} - A_{22}\theta_{,x})_{,x} + \psi A_{33}(w_{,x} - \theta) = 0
 \end{cases} \tag{16}$$

and the natural boundary conditions are of the forms

$$A_{11}u_{,x} - A_{12}\theta_{,x} = \bar{N}; \quad A_{12}u_{,x} - A_{22}\theta_{,x} = \bar{M}; \quad \psi A_{33}(w_{,x} - \theta) = \bar{Q} \quad \text{at } x=0 \text{ and } x=L. \tag{17}$$

with \bar{N} , \bar{M} , \bar{Q} are, respectively, the prescribed axial forces, moments and shear forces at the beam ends.

Since the axial displacement u , transverse displacement w and rotation θ are independent variables in Timoshenko beam theory, the interpolation functions for these variables can be chosen separately. Traditionally, linear functions are used for all the variables, but the element based on the linear functions suffers from the shear-locking, and some techniques such as the reduced integration must be applied to overcome this problem [18]. The shear-locking can also be avoided by using appropriate interpolation functions for the kinematic variables. Standard polynomial-based shape functions can be employed to approximate the displacement field of Timoshenko beam. However, the finite element formulation derived from the standard shape functions has a drawback. Since the coefficients of the polynomials are determined from the element boundary conditions, related to nodal values of the variables, totally new shape functions have to be re-determined whenever the element refinement is made [19]. The finite element formulated from the hierarchical functions, in which the higher-order shape functions contain the lower-order ones, is able to overcome the drawback. For one-dimensional beam the hierarchical shape functions are of the forms [20]

$$N_1 = \frac{1}{2}(1-\xi); N_2 = \frac{1}{2}(1+\xi); N_3 = (1-\xi^2); N_4 = \xi(1-\xi^2) \quad (18)$$

with $\xi = 2\frac{x}{l} - 1$ being the natural coordinate.

For a Timoshenko beam element, a quadratic variation of the rotation should be chosen to represent a linearly varying bending moments along the element. In addition, with the shear strain given by (7), the shape functions for the transverse displacement w should be chosen one order higher than that of θ . In this regard, the displacements and rotation can be interpolated as

$$\begin{aligned} u &= N_1 u_1 + N_2 u_2, \\ \theta &= N_1 \theta_1 + N_2 \theta_2 + N_3 \theta_3, \\ w &= N_1 w_1 + N_2 w_2 + N_3 w_3 + N_4 w_4 \end{aligned} \quad (19)$$

where $u_1, u_2, \theta_1, \theta_2, \theta_3, w_1, \dots, w_4$ are the unknown values of the variables.

A beam element can be formulated from nine degrees of freedom in (19). However, a more efficient element with less number of degrees of freedom can be derived by constraining the shear strain γ_{xz} to constant [21]. To this end, the shear strain (7) can be rewritten by using Eqs. (18) and (19) as

$$\gamma_{xz} = \left(-\frac{6}{l} w_4 + \theta_3 \right) \xi^2 - \left(\frac{4}{l} w_3 - \frac{1}{2} \theta_1 + \frac{1}{2} \theta_2 \right) \xi + \left[\frac{1}{l} (w_2 - w_1 + 2w_4) - \frac{1}{2} (\theta_1 + \theta_2 + 2\theta_3) \right] \quad (20)$$

In order to ensure $\gamma_{xz} = \text{constant}$, we need

$$-\frac{6}{l} w_4 + \theta_3 = 0, \text{ and } \frac{4}{l} w_3 - \frac{1}{2} \theta_1 + \frac{1}{2} \theta_2 = 0 \quad (21)$$

Eq. (21) gives

$$w_3 = \frac{l}{8} (\theta_1 - \theta_2), \text{ and } w_4 = \frac{l}{6} \theta_3 \quad (22)$$

Using Eqs. (18) and (22), one can write (19) in the forms

$$\begin{aligned} u &= \frac{1}{2}(1-\xi)u_1 + \frac{1}{2}(1+\xi)u_2; \\ \theta &= \frac{1}{2}(1-\xi)\theta_1 + \frac{1}{2}(1+\xi)\theta_2 + (1-\xi^2)\theta_3; \\ w &= \frac{1}{2}(1-\xi)w_1 + \frac{1}{2}(1+\xi)w_2 + \frac{l}{8}(1-\xi^2)(\theta_1 - \theta_2) + \frac{l}{6}\xi(1-\xi^2)\theta_3 \end{aligned} \quad (23)$$

The shear strain (20) is now of the form

$$\gamma_{xz} = \frac{1}{l}(w_2 - w_1) - \frac{1}{2}(\theta_1 + \theta_2) - \frac{2}{3}\theta_3 \quad (24)$$

The beam element is now derived from the displacement field in Eq. (23) and the shear strain in Eq. (24). The element vector for a generic element (\mathbf{d}) has following components

$$\mathbf{d} = \{u_1 \ w_1 \ \theta_1 \ \theta_3 \ u_2 \ w_2 \ \theta_2\}^T \quad (25)$$

In the above equation and hereafter, the superscript ‘ T ’ is used to denote the transpose of a vector or a matrix.

The axial displacement u , the transverse displacement w , and the cross-sectional rotation θ can be now written as

$$u = \mathbf{N}_u \mathbf{d}, \quad w = \mathbf{N}_w \mathbf{d}, \quad \theta = \mathbf{N}_\theta \mathbf{d} \quad (26)$$

where

$$\begin{aligned} \mathbf{N}_u &= \{N_1 \ 0 \ 0 \ 0 \ N_2 \ 0 \ 0\}^T, \\ \mathbf{N}_\theta &= \{0 \ 0 \ N_1 \ N_3 \ 0 \ 0 \ N_2\}^T, \\ \mathbf{N}_w &= \left\{0 \ N_1 \ \frac{l}{8}N_3 \ \frac{l}{6}N_4 \ 0 \ N_2 \ -\frac{l}{8}N_3\right\}^T, \end{aligned} \quad (27)$$

with N_1, N_2, N_3, N_4 are defined in Eq. (18).

From the displacement field in Eq. (23), one can rewrite the strain energy (9) in the form

$$U = \frac{1}{2} \sum_{i=1}^{ne} \mathbf{d}^T \mathbf{k} \mathbf{d}; \quad \mathbf{k} = \mathbf{k}_{uu} + \mathbf{k}_{u\theta} + \mathbf{k}_{\theta\theta} + \mathbf{k}_s \quad (28)$$

with ne is total number of the elements; \mathbf{k} is the element stiffness matrix; \mathbf{k}_{uu} , $\mathbf{k}_{u\theta}$, $\mathbf{k}_{\theta\theta}$ and \mathbf{k}_s are, respectively, the stiffness matrix stemming from the axial stretching, axial stretching-bending coupling, bending and shear deformation, and they have the following forms

$$\begin{aligned} \mathbf{k}_{uu} &= \int_0^l \mathbf{N}_{u,x}^T \left(A_{11}^{C1M1} - A_{11}^{C12M12} \left(\frac{x}{l} \right)^{n_x} \right) \mathbf{N}_{u,x} dx; \\ \mathbf{k}_{u\theta} &= - \int_0^l \mathbf{N}_{u,x}^T \left(A_{12}^{C1M1} - A_{12}^{C12M12} \left(\frac{x}{l} \right)^{n_x} \right) \mathbf{N}_{\theta,x} dx; \\ \mathbf{k}_{\theta\theta} &= \int_0^l \mathbf{N}_{\theta,x}^T \left(A_{22}^{C1M1} - A_{22}^{C12M12} \left(\frac{x}{l} \right)^{n_x} \right) \mathbf{N}_{\theta,x} dx; \\ \mathbf{k}_s &= \psi \int_0^l \left(\mathbf{N}_{w,x}^T - \mathbf{N}_\theta^T \right) \left(A_{33}^{C1M1} - A_{33}^{C12M12} \left(\frac{x}{l} \right)^{n_x} \right) \left(\mathbf{N}_{w,x} - \mathbf{N}_\theta \right) dx \end{aligned} \quad (29)$$

with $A_{ij}^{C12M12} = A_{ij}^{C1M1} - A_{ij}^{C2M2}$. Similarly, the kinetic energy (10) can also be written in the form

$$\mathcal{T} = \frac{1}{2} \sum_{i=1}^{ne} \dot{\mathbf{d}}^T \mathbf{m} \dot{\mathbf{d}}; \quad \mathbf{m} = \mathbf{m}_{uu} + \mathbf{m}_{u\theta} + \mathbf{m}_{\theta\theta} + \mathbf{m}_{ww} \quad (30)$$

with \mathbf{m} denotes the element mass matrix, and

$$\begin{aligned} \mathbf{m}_{uu} &= \int_0^l \mathbf{N}_u^T \left(I_{11}^{C1M1} - I_{11}^{C12M12} \left(\frac{x}{l} \right)^{n_x} \right) \mathbf{N}_u dx, \quad \mathbf{m}_{ww} = \int_0^l \mathbf{N}_w^T \left(I_{11}^{C1M1} - I_{11}^{C12M12} \left(\frac{x}{l} \right)^{n_x} \right) \mathbf{N}_w dx; \\ \mathbf{m}_{u\theta} &= - \int_0^l \mathbf{N}_u^T \left(I_{12}^{C1M1} - I_{12}^{C12M12} \left(\frac{x}{l} \right)^{n_x} \right) \mathbf{N}_\theta dx, \quad \mathbf{m}_{\theta\theta} = \int_0^l \mathbf{N}_\theta^T \left(I_{22}^{C1M1} - I_{22}^{C12M12} \left(\frac{x}{l} \right)^{n_x} \right) \mathbf{N}_\theta dx \end{aligned} \quad (31)$$

are, respectively, the element mass matrices resulted from the axial and transverse translations, axial translation-rotation coupling, cross-sectional rotation. As in (29), we have used the notation $I_{ij}^{C12M12} = I_{ij}^{C1M1} - I_{ij}^{C2M2}$.

Based on the derived element stiffness and mass matrices, the equations of motion for the free vibration analysis can be written in the form

$$\mathbf{M}\ddot{\mathbf{D}} + \mathbf{K}\mathbf{D} = \mathbf{0} \tag{32}$$

where \mathbf{D} , \mathbf{M} and \mathbf{K} are the global nodal displacement vector, mass and stiffness matrices, obtained by assembling the corresponding element vector and matrices over the total elements, respectively. Assuming a harmonic response for the free vibration, Eq. (32) leads to

$$(\mathbf{K} - \omega^2\mathbf{M})\bar{\mathbf{D}} = \mathbf{0} \tag{33}$$

with ω is the circular frequency, and $\bar{\mathbf{D}}$ is the vibration amplitude. Eq. (33) leads to an eigenvalue problem, which can be solved by the standard method [18].

3. NUMERICAL RESULTS AND DISCUSSION

This section presents the numerical results for the free vibration of the 2-D FGM beams. Otherwise stated, a beam with an aspect ratio $L/h = 20$, formed from stainless steel (SUS304), Titanium (Ti-6Al-4V), Silicon nitride (Si_3N_4) and Zirconia (ZrO_2) with the material properties listed in Table 1 is employed in the analysis. The SUS304, Ti-6Al-4V, Si_3N_4 and ZrO_2 are used as M1, M2, C1 and C2, respectively. In order to facility of discussion, frequency parameters defined as $\mu_i = \omega_i \frac{L^2}{h} \sqrt{\rho_{M_i} / E_{M_i}}$ with ω_i ($i = 1, 2, 3 \dots$) is the i^{th} natural frequency are introduced.

Table 1. Properties of constituent materials for the two-directional FGM beam [22].

Material	Role	E (GPa)	ρ (kg/m ³)	ν
Steel (SUS304)	M1	207.79	8166	0.3262
Titanium (Ti-6Al-4V)	M2	105.75	4420	0.2888
Silicon nitride (Si3N4)	C1	322.27	2370	0.24
Zirconia (ZrO_2)	C2	116.38	3657	0.333

Firstly, the validation and convergence of the derived formulation are examined. To this end, Table 2 shows the comparison of the fundamental frequency parameter of a simply supported (SS) 2-D FGM beam with various values of the grading indexes obtained herein with that of Ref. [13]. Very good agreement between the result of the present paper with that of Ref. [13] is seen from Table 2. Noting that the numerical result in Table 2 has been obtained for the beam with the constituent materials of Ref. [13].

Table 3 shows the convergence of the derived element in evaluating the frequency parameter μ_1 . As seen from the table, the convergence of the present beam element is fast, and the frequency parameter can be obtained by using just eighteen elements, regardless of the

grading indexes n_x and n_z . Noting that this convergence rate is almost the same as the one formulated in Ref. [13], where the Kosmatka's shape functions have been employed to interpolate the displacement field.

Table 2. Comparison of the fundamental frequency parameter (μ_1) of SS beam.

	Source	$n_x = 0$	$n_x = \frac{1}{3}$	$n_x = \frac{1}{2}$	$n_x = \frac{5}{6}$	$n_x = 1$	$n_x = \frac{4}{3}$	$n_x = \frac{3}{2}$	$n_x = 2$
$n_z = 0$	Ref. [13]	3.3018	3.7429	3.9148	4.1968	4.3139	4.5118	4.5956	4.8005
	Present	3.3018	3.7428	3.9147	4.1966	4.3138	4.5116	4.5954	4.8003
$n_z = \frac{1}{3}$	Ref. [13]	3.1542	3.505	3.6305	3.8252	3.9022	4.0277	4.0792	4.2009
	Present	3.1542	3.505	3.6305	3.8251	3.9021	4.0276	4.079	4.2007
$n_z = \frac{1}{2}$	Ref. [13]	3.1068	3.4285	3.5397	3.7087	3.7745	3.8805	3.9236	4.0245
	Present	3.1069	3.4285	3.5397	3.7086	3.7744	3.8804	3.9234	4.0244
$n_z = \frac{5}{6}$	Ref. [13]	3.0504	3.3296	3.4206	3.5548	3.6059	3.6869	3.7194	3.7947
	Present	3.0506	3.3296	3.4206	3.5548	3.6058	3.6868	3.7193	3.7946
$n_z = 1$	Ref. [13]	3.0359	3.2984	3.3819	3.5035	3.5495	3.6219	3.6508	3.7177
	Present	3.0361	3.2984	3.3819	3.5035	3.5494	3.6218	3.6508	3.7176

Table 3. Convergence of the element in evaluating frequency parameter (μ_1) of SS beam.

Grading indexes	Number of elements (ne)					
	5	10	14	16	18	20
$n_x = n_z = \frac{1}{3}$	3.5079	3.5071	3.5070	3.5070	3.5070	3.5070
$n_x = n_z = \frac{1}{2}$	3.5447	3.5438	3.5437	3.5437	3.5436	3.5436
$n_x = n_z = 1$	3.5240	3.5228	3.5226	3.5226	3.5226	3.5226
$n_x = n_z = 2$	3.4065	3.4051	3.4049	3.4049	3.4048	3.4048

Table 4 lists the fundamental frequency parameter of the SS beam for various material grading indexes n_x and n_z . As can be seen from the table, the frequency parameter increases with increasing the index n_x , irrespective of the index n_z . Furthermore, the increase of the frequency parameter in Table 4 is more significant for $n_x < 1$. For example, with $n_z = 0.2$ the frequency parameter increases 36.36 % when increasing the index n_x from 0 to 1, while this value is just 11.06 % when raising n_x from 1 to 2. However, the increase of the frequency parameter by increasing the index n_x is less pronounced for the beam with a higher index n_z .

For instance, with $n_z = 2$, μ_1 increases only 13.55 % by the increase of n_x from 0 to 1, and the corresponding value is 3.8 % when increasing n_x from 1 to 2. The increase of μ_1 by the increase of the index n_x can be explained by the fact that the percentage of Ti-6Al-4V and ZrO_2 is lower for the beam with a higher n_x , while that of SUS304 and Si_3N_4 is larger. As a result, the beam rigidities increase, and this leads to the increase of the fundamental frequency.

The effect of the grading index n_z on the fundamental frequency parameter as seen from Table 4, however is opposite to that of the index n_x . For a given value of the n_x , Table 4 shows a decrease of the parameter μ_1 by the increase of n_z . A careful examination of the table shows that the decrease of the frequency parameter is more significant for the beam associated with a higher index n_x . The numerical result of Table 4 reveals that the dependence of the fundamental frequency parameter of the 2-D FGM beam upon the grading indexes is much dependent on the value of these indexes.

Table 4. The fundamental frequency parameter (μ_1) of SS beam with various values of n_x, n_z .

	$n_x = 0$	$n_x = 0.2$	$n_x = 0.5$	$n_x = 1.0$	$n_x = 1.2$	$n_x = 1.5$	$n_x = 2.0$
$n_z = 0$	3.1726	3.6151	4.1086	4.7008	4.8854	5.1215	5.4305
$n_z = 0.2$	3.0894	3.4346	3.8008	4.2127	4.3348	4.4869	4.6788
$n_z = 0.5$	3.0146	3.2775	3.5436	3.8273	3.9084	4.0075	4.1297
$n_z = 1.0$	2.9483	3.1429	3.3310	3.5226	3.5759	3.6401	3.7182
$n_z = 1.2$	2.9316	3.1098	3.2799	3.4512	3.4985	3.5555	3.6245
$n_z = 1.5$	2.9121	3.0718	3.2219	3.3712	3.4121	3.4613	3.4613
$n_z = 2.0$	2.8890	3.0275	3.1551	3.2804	3.3146	3.3555	3.4048

The effects of the material grading indexes on the natural frequencies of the beam with various boundary conditions can be seen from Figs. 3-5, where the variation of the first four natural frequency parameters with the material indexes are depicted for the SS, CC and CF beams, respectively. Similar to the fundamental frequency, the figures also show that the higher natural frequencies increase by increasing the index n_x and decrease by increasing the index n_z , regardless of the boundary conditions. The increase of the higher natural frequencies when increasing n_x is also more significant for $n_x < 1$ for all the boundary conditions considered herein. On the other hand, the decrease of the natural frequencies by increasing the index n_z is more significant for the beam associated with higher values of the index n_z , regardless of the boundary conditions.

In order to examine the effects of the shear deformation on the natural frequencies, Table 5 lists the fundamental frequency parameter of the SS beam for various aspect ratios, namely $L/h = 5, 10$ and 30 . The effect of the aspect ratio on the fundamental frequency is clearly seen from the table, where the frequency parameter is seen to be increased by the increase of the aspect

ratio. Since the effect of the shear deformation is more significant for the beam having a lower aspect ratio, Table 5 reveals that the shear deformation which has been taken into account in the present work leads to a decrease of the frequency parameter. A careful examination of Table 5 shows that the effect of the aspect ratio is more significant for the beam with a higher index n_x , but the opposite side is hold for the index n_z . The numerical result in Table 5 also shows the ability of the present beam element in modeling the shear deformation effect of the 2-D FGM beams.

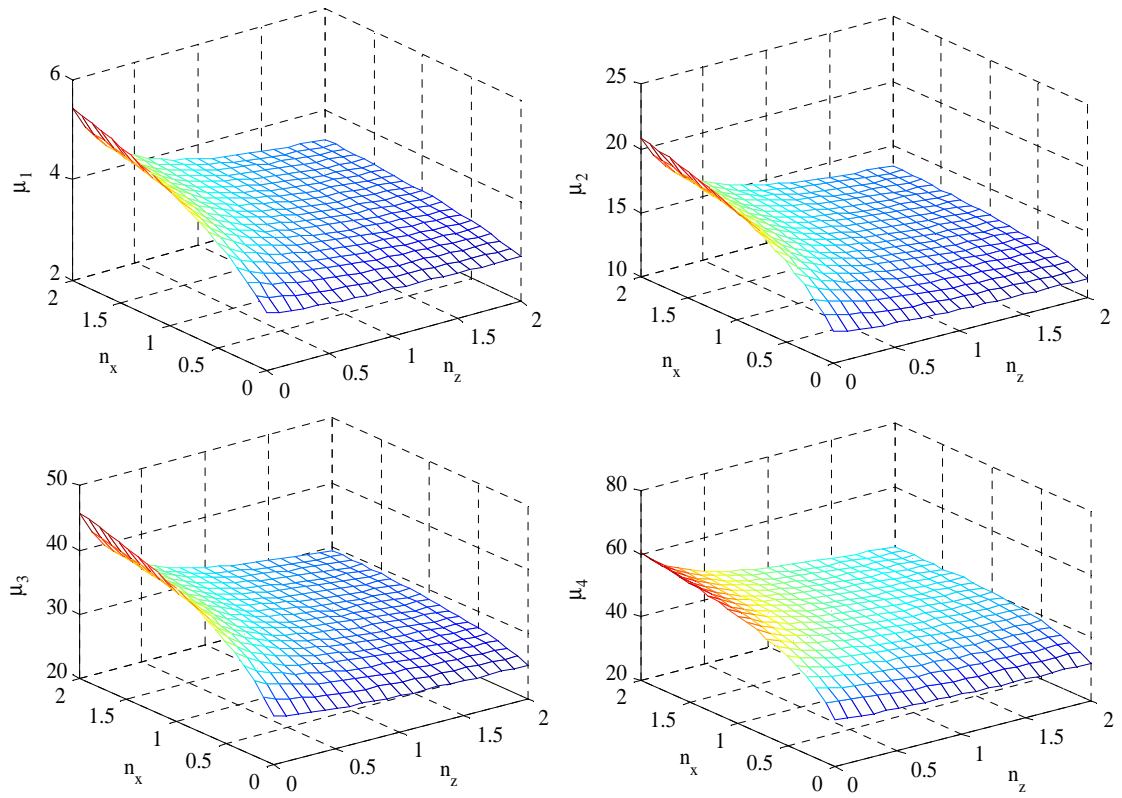


Figure 3. Variation in the first four natural frequency parameters of SS beam.

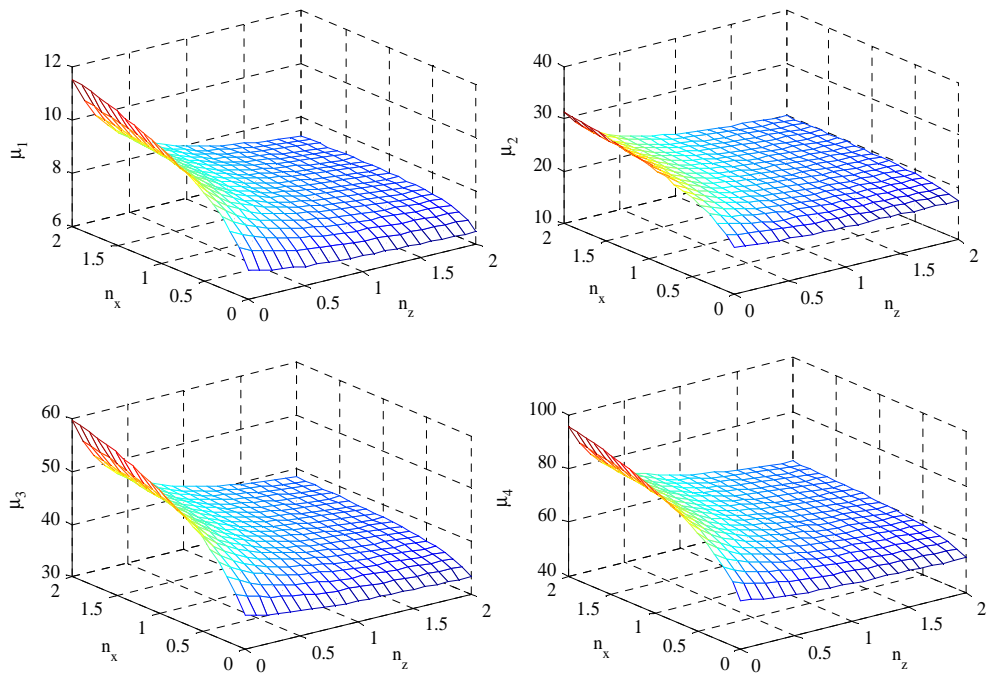


Figure 4. Variation in the first four natural frequency parameters of CC beam.

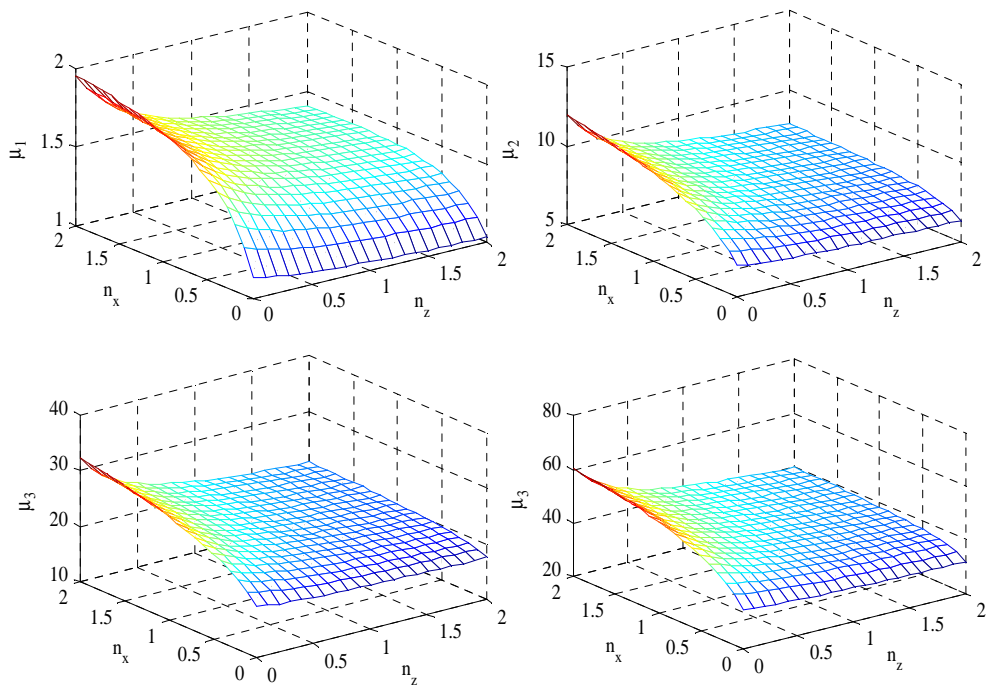


Figure 5. Variation in the first four natural frequency parameters of C-F beam.

Table 5. Fundamental frequency parameter (μ_1) of SS beam with various aspect ratios L/h .

L/h	n_z	n_x						
		0	0.2	0.5	1.0	1.2	1.5	2.0
5	0.2	2.9132	3.2445	3.5837	3.9596	4.0705	4.2086	4.3832
	0.5	2.8433	3.0961	3.3421	3.5999	3.6732	3.7626	3.8732
	1.0	2.7812	2.9686	3.1419	3.3145	3.3622	3.4195	3.4893
	1.5	2.7473	2.9011	3.0387	3.1721	3.2084	3.2517	3.3042
	2	2.7256	2.8589	2.9755	3.0866	3.1164	3.1521	3.1952
10	0.2	3.0510	3.3933	3.7535	4.1573	4.2769	4.4258	4.6139
	0.5	2.9773	3.2381	3.4997	3.7776	3.857	3.9539	4.0735
	1.0	2.9119	3.1050	3.2898	3.4772	3.5292	3.5919	3.6681
	1.5	2.8762	3.0347	3.182	3.3277	3.3676	3.4155	3.4734
	2	2.8535	2.9908	3.1160	3.2381	3.2713	3.3110	3.3589
30	0.2	3.0967	3.4425	3.8099	4.2232	4.3459	4.3459	4.6912
	0.5	3.0217	3.2850	3.5520	3.8368	3.9182	4.0177	4.1405
	1.0	2.9553	3.1502	3.3388	3.5312	3.6081	3.6493	3.7278
	1.5	2.9190	3.0789	3.2295	3.3794	3.4206	3.4701	3.5298
	2	2.8958	3.0344	3.1626	3.2884	3.3228	3.3639	3.4136

Figure 6 shows the mode shapes for w , u and θ of the SS beam with an aspect ratio $L/h = 20$. Since $n_x = 0$ the beam deduces to the unidirectional FGM beam, and thus Fig. 6(a) represents the mode shapes of the transversely unidirectional beam composed of Zirconia and Titanium. It can be seen from Fig. 6 that the vibration modes of the 2-D FGM beam, illustrated in Figure. 6(b), are very different from that of the unidirectional FGM beam. The longitudinal variation of the material properties of the 2-D FGM beam, thus has a significant influence on the vibration modes. While the mode shapes of the transverse displacement of the unidirectional FGM beam are symmetric with respect to the mid-span, that of the 2-D FGM beam are not. The difference in the mode shape of u and θ of the 2-D FGM beam with that of the unidirectional beam can also be observed from the figure, and the asymmetric of the second mode for θ with respect to the mid-span is clearly seen from Fig. 6(b).

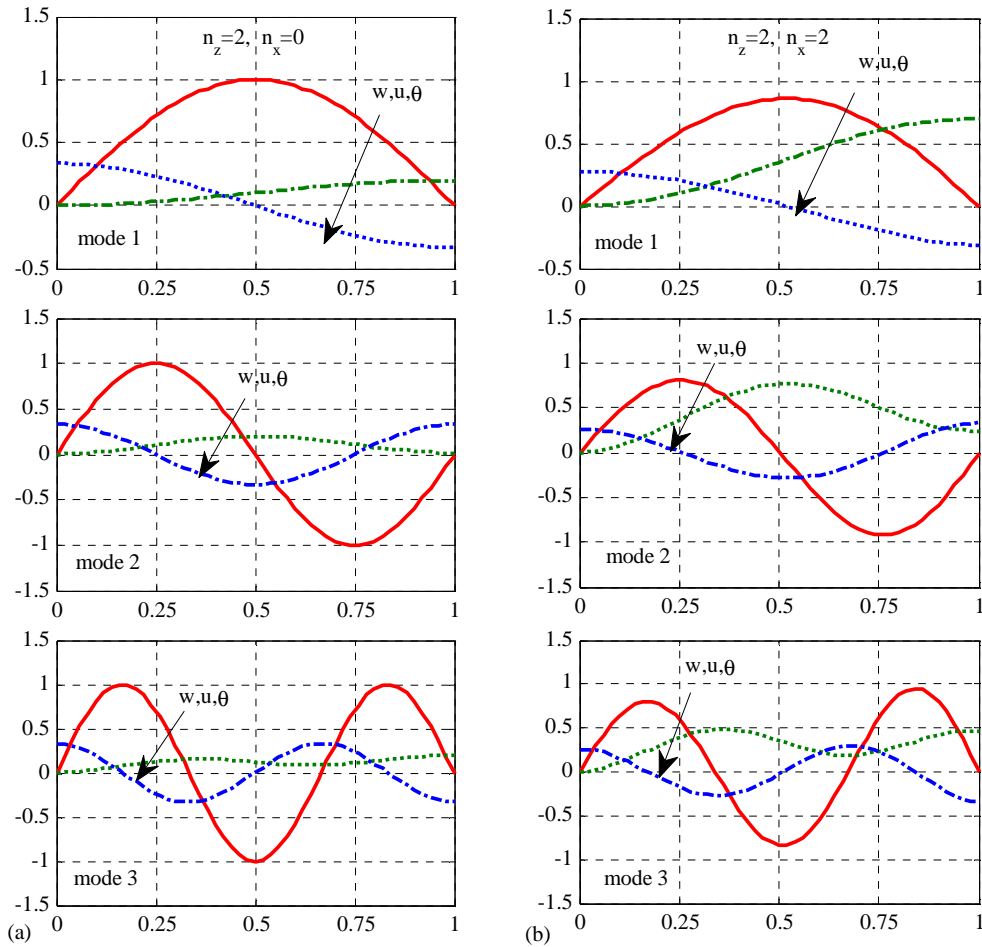


Figure 6. Mode shapes of SS beam: (a) $n_z=2, n_x=0$, (b) $n_z = n_x=2$.

4. CONCLUSION

The free vibration of 2-D FGM beams has been studied in the present paper using a finite element procedure. The material properties were assumed to be graded in both the thickness and longitudinal directions by a power-law distribution. Equations of motion based on Timoshenko beam theory are derived from Hamilton's principle. A higher-order beam element, using hierarchical functions to interpolate the displacement field, has been derived and employed to compute the vibration characteristics of the beams. The shear strain has been constrained to constant for improving the efficiency for the element. The numerical results obtained in the present work reveal that the proposed beam element is fast convergent, and it is enable to give accurate natural frequencies by using a small number of elements. It has also been shown that the derived element has good ability in modeling the shear deformation of the 2-D FGM beams. A parametric has been carried for the beams with various boundary conditions to illustrate the effects of the material distribution on vibration characteristics of the beams. The influence of the aspect ratio on the frequency of the beams has also been examined and highlighted.

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