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# Integer Solutions to Optimization Problems and Modular Sequences of Nexus Numbers

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# INTEGER SOLUTIONS TO OPTIMIZATION PROBLEMS AND MODULAR SEQUENCES OF NEXUS NUMBERS

by

## JEREMY T. DAVIS

(Under the Direction of Colton Magnant)

## ABSTRACT

In this thesis, we examine the use of integers through two ideas. As mathematics teachers, we prefer students not use calculators on assessments. In order to require this, students compute the problems by hand. We take a look at the classic Calculus I optimization box problem while restricting values to integers. In addition, sticking with the integer theme, we take a new look at the nexus numbers. Nexus numbers are extensions of the hex and rhombic dodecahedral numbers. We put these numbers into a sequence, and through a few computations of modular arithmetic, we analyze the sequences and their patterns based upon the different moduli. These patterns are specific to whether the power is even or odd. Within each power, there are other properties to this set of sequences. Depending on modulus, there are some sequences that stand out more than others.

*Index Words*: Integer, Optimization, Modular, Sequences, Nexus Numbers

# INTEGER SOLUTIONS TO OPTIMIZATION PROBLEMS AND

# MODULAR SEQUENCES OF NEXUS NUMBERS

by

**JEREMY T. DAVIS**

B.S. in Mathematics

A Thesis Submitted to the Graduate Faculty of Georgia Southern University in Partial

Fulfillment

of the Requirement for the Degree

MASTER OF SCIENCE

STATESBORO, GEORGIA

2012

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Electronic Version Approved:

December 2012

## ACKNOWLEDGMENTS

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## CHAPTER 1

## BOXES, PANS AND CANS; A DIOPHANTINE APPROACH TO OPTIMIZATION
</center>

Vincent Coll [1], Jeremy Davis[2], Martin Hall[2], Colton Magnant[2], Hua Wang[2]

<center>
"God made the integers, all else is the work of man."

- Leopold Kronecker
</center>

### 1.1   Introduction

A Diophantine equation is an indeterminate polynomial equation in which only integer (and sometimes, rational) solutions are considered. Equations approached from this point of view are usually seen in a course on number theory, where a favorite introductory Diophantine problem involves a box which contains spiders (which have 8 legs) and beetles (which have 6 legs). If we know that the box contains 46 legs, how many of each creature are there? A little algebra establishes that there are only two possible configurations: 2 spiders and 5 beetles or 5 spiders and 1 beetle. However, many Diophantine equations do not admit integer solutions. Moreover, it has been shown that there is no deterministic algorithm which can be used to show whether an arbitrary non-linear Diophantine equation even has a solution. Indeed, this classic result in logic and computability was established by Matiyasevich in 1970 and it negatively resolved the long standing 10th problem of Hilbert which asked if such an algorithm existed (see [3] and [4]).

History's most celebrated Diophantine equation is the subject of the *Last Theorem* of Fermat which he famously penned (but did not prove) in the margin of his notes in 1637. The theorem was finally established in 1994 by Andrew Wiles, fully

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358 years after Fermat insightfully said that it was true (see [**4**]). The theorem states that there are no integer solutions to the equation $x^n + y^n = z^n$, for $n \geq 3$. Of course, if $n = 2$, there are infinitely many solutions which are called *Pythagorean triples* (e.g., (3,4,5) and (5,12,13)) (see [**7**] and [**9**]).

**Problem 0.** *Create an open-top box from a rectangular sheet of cardboard by cutting squares out of each corner and folding up the sides. If the sheet is a units by b units, what is the height x of such a box with maximum volume? See Figure 1.1.*
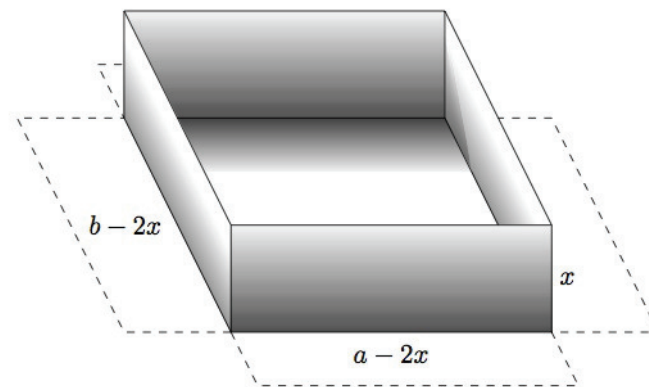


Figure 1.1: Rectangular box.

In creating a test for a first semester calculus course, the Dr. Magnant was challenged to find nice values of $a$ and $b$ in Problem 1 above. This led to the following problem.

**Problem 1.** *Classify the pairs of integers $a$ and $b$ that yield an integer solution $x$ in Problem 0.*

We can generalize Problem 1 by relaxing the condition that $x$ is an integer. This leads to the following problem.

**Problem 2.** *Classify the pairs of integers $a$ and $b$ that yield an optimized integer volume in Problem 0 where $x$ need not be an integer.*

At the outset, we remark there is no *a priori* reason to expect that there are Diophantine solutions to either Problem 2 or 3. But there are! See Table 1.1 for a small sample of solutions.

| $a$ | $b$ | $x$ | $V$ | | $a$ | $b$ | $x$ | $V$ |
|---|---|---|---|---|---|---|---|---|
| 6 | 6 | 1 | 16 | | 3 | 3 | $\frac{1}{2}$ | 2 |
| 5 | 8 | 1 | 18 | | 9 | 9 | $\frac{3}{2}$ | 54 |
| 12 | 12 | 2 | 128 | | 15 | 15 | $\frac{5}{2}$ | 250 |
| 10 | 16 | 2 | 144 | | 21 | 21 | $\frac{7}{2}$ | 686 |
| 9 | 24 | 2 | 200 | | 27 | 27 | $\frac{9}{2}$ | 1458 |
| 18 | 18 | 3 | 432 | | 21 | 45 | $\frac{9}{2}$ | 1944 |

Table 1.1: Solutions to Problem 1 (left) and Problem 2 (right).

In Section 1.2, we solve Problem 1 and Problem 2 by proving Theorems 1.2.1 and 1.2.2, respectively, and so establish the general formulae for $a$, $b$, and $x$ from which the values in Table 1 are constructed. In Section 1.3, we discuss a similar

optimization problem where the starting sheet of cardboard is in the shape of a triangle. Mirroring the rectangular case, Theorem 1.3.2 provides a classification of the possible choices for side lengths which yield an optimized integer solutions to the analogue of Problem 1. We find that similar optimization problems for cardboard sheets bounded by arbitrary $n$-gons, $n > 4$ are doable but the calculations quickly get out of hand. However, by requiring the $n$-gons be regular, we find that if solutions exist for $n = 3$, 4 and 6questions about. We establish this fact in Section 1.4, but leave the classification theorem for $n = 6$ as an exercise. We also suggest (but do not detail) that an analogue of Theorem 1.2.2 exists for the triangular and hexagonal cases. But any such characterization will likely be dependent on the angle. In Section 1.5, we move from boxes to right circular cylinders ("pans") and consider the problem of forming an open-top circular pan of maximum volume which can be made from a sheet of metal in the shape of a regular $n$-gon. Of course, since $\pi$ is involved, this will never yield an integer volume but we classify the number of sides $n$ and integer side lengths $a$ such that the resulting maximum volume is an integer multiple of $\pi$. We show that a sufficient condition to solve the optimization problem is that $n = 4$. We provide some evidence to suggest that this is also a necessary condition. Finally, in Section 1.6, we transition to constrained optimization by considering the problem of making a cylindrical can (a pan with a lid) with a fixed volume that minimizes surface area. Here, the formula for the volume of the can contains an unavoidable appearance of $\sqrt[3]{\pi}$, so, as in the previous section, we put a Diophantine spin on this problem by finding rational functions of $\sqrt[3]{\pi}$ which yield a minimized integer value for the surface area.

## 1.2  Rectangular Boxes

Given an integer $x$, we say that a set of two integers $\{a, b\}$ is a *possible pair for $x$* if the solution to Problem 2 with inputs $a$ and $b$ is $x$. See Figure 1.2.
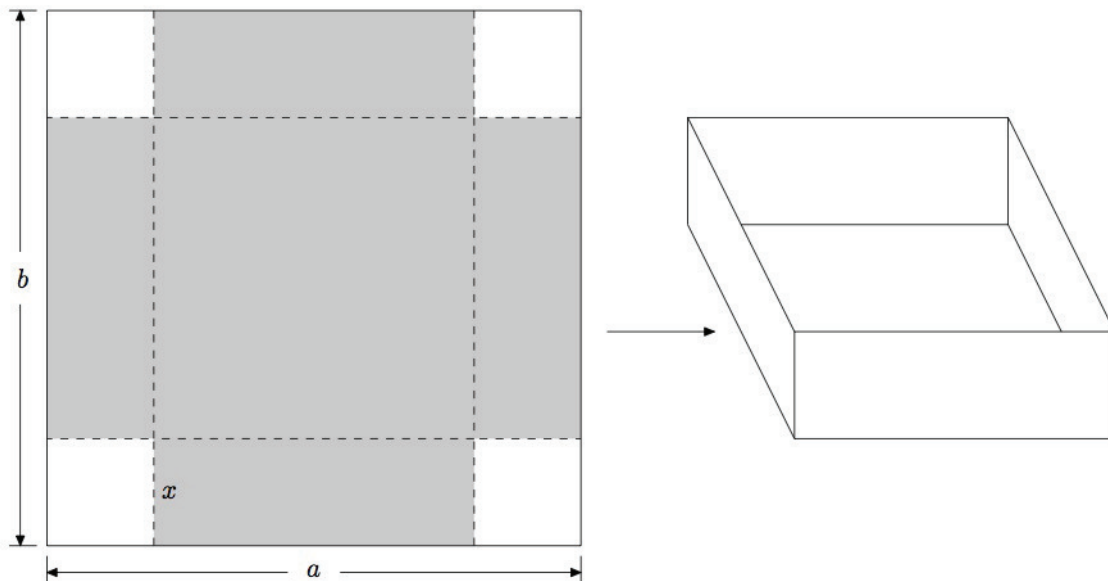


Figure 1.2: Rectangular box construction

The following Theorem resolves Problem 1.

**Theorem 1.2.1.** *Let $x \geq 1$ be an integer and let $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ be the prime factorization of $2x$. Then there are*

$$\frac{(2n_1 + 1)(2n_2 + 1) \cdots (2n_k + 1) + 1}{2}$$

*possible pairs for $x$. Furthermore, for an integer $x$, all possible pairs for $x$ are constructed by choosing $a'$ and $b'$ so that $a'b' = 4x^2$ and then setting $a = a' + 4x$ and $b = b' + 4x$.*

*Proof.* For a given pair of integers $a$ and $b$, the solution to Problem 1 begins with differentiating $(b - 2x)(a - 2x)x$ with respect to $x$ and setting it equal to zero. Thus,

our problem is concerned with finding integer solutions to

$$ab - 4ax - 4bx + 12x^2 = 0.$$

Factoring this yields

$$(a - 4x)(b - 4x) - 4x^2 = 0.$$

Setting $a' = a - 4x$ and $b' = b - 4x$, we see that $a'b' = 4x^2$. Given a value of $x$, the problem has been reduced to simply finding two integers $a'$ and $b'$ whose product is $4x^2$. Since $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ is the prime factorization of $2x$, there must be a total of $(2n_1 + 1)(2n_2 + 1) \cdots (2n_k + 1)$ choices for $a'$ and thus, as many choices for $a$. Since each corresponding value of $b'$ is uniquely determined by the choice of $a'$ and $(2n_1 + 1)(2n_2 + 1) \cdots (2n_k + 1)$ is odd, we get a total of

$$\frac{(2n_1 + 1)(2n_2 + 1) \cdots (2n_k + 1) + 1}{2}$$

choices for $a'$ and $b'$. Note that the additional of one comes from the symmetric choice of $a' = b'$. $\square$

If we relax the restriction that all dimensions of the optimal box must be integers but maintain integrality of the optimal volume, we get the following result which resolves Problem 2.

**Theorem 1.2.2.** *Let $a$ and $b$ be integers such that the maximum volume in Problem 0 is an integer. Then the height $x$ is either an integer or a rational number which, when written in reduced form, has denominator 2. Furthermore, for such a choice of $x$, all possible pairs for $x$ are constructed by choosing $a'$ and $b'$ so that $a'b' = 4x^2$ and then setting $a = a' + 4x$ and $b = b' + 4x$.*

*Proof.* By Theorem 1.2.1, we may assume $x$ is not an integer. We will first show that $x$ must be a rational number. For fixed values of $x, a$ and $b$, the volume of the box is

given by

$$V(x) = 4x^3 - (2a + 2b)x^2 + abx.$$

Differentiating this with respect to $x$ and setting it equal to zero, we get

$$x = \frac{(a + b) \pm \sqrt{a^2 + b^2 - ab}}{6}.$$

Plugging this value of $x$ into the $V(x)$ above and assuming the result is an integer, we get that $(a^2 + b^2 - ab)^{3/2}$ must be an integer. This implies that $a^2 + b^2 - ab$ must be a perfect square. Then, in the choice of $x$ above, $x$ must be a rational number. Setting $x = \frac{p}{q}$ where $p$ and $q$ are integers $\frac{p}{q}$ is in lowest terms. Assuming that $x$ is not an integer, we know $q \geq 2$ and we intend to show that $q = 2$. From the assumption that the resulting volume is an integer $c$, we get

$$ab - \frac{2ap}{q} - \frac{2bp}{q} + \frac{4p^2}{q^2} = \frac{cq}{p}. \tag{1.1}$$

Setting $V'(\frac{p}{q}) = 0$, we get

$$ab - \frac{4ap}{q} - \frac{4bp}{q} + \frac{12p^2}{q^2} = 0. \tag{1.2}$$

Subtracting twice Equation 1.1 from Equation 1.2, we get

$$-ab + \frac{4p^2}{q^2} = \frac{-2cq}{p}$$

or

$$q^2(-2cq + pab) = 4p^3.$$

Since the greatest common factor of $q$ and $p$ is 1, we must have $q^2 | 4$ which means $q = 2$, completing the proof.

Finally, note that since $2x$ is always an integer, the same classification of possible pairs $a$ and $b$ for a given value of $x$ holds. $\qquad \square$

## 1.3  Triangular Boxes

In this section, we consider the same problem as in Section 1.2 except we begin with a triangular sheet of cardboard instead of a rectangular sheet. More specifically, we have the following problem.

**Problem 3.** *Given a sheet of cardboard in the shape of a triangle with sides of integer lengths $a, b$ and $c$, find the maximum volume that can be created by cutting quadrilaterals from the corners and folding (parallel to the edges) up the sides to make an open-top triangular box. See Figure 1.3.*
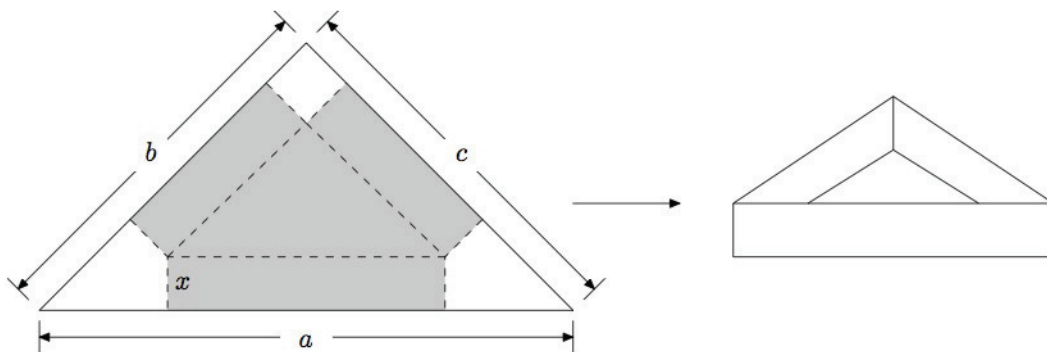


Figure 1.3: Triangular box construction

In order to discuss triangular boxes, we first recall a useful formula for computing the area of a triangle.

**Theorem 1.3.1** (Heron's Formula [5])**.** *The area of a triangle with sides of length $a, b$ and $c$ is given by*

$$A = \frac{1}{4}\sqrt{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}.$$

A complete classification of integers $a, b$ and $c$ yielding integer solutions to Problem 3 is given by the following result.

**Theorem 1.3.2.** *Integers $a, b$ and $c$ produce an integer solution for the volume in Problem 3 if and only if $54 | (a + b - c)(a - b + c)(-a + b + c)$.*

*Proof.* Consider a triangle with sides of length $a, b$ and $c$. Let $d$ be the distance from each side to the *incenter* (see Figure 1.4) of the triangle so if $A$ is the area of the triangle, we get $A = \frac{1}{2}d(a + b + c)$ or

$$d = \frac{2A}{a + b + c}.$$
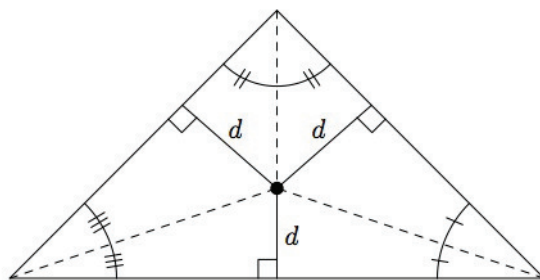


Figure 1.4: The incenter of a triangle

Let $x$ be the distance from each side where we would like to fold (cutting off the appropriate piece from each corner as in Figure 1.3). Then the volume is given by

$$V(x) = \left(\frac{d - x}{x}\right)^2 Ax.$$

Taking the derivative and setting it equal to zero, we see that $V'(x) = \frac{A}{d^2}(3x - d)(x - d)$ so since $x = d$ is a nonsensical solution, we conclude that $x = \frac{d}{3}$ must produce the maximum volume (a simple application of the second derivative test confirms this conclusion). Thus, the optimal volume is given by

$$V_{opt} = \left(\frac{d - \frac{d}{3}}{d}\right)^2 A\frac{d}{3} = \frac{4Ad}{27} = \frac{8A^2}{27(a + b + c)}.$$

By Theorem 1.3.1, we can replace $A$ and reduce this to

$$V_{opt} = \frac{(a + b - c)(a - b + c)(-a + b + c)}{54}$$

which is an integer precisely when

$$54 | (a + b - c)(a - b + c)(-a + b + c)$$

as claimed. □

In particular, if $a = b = c$ then this must be a multiple of 6 as seen in the following table containing the smallest values of $a, b$ and $c$ that are all equal.

| $a$ | $b$ | $c$ | $x$ | $V$ |
|-----|-----|-----|-----|-----|
| 6 | 6 | 6 | $\frac{4}{9\sqrt{3}}$ | 4 |
| 12 | 12 | 12 | $\frac{8}{9\sqrt{3}}$ | 32 |
| 18 | 18 | 18 | $\frac{12}{9\sqrt{3}}$ | 108 |
| 24 | 24 | 24 | $\frac{16}{9\sqrt{3}}$ | 256 |
| 30 | 30 | 30 | $\frac{20}{9\sqrt{3}}$ | 500 |
| 36 | 36 | 36 | $\frac{24}{9\sqrt{3}}$ | 864 |
| 42 | 42 | 42 | $\frac{28}{9\sqrt{3}}$ | 1372 |

Table 1.2: Solutions for Problem 3

## 1.4   Regular $n$-gon Boxes

In this section, we study analogues of Problems 1 and 3 for $n$-gonal shaped cardboard sheets. We find that imposing regularity condition on the bounding polygons allows us to produce necessary (and possibly, sufficient) conditions for the existence of Diophantine solutions. We begin with a fact from elementary plane geometry. Recall that the apothem length of a regular $n$-gon is the distance from the center of the $n$-gon to any edge.

**Fact 1.** *The area of an n-gon is $\frac{1}{2}Pa$ where $P$ is the perimeter and $a$ is the apothem length.*

**Theorem 1.4.1.** *The maximum volume of a regular $n$-gon box with side length $s$ is* $\frac{ns^3 \tan^2(\pi/n)}{54}$.

*Proof.* Given $s$ and $n$, construct a box from an $s$-regular $n$-gon by folding parallel to each of the $n$ edges to create the sides of a box of height $x$. At each corner, we remove a portion of excess material. Let $y$ be the length removed from each edge at each corner, thereby reducing the length of the edge from $s$ to $s' = s - 2y$. Using basic geometry, we see $y = \frac{x}{\tan(\alpha)}$ where $\alpha = \frac{\pi}{n}$ as seen in Figure 1.5.
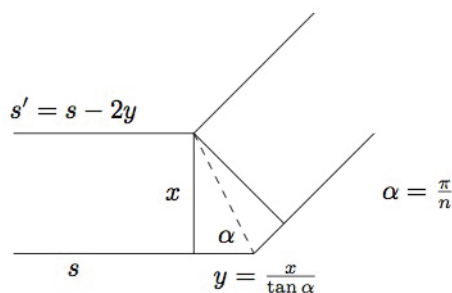


Figure 1.5: A corner of an $n$-gon.

Using Fact 1 to find the area of the base, our total volume is

$$
\begin{aligned}
V &= \frac{1}{2}n(s - 2y)\left(\frac{\tan(\alpha)(s - 2y)}{2}\right)x \\
&= \frac{nx}{4\tan(\alpha)}(s\tan(\alpha) - 2x)^2.
\end{aligned}
$$

To find the maximum volume in terms of $x$, we differentiate $V$ with respect to $x$. This yields

$$
\begin{aligned}
\frac{dV}{dx} &= \frac{n}{\tan(\alpha)}(s\tan(\alpha) - 2x)^2 + \frac{nx}{\tan(\alpha)}(s\tan(\alpha) - 2x) \\
&= \frac{1}{4}n(s(s\tan(\alpha) - 8x) + 12x^2\cot(\alpha)).
\end{aligned}
$$

Setting this derivative equal to zero and solving for $x$ gives

$$x = \frac{\tan(\alpha)\left(2ns \pm \sqrt{n^2 s^2}\right)}{6n} = \frac{\tan(\alpha)(2ns \pm ns)}{6n}$$

so $x$ can be either $\frac{s\tan(\alpha)}{2}$ or $\frac{s\tan(\alpha)}{6}$.

Substituting $\frac{\tan(\alpha)s}{2}$ into the volume equation yields a volume of zero so we conclude $x = \frac{tan(\alpha)s}{6}$. Plugging this into the volume equation, we are left with our desired solution of

$$V = \frac{ns^3 \tan^2(\alpha)}{54}.$$

$\square$

| $n$ | $s$ | $\alpha$ | $V$ |
|---|---|---|---|
| 2 | 3 | $\frac{\pi}{3}$ | 3 |
| 9 | 2 | $\frac{\pi}{3}$ | 4 |
| 4 | 3 | $\frac{\pi}{3}$ | 6 |
| 6 | 3 | $\frac{\pi}{3}$ | 9 |

| $n$ | $s$ | $\alpha$ | $V$ |
|---|---|---|---|
| 2 | 3 | $\frac{\pi}{4}$ | 1 |
| 4 | 3 | $\frac{\pi}{4}$ | 2 |
| 6 | 3 | $\frac{\pi}{4}$ | 3 |
| 2 | 6 | $\frac{\pi}{4}$ | 8 |

| $n$ | $s$ | $\alpha$ | $V$ |
|---|---|---|---|
| 6 | 3 | $\frac{\pi}{6}$ | 1 |
| 3 | 6 | $\frac{\pi}{6}$ | 12 |
| 2 | 9 | $\frac{\pi}{6}$ | 27 |
| 9 | 6 | $\frac{\pi}{6}$ | 36 |

Table 1.3: Solutions to Theorem 5 $\alpha = \frac{\pi}{3}$, $\alpha = \frac{\pi}{4}$, and $\alpha = \frac{\pi}{6}$

Our question now reduces to finding those angles whose tangent value squared is rational. Note that if one trigonometric function squared is rational then so too are the others. We trivially observe that a sufficient condition for this to happen is that the original trigonometric function is rational. The values for which the latter occurs are sparse, well-known, and first introduced into the literature by Olmsted in a Monthly article [6] in 1945 as follows:

**Theorem 1.4.2.** *If $\theta$ is rational in degree, then the only possible rational values of the trigonometric functions are:* $\sin\theta, \cos\theta = 0, \pm\frac{1}{2}, \pm1$; $\sec\theta$, $\csc\theta = \pm1, \pm2$; $\tan\theta$, $\cot\theta = 0, \pm1$.

In our case, this means that $n = 3$, 4 and 6 will solve the optimized integer volume problem. However, this does not preclude that an irrational angle may the property that its six trigonometric functions have rational squares. Indeed, the dihedral angle $\alpha$ ($\approx 70°31'44''$) of the regular tetrahedron is such that $\tan^2 \alpha = 8$. In 1993, Conway *et al* [**2**], characterized all such angles in their article entitled: *On angles whose squared trigonometric functions are rational.* They called these angles *purely geotetic* and found that the set of purely geotetic angles generates a vector space over $\mathbb{Q}$. The study of this space is beyond the scope of our investigation, but we do ask the following question:

**Question 1.** *Are there any purely geotetic angles of the form $\frac{\pi}{n}$, for $n \neq 3$, 4 or 6?*

We close this section with an Exercise.

**Exercise:** For $n = 6$, solve the analogue of Problem 1. See Figure 1.6.
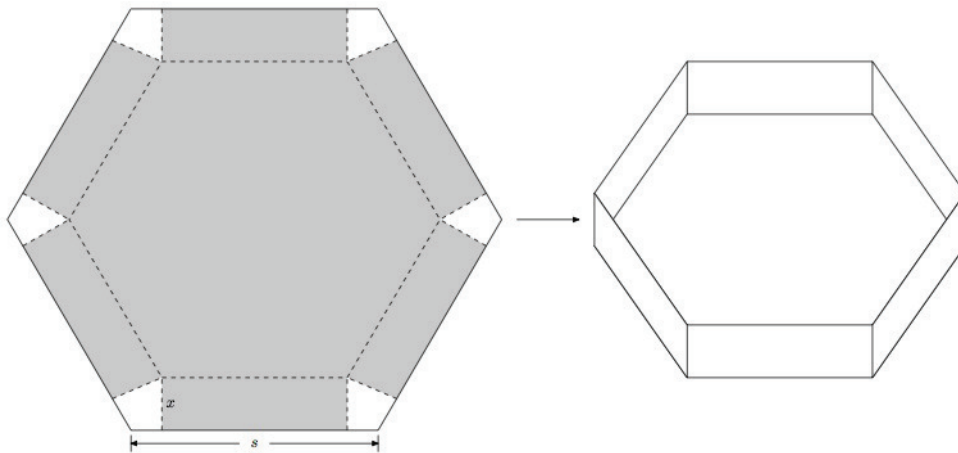


Figure 1.6: Hexagonal box construction

## 1.5   Pie Pan

When attempting to create a cyllindrical pan out of a polygonal sheet of metal with integer sides by folding up an integer height around the side, the resulting volume will necessarily be a multiple of $\pi$. This, we consider conditions that imply the result is an integer multiple of $\pi$.

**Problem 4.** *Suppose we start with a regular polygon with $n$ sides each of length $a$ and would like to create a pie pan of maximum volume by taking a circle in the interior of the polygon and folding up the sides into a cylindrical pan. See Figure 1.7. Classify the integers $n$ and $a$ such that the resulting volume is an integer multiple of $\pi$.*



Figure 1.7: Pan construction.

**Theorem 1.5.1.** *If $n = 4$ and $a^3$ is divisible by 54, then the resulting volume in Problem 4 is an integer multiple of $\pi$.*

*Proof.* The radius of a regular polygon inscribed around a circle with $n$ sides of length $a$ is

$$r'' = \frac{a}{2\cos\left(\frac{\pi}{n}\right)}.$$

This means the radius of the inscribed circle is

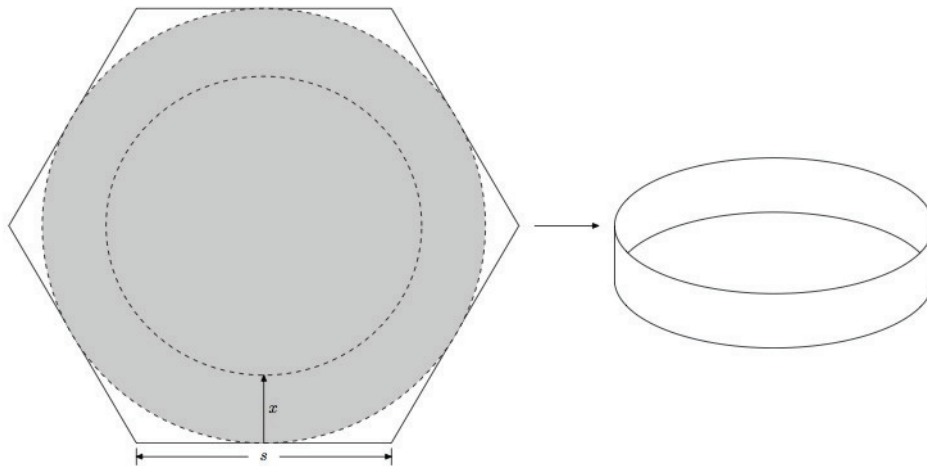$$r' = \sqrt{\frac{a^2}{4\cos^2\left(\frac{\pi}{n}\right)} - \frac{a^2}{4}} = \frac{a}{2}\sqrt{\frac{1 - \cos^2\left(\frac{\pi}{n}\right)}{\cos^2\left(\frac{\pi}{n}\right)}} = \frac{a}{2}\tan\left(\frac{\pi}{n}\right).$$

With $r''$ and $r'$ defined we much check to make sure that

$$lim_{n\to\infty} r'' = r'$$

Which it has to because as $n$ goes to infinity the polygon becomes a circle with interior radius $r'$ and exterior radius $r''$. Since we have a circle, these two radii are the same which was required, so we continue. Further, we define the volume of the total pie pan as $V = (r' - h)^2 h\pi$. Using this we find the height which maximize the volume.

$$\frac{dV}{dh} = \pi r'^2 - 4r'\pi h + 3h^2\pi = 0$$

Solving for $h$:

$$h = \frac{4r' \pm \sqrt{16r'\pi^2 - 12\pi^2 r'^2}}{6\pi} = r'\frac{2 + \pi}{3\pi}$$

This has two solutions, but subtracting the root yields a height of 0 which doesn't make physical sense, so we add giving our final result. Substituting our new expression for $h$ into our original volume equation gives

$$V = r'^3(1 - \frac{2 + \pi}{3\pi})^2(\frac{2 + \pi}{3\pi})\pi.$$

We further state that

$$\lim_{n\to\infty}\frac{dV}{dn} = 0$$

Redefining $c = \frac{2+\pi}{3\pi}$ we have

$$\frac{dV}{dn} = \frac{-(3\pi^2 a^3 c \tan^2\left(\frac{\pi}{n}\right)\sec^2\left(\frac{\pi}{n}\right)(c - 1)^2)}{8n^2}$$

since everything checks we are done.

By Theorem 1.4.2, if $n = 4$, $\cot^3(\pi/n)$ is rational. This means that $V$ is an integer multiple of $\pi$ if $n = 4$ and $a^3$ is divisible by 54. $\qquad\square$

| $a$ | $n$ | $r'$ | $(r')^3$ |
|---|---|---|---|
| 6 | 4 | 3 | 27 |
| 12 | 4 | 6 | 216 |
| 18 | 4 | 9 | 729 |
| 24 | 4 | 12 | 1728 |

Table 1.4: Solutions for Theorem 7

## 1.6 Cans

Another very standard calculus problem is making a cylindrical can with a fixed volume that minimizes surface area. We consider a slight generalization of this problem by considering not only surface area but the total cost of material where the cost of the material user for the top and bottom may be different from the cost of the material used for the side.

**Problem 5.** *Suppose we have a fixed volume $V$, a cost "a" per square unit of top and bottom material and a cost "b" per square unit of side material. Find the dimensions of a cylindrical can with an open top with volume $V$ which has minimum cost.*

Since the value of the radius in the solution to this problem contains $\sqrt{\pi}$, there is no hope that integer choices of $a, b$ and $V$ would yield an integer answer. Thus, we consider the problem of finding integers $a, b$ and $V$ such that the resulting cost is an integer multiple of $\sqrt{\pi}$.

**Theorem 1.6.1.** *The minimum cost solution to Problem 5 is an integer multiple of $\sqrt{\pi}$ if and only if $abV$ is a perfect square.*

*Proof.* Let $r$ and $h$ be the radius and height of the constructed can so the volume is

$V = \pi r^2 h$ and the cost is $C(r, h) = 2\pi r^2 a + 2\pi r h b$. This means $h = \frac{V}{\pi r^2}$ so

$$C(r) = 2\pi r^2 a + \frac{2bV}{r^2}.$$

Taking a derivative with respect to $r$, we get

$$C'(r) = 4\pi r a - \frac{4bV}{r^3}$$

and setting this equal to zero, we find that

$$r = \sqrt[4]{\frac{bV}{\pi a}}.$$

Plugging this value of $r$ back into $C(r)$, we find that the minimum cost is given by

$$C_{min} = 4\sqrt{\pi a b V}$$

which is an integer multiple of $\sqrt{\pi}$ if and only if $abV$ is a perfect square. □

## MODULAR SEQUENCES OF NEXUS NUMBERS

### 2.1 Introduction

We all remember originally learning our numbers. Your parent put items in a container and you pulled one out. After the first one, she would say, one, and you would repeat. This would continue for two, three, four, and so on. Eventually, we would take initiative and count on our own. Soon following counting, the teacher asks the child, What number comes next in the sequence? The child develops the ability to finish the sequence. By the time the student reaches high school, the student is able to find a recursive formula for a sequence. Now some play games with this by creating a recursive formula. They list the first few terms and trade with a partner. It is the partner's job to figure out the formula. Once a fascination with number sequences begins, one discovers integer patterns everywhere.

In the fall of 2008, a student entered the tutoring center for help. I forget the nature of the problem, but I do remember making a t-table for $x^6$. From there, I continued to play with the numbers by subtracting a value by the one before it. I began to notice a pattern in the differences. Each difference was congruent to 1 mod 6. Due to my enrollment in Number Theory at the time, this sparked an interest, so I began to research and grew in love with integers.

Could you imagine watching a football game on a Fall Saturday and the lines on the field are at random yard lines? In John Conway's book (see [1]), he uses figurate numbers to examine Geometry in the performance of Arithmetic and Algebraic ideas. A figurate number represents a geometric shape made from equidistant points. See Figure 2.1.

Within the figurate numbers, there are unique sets of numbers based upon the sequence $(i+1)^n - i^n$. The first four values of $n$ are easier to visualize. When $n = 1$,
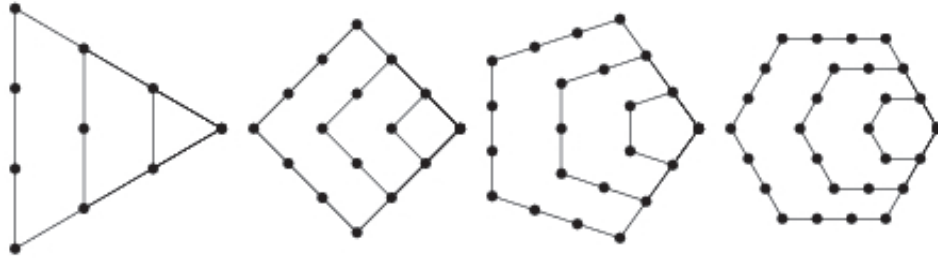
Figure 2.1: Figurate Number

we have $(i + 1)^1 - i^1 = i + 1 - i = 1$, so since each term is 1, we call this the unit sequence. For $n = 2$, we create a list of all the odd numbers since $(i+1)^2 - i^2 = 2i+1$. Moving to the second dimension we are able to visualize a honeycomb pattern, for when $n = 3$ the generating function is $3i^2 + 3i + 1$. This creation of hexagons begins with a singular hexagon $(i = 0)$. When $i$ increases, a new layer of hexagons is added around the existing image. (see Figure 2.2). Thus, this sequence is referred to as the hex numbers.
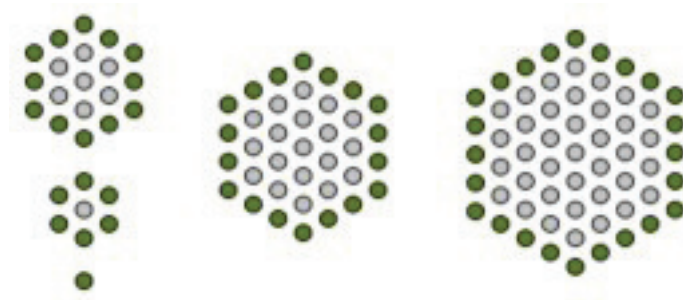


Figure 2.2: Hex Number

In the next dimension, $n = 4$, the same layering exists. Instead of hexagons though, the figure uses a rhombic dodecahedral. In the hex numbers, when $i$ increases, each side adds a new hexagon, but in the rhombic dodecahedral numbers, when $i$ increases, each face adds a new rhombic dodecahedral. As n continues to grow, geometric representation is no longer applicable because. These sequences are referred

to as the nexus numbers. After my findings in the tutoring center involving nexus numbers, I decided to investigate a general idea for the nexus numbers and different moduli.

## 2.2 What to look for?

As mentioned, previous observations showed every odd-powered nexus number is congruent to 1 mod 6. The research continued by creating tables of nexus numbers calculated by different modulos. (These tables can be found in the Appendix.) For each modulo, $k$, the sequence repeats at the $k+1$ term. If the power is odd, then the sequences are palindromic, but if the power is even, then symmetrical term's sum is k. See Appendix for the tables.

## 2.3 Definitions

The hex numbers are defined as $1 + 3x + 3x^2 = (x+1)^3 - x^3$. Likewise, the rhombic dodecahedral numbers are defined as $1+4x+6x^2+4x^3 = (x+1)^4 - x^4$. Since both hex and rhombic dodecahedral numbers can be represented geometrically, we only work with the positive integers. Thus, each term of the modular, nexus number sequence is defined as:

**Definition 2.3.1.** *Let* $a_i = [(i+1)^n - i^n] \mod k$ *for all* $i \in \mathbb{Z}^+$ *and* $0 \leq i \leq k$ .

Also, call the terms $a_i$ and $a_{k-i-1}$ symmetrical.

## 2.4 Theorems

We begin by examining the sequences where $n$ is odd. These sequences are palindromic, so the symmetrical terms are congruent.

**Theorem 2.4.1.** *If* $n$ *is odd, then* $a_i - a_{k-i-1} = 0$.

*Proof.* Assume $n$ is odd. Then

$$a_i - a_{k-i-1} = [(i+1)^n - i^n] \mod k - [(k-i)^n - (k-i-1)^n] \mod k.$$

Since $a_i$ and $a_{k-i-1}$ are both defined to be between 0 and $k-1$, if $a_i - a_{k-i-1} \equiv 0$ mod $k$, then $a_i - a_{k-i-1} = 0$. This means we may consider the quantity

$$
\begin{aligned}
(a_i - a_{k-i-1}) \mod k &= [(i+1)^n - i^n - (k-i)^n + (k-i-1)^n] \mod k \\
&= \Bigg[ (i+1)^n - i^n - \sum_{w=0}^{n} \binom{n}{w} (k)^w (i)^{n-w} (-1)^{n-w} \\
&\quad + \sum_{w=0}^{n} \binom{n}{w} (k)^w (i+1)^{n-w} (-1)^{n-w} \Bigg] \mod k.
\end{aligned}
$$

With the assumption that $n$ is odd, the first term in each sequence is negative. This means we can pull out the first terms to get

$$
\begin{aligned}
(a_i - a_{k-i-1}) \mod k &= \Bigg[ (i+1)^n - i^n + i^n - \sum_{w=1}^{n} \binom{n}{w} (k)^w (i)^{n-w} (-1)^{n-w} \\
&\quad -(i+1)^n + \sum_{w=1}^{n} \binom{n}{w} (k)^w (i+1)^{n-w} (-1)^{n-w} \Bigg] \mod k \\
&= \Bigg[ -\sum_{w=1}^{n} \binom{n}{w} (k)^w (i)^{n-w} (-1)^{n-w} \\
&\quad + \sum_{w=1}^{n} \binom{n}{w} (k)^w (i+1)^{n-w} (-1)^{n-w} \Bigg] \mod k \\
&= 0
\end{aligned}
$$

as desired since each term is a multiple of $k$. $\qquad\square$

As in the previous theorem, the sequences where $n$ is even also share a unique characteristic. In this case each pair of symmetrical terms either sum to $k$ or 0.

**Theorem 2.4.2.** *If $n$ is even, then $a_i + a_{k-i-1} = k$, unless $a_i = a_{k-i-1} = 0$.*

*Proof.* This proof is very similar in nature to the proof of Theorem 2.4.1 above. Assume $n$ is even. Then

$$a_i + a_{k-i-1} = [(i+1)^n - i^n] \mod k + [(k-i)^n - (k-i-1)^n] \mod k.$$

Since $a_i$ and $a_{k-i-1}$ are defined to be between 0 and $k-1$, if they sum to 0 mod $k$, then either they are both zero or their sum is $k$. This means we may consider the quantity

$$
\begin{aligned}
(a_i + a_{k-i-1}) \mod k &= [(i+1)^n - i^n + (k-i)^n - (k-i-1)^n] \mod k \\
&= \left[ (i+1)^n - i^n + \sum_{w=0}^{n} \binom{n}{w} (k)^w (i)^{n-w} (-1)^{n-w} \right. \\
&\quad \left. - \sum_{w=0}^{n} \binom{n}{w} (k)^w (i+1)^{n-w} (-1)^{n-w} \right] \mod k
\end{aligned}
$$

With the assumption that $n$ is even, the first term in each sequence is positive. This means we can pull out the first terms to get

$$
\begin{aligned}
(a_i + a_{k-i-1}) \mod k &= \left[ (i+1)^n - i^n + i^n + \sum_{w=1}^{n} \binom{n}{w} (k)^w (i)^{n-w} (-1)^{n-w} \right. \\
&\quad \left. - (i+1)^n - \sum_{w=1}^{n} \binom{n}{w} (k)^w (i+1)^{n-w} (-1)^{n-w} \right] \mod k \\
&= \left[ \sum_{w=1}^{n} \binom{n}{w} (k)^w (i)^{n-w} (-1)^{n-w} \right. \\
&\quad \left. - \sum_{w=1}^{n} \binom{n}{w} (k)^w (i+1)^{n-w} (-1)^{n-w} \right] \mod k \\
&= 0,
\end{aligned}
$$

as desired since each term is a multiple of $k$. □

While analyzing the tables in the appendix, one may notice the columns where the $a_i = 1$ for all $i$. This occurs multiple times for the tables where $n$ is odd, but it only occurs when $k = 2$ where $n$ is even. The explanation is in the following result.

**Theorem 2.4.3.** *If $n$ is even, then $k = 2$ is the only sequence such that $a_i = 1$ for all $i$.*

*Proof.* Since $n$ is even, we know $a_i + a_{k-i-1} = k$, unless $a_i = a_{k-i-1} = 0$. Also, each $a_0 = 1$, so $a_{k-i-1} \neq 1$ for $k \geq 3$. Therefore, the sequence where $a_i + a_{k-i-1} = k$ is $k = 2$.

$\square$

## 2.5   Conjectures

As proven in Theorem 2.3.2, we see that $k = 2$ is the only sequence where $a_i = 1$ for all $i$ when $n$ is even. But, what about when $n$ is odd? There are multiple sequences where $n$ is odd such that $a - i = 1$ for all $i$. The following conjectures provide further thought for research on these sequences.

**Conjecture 2.5.1.** *If $k$ is a prime factor of $2^n - 2$ for $k \leq n$, then $a_i = 1 \mod k$ for all $i$.*

The thought process is according to the binomial coefficients. Note, the $\binom{n}{0}$ term and the $\binom{n}{n}$ term equal one. The $\binom{n}{0}$ term is eliminated by the definition of $a_i$, but the $\binom{n}{n}$ term remains. This 1 is what we need though for our $1 \mod k$. Therefore, $k$ divides $\binom{n}{1} + \cdots + \binom{n}{n-1}$.

**Conjecture 2.5.2.** *If $a$, $b$, and $c$ be all the prime factors of $2^n - 2$, then all possible products without repetition of $a$, $b$, and $c$ are also $k$'s where $a_i = 1$ for all $i$.*

For example, consider $n = 7$. So, $2^7 - 2 = 126 = 2 \cdot 3^2 \cdot 7$. Note, when $k = 2, 3, 6, 7, 14, 21, 42$ $a_i = 1$ for all $i$, but when $k = 18$ this is not true. Even though 18 is a factor of 126, $18 = 2 \cdot 3^2$. Something about squaring the prime factor does not allow $a_i = 1$ for all $i$.

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## NEXUS NUMBER SEQUENCE TABLE

The following are tables which represent the sequences for multiple $k$ for each $n$.

| $i$ | $(i+1)^3 - i^3$ | mod 2 | mod 3 | mod 4 | mod 5 | mod 6 | mod 7 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 7 | 1 | 1 | 3 | 2 | 1 | 0 |
| 2 | 19 | 1 | 1 | 3 | 4 | 1 | 5 |
| 3 | 37 | 1 | 1 | 1 | 2 | 1 | 2 |
| 4 | 61 | 1 | 1 | 1 | 1 | 1 | 5 |
| 5 | 91 | 1 | 1 | 3 | 1 | 1 | 0 |
| 6 | 127 | 1 | 1 | 3 | 2 | 1 | 1 |
| 7 | 169 | 1 | 1 | 1 | 4 | 1 | 1 |
| 8 | 217 | 1 | 1 | 1 | 2 | 1 | 0 |
| | Sequence | 1 | 1 | 1, 3, 3, 1 | 1, 2, 4, 2, 1 | 1 | 1, 0, 5, 2, 5, 0, 1 |

Table A.1: Table of Modular Sequences for $n = 3$

| $i$ | $(i+1)^4 - i^4$ | mod 2 | mod 3 | mod 4 | mod 5 | mod 6 | mod 7 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 15 | 1 | 0 | 3 | 0 | 3 | 1 |
| 2 | 65 | 1 | 2 | 1 | 0 | 5 | 2 |
| 3 | 175 | 1 | 1 | 3 | 0 | 1 | 0 |
| 4 | 369 | 1 | 0 | 1 | 4 | 3 | 5 |
| 5 | 671 | 1 | 2 | 3 | 1 | 5 | 6 |
| 6 | 1105 | 1 | 1 | 1 | 0 | 1 | 6 |
| 7 | 1695 | 1 | 0 | 3 | 0 | 3 | 1 |
| | *Sequence* | 1 | 1, 0, 2 | 1, 3 | 1, 0, 0, 4 | 1, 3, 5 | 1, 1, 2, 0, 5, 6, 6 |

Table A.2: Table of Modular Sequences for $n = 4$

| $i$ | $(i+1)^5 - i^5$ | mod 2 | mod 3 | mod 4 | mod 5 | mod 6 | mod 7 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 31 | 1 | 1 | 3 | 1 | 1 | 3 |
| 2 | 211 | 1 | 1 | 3 | 1 | 1 | 1 |
| 3 | 781 | 1 | 1 | 1 | 1 | 1 | 4 |
| 4 | 2101 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 4651 | 1 | 1 | 3 | 1 | 1 | 3 |
| 6 | 9031 | 1 | 1 | 3 | 1 | 1 | 1 |
| 7 | 15961 | 1 | 1 | 1 | 1 | 1 | 1 |
| | *Sequence* | 1 | 1 | $1,3,3,1$ | 1 | 1 | $1,3,1,4,1,3,1$ |

Table A.3: Table of Modular Sequences for $n = 5$

| $i$ | $(i+1)^6 - i^6$ | mod 2 | mod 3 | mod 4 | mod 5 | mod 6 | mod 7 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 63 | 1 | 0 | 3 | 3 | 3 | 0 |
| 2 | 665 | 1 | 2 | 1 | 0 | 5 | 0 |
| 3 | 3367 | 1 | 1 | 3 | 2 | 1 | 0 |
| 4 | 11529 | 1 | 0 | 1 | 4 | 3 | 0 |
| 5 | 31031 | 1 | 2 | 3 | 1 | 5 | 0 |
| 6 | 70993 | 1 | 1 | 1 | 3 | 1 | 6 |
| 7 | 144495 | 1 | 0 | 3 | 0 | 3 | 1 |
| | *Sequence* | 1 | $1, 0, 2$ | $1, 3$ | $1, 3, 0, 2, 4$ | $1, 3, 5$ | $1, 0, 0, 0, 0, 6$ |

Table A.4: Table of Modular Sequences for $n = 6$

| $i$ | $(i+1)^7 - i^7$ | mod 2 | mod 3 | mod 4 | mod 5 | mod 6 | mod 7 |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 127 | 1 | 1 | 3 | 2 | 1 | 1 |
| 2 | 2059 | 1 | 1 | 3 | 4 | 1 | 1 |
| 3 | 14197 | 1 | 1 | 1 | 2 | 1 | 1 |
| 4 | 61741 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 201811 | 1 | 1 | 3 | 1 | 1 | 1 |
| 6 | 543607 | 1 | 1 | 3 | 2 | 1 | 1 |
| 7 | 1273609 | 1 | 1 | 1 | 4 | 1 | 1 |
| | *Sequence* | 1 | 1 | $1,3,3,1$ | $1,2,4,2,1$ | 1 | 1 |

Table A.5: Table of Modular Sequences for $n = 7$