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A linear matrix inequality-based approach for the computation of actuator bandwidth limits in adaptive control

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A LINEAR MATRIX INEQUALITY-BASED APPROACH FOR THE COMPUTATION OF ACTUATOR BANDWIDTH LIMITS IN ADAPTIVE CONTROL

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ABSTRACT

Linear matrix inequalities and convex optimization techniques have become popular tools to solve nontrivial problems in the field of adaptive control. Specifically, the stability of adaptive control laws in the presence of actuator dynamics remains as an important open control problem. In this thesis, we present a linear matrix inequalities-based hedging approach and evaluate it for model reference adaptive control of an uncertain dynamical system in the presence of actuator dynamics. The ideal reference dynamics are modified such that the hedging approach allows the correct adaptation without being hindered by the presence of actuator dynamics. The hedging approach is first generalized such that two cases are considered where the actuator output and control effectiveness are known and unknown. We then show the stability of the closed-loop dynamical system using Lyapunov based stability analysis tools and propose a linear matrix inequality-based framework for the computation of the minimum allowable actuator bandwidth limits such that the closed-loop dynamical system remains stable.

The results of the linear matrix inequality-based heading approach are then generalized to multiactuator systems with a new linear matrix inequality condition. The minimum actuator bandwidth solutions for closed-loop system stability are theoretically guaranteed to exist in a convex set with a partially convex constraint and then solved numerically using an algorithm in the case where there are multiple actuators. Finally, the efficacy of the results contained in this thesis are demonstrated using several illustrative numerical examples.

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1. INTRODUCTION

In recent years, the utilization of linear matrix inequalities (LMIs) and convex optimization techniques to solve engineering related problems has become increasingly popular since they are widely available [31, 32] and are capable of finding solutions that are otherwise intractable or prohibitively conservative. In the context of adaptive control, linear matrix inequalities present a new methodology to analyze the stability of systems with inherent uncertainties more rigorously such that their performance limits are guaranteed and well-posed.

1.1. COMPUTING ACTUATOR BANDWIDTH LIMITS IN ADAPTIVE CONTROL

The presence of actuator constraints, which include actuator amplitude constraints, actuator rate constraints, and actuator bandwidth constraints (i.e., actuator dynamics), can seriously limit the stability and the achievable performance of adaptive controller laws. Although actuator amplitude and rate constraints are well studied in the adaptive control literature (see, for example, [1, 2, 3, 4, 5, 6] and references therein), actuator dynamics present a serious challenge to the design and implementation of adaptive controllers. Specifically, if the actuator dynamics have sufficiently high bandwidth, then they can be neglected in the design of model reference adaptive controllers. However, if the actuator dynamics do not have sufficiently high bandwidth or the control system is used for safety-critical applications, stability verification steps must be taken in order to rigorously show the allowable bandwidth range for actuators such that adaptive controllers work correctly (i.e., the closed-loop dynamical system remains stable).

The authors of [6, 10, 11, 12, 13] present notable contributions that allow the design of model reference adaptive controllers in the presence of actuator dynamics. In particular, the authors of [10, 11, 12] present direct approaches to this problem such that the resulting closed-loop dynamical systems, which are explicitly affected by the presence of actuator dynamics, are analyzed. The framework presented in [13], while not explicitly applied to the problem of actuator dynamics, provides a novel approach using linear matrix inequalities to compute a minimum filter bandwidth and guarantee system stability. A similar analysis employed in this thesis can then be applied to the problem of actuator dynamics such that a minimum actuator bandwidth can be calculated while ensuring stability of the system.

The authors of [6] propose a novel hedging approach that enables adaptive controller laws to be designed such that their adaptation performance (i.e., their learning performances of the system uncertainties) is not affected by the presence of actuator dynamics. Specifically, this is accomplished by modifying the ideal reference model dynamics with a hedge signal such that standard adaptation dynamics are achieved even in the presence of actuator dynamics. Yet, it has not been analyzed that this modification to the ideal reference model dynamics does not yield unbounded reference model trajectories in the presence of actuator dynamics. Only the authors of [14] highlight similarities between the hedging approach and the $\mathcal{L}_1$ adaptive control approach so that methods from the latter approach can be used for the former approach to analyze the boundedness of the closed-loop dynamical system with the hedge signal. Although this is possible when the actuator output is known, an analysis is not provided in [14] and since methods from the latter approach are based on small-gain type arguments they can lead to significant conservatism in the analysis (see [13, 15] for details).

In this thesis, a novel hedging approach is developed using linear matrix inequalities and evaluated for model reference adaptive control of uncertain dynamical systems in the presence of actuator dynamics. Specifically, our first contribution is to generalize the hedging approach to cover a variety of cases in which actuator output and the control effectiveness matrix of the uncertain dynamical system are both known and unknown. Our second contribution is to show the stability of the closed-loop dynamical system, which includes the modified reference model trajectories, using tools from Lyapunov stability analysis and propose a linear matrix inequality-based framework for the computation of the minimum allowable actuator bandwidth limits such that the closed-loop dynamical system remains stable. In particular, the proposed linear matrix inequality-based hedging framework characterizes the fundamental stability interplay between the allowable system uncertainties and the bandwidth of the actuator dynamics. This allows a rigorous treatment necessary for safety-critical applications of model reference adaptive controllers. Although this thesis considers a particular model reference adaptive control formulation to present its main results, the proposed linear matrix inequality-based hedging framework can be used in a complimentary way with many other approaches in adaptive control (including but not limited to [16, 17, 18, 19, 20, 21]).

1.2. AN AFFINE QUADRATIC STABILITY CONDITION FOR A LINEAR MATRIX INEQUALITY-BASED HEDGING APPROACH TO NONCONVEX MULTIACTUATOR DYNAMICS

In the contributions stated in Section 1.1, the hedging approach is analyzed with linear matrix inequalities to compute the minimum actuator bandwidth of an uncertain dynamical in presence of actuators dynamics. However, these results are strictly limited to single, first-order actuators. This thesis also generalizes those results to multiple, independent actuator bandwidths by introducing an affine quadratic stability condition (AQS) for linear matrix inequalities with time invariant parameter uncertainties. Specifically, we convexify a generally nonconvex hedged control problem by introducing the affine quadratic stability condition and develop an algorithm that can solve the specific case where there are three actuators that have their own independent bandwidths. To the best of our knowledge, this is the first time anyone has addressed this problem using our approach.

1.3. ORGANIZATION

The organization of this thesis is as follows. Chapter 2 covers all the necessary mathematical preliminaries, Chapter 3 introduces the proposed linear matrix inequality-based hedging approach for uncertain dynamical systems subject to actuator dynamics with known and unknown outputs, Chapter 4 generalizes these results specifically to multiactuator systems, Chapter 5 contains all of the illustrative examples for all cases, and Chapter 6 contains our conclusions and some suggestions for future research.

2. NOTATION AND MATHEMATICAL PRELIMINARIES

We briefly begin by providing the notation used throughout this thesis. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (resp., $\overline{\mathbb{R}}_+$) denotes the set of positive (resp., non-negative-definite) real numbers, $\mathbb{R}_+^{n\times n}$ (resp., $\overline{\mathbb{R}}_+^{n\times n}$) denotes the set of $n \times n$ positive-definite (resp., non-negative-definite) real matrices, $\mathbb{S}^{n\times n}$ denotes the set of $n \times n$ symmetric real matrices, $\mathbb{D}^{n\times n}$ denotes the set of $n \times n$ real matrices with diagonal scalar entries, $(\cdot)^{\mathrm{T}}$ denotes the transpose operator, $(\cdot)^{-1}$ denotes the inverse operator, $\mathrm{tr}(\cdot)$ denotes the trace operator, $||\cdot||_2$ denotes the Euclidian norm, $||\cdot||_{\mathrm{F}}$ denotes the Frobenius matrix norm, $[A]_{ij}$ denotes the ij-th entry of the real matrix $A \in \mathbb{R}^{n\times m}$, $\lambda_{\min}(A)$ (resp., $\lambda_{\max}(A)$) for the minimum (resp. maximum) eigenvalue of the real matrix $A \in \mathbb{R}^{n\times m}$, and "$\triangleq$" denotes the equality by definition.

Next, we introduce some fundamental results that are needed to develop the main results of this thesis. We begin with the following definition.

Definition 1. For a convex hypercube in $\mathbb{R}^n$ defined by $\Omega = \{\theta \in \mathbb{R}^n : (\theta_i^{\min} \leq \theta_i \leq \theta_i^{\max})_{i=1,2,\ldots,n}\}$ where $(\theta_i^{\min}, \theta_i^{\max})$ represent the minimum and maximum bounds for the i^{th} component of the n-dimensional parameter vector θ. Additionally, for a sufficiently small positive constant ϵ, a second hypercube is defined by $\Omega_\epsilon = \{\theta \in \mathbb{R}^n : (\theta_i^{\min} + \epsilon \leq \theta_i \leq \theta_i^{\max} - \epsilon)_{i=1,2,\ldots,n}\}$ where $\Omega_\epsilon \subset \Omega$. Then the projection operator $\mathrm{Proj} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined compenent-wise by

$$\mathrm{Proj}(\theta, y) \triangleq \begin{cases} \left(\frac{\theta_i^{\max} - \theta_i}{\epsilon}\right) y_i, & \text{if } \theta_i > \theta_i^{\max} - \epsilon \text{ and } y_i > 0 \\ \left(\frac{\theta_i - \theta_i^{\min}}{\epsilon}\right) y_i, & \text{if } \theta_i < \theta_i^{\min} + \epsilon \text{ and } y_i < 0 \\ y_i, & \text{otherwise} \end{cases} \tag{2.1}$$

where $y \in \mathbb{R}^n$ [22].

It follows from Definition 1 that

$$(\theta - \theta^*)^{\mathrm{T}}(\mathrm{Proj}(\theta, y) - y) \leq 0, \quad \theta^* \in \mathbb{R}^n, \tag{2.2}$$

holds [22, 23].

Remark 1. Throughout the thesis, we use the generalization of this definition to matrices as

$$\mathrm{Proj_m}(\Theta, Y) = \big(\mathrm{Proj}(\mathrm{col}_1(\Theta), \mathrm{col}_1(Y)),\ \ldots,\ \mathrm{Proj}(\mathrm{col}_m(\Theta), \mathrm{col}_m(Y))\big), \tag{2.3}$$

where $\Theta \in \mathbb{R}^{n\times m}$, $Y \in \mathbb{R}^{n\times m}$, and $\mathrm{col}_i(\cdot)$ denotes the i-th column operator. In this case, for a given $\Theta^* \in \mathbb{R}^{n\times m}$, it follows from (2.2) that

$$\begin{aligned} &\mathrm{tr}\ \Big[(\Theta - \Theta^*)^{\mathrm{T}}(\mathrm{Proj_m}(\Theta, Y) - Y)\Big] \\ &\qquad = \sum_{i=1}^{m} \Big[\mathrm{col}_i(\Theta - \Theta^*)^{\mathrm{T}}(\mathrm{Proj}(\mathrm{col}_i(\Theta), \mathrm{col}_i(Y)) - \mathrm{col}_i(Y))\Big] \leq 0, \end{aligned} \tag{2.4}$$

holds.

We now briefly state the standard model reference control problem. Specifically, consider the uncertain dynamical system given by

$$\dot{x}(t) \;=\; Ax(t) + Bu(t), \quad x(0) = x_0, \tag{2.5}$$

where $x(t) \in \mathbb{R}^n$ is the state vector available for feedback, $u(t) \in \mathbb{R}^m$ is the control input restricted to the class of admissible controls consisting of measurable functions, $A \in \mathbb{R}^{n\times n}$ is an unknown system matrix, $B \in \mathbb{R}^{n\times m}$ is an unknown input matrix, and the pair (A, B) is controllable. The following assumption is standard in the model reference adaptive control literature.

Assumption 1. The unknown control input matrix is parameterized as

$$B = D\Lambda, \tag{2.6}$$

where $D \in \mathbb{R}^{n \times m}$ is a known input matrix and $\Lambda \in \mathbb{R}_+^{m \times m} \cap \mathbb{D}^{m \times m}$ is an unknown control effectiveness matrix which can be decomposed as

$$\Lambda = I + \delta\Lambda, \tag{2.7}$$

where $\delta\Lambda < I$ is unknown.

Next, consider the reference system capturing a desired, ideal closed-loop dynamical system performance given by

$$\dot{x}_{\mathrm{r}}(t) = A_{\mathrm{r}} x_{\mathrm{r}}(t) + B_{\mathrm{r}} c(t), \quad x_{\mathrm{r}}(0) = x_{\mathrm{r}0}, \tag{2.8}$$

where $x_{\mathrm{r}}(t) \in \mathbb{R}^n$ is the reference state vector, $c(t) \in \mathbb{R}^m$ is a given uniformly continuous bounded command, $A_{\mathrm{r}} \in \mathbb{R}^{n \times n}$ is the Hurwitz reference system matrix, and $B_{\mathrm{r}} \in \mathbb{R}^{n \times m}$ is the command input matrix. The objective of the model reference adaptive control problem is to construct an adaptive feedback control law $u(t)$ such that the state vector $x(t)$ asymptotically follows the reference state vector $x_{\mathrm{r}}(t)$. We now make the following assumption, which is also standard in the model reference adaptive control literature and is known as the matching condition [22, 24, 25].

Assumption 2. There exists an unknown matrix $K_1 \in \mathbb{R}^{m \times n}$ and a known matrix $K_2 \in \mathbb{R}^{m \times m}$ such that $A_{\mathrm{r}} = A + DK_1$ and $B_{\mathrm{r}} = DK_2$ hold.

Now, (2.5) subject to standard Assumptions 1 and 2 yields

$$\begin{aligned} \dot{x}(t) &= A_{\mathrm{r}} x(t) + B_{\mathrm{r}} c(t) + D(I + \delta\Lambda) u(t) + D W_1^{\mathrm{T}} x(t) - D K_2 c(t) \\ &= A_{\mathrm{r}} x(t) + B_{\mathrm{r}} c(t) + D\big[u(t) + W_1^{\mathrm{T}} x(t) + \delta\Lambda u(t) - K_2 c(t)\big], \end{aligned} \tag{2.9}$$

where $W_1 \triangleq -K_1^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ and $\delta\Lambda \in \mathbb{R}^{m\times m}$ are unknown. Let the adaptive feedback control law be given by

$$u(t) \;=\; -(I+\delta\widehat{\Lambda}(t))^{-1}\big(\widehat{W}_1^{\mathrm{T}}(t)x(t) - K_2 c(t)\big), \tag{2.10}$$

where $\widehat{W}_1(t) \in \mathbb{R}^{n\times m}$ and $\delta\widehat{\Lambda}(t) \in \mathbb{R}^{m\times m}$ are the estimates of W_1 and $\delta\Lambda$ that respectively satisfy the weight update laws

$$\dot{\widehat{W}}_1(t) \;=\; \gamma_1 \mathrm{Proj_m}\big[\widehat{W}_1(t),\; x(t)e^{\mathrm{T}}(t)PD\big], \quad \widehat{W}_1(0) = \widehat{W}_{10}, \tag{2.11}$$

$$\delta\dot{\widehat{\Lambda}}(t) \;=\; \gamma_\Lambda \mathrm{Proj_m}\big[\delta\widehat{\Lambda}(t),\; D^{\mathrm{T}}Pe(t)u^{\mathrm{T}}(t)\big], \quad \delta\widehat{\Lambda}(0) = \delta\widehat{\Lambda}_0, \tag{2.12}$$

where $\gamma_1 \in \mathbb{R}_+$ and $\gamma_\Lambda \in \mathbb{R}_+$ are the learning rate gains, $e(t) \triangleq x(t) - x_{\mathrm{r}}(t)$ is the system error state vector, and $P \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$ is a solution of the Lyapunov equation

$$0 = A_{\mathrm{r}}^{\mathrm{T}}P + PA_{\mathrm{r}} + R, \tag{2.13}$$

with $R \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$. Note that since A_{r} is Hurwitz, it follows from the converse Lyapunov theory [26] that there exists a unique P satisfying (2.13) for a given R. In addition, the projection bounds are defined such that

$$\Big|[\widehat{W}_1(t)]_{ij}\Big| \;\le\; \widehat{W}_{1,\max,i+(j-1)n}, \quad i = 1,...,n \text{ and } j = 1,...,m, \tag{2.14}$$

$$\Big|[\delta\widehat{\Lambda}(t)]_{ij}\Big| \;\le\; \delta\widehat{\Lambda}_{\max,i+(j-1)m}, \quad i = 1,...,m \text{ and } j = 1,...,m \tag{2.15}$$

where $\widehat{W}_{1,\max,i+(j-1)n} \in \mathbb{R}_+$ and $\delta\widehat{\Lambda}_{\max,i+(j-1)m} \in \mathbb{R}_+$ denote (symmetric) element-wise projection bounds. Note that the results of this thesis can be readily applied to the case when asymmetric projection bounds are considered.

Remark 2. The projection bounds on $\delta\hat{\Lambda}(t)$ are selected such that $I + \delta\hat{\Lambda}(t)$ is invertible and therefore (2.10) is implementable.

Noting that (2.10) can be given by the equivalent form

$$u(t) = -\widehat{W}_1^{\mathrm{T}}(t)x(t) - \delta\hat{\Lambda}(t)u(t) + K_2 c(t), \tag{2.16}$$

then (2.16) can be used in (2.9) to yield

$$\dot{x}(t) = A_{\mathrm{r}}x(t) + B_{\mathrm{r}}c(t) - D\big[\widetilde{W}_1^{\mathrm{T}}(t)x(t) + \delta\tilde{\Lambda}(t)u(t)\big], \tag{2.17}$$

and the system error dynamics is then given using (2.8) and (2.17) as

$$\dot{e}(t) = A_{\mathrm{r}}e(t) - D\big[\widetilde{W}_1^{\mathrm{T}}(t)x(t) + \delta\tilde{\Lambda}(t)u(t)\big], \quad e(0) = e_0, \tag{2.18}$$

where $\widetilde{W}_1(t) \triangleq \widehat{W}_1(t) - W_1 \in \mathbb{R}^{n\times m}$ and $\delta\tilde{\Lambda}(t) \triangleq \delta\hat{\Lambda}(t) - \delta\Lambda \in \mathbb{R}^{m\times m}$.

Remark 3. The weight update laws given by (2.11) and (2.12) can be derived using Lyapunov analysis by considering the Lyapunov function candidate given by (see, for example, [22, 24, 25])

$$\mathcal{V}(e, \widetilde{W}_1, \delta\tilde{\Lambda}) = e^{\mathrm{T}}Pe + \gamma_1^{-1}\mathrm{tr}\ \widetilde{W}_1^{\mathrm{T}}\widetilde{W}_1 + \gamma_\Lambda^{-1}\mathrm{tr}\ \delta\tilde{\Lambda}^{\mathrm{T}}\delta\tilde{\Lambda}. \tag{2.19}$$

Note that $\mathcal{V}(0,0,0) = 0$ and $\mathcal{V}(e, \widetilde{W}_1, \delta\tilde{\Lambda}) > 0$ for all $(e, \widetilde{W}_1, \delta\tilde{\Lambda}) \neq (0,0,0)$. Now, differentiating (2.19) yields

$$\begin{aligned}&\dot{\mathcal{V}}(e(t), \widetilde{W}_1(t), \delta\tilde{\Lambda}(t))\\ &\quad = -e^{\mathrm{T}}(t)Re(t) - 2e^{\mathrm{T}}(t)PD\widetilde{W}_1^{\mathrm{T}}(t)x(t) - 2e^{\mathrm{T}}(t)PD\delta\tilde{\Lambda}(t)u(t)\\ &\qquad +2\gamma_1^{-1}\mathrm{tr}\ \widetilde{W}_1^{\mathrm{T}}(t)\dot{\widehat{W}}_1(t) + 2\gamma_\Lambda^{-1}\mathrm{tr}\ \delta\tilde{\Lambda}^{\mathrm{T}}(t)\delta\dot{\hat{\Lambda}}(t),\end{aligned} \tag{2.20}$$

where using (2.11) and (2.12) in (2.20) results in $\dot{\mathcal{V}}(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t)) \leq -e^{\mathrm{T}}(t)Re(t) \leq 0$, which guarantees that the system error state vector $e(t)$ and the weight errors $\widetilde{W}_1(t)$ and $\delta\widetilde{\Lambda}(t)$ are Lyapunov stable, and are therefore bounded for all $t \in \overline{\mathbb{R}}_+$. Since $x(t)$ and $c(t)$ are bounded for all $t \in \overline{\mathbb{R}}_+$, it follows from (2.18) that $\dot{e}(t)$ is bounded, and hence, $\ddot{\mathcal{V}}(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t))$ is bounded for all $t \in \overline{\mathbb{R}}_+$. It then follows from Barbalat's lemma that $\lim_{t\to\infty} \dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t)\big) = 0$, which consequently shows that $e(t) \to 0$ as $t \to \infty$.

Remark 4. It should be noted that in the case B is known, the uncertainty in the control effectiveness matrix is equivalently given by $\delta\Lambda = 0$ such that Assumptions 1 and 2 hold. It then follows that the adaptive feedback control simplifies to

$$u(t) = -\widehat{W}_1^{\mathrm{T}}(t)x(t) + K_2 c(t), \tag{2.21}$$

satisfying the update law given by (2.11).

In the rest of this thesis, we resort to the mathematical preliminaries stated in this section for developing our main results.

3. COMPUTING ACTUATOR BANDWIDTH LIMITS FOR MODEL REFERENCE ADAPTIVE CONTROL

3.1. INTRODUCTION

Although model reference adaptive control theory has been used in numerous applications to achieve system performance without excessive reliance on dynamical system models, the presence of actuator dynamics can seriously limit the stability and the achievable performance of adaptive controllers. In this chapter, a linear matrix inequality-based hedging approach is developed and evaluated for model reference adaptive control of uncertain dynamical systems in the presence of actuator dynamics. The hedging method modifies the ideal reference model dynamics in order to allow correct adaptation that is not affected by the presence of actuator dynamics. Specifically, we first generalize the hedging approach to cover a variety of cases in which actuator output and the control effectiveness matrix of the uncertain dynamical system are known and unknown. We then show the stability of the closed-loop dynamical system using Lyapunov based stability analysis tools and propose a linear matrix inequality-based framework for the computation of the minimum allowable actuator bandwidth limits such that the closed-loop dynamical system remains stable.

The organization of this chapter is as follows. Section 3.2 introduces the proposed linear matrix inequality-based hedging approach for uncertain dynamical systems subject to actuator dynamics with known and unknown outputs, while Section 3.3 generalizes the results of this section to include uncertainty in the control effectiveness matrix. Lastly, the concluding remarks are stated in Section 3.4

3.2. A LINEAR MATRIX INEQUALITY-BASED HEDGING APPROACH TO ACTUATOR DYNAMICS

For the model reference adaptive control framework introduced in the previous section, we now introduce the actuator dynamics problem. Specifically, consider the uncertain system given by

$$\dot{x}(t) \;=\; Ax(t) + Bv(t), \quad x(0) = x_0, \tag{3.1}$$

where $B \in \mathbb{R}^{n\times m}$ is known for the results of this section and $v(t) \in \mathbb{R}^m$ is the actuator output given by the dynamics

$$\begin{aligned} \dot{x}_{\rm c}(t) \;&=\; -Mx_{\rm c}(t) + u(t), \quad x_{\rm c}(0) = x_{\rm c0}, \\ v(t) \;&= Mx_{\rm c}(t), \quad v(0) = v_0, \end{aligned} \tag{3.2}$$

where $x_{\rm c}(t) \in \mathbb{R}^m$ is the actuator state vector, $M \in \mathbb{R}^{m\times m} \cap \mathbb{D}^{m\times m}$ with diagonal entries $\lambda_{i,i} > 0$, $i = 1, \ldots, m$, represents the actuator bandwidth of each control channel, and $u(t) \in \mathbb{R}^m$ is the control input restricted to the class of admissible controls consisting of measurable functions. The objective of the hedging approach [6] is to construct an adaptive feedback control law such that its adaptation performance (i.e., its learning performance of system uncertainty W_1) is not affected by the presence of actuator dynamics.

Remark 5. Note that the actuator dynamics given by (3.2) can be represented in the following equivalent form

$$\begin{aligned} \dot{\bar{x}}_{\rm c}(t) \;&= M\big(-\bar{x}_{\rm c}(t) + u(t)\big), \quad \bar{x}_{\rm c}(0) = \bar{x}_{\rm c0}, \\ \bar{v}(t) \;&= \bar{x}_{\rm c}(t), \quad \bar{v}(0) = \bar{v}_0, \end{aligned} \tag{3.3}$$

where $\bar{x}_{\mathrm{c}}(t) = Mx_{\mathrm{c}}(t)$. However, the actuator dynamics given by (3.2) are used such that the resulting linear matrix inequality conditions are simplified in later analysis.

Consider now, by adding and subtracting $Bu(t)$, the following equivalent form of (3.1) subject to Assumption 2,

$$\dot{x}(t) = A_{\mathrm{r}}x(t) + B_{\mathrm{r}}c(t) + B\big[u(t) + W_1^{\mathrm{T}}x(t) - K_2c(t)\big] + B\big[v(t) - u(t)\big]. \quad (3.4)$$

Using the adaptive feedback control law given by

$$u(t) = -\widehat{W}_1^{\mathrm{T}}(t)x(t) + K_2c(t), \quad (3.5)$$

where $\widehat{W}_1(t) \in \mathbb{R}^{n\times m}$ satisfies the weight update law

$$\dot{\widehat{W}}_1(t) = \gamma_1 \mathrm{Proj}_{\mathrm{m}}\big[\widehat{W}_1(t),\ x(t)e^{\mathrm{T}}(t)PB\big], \quad \widehat{W}_1(0) = \widehat{W}_{10}, \quad (3.6)$$

with the projection bound defined by (2.14), it follows that the system dynamics (3.4) can be equivalently rewritten as

$$\dot{x}(t) = A_{\mathrm{r}}x(t) + B_{\mathrm{r}}c(t) - B\widetilde{W}_1^{\mathrm{T}}(t)x(t) + B\big[v(t) - u(t)\big]. \quad (3.7)$$

The rest of this section is broken into two subsections in which we will look into two different cases. First, it will be assumed that the actuator output is known (Section 3.2.1) and then in the second case this assumption will be removed (Section 3.2.2). For each case, a systematic proof is included.

3.2.1. Known Actuator Output Case. Using the hedging approach [6], the reference system is modified to the following

$$\dot{x}_{\mathrm{r}}(t) = A_{\mathrm{r}}x_{\mathrm{r}}(t) + B_{\mathrm{r}}c(t) + B\big[v(t) - u(t)\big], \quad x_{\mathrm{r}}(0) = x_{\mathrm{r}0}, \quad (3.8)$$

such that the system error dynamics can be given using (3.7) and (3.8) as

$$\dot{e}(t) \;=\; A_{\mathrm{r}}e(t) - B\widetilde{W}_1^{\mathrm{T}}(t)x(t), \quad e(0) = e_0. \tag{3.9}$$

The following lemma is needed for the results in this section. For this purpose, let $\underline{\lambda} \in \mathbb{R}_+$ be such that $\underline{\lambda} \leq \lambda_{i,i}$ for all $i = 1, \ldots, m$ and let $\overline{\omega} \in \mathbb{R}_+$ be such that $\widehat{W}_{1,\max,i+(j-1)n} \leq \overline{\omega}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Lemma 1. There exists a set $\kappa_1 \triangleq \big\{\underline{\lambda} : \ \underline{\lambda} \leq \lambda_{i,i}, \ i = 1, \ldots, m\big\} \bigcup \big\{\overline{\omega} : \widehat{W}_{1,\max,i+(j-1)n} \leq \overline{\omega}, \ i = 1, \ldots, n, \ j = 1, \ldots, m\big\}$ such that if $(\underline{\lambda}, \overline{\omega}) \in \kappa_1$, then

$$\mathcal{A}(\widehat{W}_1(t), M) \;=\; \begin{bmatrix} A_{\mathrm{r}} + B\widehat{W}_1^{\mathrm{T}}(t) & BM \\ -\widehat{W}_1^{\mathrm{T}}(t) & -M \end{bmatrix} \tag{3.10}$$

is quadratically stable.

Proof. We first show that there exists $\underline{\lambda}$ such that (3.10) is quadratically stable. For this purpose, consider the Lyapunov inequality given by

$$\begin{aligned} \mathcal{A}^{\mathrm{T}}(\widehat{W}_1(t), M)\mathcal{P} + \mathcal{P}\mathcal{A}(\widehat{W}_1(t), M) < 0, \\ \mathcal{P} = \mathcal{P}^{\mathrm{T}} > 0, \end{aligned} \tag{3.11}$$

with

$$\mathcal{P} = \begin{bmatrix} P & PB \\ B^{\mathrm{T}}P & B^{\mathrm{T}}PB + \rho I \end{bmatrix}, \tag{3.12}$$

where $P \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$ is a solution of the Lyapunov equation given by (2.13) with $R \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$ and $\rho \in \mathbb{R}_+$. Note that the positive-definiteness of (3.12) follows from the positive-definiteness of P and the positive-definiteness of the Schur complement of (3.12) given by

$$S_1 = B^{\mathrm{T}} P B + \rho I - B^{\mathrm{T}} P (P)^{-1} P B = \rho I > 0. \tag{3.13}$$

Next, note that

$$\begin{aligned} \mathcal{Q} &= \mathcal{A}^{\mathrm{T}}(\widehat{W}_1(t), M)\mathcal{P} + \mathcal{P}\mathcal{A}(\widehat{W}_1(t), M) \\ &= \begin{bmatrix} -R & A_{\mathrm{r}}^{\mathrm{T}} P B - \widehat{W}_1(t)\rho \\ B^{\mathrm{T}} P A_{\mathrm{r}} - \rho \widehat{W}_1^{\mathrm{T}}(t) & -2\rho M \end{bmatrix}. \end{aligned} \tag{3.14}$$

Since $-R$ is a negative-definite matrix, it follows from the Schur complement of (3.14)

$$S_2 = -2\rho M + \big(B^{\mathrm{T}} P A_{\mathrm{r}} - \rho \widehat{W}_1^{\mathrm{T}}(t)\big) R^{-1} \big(A_{\mathrm{r}}^{\mathrm{T}} P B - \widehat{W}_1(t)\rho\big), \tag{3.15}$$

that (3.15) is a negative-definite matrix when $\underline{\lambda}$ is sufficiently large, which yields to the quadratic stability of (3.10).

We next show that there exists $\overline{\varpi}$ such that (3.10) is quadratically stable. For this purpose, we note that (3.10) is quadratically stable when $\overline{\varpi} = 0$, which follows from the upper triangular structure of (3.10) in this case and the fact that A_{r} and $-M$ are Hurwitz matrices. Since (3.10) depends continuously to the variations in $0 < \widehat{W}_{1,\max,i+(j-1)n} \leq \overline{\varpi}$, the quadratic stability of (3.10) is assured when $\overline{\varpi}$ is sufficiently small. Finally, since there exist a (sufficiently large) $\underline{\lambda}$ or a (sufficiently small) $\overline{\varpi}$ such that (3.10) is quadratically stable, the existence of set κ_1 is immediate.

■

Theorem 1. Consider the uncertain dynamical system given by (3.1) subject to Assumption 2, the reference system given by (3.8), the actuator dynamics given by (3.2), the adaptive feedback control law given by (3.5) along with the update law (3.6). If $(\underline{\lambda}, \overline{\omega}) \in \kappa_1$, then the solution $(e(t), \widetilde{W}_1(t), x_{\rm r}(t), v(t))$ of the closed-loop dynamical system is bounded and $\lim_{t\to\infty} e(t) = 0$.

Proof. To show Lyapunov stability and guarantee boundedness of the system error state $e(t)$ and the weight error $\widetilde{W}_1(t)$, consider the Lyapunov function candidate

$$\mathcal{V}(e, \widetilde{W}_1) \;=\; e^{\rm T}Pe + \gamma_1^{-1}{\rm tr}\; \widetilde{W}_1^{\rm T}\widetilde{W}_1. \tag{3.16}$$

Note that $\mathcal{V}(0,0) = 0$ and $\mathcal{V}(e, \widetilde{W}_1) > 0$ for all $(e, \widetilde{W}_1) \neq (0,0)$. Then, differentiating (3.16) yields $\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t)\big) \leq -e^{\rm T}(t)Re(t) \leq 0$, which guarantees the Lyapunov stability, and hence, the boundedness of the solution $\big(e(t), \widetilde{W}_1(t)\big)$.

To show the boundedness of $x_{\rm r}(t)$ and $x_{\rm c}(t)$ (and therefore $v(t)$), consider the reference system (3.8) and the actuator dynamics (3.2) subject to (3.5) as

$$\dot{x}_{\rm r}(t) \;=\; A_{\rm r}x_{\rm r}(t) + B\big[Mx_{\rm c}(t) + \widehat{W}_1^{\rm T}(t)e(t) + \widehat{W}_1^{\rm T}(t)x_{\rm r}(t)\big], \tag{3.17}$$

$$\dot{x}_{\rm c}(t) \;=\; -Mx_{\rm c}(t) - \widehat{W}_1^{\rm T}(t)e(t) - \widehat{W}_1^{\rm T}(t)x_{\rm r}(t) + K_2c(t), \tag{3.18}$$

where (3.17) and (3.18) can be rewritten in compact form as

$$\dot{\xi}(t) \;=\; \mathcal{A}(\widehat{W}_1(t), M)\xi(t) + \omega(\cdot), \tag{3.19}$$

with $\xi(t) = [x_{\rm r}^{\rm T}(t), x_{\rm c}^{\rm T}(t)]^{\rm T}$ and

$$\omega(\cdot) \;=\; \begin{bmatrix} B\widehat{W}_1^{\rm T}(t)e(t) \\ -\widehat{W}_1^{\rm T}(t)e(t) + K_2c(t) \end{bmatrix}. \tag{3.20}$$

Note that $\omega(\cdot)$ in (3.19) is a bounded perturbation as a result of Lyapunov stability of the double $(e(t), \widetilde{W}_1(t))$. Now, it follows that since $\omega(\cdot)$ is bounded and $\mathcal{A}(\widehat{W}_1(t), M)$ is quadratically stable for $(\underline{\lambda}, \overline{\omega}) \in \kappa_1$ by Lemma 1, then $x_{\rm r}(t)$ and $x_{\rm c}(t)$ are also bounded (see, for example, [27]). This further implies that the actuator output $v(t)$ is bounded.

To show $\lim_{t\to\infty} e(t) = 0$, note that $x(t)$ is bounded as a consequence of the boundedness of $e(t)$ and $x_{\rm r}(t)$. It now follows from (3.9) that $\dot{e}(t)$ is bounded, and hence, $\ddot{\mathcal{V}}(e(t), \widetilde{W}_1(t))$ is bounded. As a consequence of the boundedness of $\ddot{\mathcal{V}}(e(t), \widetilde{W}_1(t))$ and Barbalat's lemma [27], $\lim_{t\to\infty} \dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t)\big) = 0$, and hence, $\lim_{t\to\infty} e(t) = 0$. ■

Remark 6. For the results given in Theorem 1 to hold, it is assumed that (3.10) is quadratically stable [28]. Lemma 1 shows the feasibility of this assumption when $(\underline{\lambda}, \overline{\omega}) \in \kappa_1$. Specifically, this implies the actuator dynamics are sufficiently fast (i.e., $\underline{\lambda}$ is sufficiently large such that $\lambda_{i,i}$ are, and hence, M is sufficiently large for all $i = 1, \ldots, m$) or the projection bounds on $\widehat{W}_1(t)$ are sufficiently small (i.e., $\overline{\omega}$ is sufficiently small such that $\widehat{W}_{1,\max,i+(j-1)n}$ is sufficiently small for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$). It is of practical importance to note that this reveals the fundamental stability interplay between the allowable system uncertainties (through the selection of the projection operator bounds) and the bandwidths of the actuator dynamics.

Remark 7. We now utilize linear matrix inequalities to satisfy the quadratic stability of (3.10) for given projection bounds $\widehat{W}_{1,\max}$ for the elements of $\widehat{W}_1(t)$ and the bandwidths of the actuator dynamics M. For this purpose, let $\bar{W}_{1_{i_1,\ldots,i_l}} \in \mathbb{R}^{n\times m}$ be defined as

$$\bar{W}_{1_{i_1,\ldots,i_l}} = \begin{bmatrix} (-1)^{i_1}\widehat{W}_{1,\max,1} & (-1)^{i_{1+n}}\widehat{W}_{1,\max,1+n} & \cdots & (-1)^{i_{1+(m-1)n}}\widehat{W}_{1,\max,1+(m-1)n} \\ (-1)^{i_2}\widehat{W}_{1,\max,2} & (-1)^{i_{2+n}}\widehat{W}_{1,\max,2+n} & \cdots & (-1)^{i_{2+(m-1)n}}\widehat{W}_{1,\max,2+(m-1)n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{i_n}\widehat{W}_{1,\max,n} & (-1)^{i_{2n}}\widehat{W}_{1,\max,2n} & \cdots & (-1)^{i_{mn}}\widehat{W}_{1,\max,mn} \end{bmatrix}, \tag{3.21}$$

where $i_l \in \{1,2\}$, $l \in \{1,...,mn\}$, such that $\bar{W}_{1_{i_1,...,i_l}}$ represents the corners of the hypercube defining the maximum variation of $\widehat{W}_1(t)$. Following the results in [13] and [28], if

$$\mathcal{A}_{i_1,...,i_l} = \begin{bmatrix} A_r + B\bar{W}^{\mathrm{T}}_{1_{i_1,...,i_l}} & BM \\ -\bar{W}^{\mathrm{T}}_{1_{i_1,...,i_l}} & -M \end{bmatrix}, \tag{3.22}$$

satisfies the matrix inequality

$$\mathcal{A}^{\mathrm{T}}_{i_1,...,i_l}\mathcal{P} + \mathcal{P}\mathcal{A}_{i_1,...,i_l} < 0, \quad \mathcal{P} = \mathcal{P}^{\mathrm{T}} > 0, \tag{3.23}$$

for all permutations of $\bar{W}_{1_{i_1,...,i_l}}$, then (3.10) is quadratically stable. Since (3.10) is quadratically stable for large values of M (see Remark 6), we cast (3.23) as a convex optimization problem given by

$$\begin{aligned} &\text{minimize } M, \\ &\text{subject to (3.23).} \end{aligned} \tag{3.24}$$

Therefore, we can satisfy (3.23) by minimizing M for a given projection bound.

3.2.2. Unknown Actuator Output Case. The results in Section 3.2.1 assume that the actuator output is measurable and therefore known, which may not always be the case (e.g, for some low-cost and small-in-size unmanned vehicle applications). If the actuator output is unknown, then this requires a generalization of the results presented in the previous section. For this purpose, consider the modified reference system given by

$$\dot{x}_{\mathrm{r}}(t) = A_{\mathrm{r}}x_{\mathrm{r}}(t) + B_{\mathrm{r}}c(t) + B\big[\widehat{v}(t) - u(t)\big], \quad x_{\mathrm{r}}(0) = x_{\mathrm{r}0}, \tag{3.25}$$

where $\widehat{v} \in \mathbb{R}_+$ is an estimate of the unknown actuator output $v(t)$ satisfying the update law

$$\dot{\widehat{v}}(t) = \mu B^{\mathrm{T}} P e(t) + M\big(u(t) - \widehat{v}(t)\big), \quad \widehat{v}(0) = \widehat{v}_0, \tag{3.26}$$

with $\mu = \beta M$, $\beta \in \mathbb{R}_+$, being a design parameter. In this case, the system dynamics and the adaptive feedback control law given by (3.7) and (3.5), respectively, along with the modified reference system (3.25) results in the following system error dynamics

$$\dot{e}(t) = A_{\mathrm{r}} e(t) - B\widetilde{W}_1^{\mathrm{T}}(t)x(t) - B\widetilde{v}(t), \quad e(0) = e_0, \tag{3.27}$$

where $\widetilde{v}(t) \triangleq \widehat{v}(t) - v(t) \in \mathbb{R}^m$.

Theorem 2. Consider the uncertain dynamical system given by (3.1) subject to Assumption 2, the reference system given by (3.25), the actuator dynamics given by (3.2), the feedback control law given by (3.5) along with the update laws (3.6) and (3.26). If $(\underline{\lambda}, \overline{\omega}) \in \kappa_1$, then the solution $\big(e(t), \widetilde{W}_1(t), x_{\mathrm{r}}(t), v(t), \widetilde{v}(t)\big)$ of the closed-loop dynamical system is bounded for all initial conditions and $t \in \overline{\mathbb{R}}_+$, and $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} \widetilde{v}(t) = 0$.

Proof. To show Lyapunov stability and guarantee boundedness of the system error state $e(t)$, the weight error $\widetilde{W}_1(t)$, and the actuator output error $\widetilde{v}(t)$, consider the Lyapunov function candidate

$$\mathcal{V}(e, \widetilde{W}_1, \widetilde{v}) = e^{\mathrm{T}} P e + \gamma_1^{-1} \mathrm{tr}\, \widetilde{W}_1^{\mathrm{T}} \widetilde{W}_1 + \beta^{-1} \widetilde{v}^{\mathrm{T}} M^{-1} \widetilde{v}. \tag{3.28}$$

Note that $\mathcal{V}(0,0,0) = 0$ and $\mathcal{V}\big(e, \widetilde{W}_1, \widetilde{v}\big) > 0$ for all $\big(e, \widetilde{W}_1, \widetilde{v}\big) \neq (0,0,0)$. Differentiating (3.28) yields $\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \widetilde{v}(t)\big) \leq -e^{\mathrm{T}}(t) R e(t) - 2\beta^{-1} \widetilde{v}^{\mathrm{T}}(t) \widetilde{v}(t) \leq 0$, which guarantees the Lyapunov stability, and hence, the boundedness of the solution $\big(e(t), \widetilde{W}_1(t), \widetilde{v}(t)\big)$.

Similar to the proof of Theorem 1, to show the boundedness of $x_{\mathrm{r}}(t)$ and $x_{\mathrm{c}}(t)$ (and therefore $v(t)$), consider the reference system (3.25) and the actuator dynamics (3.2) subject to (3.5) as

$$\dot{x}_{\mathrm{r}}(t) = A_{\mathrm{r}}x_{\mathrm{r}}(t) + B\big[Mx_{\mathrm{c}}(t) + \widehat{W}_1^{\mathrm{T}}(t)e(t) + \widehat{W}_1^{\mathrm{T}}(t)x_{\mathrm{r}}(t)\big] + B\widetilde{v}(t), \quad (3.29)$$

$$\dot{x}_{\mathrm{c}}(t) = -Mx_{\mathrm{c}}(t) - \widehat{W}_1^{\mathrm{T}}(t)e(t) - \widehat{W}_1^{\mathrm{T}}(t)x_{\mathrm{r}}(t) + K_2c(t), \quad (3.30)$$

where (3.29) and (3.30) can be rewritten in compact form as (3.19) with

$$\omega(\cdot) = \begin{bmatrix} B\widehat{W}_1^{\mathrm{T}}(t)e(t) + B\widetilde{v}(t) \\ -\widehat{W}_1^{\mathrm{T}}(t)e(t) + K_2c(t) \end{bmatrix}. \quad (3.31)$$

Note that $\omega(\cdot)$ in (3.31) is a bounded perturbation as a result of Lyapunov stability of the triple $\big(e(t), \widetilde{W}_1(t), \widetilde{v}(t)\big)$. Now, it follows that since $\omega(\cdot)$ is bounded and $\mathcal{A}(\widehat{W}_1(t), M)$ is quadratically stable for $(\underline{\lambda}, \overline{\omega}) \in \kappa_1$ by Lemma 1, then $x_{\mathrm{r}}(t)$ and $x_{\mathrm{c}}(t)$ are also bounded (see, for example, [27]). This further implies that the actuator output $v(t)$ is bounded. The remainder of the proof is similar to the proof of Theorem 1, and hence, omitted. ■

For the results given in Theorem 2 to hold, it is assumed that (3.10) is quadratically stable. As this is the same condition given in Section 3.2.1, it should be noted that the same discussion and results provided in Remarks 6 and 7 hold for this case of unknown actuator output.

3.3. ACTUATOR DYNAMICS WITH UNCERTAIN CONTROL EFFECTIVENESS

In this section, we generalize the results of the previous section to uncertain dynamical systems with unknown control effectiveness matrices that satisfy Assumption 1. For this purpose, consider the uncertain system given by (3.1) subject to actuator dynamics in (3.2). Using Assumptions 1 and 2, (3.1) can be equivalently written as

$$\begin{aligned} \dot{x}(t) &= A_{\mathrm{r}}x(t) + B_{\mathrm{r}}c(t) + D\big[(I+\delta\Lambda)u(t) + W_1^{\mathrm{T}}x(t) - K_2c(t)\big] \\ &\quad + D(I+\delta\Lambda)\big[v(t)-u(t)\big] \\ &= A_{\mathrm{r}}x(t) + B_{\mathrm{r}}c(t) + D\big[u(t) + W_1^{\mathrm{T}}x(t) + \delta\Lambda v(t) - K_2c(t)\big] \\ &\quad + D\big[v(t)-u(t)\big], \end{aligned} \tag{3.32}$$

where $W_1 \triangleq -K_1^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ and $\delta\Lambda \in \mathbb{R}^{m\times m}$ are unknown. Now, let the adaptive feedback control law be given by

$$u(t) = -\widehat{W}_1^{\mathrm{T}}(t)x(t) - \delta\widehat{\Lambda}(t)v(t) + K_2c(t), \tag{3.33}$$

where $\widehat{W}_1(t) \in \mathbb{R}^{n\times m}$ and $\delta\widehat{\Lambda}(t) \in \mathbb{R}^{m\times m}$ satisfy the respective weight update laws

$$\dot{\widehat{W}}_1(t) = \gamma_1 \mathrm{Proj}_{\mathrm{m}}\big[\widehat{W}_1(t),\ x(t)e^{\mathrm{T}}(t)PD\big], \quad \widehat{W}_1(0) = \widehat{W}_{10}, \tag{3.34}$$

$$\delta\dot{\widehat{\Lambda}}(t) = \gamma_\Lambda \mathrm{Proj}_{\mathrm{m}}\big[\delta\widehat{\Lambda}(t),\ D^{\mathrm{T}}Pe(t)v^{\mathrm{T}}(t)\big], \quad \delta\widehat{\Lambda}(0) = \delta\widehat{\Lambda}_0, \tag{3.35}$$

with the projection bounds defined respectively by (2.14) and (2.15), where the projection bounds of $\delta\widehat{\Lambda}(t)$ are chosen such that

$$M\delta\widehat{\Lambda}^{\mathrm{T}}(t) + \delta\widehat{\Lambda}(t)M > -2M \tag{3.36}$$

holds. It follows that using (3.33) in (3.32), the system dynamics are now given by

$$\dot{x}(t) \;=\; A_{\rm r}x(t) + B_{\rm r}c(t) - D\big[\widetilde{W}_1^{\rm T}(t)x(t) + \delta\widetilde{\Lambda}(t)v(t)\big] + D\big[v(t) - u(t)\big]. \quad (3.37)$$

Remark 8. Note that to show the condition given by (3.36) holds, we consider $\delta\bar{\Lambda}_{i_1,\ldots,i_r} \in \mathbb{R}^{m\times m}$ defined as

$$\delta\bar{\Lambda}_{i_1,\ldots,i_r} = \begin{bmatrix} (-1)^{i_1}\delta\widehat{\Lambda}_{\max,1} & (-1)^{i_{1+m}}\delta\widehat{\Lambda}_{\max,1+m} & \cdots & (-1)^{i_{1+(m-1)m}}\delta\widehat{\Lambda}_{\max,1+(m-1)m} \\ (-1)^{i_2}\delta\widehat{\Lambda}_{\max,2} & (-1)^{i_{2+m}}\delta\widehat{\Lambda}_{\max,2+m} & \cdots & (-1)^{i_{2+(m-1)m}}\delta\widehat{\Lambda}_{\max,2+(m-1)m} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{i_m}\delta\widehat{\Lambda}_{\max,m} & (-1)^{i_{2m}}\delta\widehat{\Lambda}_{\max,2m} & \cdots & (-1)^{i_{mm}}\delta\widehat{\Lambda}_{\max,mm} \end{bmatrix}, \quad (3.38)$$

where $i_r \in \{1, 2\}$, $r \in \{1, ..., mm\}$, such that $\delta\bar{\Lambda}_{i_1,\ldots,i_r}$ represents the corners of the hypercube defining the maximum variation of $\delta\widehat{\Lambda}(t)$. It then follows that if (3.38) satisfies the inequality

$$M\delta\bar{\Lambda}^{\rm T}_{i_1,\ldots,i_r} + \delta\bar{\Lambda}_{i_1,\ldots,i_r}M > -2M, \quad (3.39)$$

for all permutations of $\delta\bar{\Lambda}_{i_1,\ldots,i_r}$, then (3.36) holds.

In what follows, we first consider the case in which the actuator output is known (Section 3.3.1) and then generalize our results to the case where it is unknown (Section 3.3.2).

3.3.1. Known Actuator Output Case. Consider the following modified reference system,

$$\dot{x}_{\rm r}(t) \;=\; A_{\rm r}x_{\rm r}(t) + B_{\rm r}c(t) + D\big[v(t) - u(t)\big], \quad x_{\rm r}(0) = x_{\rm r0}, \quad (3.40)$$

and using (3.37) and (3.40) the system error dynamics are given by

$$\dot{e}(t) \;=\; A_{\rm r}e(t) - D\big[\widetilde{W}_1^{\rm T}(t)x(t) + \delta\widetilde{\Lambda}(t)v(t)\big], \quad e(0) = e_0. \tag{3.41}$$

The following lemma is needed for the results in this section. For this purpose, let $\underline{\lambda} \in \mathbb{R}_+$ be such that $\underline{\lambda} \leq \lambda_{i,i}$ for all $i = 1, \ldots, m$, let $\overline{\omega} \in \mathbb{R}_+$ be such that $\widehat{W}_{1,\max,i+(j-1)n} \leq \overline{\omega}$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$, and let $\overline{\nu} \in \mathbb{R}_+$ be such that $\delta\widehat{\Lambda}_{\max,i+(j-1)m} \leq \overline{\nu}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, m$.

Lemma 2. There exists a set $\kappa_2 \triangleq \big\{\underline{\lambda} : \ \underline{\lambda} \leq \lambda_{i,i}, \ i = 1, \ldots, m\big\} \bigcup \big\{\overline{\omega}, \ \overline{\nu} : \widehat{W}_{1,\max,i+(j-1)n} \leq \overline{\omega}, \ i = 1, \ldots, n, \ j = 1, \ldots, m, \text{ and } \delta\widehat{\Lambda}_{\max,i+(j-1)m} \leq \overline{\nu}, \ i = 1, \ldots, m, \ j = 1, \ldots, m\big\}$ such that if $(\underline{\lambda}, \overline{\omega}, \overline{\nu}) \in \kappa_2$, then

$$\mathcal{A}(\widehat{W}_1(t), \delta\widehat{\Lambda}(t), M) \;=\; \begin{bmatrix} A_{\rm r} + D\widehat{W}_1^{\rm T}(t) & D(I + \delta\widehat{\Lambda}(t))M \\ -\widehat{W}_1^{\rm T}(t) & -(I + \delta\widehat{\Lambda}(t))M \end{bmatrix} \tag{3.42}$$

is quadratically stable.

Proof. We first show that there exists $\underline{\lambda}$ such that (3.42) is quadratically stable. For this purpose, consider the Lyapunov inequality given by

$$\begin{aligned} \mathcal{A}^{\rm T}(\widehat{W}_1(t), \delta\widehat{\Lambda}(t), M)\mathcal{P} + \mathcal{P}\mathcal{A}(\widehat{W}_1(t), \delta\widehat{\Lambda}(t), M) &< 0, \\ \mathcal{P} = \mathcal{P}^{\rm T} &> 0, \end{aligned} \tag{3.43}$$

with

$$\mathcal{P} = \begin{bmatrix} P & PD \\ D^{\rm T}P & D^{\rm T}PD + \rho I \end{bmatrix}, \tag{3.44}$$

where $P \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$ is a solution of the Lyapunov equation given by (2.13) with $R \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$ and $\rho \in \mathbb{R}_+$. Note that the positive-definiteness of (3.44) follows from the positive-definiteness of P and the positive-definiteness of the Schur complement of (3.44) given by

$$S_1 = D^{\mathrm{T}}PD + \rho I - D^{\mathrm{T}}P(P)^{-1}PD = \rho I > 0. \tag{3.45}$$

Next, consider

$$\begin{aligned} \mathcal{Q} &= \mathcal{A}^{\mathrm{T}}(\widehat{W}_1(t), \delta\widehat{\Lambda}(t), M)\mathcal{P} + \mathcal{P}\mathcal{A}(\widehat{W}_1(t), \delta\widehat{\Lambda}(t), M) \\ &= \begin{bmatrix} -R & A_{\mathrm{r}}^{\mathrm{T}}PD - \rho\widehat{W}_1(t) \\ D^{\mathrm{T}}PA_{\mathrm{r}} - \rho\widehat{W}_1^{\mathrm{T}}(t) & -2\rho M - \rho(M\delta\widehat{\Lambda}^{\mathrm{T}}(t) + \delta\widehat{\Lambda}(t)M) \end{bmatrix}. \end{aligned} \tag{3.46}$$

Noting that $-R$ is negative definite, we then consider the Schur complement of (3.46) given as

$$S_2 = \left[-\rho\big(2M + M\delta\widehat{\Lambda}^{\mathrm{T}}(t) + \delta\widehat{\Lambda}(t)M\big)\right] + \left[D^{\mathrm{T}}PA_{\mathrm{r}} - \rho\widehat{W}_1^{\mathrm{T}}(t)\right]R^{-1}\left[A_{\mathrm{r}}^{\mathrm{T}}PD - \rho\widehat{W}_1(t)\right], \tag{3.47}$$

where using the condition on the projection bounds of $\delta\widehat{\Lambda}(t)$ given by (3.36) it is guaranteed that (3.47) is a negative-definite matrix when $\underline{\lambda}$ is sufficiently large, which yields to the quadratic stability of (3.42).

Note that the existence proof for sufficiently small $\overline{\omega}$ and $\overline{\nu}$ to yield quadratic stability of (3.42) is similar to the proof of Lemma 1, and hence, omitted. Finally, since there exists a (sufficiently large) $\overline{\lambda}$ or (sufficiently small) $\overline{\omega}$ and $\overline{\nu}$ such that (3.42) is quadratically stable, the existence of set κ_2 is immediate. ■

Theorem 3. Consider the uncertain dynamical system given by (3.1) subject to Assumptions 1 and 2, the reference system given by (3.40), the actuator dynamics given by (3.2), the adaptive feedback control law given by (3.33) along with the update laws (3.34) and (3.35). If $(\underline{\lambda}, \overline{\omega}, \overline{\nu}) \in \kappa_2$, then the solution $\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), x_{\mathrm{r}}(t), v(t)\big)$ of the closed-loop dynamical system is bounded and $\lim_{t\to\infty} e(t) = 0$.

Proof. To show Lyapunov stability and guarantee boundedness of the system error state $e(t)$, the weight error $\widetilde{W}_1(t)$, and the control effectiveness error $\delta\widetilde{\Lambda}(t)$, consider the Lyapunov function candidate

$$\mathcal{V}(e, \widetilde{W}_1, \delta\widetilde{\Lambda}) = e^{\mathrm{T}} P e + \gamma_1^{-1} \mathrm{tr}\ \widetilde{W}_1^{\mathrm{T}} \widetilde{W}_1 + \gamma_\Lambda^{-1} \mathrm{tr}\ \delta\widetilde{\Lambda}^{\mathrm{T}} \delta\widetilde{\Lambda}. \tag{3.48}$$

Note that $\mathcal{V}(0,0,0) = 0$ and $\mathcal{V}\big(e, \widetilde{W}_1, \delta\widetilde{\Lambda}\big) > 0$ for all $\big(e, \widetilde{W}_1, \delta\widetilde{\Lambda}\big) \neq (0,0,0)$. Differentiating (3.48) yields $\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t)\big) \leq -e^{\mathrm{T}}(t) R e(t) \leq 0$, which guarantees the Lyapunov stability and the boundedness of the solution $\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t)\big)$.

To show the boundedness of $x_{\mathrm{r}}(t)$ and $x_{\mathrm{c}}(t)$ (and therefore $v(t)$), consider the reference system (3.40) and the actuator dynamics (3.2) subject to (3.33) as

$$\dot{x}_{\mathrm{r}}(t) = A_{\mathrm{r}} x_{\mathrm{r}}(t) + D\big[M x_{\mathrm{c}}(t) + \widehat{W}_1^{\mathrm{T}}(t) e(t) + \widehat{W}_1^{\mathrm{T}}(t) x_{\mathrm{r}}(t) + \delta\widehat{\Lambda}(t) M x_{\mathrm{c}}(t)\big], \tag{3.49}$$

$$\dot{x}_{\mathrm{c}}(t) = -M x_{\mathrm{c}}(t) - \widehat{W}_1^{\mathrm{T}}(t) e(t) - \widehat{W}_1^{\mathrm{T}}(t) x_{\mathrm{r}}(t) - \delta\widehat{\Lambda}(t) M x_{\mathrm{c}}(t) + K_2 c(t). \tag{3.50}$$

Then (3.49) and (3.50) can be rewritten in compact form as

$$\dot{\xi}(t) = \mathcal{A}(\widehat{W}_1(t), \delta\widehat{\Lambda}(t), M)\xi(t) + \omega(\cdot), \tag{3.51}$$

with $\xi(t) = [x_{\mathrm{r}}^{\mathrm{T}}(t), x_{\mathrm{c}}^{\mathrm{T}}(t)]^{\mathrm{T}}$ and

$$\omega(\cdot) = \begin{bmatrix} D\widehat{W}_1^{\mathrm{T}}(t) e(t) \\ -\widehat{W}_1^{\mathrm{T}}(t) e(t) + K_2 c(t) \end{bmatrix}. \tag{3.52}$$

Note that $\omega(\cdot)$ in (3.51) is a bounded perturbation as a result of Lyapunov stability of the triple $\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t)\big)$. Now, it follows that since $\omega(\cdot)$ is bounded and $\mathcal{A}(\widehat{W}_1(t), \delta\widehat{\Lambda}(t), M)$ is quadratically stable for $(\underline{\lambda}, \overline{\omega}, \overline{\nu}) \in \kappa_2$ by Lemma 2, then $x_{\mathrm{r}}(t)$ and $x_{\mathrm{c}}(t)$ are also bounded (see, for example, [27]). This further implies that the actuator output $v(t)$ is bounded. The remainder of the proof is similar to the proof of Theorem 1, and hence, omitted. ■

Remark 9. For the results given in Theorem 3 to hold, it is assumed that (3.42) is quadratically stable. Lemma 2 shows the feasibility of this assumption when $(\underline{\lambda}, \overline{\omega}, \overline{\nu}) \in \kappa_2$. Similar to the discussion given in Remark 6, this implies that the actuator dynamics are sufficiently fast (i.e., $\underline{\lambda}$ is sufficiently large such that $\lambda_{i,i}$ are, and hence, M is sufficiently large for all $i = 1, \ldots, m$) or the projection bounds on $\widehat{W}_1(t)$ and $\delta\widehat{\Lambda}(t)$ are sufficiently small (i.e., $\overline{\omega}$ and $\overline{\nu}$ are sufficiently small such that $\widehat{W}_{1,\max,i+(j-1)n}$ and $\delta\widehat{\Lambda}_{\max,i+(j-1)m}$ are sufficiently small for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$ and $i = 1, \ldots, m$ and $j = 1, \ldots, m$, respectively). Once again, this reveals the fundamental stability interplay between the allowable system uncertainties (through the selection of the projection operator bounds) and the bandwidths of the actuator dynamics.

Remark 10. We now utilize linear matrix inequalities to satisfy the quadratic stability of (3.42) for given projection bounds $\widehat{W}_{1,\max}$ and $\delta\widehat{\Lambda}_{\max}$ for the elements of $\widehat{W}_1(t)$ and $\delta\widehat{\Lambda}(t)$, respectively, and the bandwidths of the actuator dynamics M. For this purpose, let $\bar{W}_{1_{i_1,\ldots,i_l}} \in \mathbb{R}^{n\times m}$ and $\delta\bar{\Lambda}_{1_{i_1,\ldots,i_r}} \in \mathbb{R}^{m\times m}$ be given by (3.21) and (3.38) respectively. Following the results in [13, 28], if

$$\mathcal{A}_{i_1,\ldots,i_{2^{r+l}}} = \begin{bmatrix} A_r + D\bar{W}^{\mathrm{T}}_{1_{i_1,\ldots,i_l}} & D(I + \delta\bar{\Lambda}_{i_1,\ldots,i_r})M \\ -\bar{W}^{\mathrm{T}}_{1_{i_1,\ldots,i_l}} & -(I + \delta\bar{\Lambda}_{i_1,\ldots,i_r})M \end{bmatrix}, \tag{3.53}$$

satisfies the matrix inequality

$$\mathcal{A}^{\mathrm{T}}_{i_1,\ldots,i_{2^{r+l}}}\mathcal{P} + \mathcal{P}\mathcal{A}_{i_1,\ldots,i_{2^{r+l}}} < 0, \quad \mathcal{P} = \mathcal{P}^{\mathrm{T}} > 0, \tag{3.54}$$

for all permutations of $\bar{W}_{1_{i_1,\ldots,i_l}}$ and $\delta\bar{\Lambda}_{i_1,\ldots,i_r}$, then (3.42) is quadratically stable. Since (3.42) is quadratically stable for large values of M (see Remark 9), we cast (3.54) as a convex optimization problem given by

$$\begin{aligned}&\text{minimize } M,\\&\text{subject to (3.54).}\end{aligned}\qquad(3.55)$$

Therefore, we can satisfy (3.54) by minimizing M for given projection bounds.

3.3.2. Unknown Actuator Output Case. We now extend the results of the previous section to the case of unknown actuator output. For this purpose, first consider the modified adaptive feedback control law given by

$$u(t) \;=\; -\widehat{W}_1^{\mathrm{T}}(t)x(t) - \delta\widehat{\Lambda}(t)\widehat{v}(t) + K_2 c(t), \qquad (3.56)$$

where $\widehat{W}_1(t)$ satisfies the weight update law given by (3.34) and for the case of unknown actuator output, the update law given by (3.35) is modified as

$$\delta\dot{\widehat{\Lambda}}(t) \;=\; \gamma_\Lambda \mathrm{Proj_m}\big[\delta\widehat{\Lambda}(t),\; D^{\mathrm{T}}Pe(t)\widehat{v}^{\mathrm{T}}(t)\big], \quad \delta\widehat{\Lambda}(0) = \delta\widehat{\Lambda}_0. \qquad (3.57)$$

with the projection bounds defined by (2.15). Additionally, $\widehat{v} \in \mathbb{R}^m$ is an estimate of the unknown actuator output $v(t)$, satisfying the update law,

$$\dot{\widehat{v}}(t) \;=\; \mu\big(I + \delta\widehat{\Lambda}(t)\big)D^{\mathrm{T}}Pe(t) + M\big(u(t) - \widehat{v}(t)\big), \quad \widehat{v}(0) = \widehat{v}_0, \qquad (3.58)$$

with $\mu = \beta M$, $\beta \in \mathbb{R}_+$ being a design parameter chosen such that

$$\frac{\beta}{\lambda_{\min}(R)}\big|\big|PD\big|\big|_{\mathrm{F}}^2 w_\Lambda^{*2} < 1 \qquad (3.59)$$

holds, where $\big|\big|\delta\widetilde{\Lambda}(t)\big|\big|_{\mathrm{F}} \leq w_\Lambda^*$ denotes an upper bound.

Next, consider the modified reference system given by

$$\dot{x}_{\rm r}(t) \;=\; A_{\rm r}x_{\rm r}(t) + B_{\rm r}c(t) + D\big[\widehat{v}(t) - u(t)\big], \quad x_{\rm r}(0) = x_{\rm r0}. \tag{3.60}$$

Now, using the system dynamics (3.32) with the adaptive feedback control law (3.56) and the reference system (3.60), the system error dynamics are given by

$$\begin{aligned}\dot{e}(t) \;&=\; A_{\rm r}e(t) - D\widetilde{W}_1^{\rm T}(t)x(t) + D\delta\Lambda v(t) \\ &\quad -D\delta\widehat{\Lambda}(t)\widehat{v}(t) - D\widetilde{v}(t), \quad e(0) = e_0.\end{aligned} \tag{3.61}$$

Theorem 4. Consider the uncertain dynamical system given by (3.1) subject to Assumptions 1 and 2, the reference system given by (3.60), the actuator dynamics given by (3.2), the adaptive feedback control law given by (3.56) along with the update laws (3.34), (3.57), and (3.58). If $(\underline{\lambda}, \overline{\omega}, \overline{\nu}) \in \kappa_2$, then the solution $\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), x_{\rm r}(t), v(t), \widetilde{v}(t)\big)$ of the closed-loop dynamical system is bounded for all initial conditions and $t \in \overline{\mathbb{R}}_+$, and $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} \widetilde{v}(t) = 0$.

Proof. To show Lyapunov stability and guarantee boundedness of the system error state $e(t)$, the weight error $\widetilde{W}_1(t)$, the control effectiveness error $\delta\widetilde{\Lambda}(t)$, and the actuator output error $\widetilde{v}(t)$, consider the Lyapunov function candidate

$$\mathcal{V}\big(e, \widetilde{W}_1, \delta\widetilde{\Lambda}, \widetilde{v}\big) \;=\; e^{\rm T}Pe + \gamma_1^{-1}{\rm tr}\; \widetilde{W}_1^{\rm T}\widetilde{W}_1 + \gamma_\Lambda^{-1}{\rm tr}\; \delta\widetilde{\Lambda}^{\rm T}\delta\widetilde{\Lambda} + \beta^{-1}\widetilde{v}^{\rm T}M^{-1}\widetilde{v}. \tag{3.62}$$

Note that $\mathcal{V}(0,0,0,0) = 0$ and $\mathcal{V}\big(e, \widetilde{W}_1, \delta\widetilde{\Lambda}, \widetilde{v}\big) > 0$ for all $\big(e, \widetilde{W}_1, \delta\widetilde{\Lambda}, \widetilde{v}\big) \neq (0,0,0,0)$. Differentiating (3.62) yields

$$\begin{aligned}
&\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), \widetilde{v}(t)\big) \\
&\quad = -e^{\mathrm{T}}(t)Re(t) - 2e^{\mathrm{T}}(t)PD\widetilde{W}_1^{\mathrm{T}}(t)x(t) + 2e^{\mathrm{T}}(t)PD\delta\Lambda v(t) \\
&\qquad -2e^{\mathrm{T}}(t)PD\delta\widehat{\Lambda}(t)\widehat{v}(t) - 2e^{\mathrm{T}}(t)PD\widetilde{v}(t) + 2\gamma_1^{-1}\mathrm{tr}\ \widetilde{W}_1^{\mathrm{T}}(t)\dot{\widehat{W}}_1(t) \\
&\qquad +2\gamma_\Lambda^{-1}\mathrm{tr}\ \delta\widetilde{\Lambda}^{\mathrm{T}}(t)\delta\dot{\widehat{\Lambda}}(t) + 2\beta^{-1}\widetilde{v}^{\mathrm{T}}(t)M^{-1}\dot{\widetilde{v}}(t),
\end{aligned} \tag{3.63}$$

where using (3.34), it follows that (3.63) reduces to

$$\begin{aligned}
&\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), \widetilde{v}(t)\big) \\
&\quad \leq -e^{\mathrm{T}}(t)Re(t) + 2e^{\mathrm{T}}(t)PD\delta\Lambda v(t) - 2e^{\mathrm{T}}(t)PD\delta\widehat{\Lambda}(t)\widehat{v}(t) \\
&\qquad - 2e^{\mathrm{T}}(t)PD\widetilde{v}(t) + 2\gamma_\Lambda^{-1}\mathrm{tr}\ \delta\widetilde{\Lambda}^{\mathrm{T}}(t)\delta\dot{\widehat{\Lambda}}(t) + 2\beta^{-1}\dot{\widetilde{v}}^{\mathrm{T}}(t)M^{-1}\widetilde{v}(t).
\end{aligned} \tag{3.64}$$

This can equivalently be expressed as

$$\begin{aligned}
&\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), \widetilde{v}(t)\big) \\
&\quad \leq -e^{\mathrm{T}}(t)Re(t) + 2e^{\mathrm{T}}(t)PD\delta\Lambda v(t) - 2e^{\mathrm{T}}(t)PD\delta\widehat{\Lambda}(t)\widehat{v}(t) + 2e^{\mathrm{T}}(t)PD\delta\Lambda\widehat{v}(t) \\
&\qquad -2e^{\mathrm{T}}(t)PD\delta\Lambda\widehat{v}(t) - 2e^{\mathrm{T}}(t)PD\widetilde{v}(t) + 2e^{\mathrm{T}}(t)PD\delta\widehat{\Lambda}(t)\widetilde{v}(t) \\
&\qquad -2e^{\mathrm{T}}(t)PD\delta\widehat{\Lambda}(t)\widetilde{v}(t) + 2\gamma_\Lambda^{-1}\mathrm{tr}\ \delta\widetilde{\Lambda}^{\mathrm{T}}(t)\delta\dot{\widehat{\Lambda}}(t) + 2\beta^{-1}\dot{\widetilde{v}}^{\mathrm{T}}(t)M^{-1}\widetilde{v}(t), \\
&\quad = -e^{\mathrm{T}}(t)Re(t) - 2e^{\mathrm{T}}(t)PD\delta\widetilde{\Lambda}(t)\widehat{v}(t) + 2\gamma_\Lambda^{-1}\mathrm{tr}\ \delta\widetilde{\Lambda}^{\mathrm{T}}(t)\delta\dot{\widehat{\Lambda}}(t) \\
&\qquad +2e^{\mathrm{T}}(t)PD\delta\widetilde{\Lambda}(t)\widetilde{v}(t) + 2e^{\mathrm{T}}(t)PD\big(I + \delta\widehat{\Lambda}(t)\big)\widetilde{v}(t) + 2\beta^{-1}\dot{\widetilde{v}}^{\mathrm{T}}(t)M^{-1}\widetilde{v}(t).
\end{aligned} \tag{3.65}$$

Finally, noting that $\dot{\widetilde{v}}(t) = \dot{\widehat{v}}(t) - \dot{v}(t)$, and using the actuator dynamics given by (3.2) along with the update laws (3.57) and (3.58) yields

$$\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), \widetilde{v}(t)\big)$$
$$\leq -e^{\mathrm{T}}(t)Re(t) + 2e^{\mathrm{T}}(t)PD\delta\widetilde{\Lambda}(t)\widetilde{v}(t) - 2\beta^{-1}\widetilde{v}^{\mathrm{T}}(t)\widetilde{v}(t),$$
$$\leq -\lambda_{\min}(R)\big|\big|e(t)\big|\big|_2^2 + 2\big|\big|e(t)\big|\big|_2\big|\big|PD\big|\big|_{\mathrm{F}}\ w_{\Lambda}^*\ \big|\big|\widetilde{v}(t)\big|\big|_2 - 2\beta^{-1}\big|\big|\widetilde{v}(t)\big|\big|_2^2, \quad (3.66)$$

where $\big|\big|\delta\widetilde{\Lambda}(t)\big|\big|_{\mathrm{F}} \leq w_{\Lambda}^*$ holds due to projection operator. Now, using Young's inequality [30] on the second term yields

$$\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), \widetilde{v}(t)\big)$$
$$\leq -\lambda_{\min}(R)\big|\big|e(t)\big|\big|_2^2 + \alpha\big|\big|e(t)\big|\big|_2^2 + \frac{1}{\alpha}\big|\big|PD\big|\big|_{\mathrm{F}}^2\ w_{\Lambda}^{*2}\ \big|\big|\widetilde{v}(t)\big|\big|_2^2 - 2\beta^{-1}\big|\big|\widetilde{v}(t)\big|\big|_2^2. \quad (3.67)$$

Letting $\alpha = \frac{1}{2}\lambda_{\min}(R)$, it follows that

$$\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), \widetilde{v}(t)\big)$$
$$\leq -\frac{\lambda_{\min}(R)}{2}\big|\big|e(t)\big|\big|_2^2 - 2\beta^{-1}\Big[1 - \frac{\beta}{\lambda_{\min}(R)}\big|\big|PD\big|\big|_{\mathrm{F}}^2\ w_{\Lambda}^{*2}\Big]\big|\big|\widetilde{v}(t)\big|\big|_2^2. \quad (3.68)$$

Using (3.59) in (3.68), it follows that $\dot{\mathcal{V}}\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), \widetilde{v}(t)\big) \leq 0$, which guarantees the Lyapunov stability, and hence, the boundedness of the solution $\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), \widetilde{v}(t)\big)$.

To show the boundedness of $x_{\mathrm{r}}(t)$ and $x_{\mathrm{c}}(t)$ (and therefore $v(t)$), consider the reference system (3.60) and actuator dynamics (3.2) subject to (3.56) as

$$\begin{aligned}
\dot{x}_{\mathrm{r}}(t) &= A_{\mathrm{r}}x_{\mathrm{r}}(t) + D\big[Mx_{\mathrm{c}}(t) + \delta\widehat{\Lambda}(t)Mx_{\mathrm{c}}(t) + \widehat{W}_1^{\mathrm{T}}(t)e(t) + \widehat{W}_1^{\mathrm{T}}(t)x_{\mathrm{r}}(t)\big] \\
&\quad + D\delta\widehat{\Lambda}(t)\widetilde{v}(t) + D\widetilde{v}(t), \qquad (3.69) \\
\dot{x}_{\mathrm{c}}(t) &= -Mx_{\mathrm{c}}(t) - \delta\widehat{\Lambda}(t)Mx_{\mathrm{c}}(t) - \widehat{W}_1^{\mathrm{T}}(t)e(t) - \widehat{W}_1^{\mathrm{T}}(t)x_{\mathrm{r}}(t) \\
&\quad - \delta\widehat{\Lambda}(t)\widetilde{v}(t) + K_2c(t), \qquad (3.70)
\end{aligned}$$

where (3.69) and (3.70) can be rewritten in compact form as (3.51) with

$$\omega(\cdot) = \begin{bmatrix} D\widehat{W}_1^{\mathrm{T}}(t)e(t) + D\delta\widehat{\Lambda}(t)\widetilde{v}(t) + D\widetilde{v}(t) \\ -\widehat{W}_1^{\mathrm{T}}(t)e(t) - \delta\widehat{\Lambda}(t)\widetilde{v}(t) + K_2 c(t) \end{bmatrix}. \tag{3.71}$$

Note that $\omega(\cdot)$ in (3.71) is a bounded perturbation as a result of Lyapunov stability of the quadruple $\big(e(t), \widetilde{W}_1(t), \delta\widetilde{\Lambda}(t), \widetilde{v}(t)\big)$. Now, it follows that since $\omega(\cdot)$ is bounded and $\mathcal{A}(\widehat{W}_1(t), \delta\widehat{\Lambda}(t), M)$ is quadratically stable for $(\underline{\lambda}, \overline{\omega}, \overline{\nu}) \in \kappa_2$ by Lemma 2, then $x_{\mathrm{r}}(t)$ and $x_{\mathrm{c}}(t)$ are also bounded (see, for example, [27]). This further implies that the actuator output $v(t)$ is bounded. The remainder of the proof is similar to the proof of Theorem 1, and hence, omitted. ■

For the results given in Theorem 4 to hold, it is assumed that (3.42) is quadratically stable. As this is the same condition given in Section 3.3.1, it should be noted that the same discussion and results provided in Remarks 10 and 11 hold for this case of unknown actuator output.

3.4. CONCLUDING REMARKS

It is well known that the presence of actuator dynamics can seriously limit the stability and achievable performance of model reference adaptive controllers. In this chapter, we presented a linear matrix inequality-based hedging approach to maintain stability of adaptive controllers in the presence of actuator dynamics. This approach was further generalized for several different cases in which the actuator output and the control effectiveness matrix are known and unknown. For each case, utilizing linear matrix inequalities, it was analytically proven that the closed-loop dynamical system, including the modified reference model trajectory, is stable. Although a particular model reference adaptive control formulation was considered in this chapter to present the proposed analysis, the approach can be readily extended to other approaches in adaptive control for the computation of the minimum allowable actuator bandwidth for each control channel such that the closed-loop dynamical system remains stable.

4. AN AFFINE QUADRATIC STABILITY CONDITION FOR A LINEAR MATRIX INEQUALITY-BASED HEDGING APPROACH TO NONCONVEX MULTIACTUATOR DYNAMICS

4.1. INTRODUCTION

Although a general solution for the theoretical stability of hedging in model reference adaptive control in the presence of actuator dynamics has been established using linear matrix inequalities, these solutions are limited to specific cases where convexity is guaranteed. In this chapter, we establish a new affine quadratic stability condition such that it generalizes our results to generally nonconvex cases. Specifically, we introduce a new change of coordinates and provide a means of convexifying the problem. Finally, we present an algorithm for solving a multiple actuator case using linear matrix inequalities.

The organization of this chapter is as follows. Section 4.2 introduces an affine quadratic stability condition for a the proposed linear matrix inequality-based hedging approach, and the conclusions are summarized in Section 4.3.

4.2. AN AFFINE QUADRATIC STABILITY CONDITION FOR A LINEAR MATRIX INEQUALITY-BASED HEDGING APPROACH TO NONCONVEX MULTIACTUATOR DYNAMICS

We start by introducing an affine transformation for (3.22). Consider the actuator bandwidth matrix M from (3.2), which can be rewritten as

$$M(\Delta\lambda) \quad = \quad M_0 + \sum_{j=1}^{m} \Delta\lambda_j M_j, \tag{4.1}$$

where $M_0 = -\lambda_{\text{feas}} I_{m\times m} \in \mathbb{R}^{m\times m} \cap \mathbb{D}^{m\times m}$ is chosen such that M_0 satisfies (3.23), $M_j \in \mathbb{R}^{m\times m}$ is a matrix such that $M_j(j,j) = 1$ and zero everywhere else, and $\Delta\lambda_j \in$

$\mathbb{R}_+$ belongs to the set $\Delta\lambda = (\Delta\lambda_1, \ldots, \Delta\lambda_m)$, which is defined by the parameter box

$$\Lambda \;=\; \left\{\Delta\lambda \in \mathbb{R}^m | \Delta\lambda_j \in \left[\underline{\Delta\lambda}_j, \overline{\Delta\lambda}_j\right]\right\}, \tag{4.2}$$

and is the convex hull $\Lambda = \text{conv}(\Lambda_0)$ of the corners

$$\Lambda_0 \;=\; \left\{\Delta\lambda \in \mathbb{R}^m | \Delta\lambda_j \in \{\underline{\Delta\lambda}_j, \overline{\Delta\lambda}_j\}\right\}. \tag{4.3}$$

Note that $\underline{\Delta\lambda}_j$, $\overline{\Delta\lambda}_j$ are the upper and lower bounds of each actuator bandwidth, respectively. Now, using (4.1) and letting $B = \text{col}\left([B_1, B_2, \ldots, B_m]\right)$, we can rewrite (3.53) as

$$\mathcal{A}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda) \;=\; \mathcal{A}_{0,i_1,\ldots,i_l} + \sum_{j=1}^{m} \Delta\lambda_j \mathcal{A}_j, \tag{4.4}$$

where $\mathcal{A}_{0,i_1,\ldots,i_l} = \begin{bmatrix} A_\text{r} + B\bar{W}^\text{T}_{1_{i_1,\ldots,i_l}} & BM_0 \\ -\bar{W}^\text{T}_{1_{i_1,\ldots,i_l}} & M_0 \end{bmatrix} \in \mathbb{R}^{n+m\times n+m}$, $\mathcal{A}_j \in \mathbb{R}^{n+m\times n+m}$, $\mathcal{A}_j(1:n, n+j) = B_j M_j$, $\mathcal{A}_j(n+j, n+j) = M_j$, and $\mathcal{A}_j$ is zero everywhere else.

Remark 11. To geometrically interpret (4.1) and (4.4), we can consider a three dimensional example as seen in Figure 4.1. Specifically, it shows the relationship between M, matrix M_0, and the origin such that it is always possible to recover M by evaluating (4.1) algebraically.

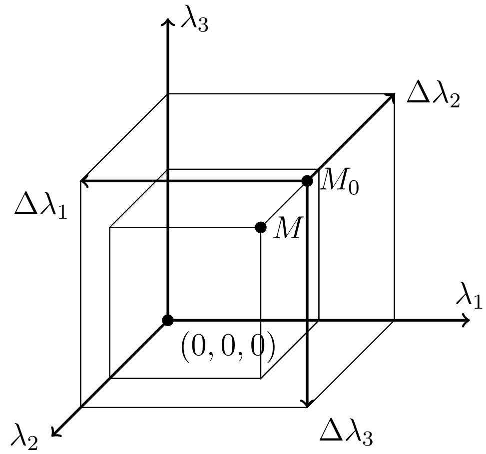

Figure 4.1: Relationship between coordinates of λ and $\Delta\lambda$

Theorem 5. Consider an affine function described by (4.4) and the parameter box $\Lambda = \operatorname{conv}(\Lambda_0)$ as defined by (4.2) and (4.3). Let M_0 to be chosen sufficiently large enough such that $\mathcal{A}_{0,i_1,\dots,i_l}$ always satisfies

$$\begin{aligned} \mathcal{A}_{0,i_1,\dots,i_l}^{\mathrm{T}}\mathcal{P}_0 + \mathcal{P}_0\mathcal{A}_{0,i_1,\dots,i_l} < 0, \\ \mathcal{P}_0 = \mathcal{P}_0^{\mathrm{T}} > 0, \end{aligned} \tag{4.5}$$

for all $\bar{W}_{1_{i_1,\dots,i_l}}$. If there exists real matrices $\mathcal{P}_0, \mathcal{P}_1, ..., \mathcal{P}_m$ where

$$\mathcal{P}(\Delta\lambda) \;\; = \;\; \mathcal{P}_0 + \sum_{j=1}^{m} \Delta\lambda_j \mathcal{P}_j, \tag{4.6}$$

such that (4.5) and the linear matrix inequality conditions

$$\begin{aligned} \mathcal{A}^{\mathrm{T}}(\bar{W}_{1_{i_1,\dots,i_l}}, \Delta\lambda)\mathcal{P}(\Delta\lambda) + \mathcal{P}(\Delta\lambda)\mathcal{A}(\bar{W}_{1_{i_1,\dots,i_l}}, \Delta\lambda) < 0, &\quad \forall \Delta\lambda \in \Lambda_0 \\ \mathcal{P}(\Delta\lambda) > 0, &\quad \forall \Delta\lambda \in \Lambda_0 \\ \mathcal{A}_j^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_j \geq 0, &\quad \text{for } i = j = 1, ..., m, \end{aligned} \tag{4.7}$$

are also satisfied, then (4.4) is affinely quadratically stable $\forall \Delta\lambda \in \Lambda$.

Proof. Consider quadratic function given by

$$\mathcal{Q}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda) = \mathcal{A}^{\mathrm{T}}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda)\mathcal{P}(\Delta\lambda) + \mathcal{P}(\Delta\lambda)\mathcal{A}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda), \quad (4.8)$$

which is negative definite at the corners according as a natural consequence of the linear matrix inequality conditions (4.7). Now, expanding $\mathcal{Q}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda)$ yields

$$\begin{aligned}
\mathcal{Q}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda) &= \left[\mathcal{A}_{0,i_1,\ldots,i_l} + \sum_{j=1}^{m} \Delta\lambda_j \mathcal{A}_j\right]^{\mathrm{T}} \left[\mathcal{P}_0 + \sum_{j=1}^{m} \Delta\lambda_j \mathcal{P}_j\right] \\
&\quad + \left[\mathcal{P}_0 + \sum_{j=1}^{m} \Delta\lambda_j \mathcal{P}_j\right]\left[\mathcal{A}_{0,i_1,\ldots,i_l} + \sum_{j=1}^{m} \Delta\lambda_j \mathcal{A}_j\right] \\
&= \mathcal{A}_{0,i_1,\ldots,i_l}^{\mathrm{T}}\mathcal{P}_0 + \mathcal{P}_0\mathcal{A}_{0,i_1,\ldots,i_l} \\
&\quad + \sum_{j=1}^{m} \Delta\lambda_j \left[\mathcal{A}_{0,i_1,\ldots,i_l}^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_{0,i_1,\ldots,i_l} + \mathcal{A}_j^{\mathrm{T}}\mathcal{P}_0 + \mathcal{P}_0\mathcal{A}_j\right] \\
&\quad + \sum_{j=1}^{m}\sum_{k=1}^{j-1} \Delta\lambda_j\Delta\lambda_k \left[\mathcal{A}_j^{\mathrm{T}}\mathcal{P}_k + \mathcal{P}_k\mathcal{A}_j + \mathcal{A}_k^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_k\right] \\
&\quad + \sum_{j=1}^{m} \Delta\lambda_j^2 \left[\mathcal{A}_j^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_j\right].
\end{aligned} \quad (4.9)$$

Next, consider the quadratic function $x^{\mathrm{T}}\mathcal{Q}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda)x$ for any vector $x \neq 0$ which can be written as

$$\begin{aligned}
x^{\mathrm{T}}\mathcal{Q}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda)x &= \alpha_{0,1_{i_1,\ldots,i_l}} + \sum_{j=1}^{m} \Delta\lambda_j \alpha_{j,1_{i_1,\ldots,i_l}} \\
&\quad + \sum_{j=1}^{m}\sum_{k=1}^{j-1} \Delta\lambda_k\Delta\lambda_j\beta_{k,j} + \sum_{j=1}^{m} \Delta\lambda_j^2\gamma_j,
\end{aligned} \quad (4.10)$$

where

$$\begin{aligned}
\alpha_{0,1_{i_1,\ldots,i_l}} &= x^{\mathrm{T}}\Big[\mathcal{A}_{0,i_1,\ldots,i_l}^{\mathrm{T}}\mathcal{P}_0 + \mathcal{P}_0\mathcal{A}_{0,i_1,\ldots,i_l}\Big]x, \\
\alpha_{j,1_{i_1,\ldots,i_l}} &= x^{\mathrm{T}}\Big[\mathcal{A}_{0,i_1,\ldots,i_l}^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_{0,i_1,\ldots,i_l} + \mathcal{A}_j^{\mathrm{T}}\mathcal{P}_0 + \mathcal{P}_0\mathcal{A}_j\Big]x, \\
\beta_{k,j} &= x^{\mathrm{T}}\Big[\mathcal{A}_j^{\mathrm{T}}\mathcal{P}_k + \mathcal{P}_k\mathcal{A}_j + \mathcal{A}_k^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_k\Big]x, \\
\gamma_j &= x^{\mathrm{T}}\Big[\mathcal{A}_j^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_j\Big]x,
\end{aligned}$$

are fixed constants. It naturally follows that the corners of (4.10) are negative whenever (4.7) is satisfied. We now only need to guarantee that the maximums of (4.10) occur at its corners. Therefore, a sufficient condition is the partial convexity constraint

$$\begin{aligned}
&\tfrac{\partial^2(x^{\mathrm{T}}\mathcal{Q}(\bar{W}_{1_{i_1,\ldots,i_l}},\Delta\lambda)x)}{\partial\Delta\lambda_j^2} = \gamma_j \geq 0 \\
&\Longrightarrow\; x^{\mathrm{T}}[\mathcal{A}_j^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_j]x \geq 0,
\end{aligned} \tag{4.11}$$

for $j = 1,\ldots,m$. Since we defined x as arbitrary, we obtain that

$$\mathcal{A}_j^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_j \geq 0. \tag{4.12}$$

The results of Theorem 3.1 of [9] and Theorem 5.7 of [29] guarantee that the linear matrix inequalities (4.7) hold for $\Lambda = \mathrm{conv}\,(\Lambda_0)$ since $x^{\mathrm{T}}\mathcal{Q}(\bar{W}_{1_{i_1,\ldots,i_l}},\Delta\lambda)x$ always obtains its maximums at some corner of the parameter box Λ_0. ■

Remark 12. We now utilize linear matrix inequalities to satisfy the affine quadratic stability condition of (4.7) for given projection bounds of $\widehat{W}_{1,\max}$ for the elements $\widehat{W}_{1,}(t)$, respectively, and the change in actuator bandwidth limits contained within the paramter box Λ_0. For this purpose, let $\bar{W}_{1_{i_1,\ldots,i_l}} \in \mathbb{R}^{n\times m}$ be given by (3.21).

Following the results of Theorem 5, if

$$\mathcal{A}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda) \;=\; \mathcal{A}_{0,i_1,\ldots,i_l} + \sum_{j=1}^{m} \Delta\lambda_j \mathcal{A}_j, \tag{4.13}$$

satisfies the linear matrix inequalities

$$\begin{aligned} \mathcal{A}^{\mathrm{T}}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda)\mathcal{P}(\Delta\lambda) + \mathcal{P}(\Delta\lambda)\mathcal{A}(\bar{W}_{1_{i_1,\ldots,i_l}}, \Delta\lambda) < 0, &\quad \forall \Delta\lambda \in \Lambda_0 \\ \mathcal{P}(\Delta\lambda) > 0, &\quad \forall \Delta\lambda \in \Lambda_0 \\ \mathcal{A}_j^{\mathrm{T}}\mathcal{P}_j + \mathcal{P}_j\mathcal{A}_j \geq 0, &\quad \text{for } i = j = 1, \ldots, m, \end{aligned} \tag{4.14}$$

for all permutations of $\bar{W}_{1_{i_1,\ldots,i_l}}$, then (3.22) is affinely quadratically stable. Since (3.22) is feasible for large values of M (see Lemma 1), we can then recast (4.14) as the general eigenvalue problem given by

$$\begin{aligned} &\text{maxmimize } \Lambda_0, \\ &\text{subject to (4.14).} \end{aligned} \tag{4.15}$$

We can therefore satisfy (4.14) by maximizing Λ_0.

Remark 14. Since (4.14) is affinely quadratically stable, one can evaluate (4.15) in a finite number of iterations. The following algorithm describes a way to evaluate (4.15) by expanding the corners Λ_0 of the parameter box Λ. Algorithm 1 introduces the new term ϵ_{LMI}, which is a specified step tolerance. The generalized eigenvalue problem (4.15) were solved using YALMIP [31], but other solvers can also be used [9]. It is apparent that it becomes exhaustively difficult to evaluate every possible combination of (4.4), especially in evaluating cases where there are more than three actuators. For purposes of brevity, Algorithm 1 is restricted to the three actuator case illustrated in Chapter 5.

Data: $A_{\mathrm{r}}, B, M_0, \widehat{W}_{1,\max}, \epsilon_{\mathrm{LMI}}$,
Result: $\Delta\lambda_{\max}$
for $\Delta\lambda_1 = 0 : \epsilon_{\mathrm{LMI}} : \Delta\lambda_{\mathrm{feas}}$ **do**
  for $\Delta\lambda_2 = 0 : \epsilon_{\mathrm{LMI}} : \Delta\lambda_{\mathrm{feas}}$ **do**
    ⋮
    for $\Delta\lambda_m = 0 : \epsilon_{\mathrm{LMI}} : \Delta\lambda_{\mathrm{feas}}$ **do**
      if (4.14) *is feasible* **then**
        Continue
      else if (4.14) *is infeasible* **then**
        Bisect using (4.14)
      end
    end
    ⋮
  end
end

Algorithm 1: General search algorithm

Table 4.1 and Figure 4.2 describe the process in which a shaded feasible region for three actuators is approximated by Algorithm 1 when $\epsilon_{\mathrm{LMI}} = \lambda_{\mathrm{feas}}$. For this case, the search is done in a total of eight steps. Algorithm 1 can only approximate a feasible space since there are an infinite amount of combinations of the upper bounds defined in Λ_0, but by decreasing ϵ_{LMI} one can easily find a better estimate for the feasible region at the expense of computation time.

Table 4.1: Algorithm description for three actuators ($\epsilon_{\mathrm{LMI}} = \lambda_{\mathrm{feas}}$)

Step	Description
1	If $\Delta\lambda_3 = \Delta\lambda_2 = \Delta\lambda_1 = 0$ is feasible, continue to next step
2	If $\Delta\lambda_3 = \lambda_{\mathrm{feas}}$ and $\Delta\lambda_2 = \Delta\lambda_1 = 0$ is infeasible, then bisect until feasible
3	If $\Delta\lambda_2 = \lambda_{\mathrm{feas}}$ and $\Delta\lambda_3 = \Delta\lambda_1 = 0$ is infeasible, then bisect until feasible
4	If $\Delta\lambda_3 = \Delta\lambda_2 = \lambda_{\mathrm{feas}}$ and $\Delta\lambda_1 = 0$ is infeasible, then bisect until feasible
5	If $\Delta\lambda_1 = \lambda_{\mathrm{feas}}$ and $\Delta\lambda_3 = \Delta\lambda_1 = 0$ is infeasible, then bisect until feasible
6	If $\Delta\lambda_3 = \Delta\lambda_1 = \lambda_{\mathrm{feas}}$ and $\Delta\lambda_2 = 0$ is infeasible, then bisect until feasible
7	If $\Delta\lambda_2 = \Delta\lambda_1 = \lambda_{\mathrm{feas}}$ and $\Delta\lambda_1 = 0$ is infeasible, then bisect until feasible
8	If $\Delta\lambda_3 = \Delta\lambda_2 = \Delta\lambda_1 = \lambda_{\mathrm{feas}}$ is infeasible, then bisect until feasible

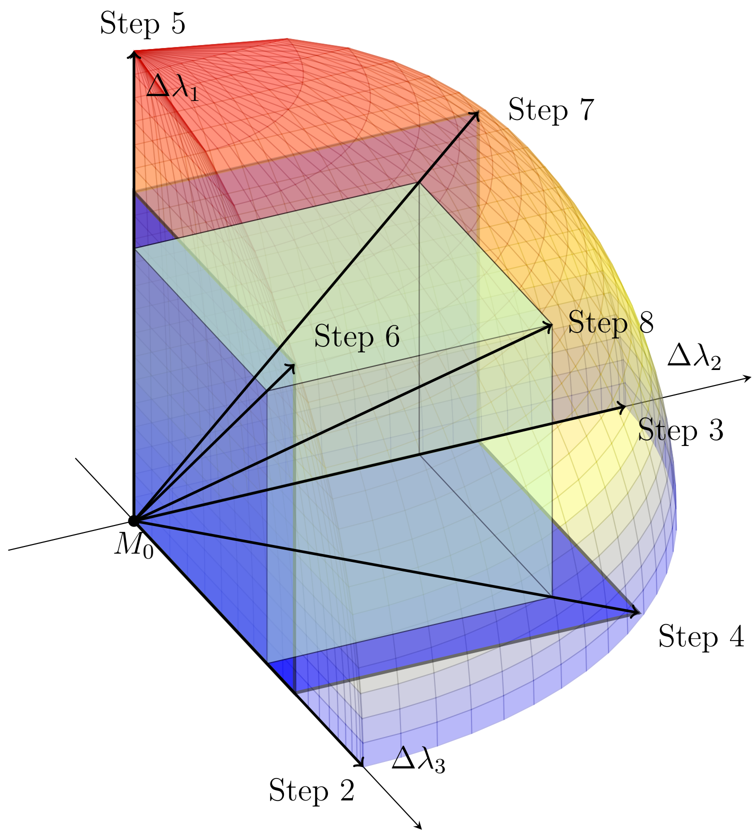

Figure 4.2: Algorithm 1 solving three actuators ($\epsilon_{\text{LMI}} = \lambda_{\text{feas}}$)

Remark 15. Algorithm 1 will always produce a less conservative solution than the case where (3.24) is solved using bisection and a positive definite constraint on the $\mathcal{P}$ matrix from (3.12). To illustrate this point, consider a example where $M \in \mathbb{R}_+$, $A_{\text{r}} = -1$ and $B = 1$. Figure 4.3 shows the differences when searching for a minimum M when given a range of $\widehat{W}_{1,\max}$.

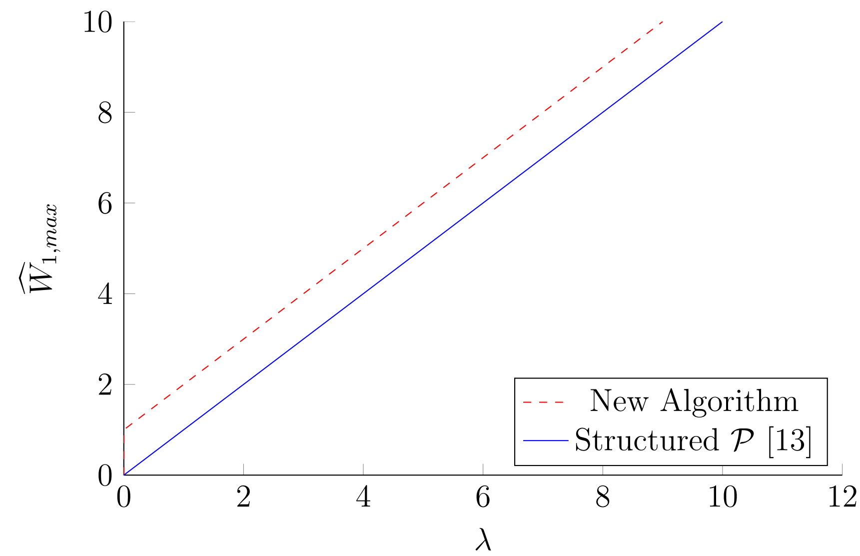

Figure 4.3: Linear matrix inequality comparison plot ($A_r = -1$ and $B = 1$)

4.3. CONCLUDING REMARKS

In this chapter we expressed a new method for evaluating the robustness of model reference adaptive controls in the presence of multiactuator dynamics. An algorithm was presented and it was shown that the results were recoverable regardless of the starting plane. In comparison with other methods, our results extended out to multiactuator systems and solve them efficiently.

5. ILLUSTRATIVE NUMERICAL EXAMPLES

To show the efficacy of our results in Chapters 3 and 4, we consider several illustrative numerical examples. Specifically, Section 5.1 shows us our numerical results for Chapter 3 and Section 5.2 shows the results from evaluating Algorithm 1 in Chapter 4.

5.1. COMPUTING ACTUATOR BANDWIDTHS IN ADAPTIVE CONTROL

In order to illustrate the proposed adaptive control architecture with actuator dynamics, we consider the second-order system given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Lambda v(t), \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.1}$$

where Λ denotes the control effectiveness matrix. For the following examples, let $x_1(t)$ represent the angle in radians and $x_2(t)$ represent the angular rate of change in radians per second. We use a filtered tracking command $c(t)$ and consider a single channel actuator for the control input such that $M = \lambda,\ \lambda \in \mathbb{R}_+$. In addition, we set $R = I_2$ from (2.13), for the proposed adaptive controller designs and select a reference system with zero initial conditions, a natural frequency of $\omega_{\mathrm{n}} = 0.7$ rad/s, and a damping ratio $\zeta = 1.0$, which yields

$$A_{\mathrm{r}} = \begin{bmatrix} 0 & 1 \\ -0.49 & -1.4 \end{bmatrix}, \quad B_{\mathrm{r}} = \begin{bmatrix} 0 \\ 0.49 \end{bmatrix}. \tag{5.2}$$

For the proposed adaptive controller configurations, we now present four examples.

Example 5.1.1 (Known Control Effectiveness Matrix and Known Actuator Output). In this example, we assume the control effectiveness matrix is known. This implies $\Lambda = 1$, so the results of Section 3.2 apply. Using the rectangular projection operator, the bounds on the uncertainty are set element-wise such that $\left|[\widehat{W}_1(t)]_{1,1}\right| \leq 0.5$ and $\left|[\widehat{W}_1(t)]_{2,1}\right| \leq 1.5$ (such that we set all initial conditions to zero). Then using the bounds on $\widehat{W}_1(t)$ in the linear matrix inequality analysis highlighted in Remark 7, the minimum allowable actuator bandwidth is calculated as $\lambda_{\min} = 0.77$. Figures 5.1–5.3 show the proposed adaptive controller design performance in the presence of actuator dynamics using a range of actuator bandwidth settings. Since it is calculated that the minimum actuator bandwidth allowed for the actuator dynamics is 0.77, it is expected that the system performances are guaranteed to be bounded for actuator bandwidths greater than and equal to the calculated minimum. This can be seen from Figures 5.1 and 5.2 in which actuator bandwidths of $\lambda = 25$ and $\lambda = 0.77$ are used, respectively. In Figure 5.3, we let the actuator bandwidth be smaller than 0.77 to show that the closed-loop system remains bounded until the actuator bandwidth reaches a value of $\lambda = 0.35$. This is consistent with the presented theory, as we provide a (conservative) upper bound on the allowable actuator bandwidth such that the closed-loop system remains bounded.

Example 5.1.2 (Known Control Effectiveness Matrix and Unknown Actuator Output). Once again, since the control effectiveness matrix is known, the linear matrix inequality analysis of Remark 7 still holds with $\lambda_{\min} = 0.77$. In addition, we use the same projection operator bounds and initial conditions in Example 5.1.1. For the proposed adaptive controller design in this example, we use the results of Theorem 2, since the actuator output is unknown. Figures 5.4–5.6 show the proposed adaptive controller performance with the same actuator bandwidth values used in Example 5.1.2. Once again, since it is calculated that the minimum actuator bandwidth allowed for the actuator dynamics is 0.77, it is expected that the system performances are

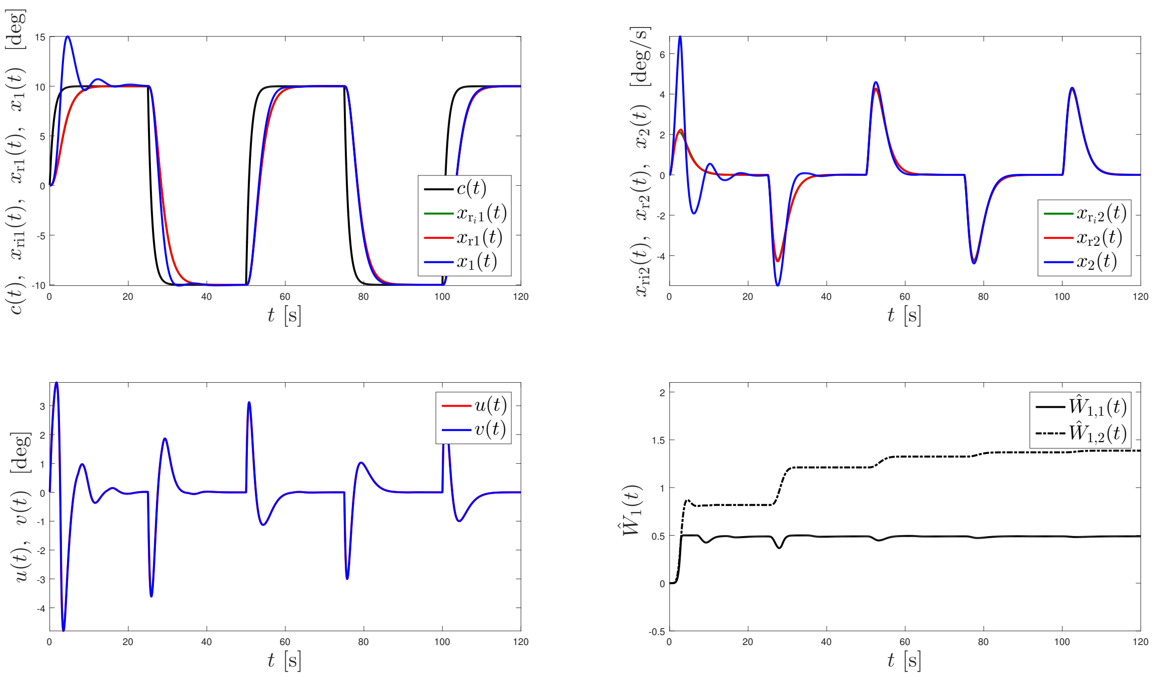

Figure 5.1: Proposed controller performance in Example 5.1.1 with actuator dynamics ($\lambda = 25$ and $\gamma_1 = 25$).

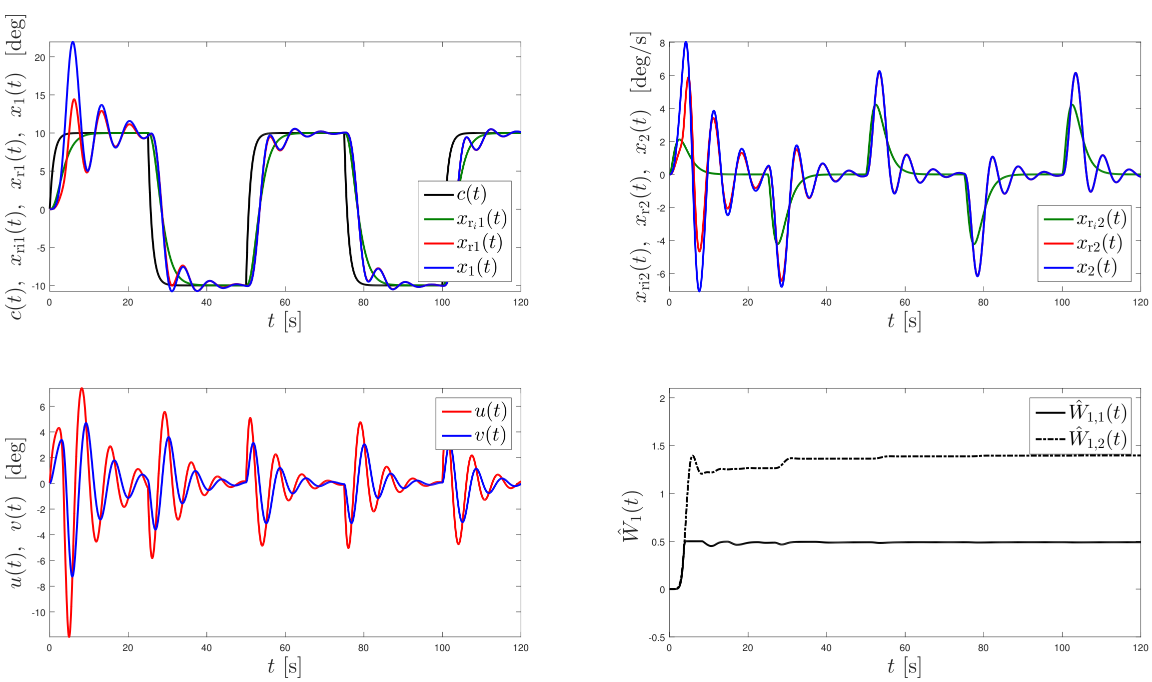

Figure 5.2: Proposed controller performance in Example 5.1.1 with actuator dynamics ($\lambda = 0.77$ and $\gamma_1 = 25$).

guaranteed to be bounded for actuator bandwidth values greater than or equal to the calculated minimum, where Figures 5.4 and 5.5 illustrate this statement. In

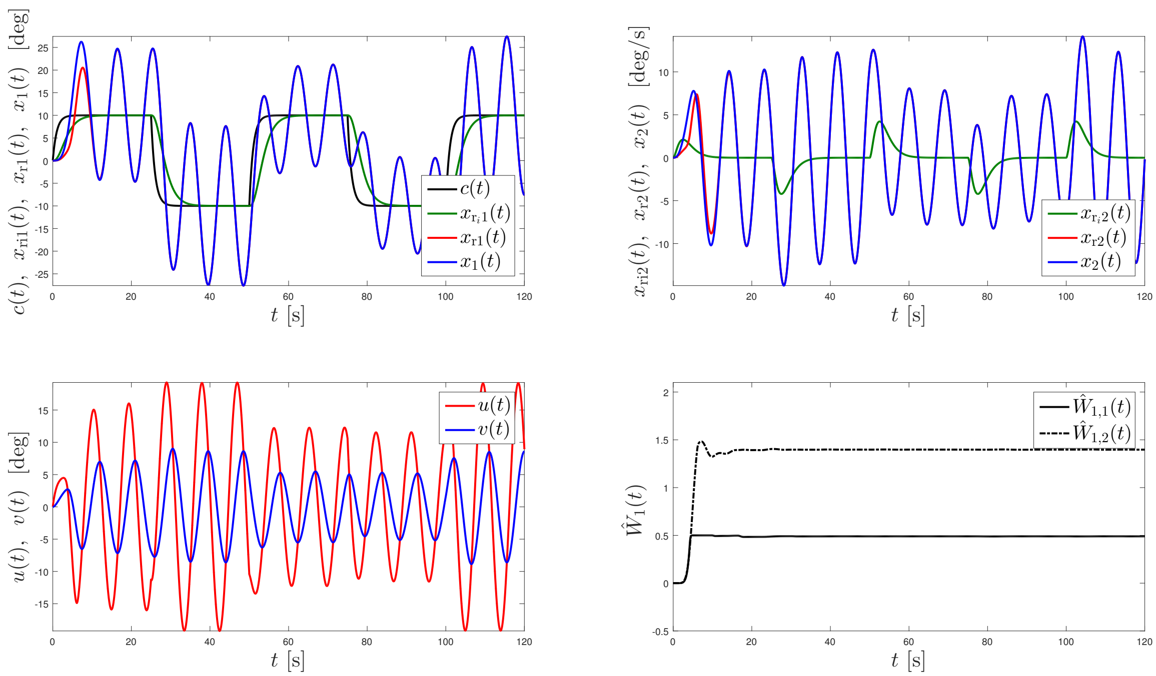

Figure 5.3: Proposed controller performance in Example 5.1.1 with actuator dynamics ($\lambda = 0.35$ and $\gamma_1 = 25$).

Figure 5.6, it is shown that the system becomes unstable for $\lambda = 0.35$. Once again, this is consistent with the presented theory in that we provide a (conservative) upper bound on the allowable actuator bandwidth such that the closed-loop system remains bounded.

Example 5.1.3 (Unknown Control Effectiveness Matrix and Known Actuator Output). We now consider a case that the control effectiveness matrix is unknown, assuming that $\Lambda = 0.5$. This assumption corresponds to the results in Section 3.3. Using this formulation, we set the projection operator bounds such that $\left|[\widehat{W}_1(t)]_{1,1}\right| \leq 0.5$ and $\left|[\widehat{W}_1(t)]_{2,1}\right| \leq 1.5$, and for the unknown control effectiveness $\left|\delta\widehat{\Lambda}(t)\right| \leq 0.6$ (such that we set all initial conditions to zero). Then using the linear matrix inequality analysis highlighted in Remark 10, the minimum allowable actuator bandwidth is calculated as $\lambda_{\min} = 2.3$. Figures 5.7–5.9 show the proposed adaptive controller performance in the presence of actuator dynamics using a range of actuator bandwidth settings. Since it is calculated that the minimum actuator bandwidth in this case

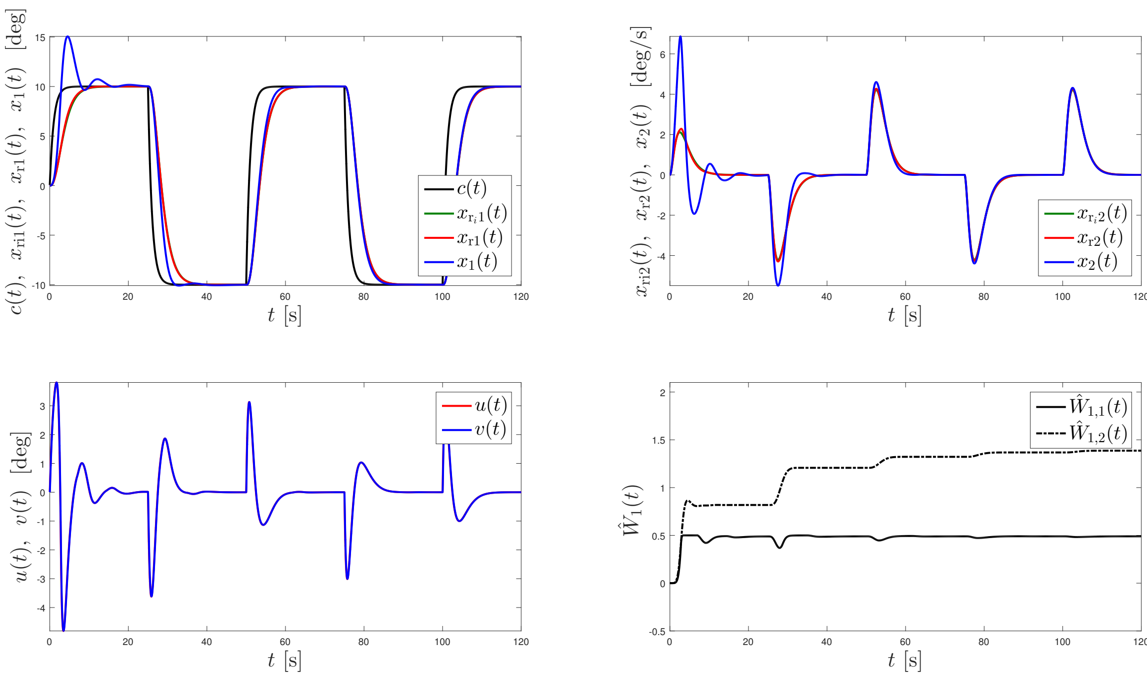

Figure 5.4: Proposed controller performance in Example 5.1.2 with actuator dynamics ($\lambda = 25$, $\gamma_1 = 25$, and $\beta = 0.015$).

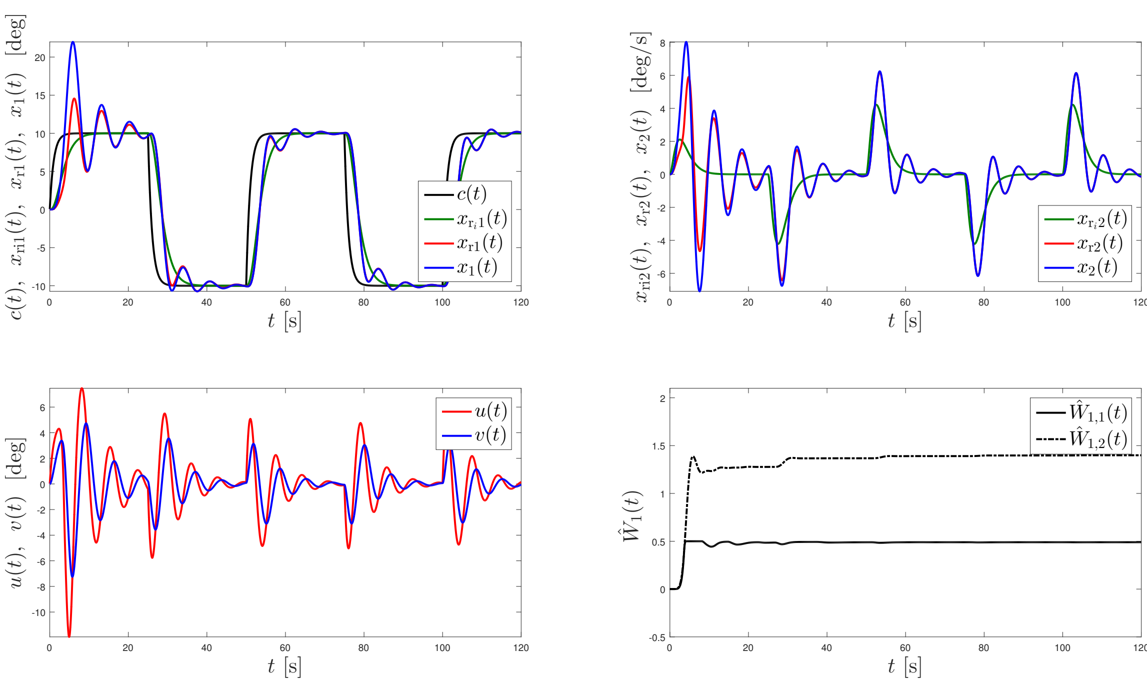

Figure 5.5: Proposed controller performance in Example 5.1.2 with actuator dynamics ($\lambda = 0.77$, $\gamma_1 = 25$, and $\beta = 0.015$).

is 2.3, it is expected that the system performance is guaranteed bounded until the

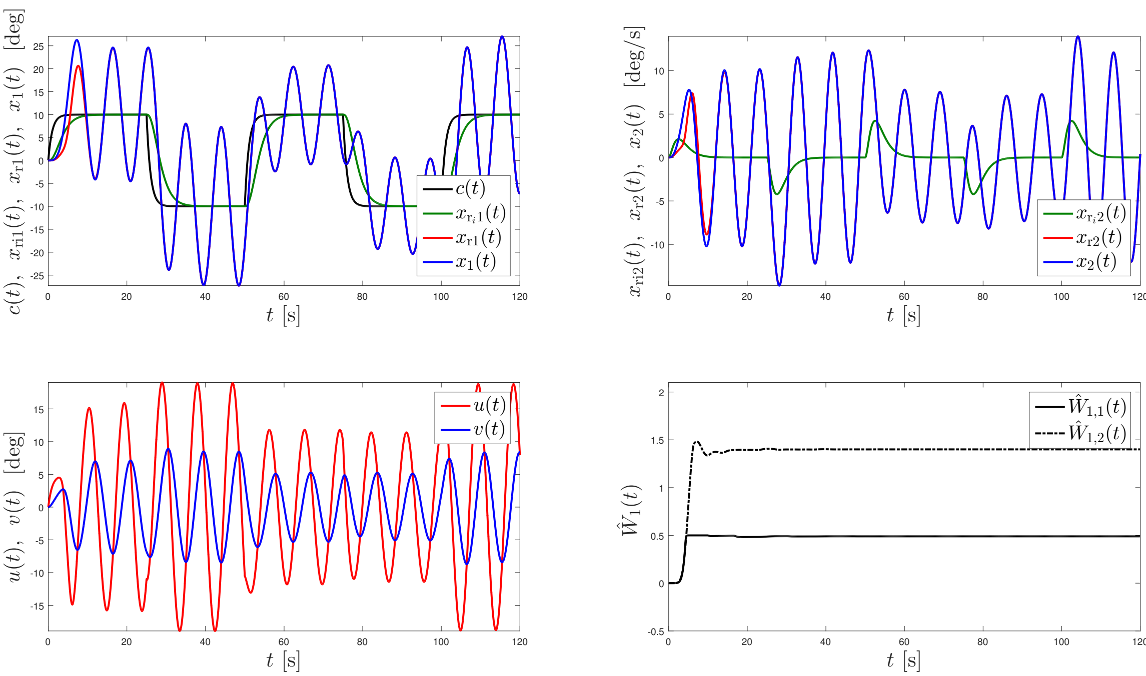

Figure 5.6: Proposed controller performance in Example 5.1.2 with actuator dynamics ($\lambda = 0.35$, $\gamma_1 = 25$, and $\beta = 0.015$).

bandwidth drops below this value. This is consistent with the results shown in Figures 5.7 and 5.8. In Figure 5.9, it is shown that the system becomes unstable for $\lambda = 0.7$. As highlighted before, this is consistent with the presented theory in that we provide a (conservative) upper bound on the allowable actuator bandwidth such that the closed-loop system remains bounded.

Example 5.1.4 (Unknown Control Effectiveness Matrix and Unknown Actuator Output). Since the control effectiveness matrix is unknown, as in the previous example, the linear matrix inequality analysis of Remark 10 still holds with $\lambda_{\min} = 2.3$. In addition, we choose $\beta = 0.015$ such that (3.59) holds and we use the same projection operator bounds and initial conditions in Example 5.1.3. For the proposed adaptive controller design, we use the results of Theorem 4 (the most general case considered). Figures 5.10–5.12 show the proposed adaptive controller design with the same actuator bandwidth values as in Example 5.1.3. Once again, since it is

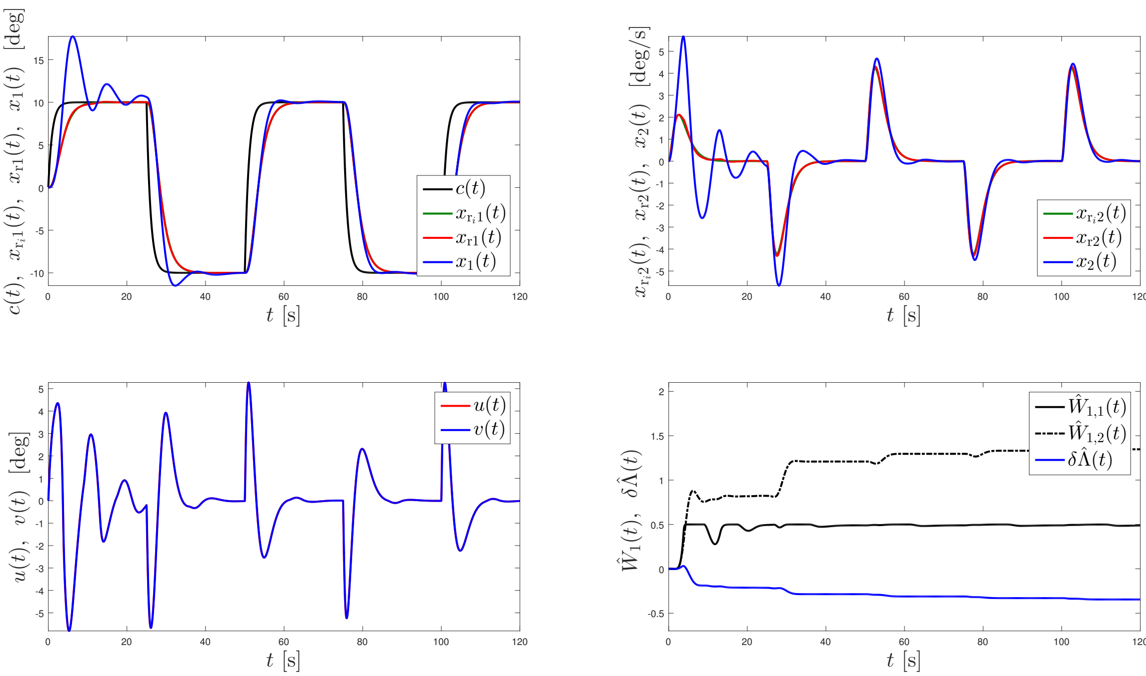

Figure 5.7: Proposed controller performance in Example 5.1.3 with actuator dynamics ($\lambda = 25$, $\gamma_1 = 25$, and $\gamma_\Lambda = 5$).

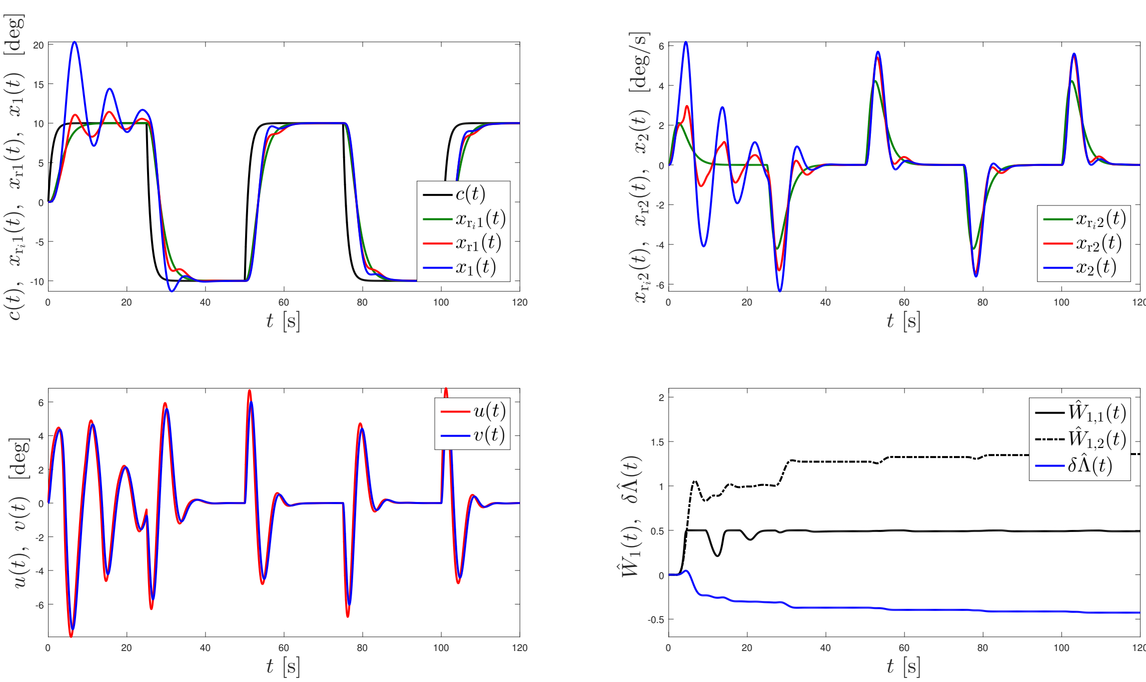

Figure 5.8: Proposed controller performance in Example 5.1.3 with actuator dynamics ($\lambda = 2.3$, $\gamma_1 = 25$, and $\gamma_\Lambda = 5$).

calculated that the minimum actuator bandwidth allowed for the actuator dynamics is 2.3, it is expected that the system performances are guaranteed to be bounded for

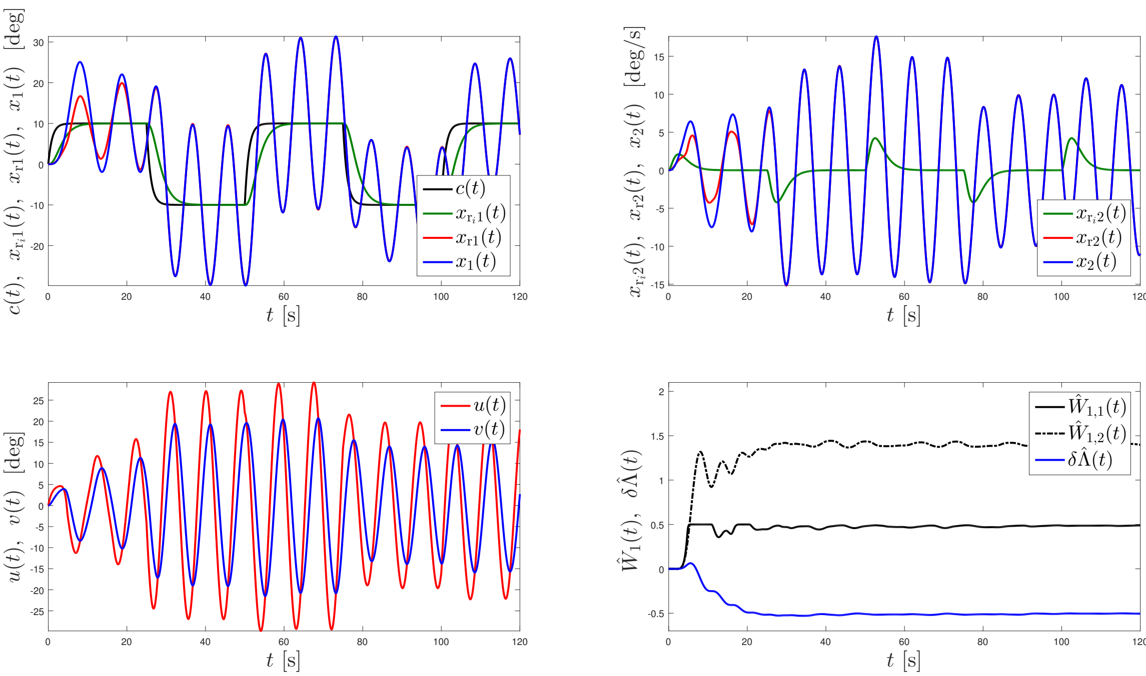

Figure 5.9: Proposed controller performance in Example 5.1.3 with actuator dynamics ($\lambda = 0.7$, $\gamma_1 = 25$, and $\gamma_\Lambda = 5$).

actuator bandwidth values greater than or equal to the calculated minimum, where Figures 5.10 and 5.11 illustrate this statement. In Figure 5.12, it is shown that the system becomes unstable for $\lambda = 0.7$. As in the previous examples, this is consistent with the presented theory in that we provide a (conservative) upper bound on the allowable actuator bandwidth such that the closed-loop system remains bounded.

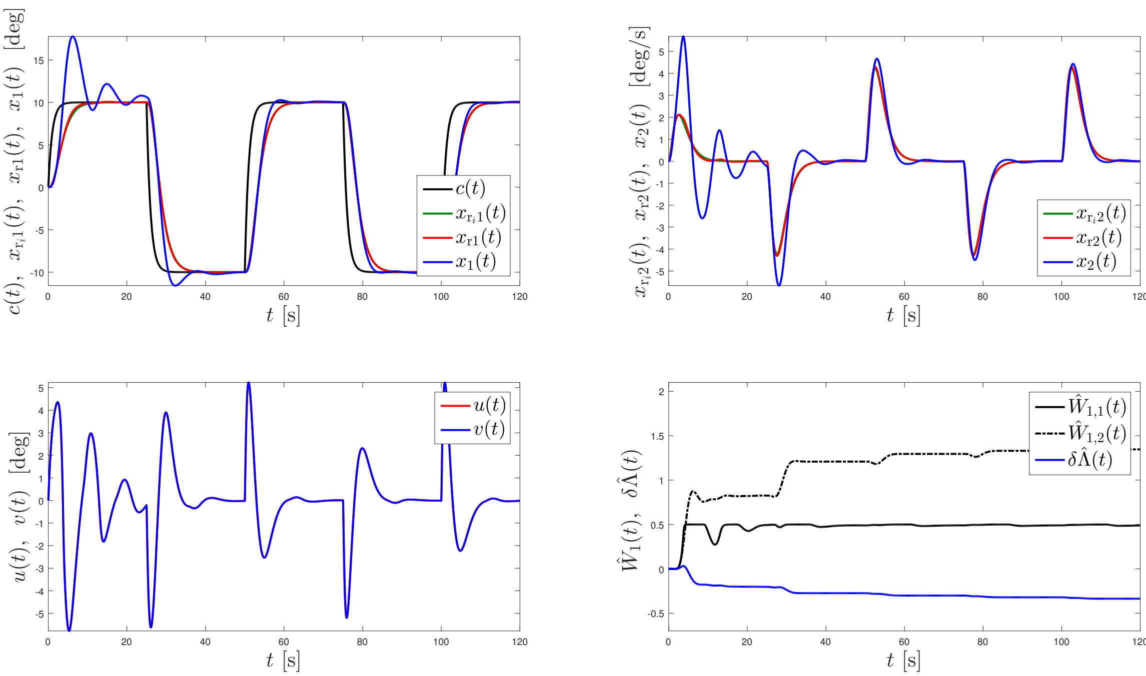

Figure 5.10: Proposed controller performance in Example 5.1.4 with actuator dynamics ($\lambda = 25$, $\gamma_1 = 25$, $\gamma_\Lambda = 5$, and $\beta = 0.015$).

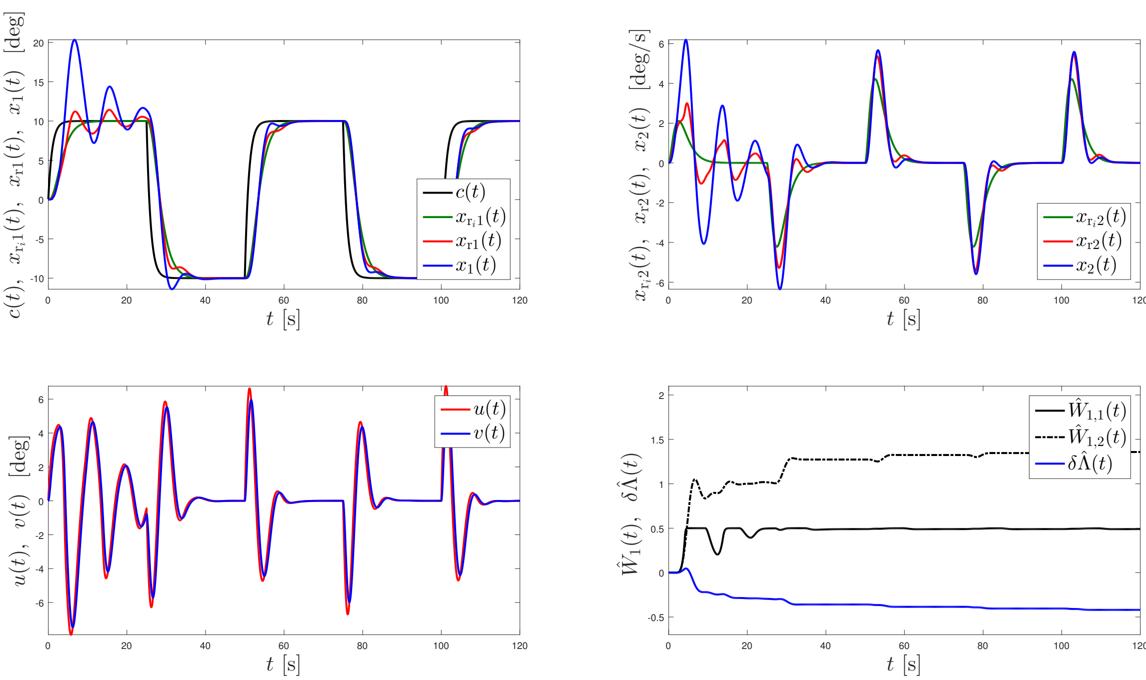

Figure 5.11: Proposed controller performance in Example 5.1.4 with actuator dynamics ($\lambda = 2.3$, $\gamma_1 = 25$, $\gamma_\Lambda = 5$, and $\beta = 0.015$).

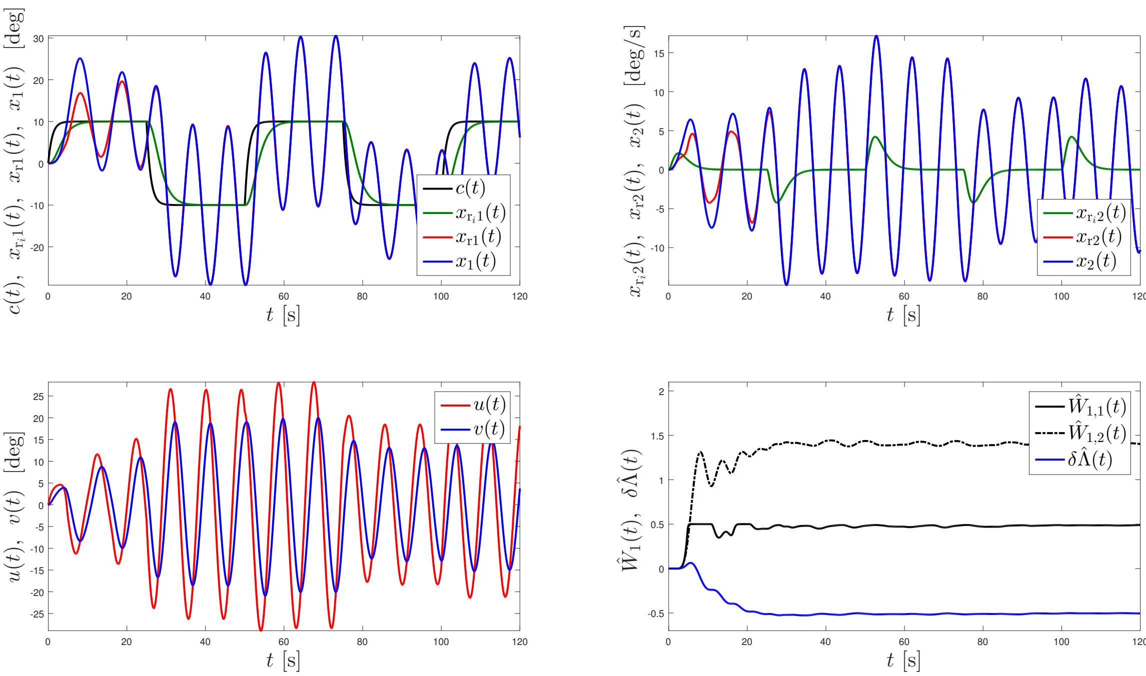

Figure 5.12: Proposed controller performance in Example 5.1.4 with actuator dynamics ($\lambda = 0.7$, $\gamma_1 = 25$, $\gamma_\Lambda = 5$, and $\beta = 0.015$).

5.2. AN AFFINE QUADRATIC STABILITY CONDITION FOR A LINEAR MATRIX INEQUALITY-BASED HEDGING APPROACH TO NONCONVEX MULTIACTUATOR DYNAMICS

For the following multiactuator examples, we consider three separate cases in which three actuators are used.

Example 5.2.1 (Scalar Reference Dynamics). The first case considers variations in the reference model matrix A_r. The feasible point M_0 was chosen arbitrarily such that $M_0 = 30I_{3\times3}$. We also let $\epsilon_{LMI} = 0.2$. Figure 5.13 shows the results of evaluating Algorithm 1 when $A_\mathrm{r} = -1$ and $B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. The shaded volume shows us the polytope representing the feasible regions of $\Delta\lambda$. As $\Delta\lambda_3$ gets larger, the region in which $\Delta\lambda_1$ and $\Delta\lambda_2$ are feasible gets smaller. Figure 5.13 shows us the results whenever $A_\mathrm{r} = -1$ and $|\widehat{W}_{1,\max}| \leq 1$. Figure 5.14 shows us similar results with $A_\mathrm{r} = -2$ and $|\widehat{W}_{1,\max}| \leq 1$.

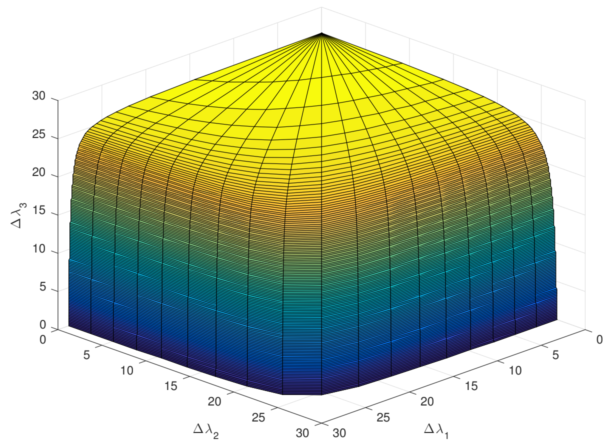

Figure 5.13: Proposed search algorithm solving Example 5.2.1 ($A_{\mathrm{r}} = -1$, $|\widehat{W}_{1,\max}| \leq 1$, $M_0 = 30I_{3\times3}$, and $\epsilon_{LMI} = 0.2$).

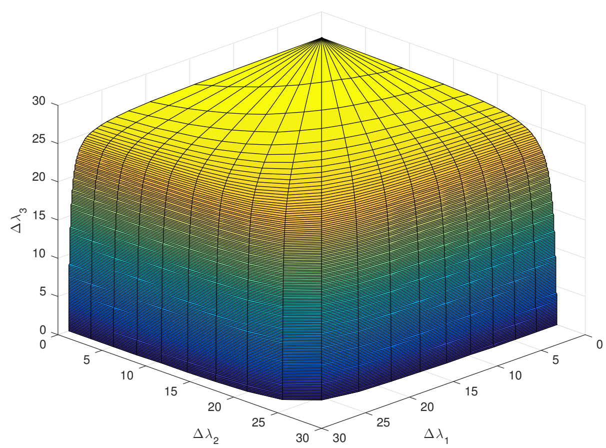

Figure 5.14: Proposed search algorithm solving Example 5.2.1 ($A_{\mathrm{r}} = -2$, $|\widehat{W}_{1,\max}| \leq 1$, $M_0 = 30I_{3\times3}$, and $\epsilon_{LMI} = 0.2$).

Example 5.2.2 (Second Order Reference Dynamics). Now consider the case where A_{r} and B are

$$A_{\mathrm{r}} = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}. \tag{5.3}$$

The algorithm tolerances are $\epsilon_{\mathrm{LMI}} = 0.2$. The initial conditions of matrix $M_0 \in \mathbb{R}^{3\times3}$ are given such that $M_0 = 30I_{3\times3}$. Figure 5.15 considers the case when $|\widehat{W}_{1,\max}| \leq 1$.

Figure 5.16 considers the case when $|\widehat{W}_{1,\max}| \leq 2$. Intuitively, we can grasp that by increasing the bounds on $|\widehat{W}_{1,\max}|$, we decrease the region in which (4.15) is feasible.

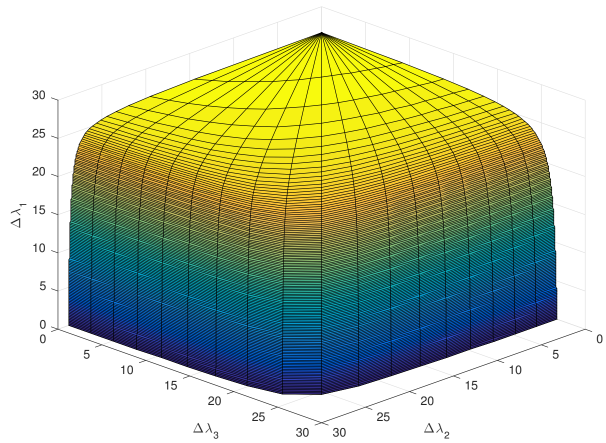

Figure 5.15: Proposed search algorithm solving Example 5.2.2 ($|\widehat{W}_{1,\max}| \leq 1$, $M_0 = 30I_{3\times 3}$, and $\epsilon_{LMI} = 0.2$).

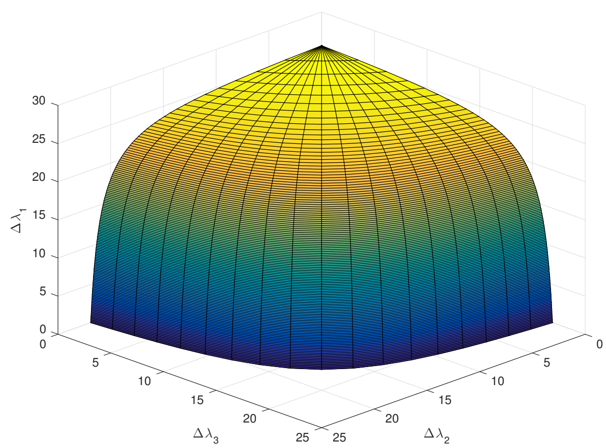

Figure 5.16: Proposed search algorithm solving Example 5.2.2 ($|\widehat{W}_{1,\max}| \leq 2$, $M_0 = 30I_{3\times 3}$, and $\epsilon_{LMI} = 0.2$).

Example 5.2.3 (Algorithm Verification). A major concern was whether or not the algorithms holds when evaluating (4.15) when a new search direction is used. In other words, can we produce the same results if we ran our search algorithm, for example, from the $\Delta\lambda_2$ and $\Delta\lambda_1$ plane? To verify this, we modified Algorithm 1 such that it began its search from this new plane and gradually increases $\Delta\lambda_3$ until the system is infeasible. Figure 5.17 considers when $|\widehat{W}_{1,\max}| \leq 1$ and has the same results as Figure 5.15 from a different orientation. Similarly, Figure 5.18 considers $|\widehat{W}_{1,\max}| \leq 2$ and has the same results Figure 5.16 from a different orientation.

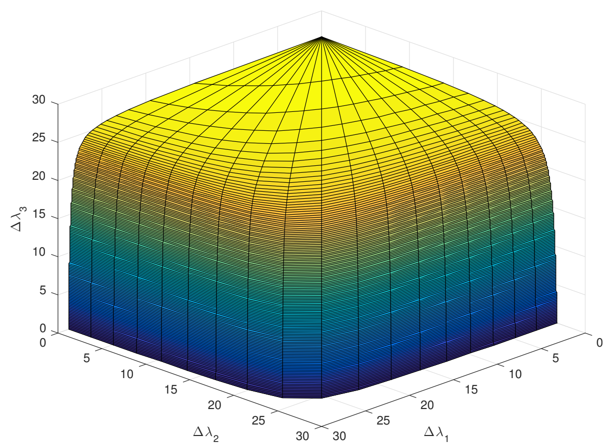

Figure 5.17: Proposed search algorithm solving Example 5.2.3 ($|\widehat{W}_{1,\max}| \leq 1$, $M_0 = 30I_{3\times3}$, and $\epsilon_{LMI} = 0.2$).

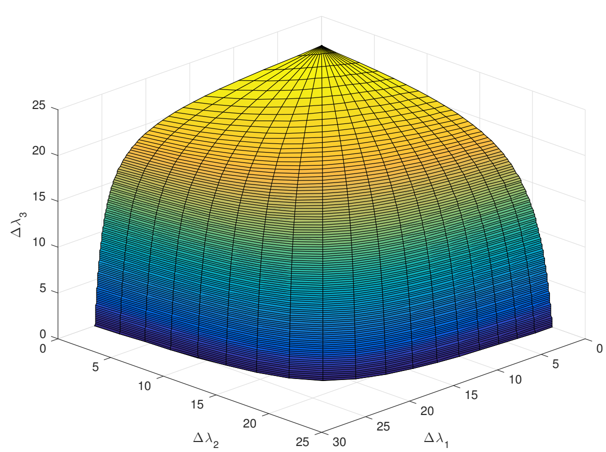

Figure 5.18: Proposed search algorithm solving Example 5.2.3 ($|\widehat{W}_{1,\max}| \leq 2$, $M_0 = 30I_{3\times3}$, and $\epsilon_{LMI} = 0.2$).

6. CONCLUSIONS

In this thesis, we utilized linear matrix inequalities and convex optimization techniques to solve the actuator dynamics problem in adaptive control literature. Specifically, we presented a linear matrix inequality-based hedging approach such that the stability of the system is maintained in the presence of actuator dynamics. This approach was generalized to cases in which actuator output and the control effectiveness matrix are known and unknown. For all cases, the existence of a feasible solutions was proven and, by consequence, analytically guaranteed the stability of the closed-loop dynamical system, including the modified reference model trajectory. Although a particular model reference adaptive control formulation was considered in this thesis to present the proposed analysis, the approach can be readily extended to other approaches in adaptive control for the computation of the minimum allowable actuator bandwidth for each control channel such that the closed-loop dynamical system remains stable.

The results of the linear matrix inequality-based hedging approach were generalized to cases where there are two or more first order actuators. A new linear matrix inequality condition was introduced such that the solution sets of the minimum actuator bandwidth of each actuator are convex, the stability of the system was theoretically guaranteed by introducing the partially convex linear matrix inequality constraint, and then demonstrated a case where there are three first order actuator using a new algorithm specifically developed to solve them while observing the new linear matrix inequality conditions. We considered a wide variety of cases to prove our results and have shown them as being less conservative.

For future work, these results can be generalized to cases where the actuators are not of the first order. Specifically, the observation of frequency response of a second or third order actuator dynamics by utilization of linear matrix inequalities remain practical problems to consider.

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VITA

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