1963

Digital computer design of compensation for linear control systems

Larry C. Amsler

DIGITAL COMPUTER DESIGN OF COMPENSATION FOR LINEAR CONTROL SYSTEMS

BY

LARRY C. AMSLER

A

THESIS

submitted to the faculty of the

SCHOOL OF MINES AND METALLURGY OF THE UNIVERSITY OF MISSOURI

in partial fulfillment of the work required for the

Degree of

MASTER OF SCIENCE IN ELECTRICAL ENGINEERING

Rolla, Missouri

1963

Approved by

Robert D. Chenoweth (Advisor) Earl Richards

Ralph E. Lee Charles E. Antle

ABSTRACT

If a control system is to be synthesized, it is inferred that a plant or process is present which must be controlled, and the problem of how to design the control system then arises. The first step is to decide on performance specifications to which the complete system must conform. These specifications may involve such things as the system steady state response, transient response, or frequency response. Any of several synthesis methods may then be applied to complete the system design.

In this study a synthesis method is developed for single-loop linear feedback systems. First, the number of compensating poles and zeros and the approximate location of each is determined by conventional methods. A set of functions, one for each specification and one involving each plant pole, is written in terms of the system singularities with the compensation singularity positions as variables and each such function is equated to zero. Linear approximations of each of these generally non-linear functions are obtained by expanding each function with a multivariable Taylor series and retaining only linear terms. Expansion is about a point described by the approximate singularity values. This linear set of equations is solved by the Gauss-Jordan elimination method. Due to truncation of the Taylor series, this does not give an exact solution to the original specification equations but will serve as a second approximation which is used as a new point of Taylor series expansion. This iterative process is repeated until a satisfactory solution is found.

This entire iterative technique is adapted for digital computer programming and flow charts for such a program are drawn.

ACKNOWLEDGEMENT

The author wishes to thank Dr. R. D. Chenoweth for his help with problems regarding this study and for suggestions in writing the thesis. He also wishes to thank Mr. E. F. Richards for his recommendations for the thesis writing.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

A. Statement of the problem.

Assume a plant or process exists whose components are unalterable. If the performance of this plant is not satisfactory, an attempt is made to synthesize a control system by adding additional elements in an appropriate configuration to meet performance specifications. The plant must remain a part of this control system.

In the past, most design procedures have involved cut-and- try methods where various charts and graphs are often employed. The purpose of this study was (1) to investigate an automatic method of linear control system compensation design which was outlined by Zabroszky and Marsh[1]; (2) to develop the technique into a form suitable for programming and write a generalized digital computer program to carry out the design.

B. Importance of the problem.

For a designer to resort to cut-and-try methods makes both design results and time spent in arriving at the results depend largely on the experience of the designer.

Using the computer technique proposed, it is only necessary to make an initial estimate of the compensation needed and to supply this information with a description of the plant and performance specifications into the computer as data. The initial estimate is then refined by the computer until specifications are satisfied. This reduces the tedious routine work of a synthesis problem and releases the designer's time for other phases of the problem.

C. Organization of remainder of the thesis.

Chapter II is a review of automatic design methods previously introduced. The outset of Chapter III introduces the notation to be used in this study. Generalized forward and feedback-transfer functions are written in terms of singularities. The equations relating several performance specifications to these transfer function singularities are developed. A method of solving these equations is then given and finally, a computer flow chart of a program for executing the design is constructed.

CHAPTER II

REVIEW OF THE LITERATURE

A number of ways have been contrived to make control system synthesis, at least to some extent, automatic. Using a method presented by Aseltine[2], some closed-loop poles, some real open-loop poles, and a velocity constant are specified. A rough inverse root locus plot is made to see if the specifications are consistent and if so, a set of algebraic equations based on the inverse root locus geometry are written to impose the open-loop constraints. The solution of these equations gives an open-loop transfer function containing the prescribed poles. The open-loop plant poles must be real due to the fact that a segment of the real axis can always be made part of the inverse root locus while this is not true for complex poles. Also, the predominant closed-loop poles which will exist in the compensated system are found from design curves or from an analog computer study before applying the method. The method is laborious if the system under consideration has more than four poles because the equations written for such a system are nonlinear.

A similar method was proposed by Zaborszky[6] using 180-degree and 0-degree root loci alternately to satisfy open- and closed-loop requirements.

Later a paper by Zaborszky and Marsh[1] introduced a method that could be adapted to the digital computer which is the method incorporated in this thesis.

CHAPTER III

MATHEMATICAL DEVELOPMENT

A. Notation.

The following synthesis method is based on properties of the open- and closed-loop transfer functions of the system. A block diagram of the single-loop system considered is shown in Figure 1. This system can be described mathematically by the equations

$$G_p(s) = K_1 \frac{\prod_{i=1}^{\alpha} (s - z_i)}{\prod_{i=1}^{\beta} (s - p_i)} \tag{1}$$

$$G_c(s) = K_2 \frac{\prod_{i=1}^{\gamma} (s - t_i)}{\prod_{i=1}^{\delta} (s - v_i)} \tag{2}$$

$$H(s) = K_n \frac{\prod_{i=1}^{\ell} (s - y_i)}{\prod_{i=1}^{j} (s - r_i)} \tag{3}$$

$$K(s) = \frac{C(s)}{R(s)} = K_1 K_2 \frac{\prod_{i=1}^{\alpha} (s - z_i) \prod_{i=1}^{\gamma} (s - t_i) \prod_{i=1}^{j} (s - r_i)}{\prod_{i=1}^{\tau} (s - g_i)} \tag{4}$$

Where $\tau = \beta + \delta + j$ and it is assumed in the last expression that $\beta + \delta + j > \alpha + \gamma + \ell$. This inequality must be satisfied or the constant $K_1 K_2$ in $K(s)$ will not be correct.

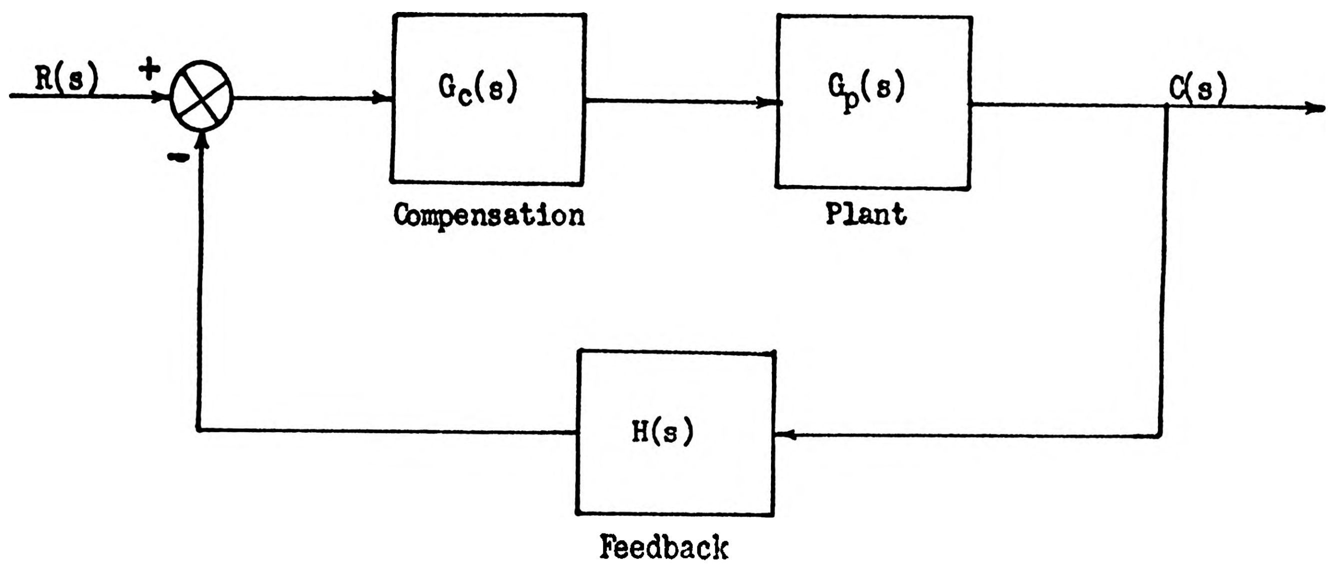


FIGURE 1

A GENERAL SINGLE-LOOP FEEDBACK SYSTEM

B. System specifications.

Any number of closed-loop specifications can be made as long as they can be written mathematically in terms of system singularities. These specifications may be for steady state response, transient response, frequency response, etc.

Two specifications for the transient response to a unit step input and one steady state response requirement are used in this study. They are the peak time, per cent overshoot, and velocity constant respectively.

System stability will be considered a requirement. This will be discussed in Chapter IV.

1. The equation for peak time.

The system output for a unit step input is

$$C(s) = \frac{K_1 K_2}{s} \cdot \frac{\prod_{i=1}^{\alpha}(s-z_i)\prod_{i=1}^{\gamma}(s-t_i)\prod_{i=1}^{j}(s-r_i)}{\prod_{i=1}^{\tau}(s-q_i)} \qquad (5)$$

It is necessary at this point to assume K(s) to have a predominant pair of poles, $q_1 = \sigma_1 + j\omega_1$ and $q_2 = q_1^* = \sigma_1 - j\omega_1$. In particular, let it be required that the real part of all other q_i be at least three times the real part of q_1. By making these restrictions, all components of the time response except that due to the predominant pole pair are damped to small values which are negligible in comparison to the predominant component at peak time. This will simplify the following work. Expanding by partial fractions, C(s) becomes

$$C(s) = \frac{A}{s} + \frac{B}{s-q_1} + \frac{B^*}{s-q_1^*} + \cdots \qquad (6)$$

The inverse Laplace transform results in the time response

$$c(t) = A + 2|B| \epsilon^{\sigma_1 t} \cos[\omega_1 t + \angle B] + \cdots \tag{7}$$

Figure 2 shows the general form of this time function. To find the peak time it is necessary to set the time derivative of c(t) equal to zero and solve for t. In doing this the terms not written out in Eq. (7) are neglected since they are almost completely damped out before t_p is reached. The result is

$$\left.\frac{d[c(t)]}{dt}\right|_{t_p} = 2|B|\sigma_1 \epsilon^{\sigma_1 t_p} \cos[\omega_1 t_p + \angle B] - 2|B|\omega_1 \epsilon^{\sigma_1 t_p} \sin[\omega_1 t_p + \angle B] = 0 \tag{8}$$

Solving Eq. (8) gives

$$\tan[\omega_1 t_p + \angle B] = \frac{\sigma_1}{\omega_1} \tag{9}$$

or

$$t_p = \frac{1}{\omega_1}\left[\tan^{-1}\frac{\sigma_1}{\omega_1} - \angle B\right] \tag{10}$$

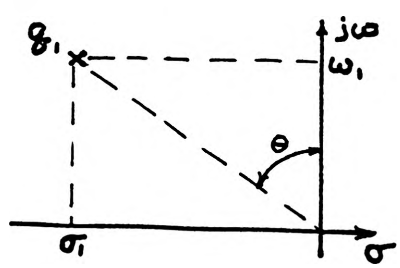


FIGURE 3

From Figure 3 it is seen that

$$\tan^{-1}\frac{\sigma_1}{\omega_1} = -\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

and

$$\angle s_1 = \frac{\pi}{2} + \theta = \frac{\pi}{2} - \tan^{-1}\frac{\sigma_1}{\omega_1}$$

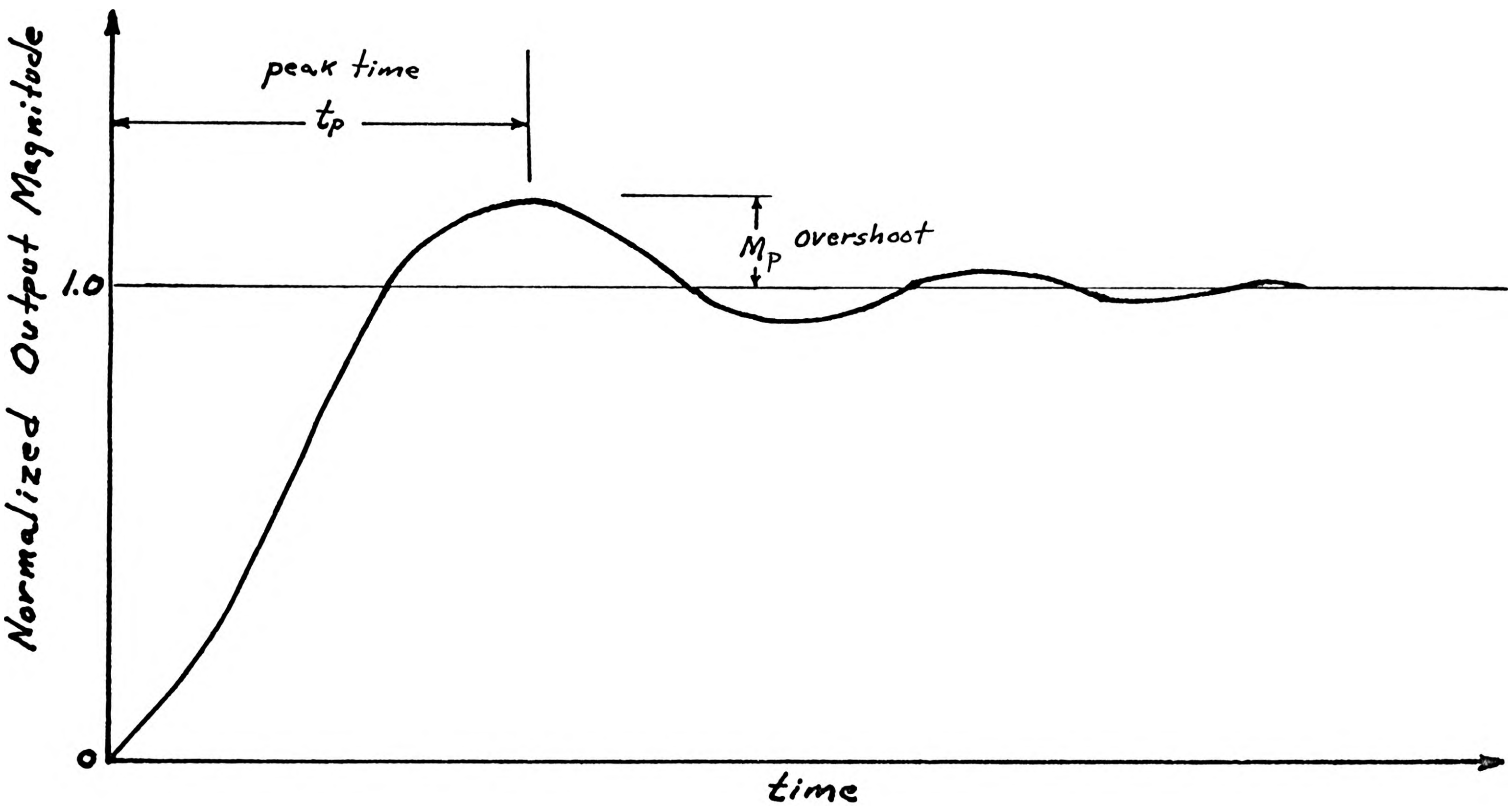


FIGURE 2

THE TIME RESPONSE TO A STEP INPUT

therefore

$$\tan^{-1}\frac{\sigma_1}{\omega_1} = \frac{\pi}{2} - \angle g_1 \qquad (11)$$

The constants A and B will now be evaluated. They are

$$A = C(s)\cdot s\Big|_{s=0} = K_1 K_2 \frac{\prod_{i=1}^{\alpha}(-z_i)\prod_{i=1}^{\gamma}(-t_i)\prod_{i=1}^{j}(-r_i)}{\prod_{i=1}^{\tau}(-g_i)} \qquad (12)$$

and

$$B = C(s)\cdot(s-g_1)\Big|_{s=g_1} = \frac{K_1 K_2}{g_1}\,\frac{\prod_{i=1}^{\alpha}(g_1-z_i)\prod_{i=1}^{\gamma}(g_1-t_i)\prod_{i=1}^{j}(g_1-r_i)}{\prod_{i=2}^{\tau}(g_1-g_i)} \qquad (13)$$

The angle of B is

$$\angle B = \sum_{i=1}^{\alpha}\angle g_1-z_i + \sum_{i=1}^{\gamma}\angle g_1-t_i + \sum_{i=1}^{j}\angle g_1-r_i - \sum_{i=2}^{\tau}\angle g_1-g_i - \angle g_1 \qquad (14)$$

Substituting Eq. (11) and Eq. (14) into Eq. (9)

$$t_p = \frac{1}{\omega_1}\left[\frac{\pi}{2} - \sum_{i=1}^{\alpha}\angle g_1-z_i - \sum_{i=1}^{\gamma}\angle g_1-t_i - \sum_{i=1}^{j}\angle g_1-r_i + \sum_{i=2}^{\tau}\angle g_1-g_i\right] \qquad (15)$$

One term of the last summation can be replaced as $\angle g_1 - g_2 = \angle g_1 - g_1^{*} = \frac{\pi}{2}$ so that finally

$$t_p = \frac{1}{\omega_1}\left[\frac{\pi}{2} - \sum_{i=1}^{\alpha}\angle g_1-z_i - \sum_{i=1}^{\gamma}\angle g_1-t_i - \sum_{i=1}^{j}\angle g_1-r_i + \sum_{i=3}^{\tau}\angle g_1-g_i\right] \qquad (16)$$

The function, f_p, will be defined as

$$f_p = \frac{1}{\omega_1}\left[\frac{\pi}{2} - \sum_{i=1}^{\alpha}\angle g_1-z_i - \sum_{i=1}^{\gamma}\angle g_1-t_i - \sum_{i=1}^{j}\angle g_1-r_i + \sum_{i=3}^{\tau}\angle g_1-g_i\right] - T_p \qquad (17)$$

where T_p is the specified peak time. This form will be convenient in solving the set of condition equations.

2. The equation for overshoot.

Substituting the constants from Eq. (12) and Eq. (13) into Eq. (7) and retaining only the predominant terms gives

$$C(t_p) = K_1 K_2 \frac{\prod_{i=1}^{\alpha}(-z_i)\prod_{i=1}^{\gamma}(-t_i)\prod_{i=1}^{j}(-r_i)}{\prod_{i=1}^{\tau}(-g_i)} \tag{18}$$

$$+ 2\left[\frac{K_1 K_2}{g_1} \cdot \frac{\prod^{\alpha}(g_1-z_i)\prod^{\gamma}(g_1-t_i)\prod^{j}(g_1-r_i)}{\prod_{i=2}^{\tau}(g_1-g_i)}\right]\epsilon^{\sigma_1 t_p}\cos\left[\omega_1 t_p - \angle B\right]$$

where t_p is given in Eq. (16). From Eq. (9) and consideration of Figure 4 it follows that

$$\cos\left[\omega_1 t_p - \angle B\right] = \frac{\omega_1}{\sqrt{\omega_1^2+\sigma_1^2}} = \frac{\omega_1}{|g_1|} \tag{19}$$

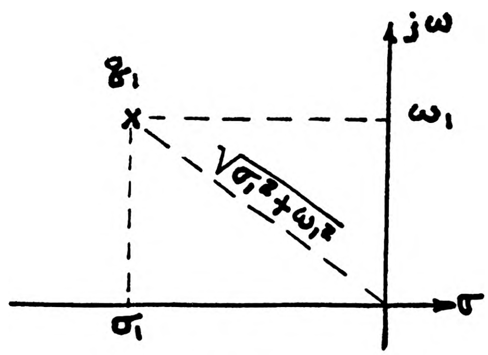


FIGURE 4

Substituting Eq. (19) into Eq. (18) and dividing the cosine term by the magnitude of the first term gives the overshoot, M_p, as a fraction of the steady state output magnitude.

$$M_p = \frac{2\omega_1}{|q_1|^2} \cdot \frac{\prod_{i=1}^{\alpha} |q_1 - z_i| \cdot \prod_{i=1}^{\gamma} |q_1 - t_i| \cdot \prod_{i=1}^{j} |q_1 - r_i| \cdot \prod_{i=1}^{\tau} |q_i|}{\prod_{i=2}^{\tau} |q_1 - q_i| \cdot \prod_{i=1}^{\alpha} |z_i| \cdot \prod_{i=1}^{\gamma} |t_i| \cdot \prod_{i=1}^{j} |r_i|} \epsilon^{\sigma_1 t_p} \qquad (20)$$

Since $|q_1| = |q_2|$, it follows that $|q_1|^2 = |q_1| \cdot |q_2|$, and this will cancel the first two terms in the product series of q_i. Also, in the denominator $|q_1 - q_2| = 2\omega_1$. Using these reductions gives

$$M_p = \frac{\prod_{i=1}^{\alpha} |q_1 - z_i| \cdot \prod_{i=1}^{\gamma} |q_1 - t_i| \cdot \prod_{i=1}^{j} |q_1 - r_i| \cdot \prod_{i=3}^{\tau} |q_i|}{\prod_{i=3}^{\tau} |q_1 - q_i| \cdot \prod_{i=1}^{\alpha} |z_i| \cdot \prod_{i=1}^{\gamma} |t_i| \cdot \prod_{i=1}^{j} |r_i|} \epsilon^{\sigma_1 t_p} \qquad (21)$$

or in natural logarithmic form

$$\sum_{i=1}^{\alpha} \ln|q_1 - z_i| + \sum_{i=1}^{\gamma} \ln|q_1 - t_i| + \sum_{i=1}^{j} \ln|q_1 - r_i| + \sum_{i=3}^{\tau} \ln|q_i| - \sum_{i=3}^{\tau} \ln|q_1 - q_i|$$
$$\sum_{i=1}^{\alpha} \ln|z_i| - \sum_{i=1}^{\gamma} \ln|t_i| - \sum_{i=1}^{j} \ln|r_i| + \sigma_1 t_p = \ln M_p \qquad (22)$$

The function f_m will be defined as

$$f_m = \sum_{i=1}^{\alpha} (\ln|q_1 - z_i| - \ln|z_i|) + \sum_{i=1}^{\gamma} (\ln|q_1 - t_i| - \ln|t_i|) + \sum_{i=1}^{j} (\ln|q_1 - r_i|$$
$$- \ln|r_i|) - \sum_{i=3}^{\tau} (\ln|q_1 - q_i| - \ln|q_i|) + \sigma_1 t_p - \ln M \qquad (23)$$

where M is the specified overshoot.

3. The equation for the velocity constant.

Truxal[4] developed an equation relating the velocity constant to system poles and zeros. It is

$$K_v = 1 \Big/ \left[\sum_{i=1}^{\alpha} \frac{1}{z_i} + \sum_{i=1}^{\gamma} \frac{1}{t_i} + \sum_{i=1}^{j} \frac{1}{r_i} + \sum_{i=1}^{\tau} \frac{-1}{g_i} \right] \tag{24}$$

The function f_k is now defined as

$$f_K = \left[\sum_{i=1}^{\alpha} \frac{1}{z_i} + \sum_{i=1}^{\gamma} \frac{1}{t_i} + \sum_{i=1}^{j} \frac{1}{r_i} - \sum_{i=1}^{\tau} \frac{1}{g_i} \right] K_v - 1 \tag{25}$$

The only restriction is that $C(s)/R(s)|_{s=0} = 1$. This infers that the system be type 1 if it has unity feedback. If the system does not have unity feedback, its unity feedback equivalent must be type 1.

4. <u>Equations for the plant poles.</u>

The closed-loop transfer function can be written as

$$K(s) = \frac{G_c(s)\, G_p(s)}{1 + G_c(s) G_p(s)\, H(s)} \tag{26}$$

Solving for the forward-loop function gives

$$G_p(s)\, G_c(s) = \frac{K(s)}{1 - K(s) H(s)}$$

Hence for each plant pole, p_e, $1-K(s)H(s)$ must be zero. Then

$$1 - K_1 K_2 K_n \cdot \frac{\prod_{i=1}^{\ell} (P_e - y_i)}{\prod_{i=1}^{j} (P_e - r_i)} \cdot \frac{\prod_{i=1}^{\alpha} (P_e - z_i) \prod_{i=1}^{\gamma} (P_e - t_i) \prod_{i=1}^{j} (P_e - r_i)}{\prod_{i=1}^{\tau} (P_e - g_i)} = 0 \tag{27}$$

Both magnitude and angle of this expression must equal zero. Separate expressions will, therefore, be written for these two requirements. Writing Eq. (27) in logarithmic form gives

$$\sum_{i=1}^{\tau} \ln |P_e - g_i| - \ln K_1 K_2 K_n - \sum_{i=1}^{\ell} \ln |P_e - y_i| - \sum_{i=1}^{\alpha} \ln |P_e - z_i| - \sum_{i=1}^{\gamma} \ln |P_e - t_i| = 0 \tag{28}$$

which must be satisfied for the magnitude requirement. The function f_e shall be defined as

$$f_e = \sum_{i=1}^{\tau} \ln|p_e - q_i| - \ln K_1 K_2 K_n - \sum_{i=1}^{\ell} \ln|p_e - y_i| - \sum_{i=1}^{\alpha} \ln|p_e - z_i| - \sum_{i=1}^{\gamma} \ln|p_e - t_i| \qquad (29)$$

The equation

$$\sum_{i=1}^{\ell} \angle p_e - y_i + \sum_{i=1}^{\gamma} \angle p_e - t_i + \sum_{i=1}^{\alpha} \angle p_e - z_i - \sum_{i=1}^{\tau} \angle p_e - q_i = a\,2\pi \qquad (30)$$

$$a = 0, \pm 1, \pm 2, \cdots$$

must be satisfied for the angle requirement. The function g_e shall be defined as

$$g_e = \sum_{i=1}^{\ell} \angle p_e - y_i + \sum_{i=1}^{\gamma} \angle p_e - t_i + \sum_{i=1}^{\alpha} \angle p_e - z_i - \sum_{i=1}^{\tau} \angle p_e - q_i - a\,2\pi \qquad (31)$$

It will now be shown that it is unnecessary to write equations like Eq. (27) and Eq. (30) for both poles of a complex pair. Any complex pair of plant poles, say p_e and $p_{e+1} = p_e^*$, can be written in terms of the real and imaginary parts as $p_e = p_{er} + j\,p_{ej}$ and $p_{e+1} = p_{er} - j\,p_{ej}$. For example, applying this to any one of the summations of f_e, given in equation (28), yields

$$\sum_{i=1}^{\tau} \ln|p_e - q_i| = \sum_{i=1}^{\tau} \frac{1}{2} \ln\left[(p_{er} - q_{ir})^2 + (p_{ej} - q_{ij})^2\right]$$

For a real q_i, $q_{ij} = 0$ and a term of this summation is

$$\frac{1}{2} \ln (p_{er} - q_{ir})^2$$

For a complex conjugate pair of q_i the terms are

$$\frac{1}{2} \ln\left[(p_{er} - q_{ir})^2 + (p_{ej} - q_{ij})^2\right]$$

and

$$\frac{1}{2}\ln\left[(p_{er}-g_{ir})^2+(p_{ej}+g_{ij})^2\right]$$

Now for the equation involving the conjugate of p_e these last three terms are

$$\frac{1}{2}\ln\,(p_{er}-g_{ir})^2$$

$$\frac{1}{2}\ln\left[(p_{er}-g_{ir})^2+(p_{ej}+g_{ij})^2\right]$$

$$\frac{1}{2}\ln\left[(p_{er}-g_{ir})^2+(p_{ej}-g_{ij})^2\right]$$

Notice the terms in p_{e+1} correspond exactly to those in p_e. Hence both function f_e and f_{e+1} are identical. Only the upper half plane pole of a conjugate pair needs to be used in writing equations like Eq. (29). A similar procedure will show the functions g_e and g_{e+1}, which are written for p_e and $p_{e+1} = p_e^*$, are also identical, and only the upper half plane pole needs to be used.

In addition, consider Eq. (30) written for a real pole. One summation in the equation is

$$\sum_{i=1}^{\alpha} \underline{/p_e - z_i} = \sum_{i=1}^{\alpha} \tan^{-1}\frac{0-z_{ij}}{p_{er}-z_{ir}}$$

For a complex pair of z_i the terms are

$$\tan^{-1}\frac{-z_{ij}}{p_{re}-z_{ir}} \quad \text{and} \quad \tan^{-1}\frac{z_{ij}}{p_{er}-z_{ir}}$$

which cancel each other. For each real z_i

$$\tan^{-1}\frac{0}{p_{er}-z_{ir}} = 90^\circ$$

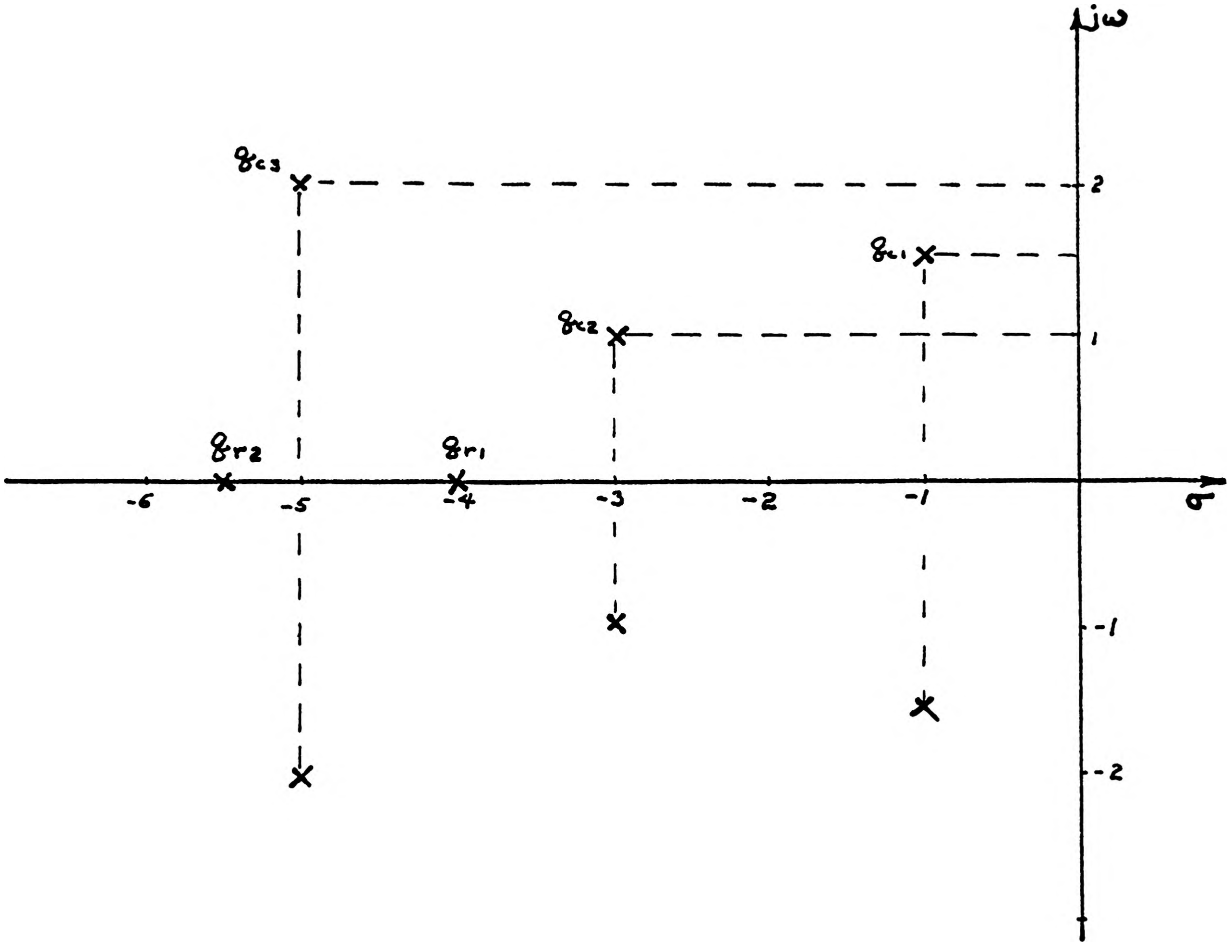
Therefore where p_e is real, Eq. (30) is not dependent on numerical values of other system singularities. It is then not necessary to write equations like Eq. (30) for real p_e.

C. Modified notation.

It is now convenient to change notation for programming purposes as follows:

z_{ri}	are	the					α_R	real	zeros	of	$G_p(s)$
p_{ri}	"	"					β_R	"	poles	"	$G_p(s)$
t_{ri}	"	"					γ_R	"	zeros	"	$G_c(s)$
y_{ri}	"	"					λ_R	"	zeros	"	$H(s)$
r_{ri}	"	"					j_R	"	poles	"	$H(s)$
q_{ri}	"	"					τ_R	"	poles	"	$K(s)$
z_{cri}	are	the	real	parts	of	the	α_c	complex	zeros	of	$G_p(s)$
p_{cri}	"	"	"	"	"	"	β_c	"	poles	of	$G_p(s)$
t_{cri}	"	"	"	"	"	"	γ_c	"	zeros	of	$G_c(s)$
y_{cri}	"	"	"	"	"	"	λ_c	"	zeros	of	$H(s)$
r_{cri}	"	"	"	"	"	"	j_c	"	poles	of	$H(s)$
q_{cri}	"	"	"	"	"	"	τ_c	"	poles	of	$K(s)$
z_{cji}	are	the	imaginary	parts	of	the	α_c	complex	zeros	of	$G_p(s)$
r_{cri}	"	"	"	"	"	"	β_c	"	poles	of	$G_p(s)$
t_{cri}	"	"	"	"	"	"	γ_c	"	zeros	of	$G_c(s)$
y_{cri}	"	"	"	"	"	"	λ_c	"	zeros	of	$H(s)$
r_{cri}	"	"	"	"	"	"	j_c	"	poles	of	$H(s)$
q_{cri}	"	"	"	"	"	"	τ_c	"	poles	of	$K(s)$

Where $q_{cr1} + j\, q_{cj1}$ is the predominant closed-loop pole. Only the real and upper half plane singularities are counted and given numbers. For example the poles of K(s) would be numbered as shown in Figure 5.



$q_{cr1} = -1.0$

$q_{cj1} = 1.5$

$q_{cr2} = -3.0$

$q_{cj2} = 1.0$

$q_{cr3} = -5.0$

$q_{cj3} = 2.0$

$q_{r1} = -4.0$

$q_{r2} = -5.5$

FIGURE 5

NOTATION FOR THE POLES OF K(s)

The singularities were divided into real and complex types because each type is handled differently in programming. For physical systems, complex poles have to exist in conjugate pairs and they cannot, therefore, be treated independently. In rewriting the condition equations, the summations will be divided into two parts; one involving real and the other involving complex singularities. For complex singularities, only the upper half plane components are used but are used two times, first with the imaginary part positive and then with it negative, to account for both the singularity and its conjugate. With the new notation, a conjugate pair of complex variables has been changed to two real variables with the total number of variables remaining unchanged. This eliminates the possibility of a pair of conjugate singularities varying independently.

Using the new notation the condition functions can be written as follows.

$$
\begin{aligned}
f_p = \frac{1}{g_{cj1}}\Bigg[\pi &+ \sum_{i=1}^{\tau_R} \tan^{-1}\frac{g_{cj1}}{g_{cr1}-g_{ri}} - \sum_{i=1}^{\alpha_R} \tan^{-1}\frac{g_{cj1}}{g_{cr1}-Z_{ri}} \\
&- \sum_{i=1}^{\gamma_R} \tan^{-1}\frac{g_{cj1}}{g_{cr1}-t_{ri}} - \sum_{i=1}^{j_R} \tan^{-1}\frac{g_{cj1}}{g_{cr1}-r_{ri}} \\
&+ \sum_{i=2}^{\tau_c}\left(\tan^{-1}\frac{g_{cj1}-g_{cji}}{g_{cr1}-g_{cri}} + \tan^{-1}\frac{g_{cj1}+g_{cji}}{g_{cr1}-g_{cri}}\right) \\
&- \sum_{i=1}^{\alpha_c}\left(\tan^{-1}\frac{g_{cj1}-Z_{cji}}{g_{cr1}-Z_{cri}} + \tan^{-1}\frac{g_{cj1}+Z_{cji}}{g_{cr1}-g_{cri}}\right) \\
&- \sum_{i=1}^{\gamma_c}\left(\tan^{-1}\frac{g_{cj1}-t_{cji}}{g_{cr1}-t_{cri}} + \tan^{-1}\frac{g_{cj1}+t_{cji}}{g_{cr1}-t_{cri}}\right) \\
&- \sum_{i=1}^{j_c}\left(\tan^{-1}\frac{g_{cj1}-r_{cji}}{g_{cr1}-r_{cri}} + \tan^{-1}\frac{g_{cj1}+r_{cji}}{g_{cr1}-r_{cri}}\right)\Bigg] - T_p \qquad (32)
\end{aligned}
$$

$$
\begin{aligned}
f_m = & \sum_{i=1}^{\alpha_R} \left\{ \frac{1}{2} \ln \left[(g_{cr1} - z_{ri})^2 + g_{cj1}^2 \right] + \frac{1}{2} \ln z_{ri}^2 \right\} \\
& + \sum_{i=1}^{\gamma_R} \left\{ \frac{1}{2} \ln \left[(g_{cr1} - t_{ri})^2 + g_{cj1}^2 \right] + \frac{1}{2} \ln t_{ri}^2 \right\} \\
& + \sum_{i=1}^{j_R} \left\{ \frac{1}{2} \ln \left[(g_{cr1} - r_{ri})^2 + g_{cj1}^2 \right] + \frac{1}{2} \ln r_{ri}^2 \right\} \\
& - \sum_{i=1}^{\tau_R} \left\{ \frac{1}{2} \ln \left[(g_{cr1} - g_{ri})^2 + g_{cj1}^2 \right] + \frac{1}{2} \ln g_{ri}^2 \right\} \\
& + \sum_{i=1}^{\alpha_c} \left\{ \frac{1}{2} \ln \left[(g_{cr1} - z_{cri})^2 + (g_{cj1} - z_{cji})^2 \right] + \ln (z_{cri}^2 + z_{cji}^2) \right. \\
& \qquad \left. + \frac{1}{2} \ln \left[(g_{cr1} - z_{cri})^2 + (g_{cj1} + z_{cji})^2 \right] \right\} \\
& + \sum_{i=1}^{\gamma_c} \left\{ \frac{1}{2} \ln \left[(g_{cr1} - t_{cri})^2 + (g_{cj1} - t_{cji})^2 \right] + \ln (t_{cri}^2 + t_{cji}^2) \right. \\
& \qquad \left. + \frac{1}{2} \ln \left[(g_{cr1} - t_{cri})^2 + (g_{cj1} + t_{cji})^2 \right] \right\} \\
& + \sum_{i=1}^{j_c} \left\{ \frac{1}{2} \ln \left[(g_{cr1} - r_{cri})^2 + (g_{cj1} - r_{cri})^2 \right] + \ln (r_{cri}^2 + r_{cji}^2) \right. \\
& \qquad \left. + \frac{1}{2} \ln \left[(g_{cr1} - r_{cri})^2 + (g_{cj1} + r_{cri})^2 \right] \right\} \\
& - \sum_{i=2}^{\tau_c} \left\{ \frac{1}{2} \ln \left[(g_{cr1} - g_{cri})^2 + (g_{cj1} - g_{cji})^2 \right] - \ln (g_{cri}^2 + g_{cji}^2) \right. \\
& \qquad \left. + \frac{1}{2} \ln \left[(g_{cr1} - g_{cri})^2 + (g_{cj1} + g_{cji})^2 \right] \right\} \\
& \qquad + g_{cr1} (f_p + T_p) - \ln M
\end{aligned}
\tag{33}
$$

$$f_K = \left[\sum_{i=1}^{\alpha_R} \frac{1}{Z_{ri}} + \sum_{i=1}^{\alpha_c} \frac{2 Z_{cri}}{Z_{cri}^2 + Z_{cji}^2} + \sum_{i=1}^{\gamma_R} \frac{1}{t_{ri}} + \sum_{i=1}^{\gamma_c} \frac{2 t_{cri}}{t_{cri}^2 + t_{cji}^2} \right.$$

$$\left. + \sum_{i=1}^{j_R} \frac{1}{r_{ri}} + \sum_{i=1}^{j_c} \frac{2 r_{cri}}{r_{cri}^2 + r_{cji}^2} - \sum_{i=1}^{\tau_R} \frac{1}{g_{ri}} - \sum_{i=1}^{\tau_c} \frac{2 g_{cri}}{g_{cri}^2 + g_{cji}^2} \right] K_v - 1 \qquad (34)$$

Since the form of Eq. (29) will be slightly different for real and complex p_e, i.e., there is no imaginary part of p_e to be considered for p_e real, the equations for each will be defined separately. For each real pole of $G_p(s)$ write the equation

$$f_{Re} = \sum_{i=1}^{\tau_R} \frac{1}{2} \ln (P_{re} - g_{ri})^2 + \sum_{i=1}^{\tau_c} \ln [(P_{re} - g_{cri})^2 + g_{cji}^2]$$

$$- \sum_{i=1}^{\alpha_R} \frac{1}{2} \ln (P_{re} - Z_{ri})^2 - \sum_{i=1}^{\alpha_c} \ln [(P_{re} - Z_{cri})^2 + Z_{cji}^2]$$

$$- \sum_{i=1}^{\gamma_R} \frac{1}{2} \ln (P_{re} - t_{ri})^2 - \sum_{i=1}^{\gamma_c} \ln [(P_{re} - t_{cri})^2 + t_{cji}^2]$$

$$- \sum_{i=1}^{l_R} \frac{1}{2} \ln (P_{re} - y_{ri})^2 - \sum_{i=1}^{l_c} \ln [(P_{re} - y_{cri})^2 + y_{cji}^2] - \ln K_1 K_2 K_n \qquad (35)$$

For each upper half plane complex pole of $G_p(s)$ write the equation

$$f_{ce} = \sum_{i=1}^{\tau_R} \frac{1}{2} \ln [(P_{cre} - g_{ri})^2 + P_{cje}^2] - \sum_{i=1}^{\alpha_R} \frac{1}{2} \ln [(P_{cre} - Z_{ri})^2 + P_{cje}^2]$$

$$- \sum_{i=1}^{\gamma_R} \frac{1}{2} \ln [(P_{cre} - t_{ri})^2 + P_{cje}^2] - \sum_{i=1}^{l_R} \frac{1}{2} \ln [(P_{cre} - y_{ri})^2 + P_{cje}^2]$$

$$+ \sum_{i=1}^{\tau_c} \frac{1}{2} \left\{ \ln [(P_{cre} - g_{cri})^2 + (P_{cje} - g_{cji})^2] + \ln [(P_{cre} - g_{cri})^2 + (P_{cje} + g_{cji})^2] \right\}$$

$$- \sum_{i=1}^{\alpha_c} \frac{1}{2} \left\{ \ln [(P_{cre} - Z_{cri})^2 + (P_{cje} - Z_{cji})^2] + \ln [(P_{cre} - Z_{cri})^2 + (P_{cje} + Z_{cji})^2] \right\}$$

$$-\sum_{i=1}^{\gamma_c}\frac{1}{2}\left\{\ln\left[(P_{cre}-t_{cri})^2+(P_{cje}-t_{cji})^2\right]+\ln\left[(P_{cre}-t_{cri})^2+(P_{cje}+t_{cji})^2\right]\right\}$$

$$-\sum_{i=1}^{\ell_c}\frac{1}{2}\left\{\ln\left[(P_{cre}-y_{cri})^2+(P_{cje}-y_{cji})^2\right]+\ln\left[(P_{cre}-y_{cri})^2+(P_{cje}+y_{cji})^2\right]\right\}$$

$$-\ln K_1 K_2 K_n \tag{36}$$

Also, for each upper half plane complex pole of $G_p(s)$ write

$$g_{ce}=\sum_{i=1}^{\alpha_R}\tan^{-1}\frac{P_{cje}}{P_{cre}-Z_{ri}}+\sum_{i=1}^{\alpha_c}\left(\tan^{-1}\frac{P_{cje}-Z_{cji}}{P_{cre}-Z_{cri}}+\tan^{-1}\frac{P_{cje}+Z_{cji}}{P_{cre}-Z_{cri}}\right)$$

$$+\sum_{i=1}^{\gamma_R}\tan^{-1}\frac{P_{cje}}{P_{cre}-t_{ri}}+\sum_{i=1}^{\gamma_c}\left(\tan^{-1}\frac{P_{cje}-t_{cji}}{P_{cre}-t_{cri}}+\tan^{-1}\frac{P_{cje}+t_{cji}}{P_{cre}-t_{cri}}\right)$$

$$+\sum_{i=1}^{\ell_R}\tan^{-1}\frac{P_{cje}}{P_{cre}-y_{ri}}+\sum_{i=1}^{\ell_c}\left(\tan^{-1}\frac{P_{cje}-y_{cji}}{P_{cre}-y_{cri}}+\tan^{-1}\frac{P_{cje}+y_{cji}}{P_{cre}-y_{cri}}\right)$$

$$-\sum_{i=1}^{\pi_R}\tan^{-1}\frac{P_{cje}}{P_{cre}-g_{ri}}-\sum_{i=1}^{\pi_c}\left(\tan^{-1}\frac{P_{cje}-g_{cji}}{P_{cre}-g_{cri}}+\tan^{-1}\frac{P_{cje}+g_{cji}}{P_{cre}-g_{cri}}\right)$$

$$-a\,2\pi \quad \text{where} \quad a=0,\pm 1,\pm 2,\ldots \tag{37}$$

Since three equations for response specifications, β_r equations for the real plant poles, and $(2\cdot\beta_c)$ equations for the upper-half-plane complex plant poles are written, the total number of equations is finally $(3+\beta_r)+(2\cdot\beta_c)$.

D. Solving the equations.

This preceeding set of equations must be satisfied for a system to give the desired performance. It was assumed initially that only the plant poles and zeros were known, leaving the remaining parameters to be selected. The design must provide as many parameters as there are equations in order for a solution to exist.

A closed form solution of these equations is not generally possible since they are not linear. To solve the equations an approximate solution will first be found by any conventional method of synthesis and then refined by the following procedure.

Let the unknown system parameters be variables, x_1, x_2, x_3, ..., x_n, in the set of condition functions, and let the approximate solution be $x_1 = a_1$, $x_2 = a_2$, $x_3 = a_3, \ldots, x_n = a_n$. Using an n-dimensional Taylor series the functions can be expanded about some n-dimensional point. If this point is chosen as the approximated solution, and only the linear terms of the expansion are retained, the result for one function is

$$f(x_1, x_2, \cdots, x_n) = f(a_1, a_2, \cdots, a_n) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\bigg|_{a_1, a_2, \cdots, a_n} (x_i - a_i) \doteq 0 \qquad (38)$$

or

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\bigg|_{a_1, a_2, \cdots, a_n} (x_i - a_i) \doteq -f(a_1, a_2, \cdots, a_n) \qquad (39)$$

By treating each condition function in this manner a set of linear equations results which can be solved for the x_i. This would be an exact solution for the condition equations except for Taylor series truncation error. Using the new values, the process can be repeated several times until a solution of any desired accuracy is achieved. Convergence is not assured but should present no problem if the initial solution is reasonably accurate.

Expressions for the partial derivatives of each function with respect to each variable are given in the appendix.

Notice that all singularity values except the v_i will appear directly in the solution of the condition equations. Since the forward-loop compensating poles (the v_i) do not appear as variables, and hence do not appear in the solution they are found after solving the condition equations. Substituting into Eq. (26) the expressions for $G_p(s)$, $G_c(s)$, and $H(s)$, as given in Eqs. (1), (2), and (3), yields

$$K(s) = \frac{K_1 K_2 \prod_{i=1}^{\alpha}(s-z_i)\prod_{i=1}^{\gamma}(s-t_i)\prod_{i=1}^{j}(s-r_i)}{\prod_{i=1}^{\beta}(s-p_i)\prod_{i=1}^{\delta}(s-v_i)\prod_{i=1}^{j}(s-r_i) + K_1 K_2 K_n \prod_{i=1}^{\alpha}(s-z_i)\prod_{i=1}^{\gamma}(s-t_i)\prod_{i=1}^{\ell}(s-y_i)} \tag{40}$$

The denominator of Eq. (40) can be equated to the denominator of Eq. (4) giving

$$\begin{aligned}\prod_{i=1}^{\tau}(s-g_i) = {} & \prod_{i=1}^{\beta}(s-p_i)\prod_{i=1}^{\delta}(s-v_i)\prod_{i=1}^{j}(s-r_i) \\ & + K_1 K_2 K_n \prod_{i=1}^{\alpha}(s-z_i)\prod_{i=1}^{\gamma}(s-t_i)\prod_{i=1}^{\ell}(s-y_i)\end{aligned} \tag{41}$$

from which

$$\prod_{i=1}^{\delta}(s-v_i) = \frac{\prod_{i=1}^{\tau}(s-g_i) - K_1 K_2 K_n \prod_{i=1}^{\alpha}(s-z_i)\prod_{i=1}^{\gamma}(s-t_i)\prod_{i=1}^{\ell}(s-y_i)}{\prod_{i=1}^{\beta}(s-p_i)\prod_{i=1}^{j}(s-r_i)} \tag{42}$$

The right hand side of Eq. (42) will result in a polynomial whose roots are the poles of $G_c(s)$.

CHAPTER IV

PROGRAMMING THE PROBIEM

Programming consists of two primary parts: first evaluation of the condition functions and their partial derivatives to obtain the linear equation coefficients; second, solving these linear equations for the next approximation and looping back to part one. The approximation obtained by conventional design methods is used to initiate the process. A number of other steps are, however, involved in the program. They are: (1) Test the solution after each iteration to see if it is near enough to the correct solution. This is accomplished by evaluating each function and testing to see if its value is near enough to zero. The figure for comparison which will be used is 0.01. This will permit, for example, a peak time which is within ± 0.01 seconds of that specified, or an overshoot within $\pm 1\%$ of that specified; (2) check the closed-loop poles after each iteration to be sure they have not moved into the right half of the s-plane; this would cause instability in the system; and (3) check the conditions $|g_{cr2}| \geq 3|g_{cr1}|$ and $|g_{r2}| \geq 3|g_{cr1}|$.

A flow diagram, indicating the basic parts of this program, is given in Figure 6. A general program for the solution is quite lengthy, but programming is principally straight forward. If extensive use is to be made of this technique, it is time saving to write a generalized program rather than to program each problem individually.

Two points warrant mentioning. First, notice that in the functions g_{ce} some multiple of 2π is subtracted. This multiple will not generally be the same for any two of these functions and may vary from one iteration to the next. This operation can be approached as follows. Each time one

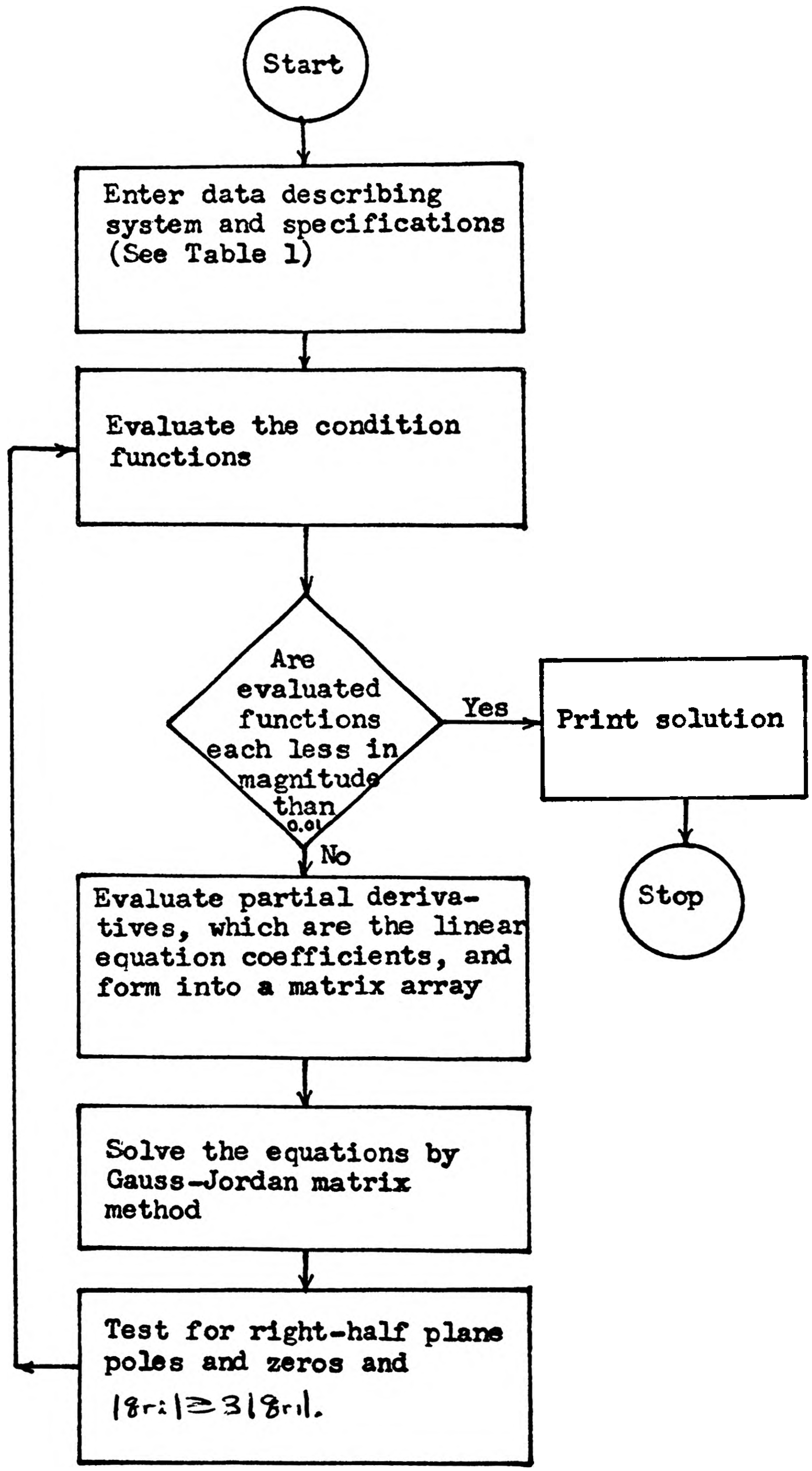


FIGURE 6

THE BASIC FLOW DIAGRAM

TABLE 1

DATA SUPPLIED TO THE PROGRAM

(see Flow Chart in Figure 6)

VARIABLES--specify initial values

q_{ri}	r_{ri}	t_{ri}	y_{ri}
q_{cri}	r_{cri}	t_{cri}	y_{cri}
q_{cji}	r_{cji}	t_{cji}	y_{cji}
K_2	K_n		

CONSTANTS

z_{ri}	p_{ri}	K_v
z_{cri}	p_{cri}	T_p
z_{cji}	p_{cji}	M

Constants to be used to set up the number of loops in a multi-loop calculation.

α_R	β_R	γ_R	ℓ_R	j_R	τ_R
α_c	β_c	γ_c	ℓ_c	j_c	τ_c

The notation is explained in Chapter III.

of these functions is to be evaluated, 2π is subtracted repeatedly until a residual equal to or less in magnitude than π remains. This residual is the value of the function. Second, many computer languages have a command for finding the arctangent but only give the principal value. Usage in this problem requires the angle to be located in the correct quadrant. A routine for this purpose is shown in Figure 7. It is only necessary to define N and D and then transfer to this routine.

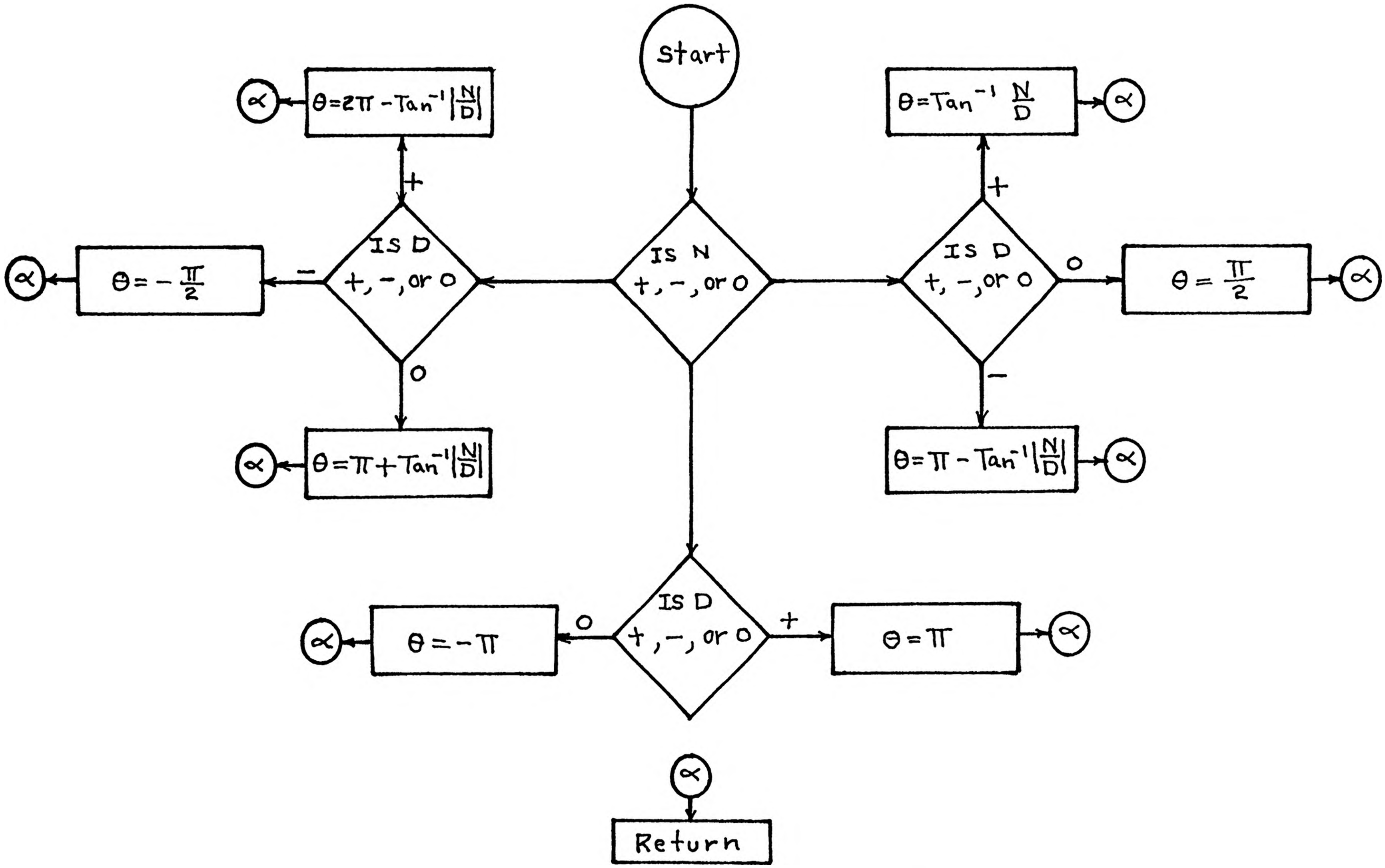


FIGURE 7

A ROUTINE FOR COMPUTING $\theta = \tan^{-1}\frac{N}{D}$

CHAPTER V

ILLUSTRATIVE EXAMPLE

A relatively simple compensation problem will now be presented to illustrate the method and results. A plant with transfer function

$$G_p(s) = \frac{10}{s(s+2)}$$

will be considered. When used in a unity feedback single-loop configuration, as shown in Figure 8, the response characteristics of concern to this example are found by analysis to be

$$K_v = 5$$
$$t_p = 1.045 \text{ seconds}$$
$$M = 0.351$$

Assume that a shorter peak time, less overshoot, and a higher velocity constant are required for satisfactory system operation. In particular, let the specifications be

$$K_v = 10$$
$$T_p = 0.613 \text{ seconds}$$
$$M_p = 0.127$$

The uncompensated system root locus is shown in Figure 9. By adding derivative compensation and additional forward path gain it should be possible to meet the specifications. An approximate root locus plot for the system with this compensation is shown in Figure 10. Initial values for the five variables of the new system are estimated from this plot as

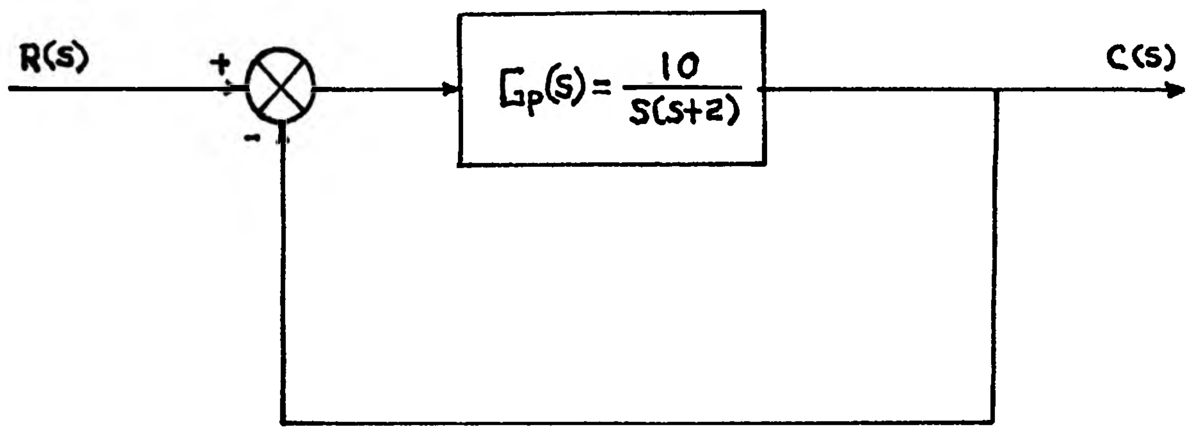


FIGURE 8

THE UNCOMPENSATED CLOSED-LOOP SYSTEM

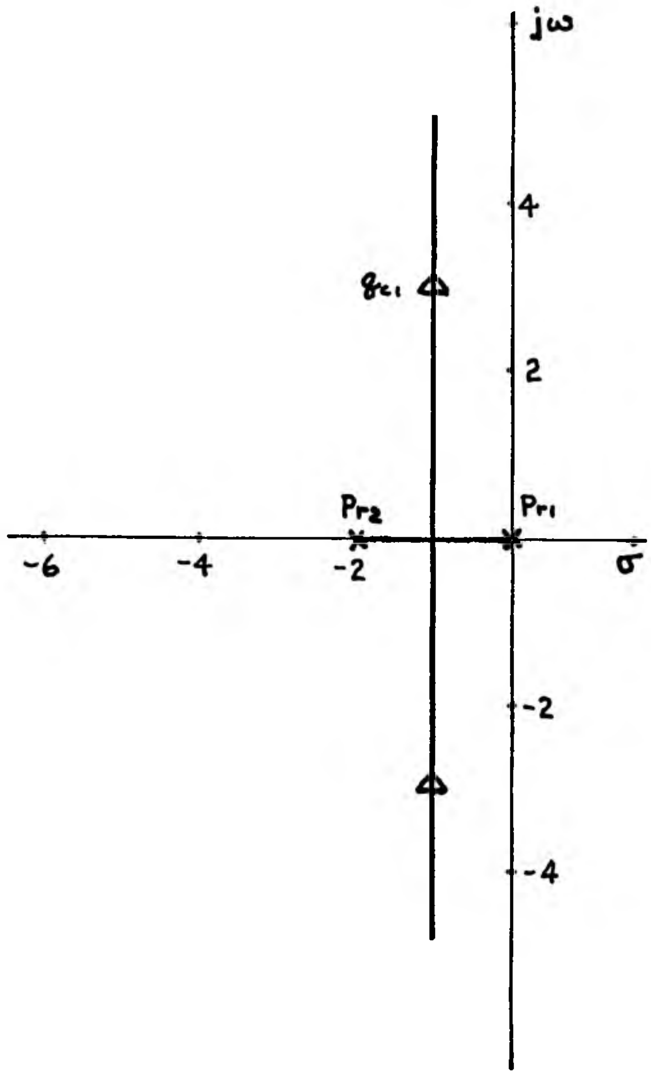


FIGURE 9

ROOT LOCUS PLOT OF THE UNCOMPENSATED SYSTEM

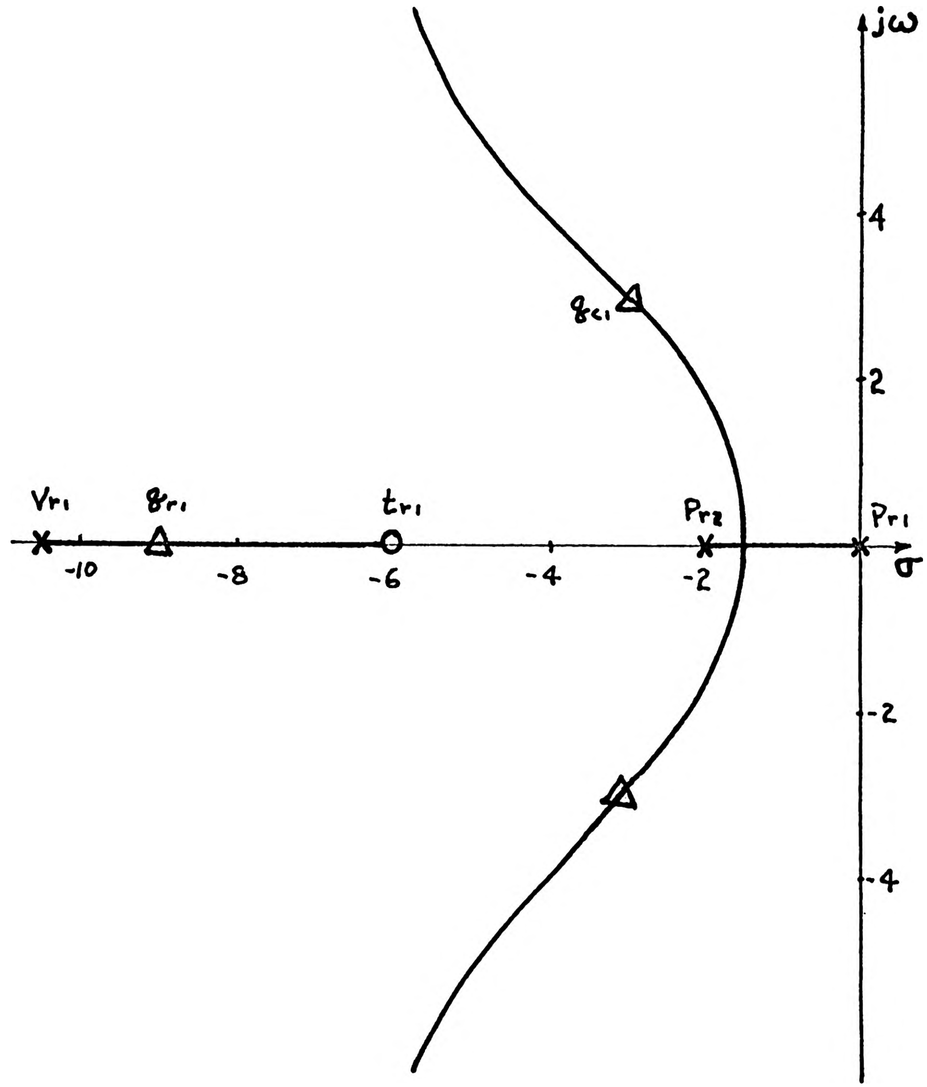


FIGURE 10

APPROXIMATE ROOT LOCUS PLOT OF THE COMPENSATED SYSTEM

$q_{cr1} = -3$ $\quad$ $t_{r1} = -6$ $\quad$ $K_2 = 5$

$q_{cj1} = 3$ $\quad$ $q_{r1} = -9$

Constants of the plant and its control system are

$K_1 = 10$ $\quad$ $\beta_R = 2$

$p_{r1} = 0$ $\quad$ $\gamma_R, \tau_R, \tau_c = 1$

$p_{r2} = -2$ $\quad$ $\alpha_R, \alpha_c, \beta_c, \gamma_c, \ell_R, \ell_c, j_R, j_c = 0$

$K_n = 1$

The set of equations for this problem would be

$$f_p = \frac{1}{q_{cj1}}\left[\pi + \tan^{-1}\frac{q_{cj1}}{q_{cr1}-q_{r1}} - \tan^{-1}\frac{q_{cj1}}{q_{cr1}-t_{r1}}\right] - 0.613$$

$$f_m = \tfrac{1}{2}\ln\left[(q_{cr1}-t_{r1})^2 + q_{cj1}^2\right] + \tfrac{1}{2}\ln t_{r1}^2$$

$$-\tfrac{1}{2}\ln\left[(q_{cr1}-q_{r1})^2 + q_{cj1}^2\right] + \tfrac{1}{2}\ln q_{r1}^2$$

$$-q_{cr1}(f_p + 0.613) - \ln 0.127$$

$$f_K = \left[\frac{1}{t_{r1}} - \frac{1}{q_{r1}} - \frac{2q_{cr1}}{q_{cr1}^2 + q_{cj1}^2}\right] \times 10 - 1$$

$$f_{R1} = \tfrac{1}{2}\ln\left[-q_{r1}\right]^2 + \ln\left[(-q_{cr1})^2 + q_{cj1}^2\right] - \tfrac{1}{2}\ln\left[-t_{r1}\right]^2 - \ln 10 \cdot K_2$$

$$f_{R2} = \tfrac{1}{2}\ln\left[-2-q_{r1}\right]^2 + \ln\left[(-2-q_{cr1})^2 + q_{cj1}^2\right] - \tfrac{1}{2}\ln\left[-2-t_{r1}\right]^2 - \ln 10 \cdot K_2$$

Recall that it is not necessary to write out these equations when a generalized program is used. They are written here for illustration.

The computer, after ten iterations, gave these values for the variables.

$$t_{r1} = -5$$

$$q_{cr1} = -2.3$$

$$q_{cj1} = 4.6$$

$$q_{r1} = -7.55$$

$$K_2 = 4$$

An analysis of the system using these values shows the specifications are satisfied. The compensation zero position is known but the pole position must be found from Eq. (42).

$$S - V_1 = \frac{(S - q_{c1})(S - q_1^*)(S - q_{r1}) - 10 \cdot K_2 (S - t_{r1})}{(S - P_{r1})(S - P_{r2})}$$

$$= \frac{(S + 2.3 - j4.6)(S + 2.3 + j4.6)(S + 7.55) - 10 \cdot 4 (S + 5)}{S(S + 2)}$$

$$= \frac{S^2 + 12.15\,S + 21.3}{S + 2}$$

$$= S + 10.15$$

Therefore, the compensation network required is

$$G_c(S) = \frac{4(S+5)}{(S + 10.5)}$$

CHAPTER VI

DISCUSSION AND CONCLUSIONS

A. Discussion

Certain restrictions will be pointed out and discussed briefly. First, the system considered is required to have a predominant pair of complex poles which is equivalent to saying that it must behave as a second order system. This permitted neglecting all but the predominant terms when writing equations for peak time and overshoot. The error due to this assumption, being in the order of 5%, is admissible in most cases. Including other modes would complicate the mathematics considerably.

Specifying a critically damped or underdamped system is not allowed when using the given condition equations. However, other equations could be derived for this purpose.

The work presented here does not terminate the possibilities of this method. Many other system specifications could be used. Settling time, bandwidth, damping factor, or the transient response to excitation other than a unit step could be incorporated. Also, the specifications used in this development could have been written with inequalities as

$$\text{peak time} \leq T_p$$

$$\text{overshoot} \leq M$$

$$\text{velocity constant} \geq K_v$$

instead of with equalities. Perhaps some method could be contrived to solve the equations in terms of inequalities and, in doing so, reduce the amount of compensation required.

B. Conclusions

The proposed digital computer method will refine an initial estimate of the plant compensation needed to meet certain closed loop specifications. While it is subject to a few limitations, and still requires some judgment and preparation by the designer, it is an aid to design which should prove useful.

Emphasis has been placed on meeting a set of performance specifications. Compensation may also be required for a system which is unstable. Since it is not necessary for the uncompensated system to be stable when using the computer technique, an additional benefit arises in that stability may be achieved concurrently with meeting the other specifications.

The digital computer is seen to be a convenient tool for linear control system design and many possibilities exist for extending the work presented here.

BIBLIOGRAPHY

1. A Computer-Oriented Iterative Design of Linear Control Systems Subject to Open- and Closed-Loop Specifications, J. Zaborszky and R. E. Marsh. AIEE Transactions, pt. II (Applications and Industry), vol. 81, 1962, pp 129-132.

2. On The Synthesis of Feedback Systems With Open-Loop Constraints, J. A. Aseltine. IRE Transactions (Automatic Control), vol. AC-4, no. 1, 1959, pp 31-37.

3. Synthesis of Feedback Control Systems By Phase Angle Loci, Yaohan Chu. AIEE Transactions (Applications and Industry), vol. 71, 1952, pp 330-339.

4. Automatic Feedback Control System Synthesis, J. G. Truxal. McGraw-Hill Book Company, Inc., New York, N. Y., 1955, chp. 5.

5. Feedback System Synthesis By The Inverse Root Locus Method, J. A. Aseltine. IRE National Convention Record, vol. 4, pt. 2, 1956, pp 13-16.

6. Integrated s-Plane Synthesis Using 2-Way Root Locus, J. Zaborszky. AIEE Transactions (Applications and Industry), vol. 75, 1957, pp 797-801.

APPENDIX

THE PARTIAL DERIVATIVES OF THE CONDITION EQUATIONS

$$
\begin{aligned}
\frac{\partial f_p}{\partial g_{cr1}} = \frac{1}{g_{cj1}} \Bigg[& -\sum_{i=1}^{\tau_R} \frac{g_{cj1}}{(g_{cr1}-g_{ri})^2 + g_{cj1}{}^2} + \sum_{i=1}^{\alpha_R} \frac{g_{cj1}}{(g_{cr1}-Z_{ri})^2 + g_{cj1}{}^2} \\
& + \sum_{i=1}^{\gamma_R} \frac{g_{cj1}}{(g_{cr1}-t_{ri})^2 + g_{cj1}{}^2} - \sum_{i=1}^{j_R} \frac{g_{cj1}}{(g_{cr1}-r_{ri})^2 + g_{cj1}{}^2} \\
& - \sum_{i=2}^{\tau_c} \left\{ \frac{g_{cj1}-g_{cji}}{(g_{cr1}-g_{cri})^2+(g_{cj1}-g_{cji})^2} + \frac{g_{cj1}+g_{cji}}{(g_{cr1}-g_{cri})^2+(g_{cj1}+g_{cji})^2} \right\} \\
& + \sum_{i=1}^{\alpha_c} \left\{ \frac{g_{cj1}-Z_{cji}}{(g_{cr1}-Z_{cri})^2+(g_{cj1}-Z_{cji})^2} + \frac{g_{cj1}+Z_{cji}}{(g_{cr1}-Z_{cri})^2+(g_{cj1}+Z_{cji})^2} \right\} \\
& + \sum_{i=1}^{\gamma_c} \left\{ \frac{g_{cj1}-t_{cji}}{(g_{cr1}-t_{cri})^2+(g_{cj1}-t_{cji})^2} + \frac{g_{cj1}+t_{cji}}{(g_{cr1}-t_{cri})^2+(g_{cj1}+t_{cji})^2} \right\} \\
& + \sum_{i=1}^{j_c} \left\{ \frac{g_{cj1}-r_{cji}}{(g_{cr1}-r_{cri})^2+(g_{cj1}-r_{cji})^2} + \frac{g_{cj1}+r_{cji}}{(g_{cr1}-r_{cri})^2+(g_{cj1}+r_{cji})^2} \right\} \Bigg]
\end{aligned}
$$

$$
\frac{\partial f_p}{\partial g_{cri}} = \frac{1}{g_{cj1}} \left[\frac{g_{cj1}-g_{cji}}{(g_{cr1}-g_{cri})^2+(g_{cj1}-g_{cji})^2} + \frac{g_{cj1}+g_{cji}}{(g_{c1r}-g_{cri})^2+(g_{cj1}+g_{cji})^2} \right]_{i \geq 2}
$$

$$
\frac{\partial f_p}{\partial g_{cji}} = \frac{1}{g_{cj1}} \left[\frac{-(g_{cr1}-g_{cri})}{(g_{cr1}-g_{cri})^2+(g_{cj1}-g_{cji})^2} + \frac{g_{cr1}-g_{cri}}{(g_{cr1}-g_{cri})^2+(g_{cj1}+g_{cji})^2} \right]_{i \geq 2}
$$

$$
\frac{\partial f_p}{\partial g_{ri}} = \frac{1}{(g_{cr1}-g_{ri})^2 + g_{cj1}{}^2}
$$

$$\frac{\partial f_p}{\partial g_{cj1}} = \frac{1}{g_{cj1}}\left[\sum_{i=1}^{\tau_R}\frac{g_{cr1}-g_{ri}}{(g_{cr1}-g_{ri})^2+g_{cj1}^2} - \sum_{i=1}^{\alpha_R}\frac{g_{cr1}-Z_{ri}}{(g_{cr1}-Z_{ri})^2+g_{cj1}^2}\right.$$

$$-\sum_{i=1}^{\gamma_R}\frac{g_{cr1}-t_{ri}}{(g_{cr1}-r_{ri})^2+g_{cj1}^2} - \sum_{i=1}^{j_R}\frac{g_{cr1}-r_{ri}}{(g_{cr1}-r_{ri})^2+g_{cj1}^2}$$

$$+\sum_{i=2}^{\tau_c}\left\{\frac{g_{cr1}-g_{cri}}{(g_{cr1}-g_{cri})^2+(g_{cj1}-g_{cji})^2}+\frac{g_{cr1}-g_{cri}}{(g_{cr1}-g_{cri})^2+(g_{cj1}+g_{cji})^2}\right\}$$

$$-\sum_{i=1}^{\alpha_c}\left\{\frac{g_{cr1}-Z_{cri}}{(g_{cr1}-Z_{cri})^2+(g_{cj1}-Z_{cji})^2}+\frac{g_{cr1}-Z_{cri}}{(g_{cr1}-Z_{cri})^2+(g_{cj1}+Z_{cji})^2}\right\}$$

$$-\sum_{i=1}^{\gamma_c}\left\{\frac{g_{cr1}-t_{cri}}{(g_{cr1}-t_{cri})^2+(g_{cj1}-t_{cji})^2}+\frac{g_{cr1}-t_{cri}}{(g_{cr1}-t_{cri})^2+(g_{cj1}+t_{cri})^2}\right\}$$

$$\left.-\sum_{i=1}^{j_c}\left\{\frac{g_{cr1}-r_{cri}}{(g_{cr1}-r_{cri})^2+(g_{cj1}-r_{cji})^2}+\frac{g_{cr1}-r_{cri}}{(g_{cr1}-r_{cri})^2+(g_{cj1}+r_{cri})^2}\right\}\right]$$

$$-\frac{1}{g_{cj1}}(f_p+T_p)$$

$$\frac{\partial f_p}{\partial t_{cri}} = \frac{1}{g_{cj1}}\left[\frac{g_{cj1}-t_{cji}}{(g_{cr1}-t_{cri})^2+(g_{cj1}-g_{cji})^2}+\frac{g_{cj1}+t_{cji}}{(g_{cr1}-t_{cri})^2+(g_{cj1}+t_{cji})^2}\right]$$

$$\frac{\partial f_p}{\partial t_{cji}} = \frac{1}{g_{cj1}}\left[\frac{g_{cr1}-t_{cr1}}{(g_{cr1}-t_{cr1})^2+(g_{cj1}-t_{cji})^2}-\frac{g_{cr1}-t_{cr1}}{(g_{cr1}-t_{cri})^2+(g_{cj1}+t_{cji})^2}\right]$$

$$\frac{\partial f_p}{\partial t_{ri}} = \frac{-1}{(g_{cr1}-t_{ri})^2+g_{cj1}^2}$$

$$\frac{\partial f_p}{\partial r_{ri}} = \frac{-1}{(g_{cr1} - r_{ri})^2 + g_{cj1}^2}$$

$$\frac{\partial f_p}{\partial r_{cri}} = \frac{-1}{g_{cj1}} \left[\frac{g_{cj1} - r_{cj1}}{(g_{cr1} - r_{cri})^2 + (g_{cj1} - r_{cji})^2} + \frac{g_{cj1} + r_{cji}}{(g_{cr1} - r_{cri})^2 + (g_{cj1} + r_{cji})^2} \right]$$

$$\frac{\partial f_p}{\partial r_{cji}} = \frac{1}{g_{cj1}} \left[\frac{g_{cr1} - r_{cr1}}{(g_{cr1} - r_{cri})^2 + (g_{cj1} - r_{cji})^2} - \frac{g_{cr1} - r_{cr1}}{(g_{cr1} - r_{cri})^2 + (g_{cj1} + r_{cji})^2} \right]$$

$$\frac{\partial f_m}{\partial g_{cj1}} = \sum_{i=1}^{\alpha_R} \frac{g_{cj1}}{(g_{cr1} - Z_{ri})^2 + g_{cj1}^2} + \sum_{i=1}^{\gamma_R} \frac{g_{cj1}}{(g_{cr1} - t_{ri})^2 + g_{cj0}^2}$$

$$+ \sum_{i=1}^{j_R} \frac{g_{cj1}}{(g_{cr1} - r_{ri})^2 + g_{cj1}^2} - \sum_{i=1}^{\tau_R} \frac{g_{cj1}}{(g_{cr1} - g_{ri})^2 + g_{cj1}^2}$$

$$+ \sum_{i=1}^{\alpha_c} \left\{ \frac{g_{cj1} - Z_{cji}}{(g_{cr1} - Z_{cri})^2 + (g_{cj1} - Z_{cji})^2} + \frac{g_{cj1} + Z_{cji}}{(g_{cr1} - Z_{cri})^2 + (g_{cj1} + Z_{cji})^2} \right\}$$

$$+ \sum_{i=1}^{\gamma_c} \left\{ \frac{g_{cj1} - t_{cji}}{(g_{cr1} - t_{cri})^2 + (g_{cj1} - t_{cji})^2} + \frac{g_{cj1} + t_{cji}}{(g_{cr1} - t_{cri})^2 + (g_{cj1} + t_{cj1})^2} \right\}$$

$$- \sum_{i=2}^{\tau_c} \left\{ \frac{g_{cj1} - g_{cji}}{(g_{cr1} - g_{cri})^2 + (g_{cj1} - g_{cji})^2} + \frac{g_{cj1} + g_{cji}}{(g_{cr1} - g_{cri})^2 + (g_{cj1} + g_{cji})^2} \right\}$$

$$+ \sum_{i=1}^{j_c} \left\{ \frac{g_{cj1} - r_{cji}}{(g_{cr1} - r_{cri})^2 + (g_{cj1} - r_{cji})^2} + \frac{g_{cj1} + r_{cji}}{(g_{cr1} - r_{cri})^2 + (g_{cj1} + r_{cji})^2} \right\}$$

$$+ g_{cr1} \cdot \frac{\partial f_p}{\partial g_{cj1}}$$

$$\frac{\partial f_m}{\partial g_{cr1}} = \sum_{i=1}^{\alpha_R} \frac{g_{cr1} - Z_{ri}}{(g_{cr1} - Z_{ri})^2 + g_{cj1}{}^2} + \sum_{i=1}^{\gamma_R} \frac{g_{cr1} - t_{ri}}{(g_{cr1} - t_{ri})^2 + g_{cj1}{}^2}$$

$$+ \sum_{i=1}^{j_R} \frac{g_{cr1} - r_{ri}}{(g_{cr1} - r_{ri})^2 + g_{cj1}{}^2} - \sum_{i=1}^{\tau_R} \frac{g_{cr1} - g_{ri}}{(g_{cr1} - g_{ri})^2 + g_{cj1}{}^2}$$

$$+ \sum_{i=1}^{\alpha_c} \left\{ \frac{g_{cr1} - Z_{cri}}{(g_{cr1} - Z_{cri})^2 + (g_{cj1} - Z_{cji})^2} + \frac{g_{cr1} - Z_{cri}}{(g_{cr1} - Z_{cri})^2 + (g_{cj1} + Z_{cji})^2} \right\}$$

$$+ \sum_{i=1}^{\gamma_c} \left\{ \frac{g_{cr1} - t_{cri}}{(g_{cr1} - t_{cri})^2 + (g_{cj1} - t_{cji})^2} + \frac{g_{cr1} - t_{cri}}{(g_{cr1} - t_{cri})^2 + (g_{cj1} + t_{cji})^2} \right\}$$

$$+ \sum_{i=1}^{j_c} \left\{ \frac{g_{cr1} - r_{cri}}{(g_{cr1} - r_{cri})^2 + (g_{cj1} - r_{cji})^2} + \frac{g_{cr1} - r_{cri}}{(g_{cr1} - r_{cri})^2 + (g_{cj1} + r_{cji})^2} \right\}$$

$$- \sum_{i=2}^{\tau_c} \left\{ \frac{g_{cr1} - g_{cri}}{(g_{cr1} - g_{cri})^2 + (g_{cj1} - g_{cji})^2} + \frac{g_{cr1} - g_{cri}}{(g_{cr1} - g_{cri})^2 + (g_{cj1} + g_{cji})^2} \right\}$$

$$+ g_{cr1} \frac{\partial f_p}{\partial g_{cr1}} + f_p + T_p$$

$$\frac{\partial f_m}{\partial g_{ri}} = \frac{1}{g_{ri}} + \frac{g_{cr1} - g_{ri}}{(g_{cr1} - g_{ri})^2 + g_{cj1}{}^2} + g_{cr1} \frac{\partial f_p}{\partial g_{ri}}$$

$$\frac{\partial f_m}{\partial g_{cri}} = \frac{2 g_{cri}}{g_{cri}{}^2 + g_{cji}{}^2} + \frac{g_{cr1} - g_{cri}}{(g_{cr1} - g_{cri})^2 + (g_{cj1} - g_{cji})} + \frac{\partial f_p}{\partial g_{ri}} \cdot g_{cr1}$$

$$+ \frac{g_{cr1} - g_{cri}}{(g_{cr1} - g_{cri})^2 + (g_{cj1} + g_{cji})} \qquad i \geq 2$$

$$\frac{\partial f_m}{\partial g_{cji}} = \frac{2\,g_{cji}}{g_{cri}^{2}+g_{cji}^{2}} + \frac{g_{cj1}-g_{cji}}{(g_{cr1}-g_{cri})^2+(g_{cj1}-g_{cji})^2}$$

$$-\frac{g_{cj1}+g_{cji}}{(g_{cr1}-g_{cri})^2+(g_{cj1}+g_{cji})^2} + g_{cr1}\,\frac{\partial f_p}{\partial g_{cji}} \qquad i \geq 2$$

$$\frac{\partial f_m}{\partial t_{ri}} = \frac{-(g_{cr1}-t_{ri})}{(g_{cr1}-t_{ri})^2+g_{cj1}^{2}} + \frac{1}{t_{ri}} + g_{cr1}\,\frac{\partial f_p}{\partial t_{ri}}$$

$$\frac{\partial f_m}{\partial t_{cri}} = \frac{-(g_{cr1}-t_{cri})}{(g_{cr1}-t_{cri})^2+(g_{cj1}-t_{cji})^2} - \frac{g_{cr1}-t_{cri}}{(g_{cr1}-t_{cri})^2+(g_{cj1}+t_{cji})^2}$$

$$+\frac{2\,t_{cji}}{t_{cri}^{2}+t_{cji}^{2}} + g_{cr1}\,\frac{\partial f_p}{\partial t_{cri}}$$

$$\frac{\partial f_m}{\partial t_{cji}} = \frac{-(g_{cj1}-t_{cji})}{(g_{cr1}-t_{cri})^2+(g_{cj1}-t_{cji})^2} + \frac{g_{cj1}+t_{cji}}{(g_{cr1}-t_{cri})^2+(g_{cj1}+t_{cji})^2}$$

$$+\frac{2\,t_{cri}}{(t_{cri}^{2}+t_{cji}^{2})} + g_{cr1}\,\frac{\partial f_p}{\partial t_{cji}}$$

$$\frac{\partial f_m}{\partial r_{ri}} = \frac{-(g_{cr1}-r_{ri})}{(g_{cr1}-r_{ri})^2+g_{cj1}^{2}} + \frac{1}{r_{ri}} + g_{cr1}\,\frac{\partial f_p}{\partial r_{ri}}$$

$$\frac{\partial f_m}{\partial r_{cri}} = \frac{g_{cr1}-r_{cri}}{(g_{cr1}-r_{cri})^2+(g_{cj1}-r_{cji})^2} - \frac{g_{cr1}-r_{cri}}{(g_{cr1}-r_{cri})^2+(g_{cj1}+r_{cji})^2}$$

$$+\frac{2\,r_{cri}}{r_{cri}^{2}+r_{cji}^{2}} + g_{cr1}\,\frac{\partial f_p}{\partial r_{cri}}$$

$$\frac{\partial f_m}{\partial r_{cji}} = \frac{-(g_{cj1} - r_{cji})}{(g_{cr1} - r_{cri})^2 + (g_{cj1} - r_{cji})^2} + \frac{g_{cj1} - r_{cji}}{(g_{cr1} - r_{cri})^2 + (g_{cj1} + r_{cji})^2}$$

$$+ \frac{2\, r_{cji}}{r_{cri}^2 + r_{cji}^2} + g_{cr1} \frac{\partial f_p}{\partial r_{cji}}$$

$$\frac{\partial f_k}{\partial g_{ri}} = \frac{K_v}{g_{ri}^2}$$

$$\frac{\partial f_k}{\partial t_{ri}} = - \frac{K_v}{t_{ri}^2}$$

$$\frac{\partial f_k}{\partial r_{ri}} = - \frac{K_v}{r_{ri}^2}$$

$$\frac{\partial f_k}{\partial g_{cji}} = 4 K_v \frac{g_{cri}\, g_{cji}}{(g_{cri}^2 + g_{cji}^2)^2}$$

$$\frac{\partial f_k}{\partial t_{cji}} = - 4 K_v \frac{t_{cri}\, t_{cji}}{(t_{cri}^2 + t_{cji}^2)^2}$$

$$\frac{\partial f_k}{\partial r_{cji}} = - 4 K_v \frac{r_{cri}\, r_{cji}}{(r_{cri}^2 + r_{cji}^2)^2}$$

$$\frac{\partial f_k}{\partial g_{cri}} = 2 K_v \frac{(g_{cri}^2 - g_{cji}^2)}{(g_{cri}^2 + g_{cji}^2)^2}$$

$$\frac{\partial f_k}{\partial t_{cri}} = -2 K_v \frac{(t_{cri}^2 - t_{cji}^2)}{(t_{cri}^2 + t_{cji}^2)^2}$$

$$\frac{\partial f_k}{\partial r_{cri}} = -2 K_v \frac{(r_{cri}^2 - r_{cji}^2)}{(r_{cri}^2 + r_{cji}^2)^2}$$

$$\frac{\partial f_{RE}}{\partial g_{ri}} = \frac{-1}{P_{re} - g_{ri}}$$

$$\frac{\partial f_{RE}}{\partial t_{ri}} = \frac{1}{P_{re} - t_{ri}}$$

$$\frac{\partial f_{RE}}{\partial y_{ri}} = \frac{1}{P_{re} - y_{ri}}$$

$$\frac{\partial f_{RE}}{\partial g_{cri}} = \frac{-(P_{re} - g_{cri})}{(P_{re} - g_{cri})^2 + g_{cji}^2}$$

$$\frac{\partial f_{RE}}{\partial t_{cri}} = \frac{(P_{re} - t_{cri})}{(P_{re} - t_{cri})^2 + t_{cji}^2}$$

$$\frac{\partial f_{RE}}{\partial y_{cri}} = \frac{(P_{re} - y_{cri})}{(P_{re} - y_{cri})^2 + y_{cji}^2}$$

$$\frac{\partial f_{RE}}{\partial g_{cji}} = \frac{g_{cji}}{(P_{re} - g_{cri})^2 + g_{cji}^2}$$

$$\frac{\partial f_{RE}}{\partial t_{cji}} = \frac{-t_{cji}}{(P_{re} - t_{cri})^2 + t_{cji}^2}$$

$$\frac{\partial f_{RE}}{\partial y_{cji}} = \frac{-y_{cji}}{(P_{re} - y_{cri})^2 + y_{cji}^2}$$

$$\frac{\partial f_{RE}}{\partial K_n} = -\frac{1}{K_n}$$

$$\frac{\partial f_{RE}}{\partial K_2} = -\frac{1}{K_2}$$

$$\frac{\partial f_{ce}}{\partial g_{ri}} = \frac{-(P_{cre} - g_{ri})}{(P_{cre} - g_{ri})^2 + P_{cje}^2}$$

$$\frac{\partial f_{ce}}{\partial t_{ri}} = \frac{P_{cre} - t_{ri}}{(P_{cre} - t_{ri})^2 + P_{cje}^2}$$

$$\frac{\partial f_{ce}}{\partial y_{ri}} = \frac{P_{cre} - y_{ri}}{(P_{cre} - y_{ri})^2 + P_{cje}^2}$$

$$\frac{\partial f_{ce}}{\partial g_{cri}} = \frac{-(P_{cre} - g_{cri})}{(P_{cre} - g_{cri})^2 + (P_{cje} - g_{cji})^2} - \frac{P_{cre} - g_{cri}}{(P_{cre} - g_{cri})^2 + (P_{cje} + g_{cji})^2}$$

$$\frac{\partial f_{ce}}{\partial g_{cji}} = \frac{-(P_{cje} - g_{cji})}{(P_{cre} - g_{cri})^2 + (P_{cje} - g_{cji})^2} + \frac{P_{cje} + g_{cji}}{(P_{cre} - g_{cri})^2 + (P_{cje} + g_{cji})^2}$$

$$\frac{\partial f_{ce}}{\partial t_{cri}} = \frac{P_{cre} - t_{cri}}{(P_{cre} - t_{cri})^2 + (P_{cje} - t_{cji})^2} + \frac{P_{cre} - t_{cri}}{(P_{cre} - t_{cri})^2 + (P_{cje} + t_{cji})^2}$$

$$\frac{\partial f_{ce}}{\partial t_{cji}} = \frac{P_{cje} - t_{cji}}{(P_{cre} - t_{cri})^2 + (P_{cje} - t_{cji})^2} - \frac{P_{cje} + t_{cji}}{(P_{cre} - t_{cri})^2 + (P_{cje} + t_{cji})^2}$$

$$\frac{\partial f_{ce}}{\partial y_{cri}} = \frac{P_{cre} - y_{cri}}{(P_{cre} - y_{cri})^2 + (P_{cje} - y_{cji})^2} + \frac{P_{cre} - y_{cri}}{(P_{cre} - y_{cri})^2 + (P_{cje} + y_{cji})^2}$$

$$\frac{\partial f_{ce}}{\partial y_{cji}} = \frac{P_{cje} - y_{cji}}{(P_{cre} - y_{cri})^2 + (P_{cje} - y_{cji})^2} - \frac{P_{cje} + y_{cji}}{(P_{cre} - y_{cri})^2 + (P_{cje} + y_{cji})^2}$$

$$\frac{\partial f_{ce}}{\partial K_n} = -\frac{1}{K_n}$$

$$\frac{\partial f_{ce}}{\partial K_z} = -\frac{1}{K_z}$$

$$\frac{\partial g_{ce}}{\partial g_{ri}} = \frac{-P_{cje}}{(P_{cre} - g_{ri})^2 + P_{cje}^2}$$

$$\frac{\partial g_{ce}}{\partial t_{ri}} = \frac{P_{cej}}{(P_{cre} - t_{ri})^2 + P_{cje}^2}$$

$$\frac{\partial g_{ce}}{\partial y_{ri}} = \frac{P_{cej}}{(P_{cre} - y_{ri})^2 + P_{cje}^2}$$

$$\frac{\partial g_{ce}}{\partial g_{cri}} = \frac{-(P_{cje} - g_{cji})}{(P_{cre} - g_{cri})^2 + (P_{cje} - g_{cji})^2} - \frac{P_{cje} + g_{cji}}{(P_{cre} - g_{cri})^2 + (P_{cje} + g_{cji})^2}$$

$$\frac{\partial g_{ce}}{\partial g_{cji}} = \frac{P_{cre} - g_{cri}}{(P_{cre} - g_{cri})^2 + (P_{cje} - g_{cji})^2} - \frac{P_{cre} - g_{cri}}{(P_{cre} - g_{cri})^2 + (P_{cje} + g_{cji})^2}$$

$$\frac{\partial g_{ce}}{\partial t_{cri}} = \frac{P_{cje} - t_{cji}}{(P_{cre} - t_{cri})^2 + (P_{cje} - t_{cji})^2} + \frac{P_{cje} + t_{cji}}{(P_{cre} - t_{cri})^2 + (P_{cje} + t_{cji})^2}$$

$$\frac{\partial g_{ce}}{\partial t_{cji}} = \frac{-(P_{cre} - t_{cri})}{(P_{cre} - t_{cri})^2 + (P_{cje} - t_{cji})^2} + \frac{P_{cre} - t_{cri}}{(P_{cre} - t_{cri})^2 + (P_{cje} + t_{cji})^2}$$

$$\frac{\partial g_{ce}}{\partial y_{cri}} = \frac{P_{cje} - y_{cji}}{(P_{cre} - y_{cri})^2 + (P_{cje} - y_{cji})} + \frac{P_{cje} + y_{cji}}{(P_{cre} - y_{cri})^2 + (P_{cje} + y_{cji})^2}$$

$$\frac{\partial g_{ce}}{\partial y_{cji}} = \frac{-(P_{cre} - y_{cri})}{(P_{cre} - y_{cri})^2 + (P_{cje} - y_{cji})^2} + \frac{P_{cre} - y_{cri}}{(P_{cre} - y_{cri})^2 + (P_{cje} + y_{cji})^2}$$

The partial derivatives not shown are zero.

VITA

The author was born on May 26, 1939, in Springfield, Missouri, and received his primary and secondary education there. He entered the University of Missouri School of Mines and Metallurgy in September, 1957, and received the degree of Bachelor of Science in Electrical Engineering in May, 1961. He has been enrolled in the Graduate School of the Missouri School of Mines and Metallurgy since September, 1961, and has held the position of Graduate Assistant for the period September, 1961, to June, 1962.

The author is a member of the Institute of Electrical and Electronics Engineers, the Missouri Society of Professional Engineers, Eta Kappa Nu, and Kappa Mu Epsilon.