Stable predictor-corrector methods for first order ordinary differential equations

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STABLE PREDICTOR-CORRECTOR METHODS FOR FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

Because of the wide variety of differential equations, there seems to be no numerical method which will affect the solution best for all problems. Predictor-corrector methods have been developed which utilize more ordinates in the predictor and corrector equations in the search for a better method.

These methods are compared for stability and convergence with the well known methods of Milne, Adams, and Hamming.

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CHAPTER I

INTRODUCTION

In most areas of numerical analysis the first step in the solution of a particular problem is to find one technique which can be used to obtain the solution to the problem. The numerical solution of ordinary differential equations is somewhat different. More often the first step is to choose that technique among the many available which will serve the purpose best. There seems to be no method which is "best" in all situations. For example, there is usually a different criterion on the error when a problem has a solution which tends toward a constant rather than an exponential type solution. This phenomenon will be explained in more detail in the next chapter.

A number of factors must be kept in mind when trying to choose the best method for a particular problem in the numerical solution of ordinary differential equations. They are:

1. The degree of accuracy required. The error in the final result depends both on the error incurred at each step of the integration and on how the error in earlier steps propogate into latter steps. The first type of error is due to truncation and round-off, while the later is determined by the stability properties of the particular method.

2. The effort required to find an estimation of the error at each step. Since the error in each step is a function of the integration step size, it is

essential to be able to estimate the error at each step to determine when to change the interval between steps. If the error is smaller than required, it is advised to increase the step size to avoid unnecessary waste of machine time. On the other hand the error may be larger than desired and the step size should be decreased.

3. The speed of computation. Since some equations may require a large amount of machine time even on the fastest computer, this must be an important factor to consider.

4. The ease with which a method can be adapted to machine use, or programmed. This depends on such considerations as the ease in which the method can be started and the difficulty involved in changing the interval between steps.[1]

Predictor-corrector methods for integrating ordinary differential equations, which are to be analyzed in this study, are widely used because of the following advantages:

1. One measure of the error being made at each step is provided by taking the difference between the predicted and corrected values. This provides a relatively simple means of controlling the step size employed in the integration.

2. The derivative needs to be computed only two or three times, compared with four or more for the various Runge-Kutta methods. This can save considerable computing time and effort on high order systems.

3. Various types of machine failures are easily caught.

Of course, there are also disadvantages or sources of trouble associated with predictor-corrector methods. The main disadvantages are:

1. Finite approximations for the derivatives cause a certain amount of truncation error.
2. Propagation errors may arise from solutions of the approximate difference equations which do not correspond to solutions of the differential equation.
3. Certain combinations of finite difference formula coefficients may cause amplification of roundoff errors.
4. The computation must be started by another method.(2)

The type of problem being considered in this study is the first order initial value problem which appears in the form

$$y' = f(x,y), \quad y(a) = n$$

where n is a constant. It will be assumed that the function $f(x,y)$ is defined for $x \in (a,b)$ and for all finite y.

Although textbooks on differential equations often give the impression that most differential equations can be solved in closed form, it should not be overlooked that even if the explicit solution exists it may be no easy task to find its numerical values. To some extent, this is true even of the trivial initial value problem $y' = y$, $y(0) = 1$, where, in order to find numerical values, one has either to calculate or to look up in a table and possibly interpolate values of e^x.

Another example is the solution $y(x) = e^{-x^2}\int_0^x e^{t^2}dt$ of the equation $y' = 1 - 2xy$; in order to determine values of $y(x)$ one has to calculate an integral that is not expressible in terms of elementary functions and is not adequately tabulated.

Predictor-corrector methods are probably used more universally than any other method. In the case of the initial value problem, the only requirement is the ability to calculate a good approximation to the value of $f(x,y)$ for a given x and y. Although, in order to keep the error sufficiently small, the function f may have to be evaluated a large number of times. There is no reason to be concerned over this fact today, since large numbers of exactly this type of repetitive calculations can be performed efficiently and reliably on automatic digital computers.

The stability properties of the most commonly used predictor-corrector equations for the numerical solution of ordinary differential equations are fairly well known today, due to the advent of modern computers. In this study formulas are developed with error terms of orders $(O)h^5$, $(O)h^6$, and $(O)h^7$, where the notation, (O), signifies a constant times the indicated power of the integration step length. The stability properties of these formulas are compared with the most commonly used formulas due to Milne, Hamming, and Adams.

CHAPTER II

REVIEW OF LITERATURE

Although there is a large amount of literature pertaining to the numerical solution of differential equations, this subject seems to have been largely neglected by modern mathematicians and numerical analysts. Recent publications, introducing new methods, have been contributed by Hamming(6), Milne-Reynolds(1), Crane-Klopfenstein(3), and others.

The difficulties of this topic and the need for more research is described by Fox(12):

> "There is no single numerical method which is applicable to every differential equation, or even to every ordinary differential equation, or which is 'best possible' for every member of even the much smaller class of ordinary linear equations. The field is very large, and for the most economic use of our computing machine, coupled with the necessity for producing accurate answers, we need a variety of methods, each appropriate to its particular and rather small class of problems."

The general form of the numerical integration formulas used in this study are

$$y_{n+1} = a_o y_n + a_1 y_{n-1} + \cdots + a_p y_{n-p}$$

$$+ h(b_{-1} y'_{n+1} + b_o y'_n + \cdots + b_p y'_{n-p}) + E_n \qquad (2.1)$$

where, in the predictor formulas, $b_{-1} = 0$. Some of the other coefficients may also be zero in the predictor or the corrector equation.

Also, in this study, the various stability analyses are performed under the assumption that the differential equation to be solved is of the form

$$y' = f(x,y) \quad . \tag{2.2}$$

In selecting a predictor-corrector algorithm, the stability of the particular method is one of the key factors to be considered. This is very important when the differential equations being solved correspond to a system with a forcing function whose time duration or period is relatively long compared to the transient time constants of the system[2]. Some effort has been directed toward the development of algorithms having improved stability characteristics[11,6].

Ralston-Wilf[10] define stability as follows: "A numerical integration procedure is said to be stable if, when $f_y = \partial f(x,y)/\partial y < 0$, the error, measured by the difference between the true solution and the numerical solution, decreases in magnitude on the average with increasing n (i.e., as the integration proceeds step by step)."

In the case when $f_y > 0$, the solution itself and usually the error are increasing exponentially. It is then desired to use the term "relatively stable", which implies that the rate of change of the error is less than the rate of change of the solution with respect to the number of integration steps[10]. In other words, if the integration step length is held fixed, the asymptotic characteristics of the finite-difference solution are the same as that of the true solution of the differential equation, after a large number of steps.[12]

Although some knowledge of the local truncation error is valuable for various purposes, it is insufficient for the analysis of the extended integration process. Since the value of one integration step forward requires a knowledge of previously calculated values, all future values will contain the effects of truncation and rounding errors of previous steps. It is of utmost importance to know how these errors are propagated into the values calculated in later steps. They may have the undesirable tendency to accumulate rapidly, in which case, will give rise to an unstable integration process.[12]

When an iterative formula is used to approximate the solution of an ordinary differential equation, it is important to determine conditions under which this iteration converges and to investigate the rate of convergence. By convergence it is meant that, as the interval of integration approaches zero, the numerical or finite-difference solution tends to the true solution of the differential equation at each particular point x.[12]

When an iterative predictor-corrector process is used to find the numerical solution of an ordinary differential equation, Hildebrand[8] shows by induction the conditions to be met for the corrector equation to converge to a solution. The error in the i^{th} iteration of the corrector equation tends to zero, as i increases, if h is sufficiently small to insure that

$$h < \frac{1}{|b_{-1}| \left| \frac{\partial f(x,y)}{\partial y} \right|} \qquad (2.3)$$

where h is the interval of integration and b_{-1} is the coefficient of y'_{n+1} in the corrector equation.

The ratio of the magnitude of errors between successive iterates, or the rate of convergence, is approximated by the absolute value of the convergence factor, Θ, such that

$$\Theta_n = hb_{-1} \frac{\partial f(x_n, y_n)}{\partial y} \; . \qquad (2.4)$$

Because of the way stability has been defined, it is important to know when a method is both stable and relatively stable. A method with these desirable qualities has a more general application appeal.[10]

There are two distinct modes of application for predictor-corrector algorithms. In the first mode, an estimate of the next value of the dependent variable is obtained by using the predictor formula once and the corrector formula is applied in an iterative fashion until convergence is obtained. The number of derivative evaluations may in fact exceed the number required by a Runge-Kutta algorithm since the number of evaluations exceeds the number of corrector iterations by one. However in this case, the stability properties of the algorithm are completely determined by the corrector formula and the predictor formula only determines the number of iterations required. With an accurate predictor, usually only one or two iterations are required for convergence.[2]

In the second commonly used mode, one application of the predictor and corrector equation is used and the values of the dependent variables are accepted as the final values. The predicted and corrected values are compared to obtain an estimation of the truncation error associated with the integration step. If the estimated error does not exceed a specified maximum value, the corrected values are accepted. Starting from the last accepted point, the interval of integration may be reduced or increased depending on whether the estimation of the error is too large or too small relative to the specified limiting value. In this mode of application, only two derivative evaluations are required per integration step. However, the stability properties of the predictor-corrector algorithm depend not only upon the corrector equation, but also the predictor equation.

The following terminology is used to avoid confusion in the description of the various predictor-corrector processes:[2]

1. An iterative method refers to the use of the predictor equation once and then applying the corrector equation in an iterative manner until convergence is obtained.
2. A predictor-corrector method refers to the use of the predictor and corrector equations only once.
3. The incorporation of the error estimates, as suggested in the paper by Hamming[6], with one application of

the predictor and corrector equations is referred to as a modified, or modified predictor-corrector method.

All of the methods developed in this study will be used and analyzed as iterative methods, in which the following steps are performed in obtaining the solution[14]:

1. Predict y_{n+1}, using a selected predictor equation.
2. Calculate y'_{n+1} by the differential equation $y' = f(x,y)$.
3. Correct Y_{n+1} by the selected corrector equation, using the value y'_{n+1} just obtained for Y'_{n+1}.
4. Correct Y'_{n+1} by the differential equation.
5. Repeat steps 3 and 4, if necessary, until no further change occurs.

The relative merits of the different methods of solution can be displayed by a comparison of the root loci of their various characteristic equations. They provide good information as to the behavior of the error in the solution when that method is used to solve an ordinary differential equation. This follows from the fact that the general solution of the difference equation for distinct roots different from unity is

$$\epsilon_n = K_1\rho_1^n + K_2\rho_2^n + \cdots + K_i\rho_i^n + K_{i+1} , \tag{2.5}$$

where i is the number of roots in the characteristic equation. The form of the solution is modified, however, if the roots are not distinct. For example, when $\rho_1 = \rho_2 = \rho$, then $K_1\rho_1^n + K_2\rho_2^n$ in the general solution is replaced by $(K_1 + K_2 n)\rho^n$. In the case of one or more roots being equal to positive one,

K_{i+1} may be a function of n rather than a constant. In any situation, the root loci of the characteristic equation is important in the determination of the behavior of the error. Where

$$H = \frac{\partial f}{\partial y} h \,, \tag{2.6}$$

the regions of positive and negative H have been studied separately, since the stability requirements to be met are different in each case. Where H is negative, and the true solution tends toward a constant, the conditions to be met for stability are that the magnitude of the roots be less than one. In the interval of positive H, the curve e^H gives more information as to stability. In this case the solution is increasing exponentially and relative accuracy is maintained if the error does not grow more rapidly than the solution.[2]

There are two distinct methods for deriving the formulas used in this study. The first technique is that used by Adams[8], Euler[8], Milne and Reynolds[1], and others. In particular, use is made of the relation

$$y_{n+1} = y_n + \int_{x_n}^{x_n+h} y'(x)dx \,. \tag{2.7}$$

The ordinates y_n, y_{n-1}, ... y_1, and y_o are known either from an appropriate starting method, or from the method itself. The corresponding $y'(x)$ values can be calculated from the formula

$$y'_k = y'(x_k) = f(x_k, y_k) \; . \tag{2.8}$$

The value of $y'(x)$ is approximated by the polynomial of degree N which takes on the calculated values at the N+1 points x_n, x_{n-1}, ..., and x_{n-N}, by using the Newton backward-difference formula

$$y'_{n+s} \approx y'_n + s\nabla y'_n + \frac{s(s+1)}{2!}\nabla^2 y'_n + \ldots$$

$$+ \frac{s(s+1)\ \ldots\ (s+N-1)}{N!}\nabla^n y'_n \tag{2.9}$$

where

$$s = \frac{x - x_n}{h} \; .$$

The integration indicated in (2.7) is affected by using this polynomial to extrapolate $y'(x)$ over the interval (x_n, x_n+h).

The result of this calculation is[10]

$$y_{n+1} = y_n + h\int_o^1 y'_{n+s}\,ds \approx y_n + h\sum_{k=o}^{n} a_k \nabla^k y'_n \; , \tag{2.10}$$

where

$$a_k = \int_o^1 \frac{s(s+1)\ \ldots\ (s+k-1)}{k!}\,ds \; . \tag{2.11}$$

Hildebrand[10] gives the error term in the form

$$E = h^{n+2}\int_o^1 \frac{s(s+1)\ \ldots\ (s+N)}{(N+1)!}\, y^{(N+2)}(\eta)\,ds \tag{2.12}$$

where

$$x_{n+1} > \eta > x_{n-N} .$$

For more general formulas, (2.9) is used in the relation

$$y_{n+1} = y_{n-p} + h \int_{-p}^{1} y'_{n+s} \, ds \ , \tag{2.13}$$

where p is any positive integer. The ordinate following the n^{th} is expressed in terms of the ordinate calculated p steps previously and the N+1 already calculated values of y'.[10]

From these relationships the predictor equations using this method are developed. The corrector equation, which makes use of the unknown slope y'_{n+1}, is obtained by replacing the right-hand member of (2.9) by the interpolation polynomial agreeing with y'(x) at x_{n+1}, x_n, ..., x_{n-N+1}. That is,

$$y'_{n+s} \approx y'_{n+1} + (x-1)\nabla y'_{n+1} + \frac{s(s-1)s}{2!} \nabla^2 y'_{n+1} + \ldots$$

$$+ \frac{(s-1)s(s+1) \ \ldots \ (s+N-2)}{n!} \tag{2.14}$$

where again s is defined as before[10]

$$s = \frac{x-x_n}{h} .$$

Due to the length of the difference equations derived by this method, they have been listed in Appendix II for the readers convenience.

The second method for deriving formulas for the numerical solution of differential equations is used almost exclusively by Hamming[6,7]. His approach has a more general philosophy than the previously discussed method. Instead of examining many special formulas, a whole class of formulas can be investigated at once.

The theory is illustrated by developing a corrector which has an error term of order $(O)h^5$. The equation used is of the form

$$y_{n+1} = a_o y_n + a_1 y_{n-1} + a_2 y_{n-2} + h(b_{-1} y'_{n+1} + b_o y'_n + b_1 y'_{n+1} + b_2 y'_{n-2}) + E_5 \frac{h^5 y^{(5)}(\theta)}{5!} \qquad (2.15)$$

where $x_{n-2} < \theta < x_{n+1}$. This formula uses three old values of the integral together with one new and three old values of the integrand. Formula (2.15) as written implies it is exact for polynomials through degree 4.

The word exact has the meaning that if $y(x)$ is a polynomial of degree 4 or less and the values on the right-hand side of (2.15) are true values of the solution then, except for roundoff, y_{n+1} will also be a true value.[13]

Taking (2.15) for example, there are seven coefficients. If they are determined so that (2.15) is exact for polynomials of degree 4 or less, then there will be two free parameters which may be used for one or more of the following purposes:[13]

1. Make the error term coefficient small.

2. Make the stability characteristics as desirable as possible.
3. Force the formula to have certain other desirable computational properties such as zero coefficients.

Formula (2.15) can be made exact through degree 4 by substituting $y_{n+1} = x_n+h$, $y_n = x$, $y_{n-1} = x-h$, $y_{n-2} = x-2h$, and $y_{n-3} = x-3h$, or by expanding the y's in a Taylor Series. By the former method the following expression results:

$$(x+h)^4 = a_o x^4 + a_1(x-h)^4 + a_2(x-2h)^4 + h\Big[4b_{-1}(x+h)^3 + 4b_o x^3 + 4b_1(x-h)^3 + 4b_2(x-2h^3)\Big] . \qquad (2.16)$$

Equating like powers of x and h results in the following equations:

$$\begin{aligned} 1 &= a_o + a_1 + a_2 \\ 4 &= -4a_1 - 8a_2 + 4b_{-1} + 4b_o + 4b_1 + 4b_2 \\ 6 &= 6a_1 + 24a_2 + 12b_{-1} - 12b_1 - 24b_2 \\ 4 &= -4a_1 - 32a_2 + 12b_{-1} + 12b_1 + 48b_2 \\ 1 &= a_1 + 16a_2 + 4b_{-1} - 4b_1 - 32b_2 \end{aligned} \qquad (2.17)$$

Using these five conditions and taking a_1 and a_2 as parameters results in the coefficients

$$a_o = 1-a_1-a_2 \qquad b_o = \frac{1}{24}(19 + 13a_1 + 8a_2)$$
$$a_1 = a_1 \qquad b_1 = \frac{1}{24}(-5 + 13a_1 + 32a_2) \qquad (2.18)$$
$$a_2 = a_2 \qquad b_2 = \frac{1}{24}(1 - a_1 + 8a_2)$$
$$b_{-1} = \frac{1}{24}(9 - a_1) \qquad E_5 = \frac{1}{6}(-19 + 11a_1 - 8a_2)$$

where E_5 is calculated by making equation (2.15) exact for degree 5.

Next we must examine the error term of such a polynomial approximation for formulas of the type

$$y_{n+1} = \sum_{i=o}^{p} a_i y_{n-i} + h \sum_{i=-1}^{p} b_i y'_{n-i} \; . \qquad (2.19)$$

If the numerical integration method is of this form, then the true solution must satisfy the equation

$$Y_{n+1} = \sum_{i=o}^{p} a_i Y_{n-i} + h \sum_{i=-1}^{p} b_i Y'_{n-i} + T_n \; , \qquad (2.20)$$

where T_n denotes the truncation error in the step x_n to x_{n+1}.

Since

$$T_n = Ch^{(r+1)} Y^{(r+1)}(\theta) \; , \qquad (2.21)$$

where r is the order of accuracy and C is a constant, is the general expression for the error of the previously discussed method, it is tempting to assume a similar error term in this case.

As is shown in the following derivation, there are certain conditions to be met for an error term of the form (2.21).[13]

To determine the expression for T_n, each Y_i and Y'_i in (2.20) is expanded in a Taylor Series about the point x_n. This results in

$$Y_{n-i} = Y_n - ihY'_n + \frac{i^2h^2}{2!} Y''_n + \cdots + \frac{(-1)^r i^r h^r}{r!} Y_n^{(r)} + \frac{1}{r!}\int_{x_n}^{x_{n-i}} (x_{n-i}-s)^r Y^{(r+1)}(s)ds \tag{2.22}$$

and

$$Y'_{n-i} = Y'_n - ihY''_n + \cdots + \frac{(-1)^{r-1} i^{r-1} h^{r-1}}{(r-1)!} Y_n^{(r)} + \frac{1}{(r-1)!}\int_{x_n}^{x_{n-i}} (x_{n-i}-s)^{r-1} Y^{(r+1)}(s)ds \tag{2.23}$$

where r is the order of accuracy of (2.20). Substituting these equations into (2.20) and remembering that it is exact when Y(x) is a polynomial of degree r or less, T_n is given as

$$T_n = \frac{1}{r!}\Big[\int_{x_n}^{x_{n+1}} (x_{n+1}-s)^r Y^{(r+1)}(s)ds - \sum_{i=o}^{p} a_i \int_{x_n}^{x_{n-i}} (x_{n-i}-s)^r Y^{(r+1)}(s)ds - rh \sum_{i=-1}^{p} b_i \int_{x_n}^{x_{n-i}} (x_{n-i}-s)^{r-1} Y^{(r+1)}(s)ds\Big] \tag{2.24}$$

which is rewritten as

$$T_n = \frac{1}{r!}\int_{x_{n-p}}^{x_{n+1}} \left\{ \overline{(x_{n+1}-s)}^r - rhb_{-1}\overline{(x_{n+1}-s)}^{r-1} \right.$$

$$\left. - \sum_{i=1}^{p}\left[a_i\overline{(x_{n-i}-s)}^r + rhb_i\overline{(x_{n-i}-s)}^{r-1}\right]\right\} Y^{(r+1)}(s)ds$$

$$= \frac{1}{r!}\int_{x_{n-p}}^{x_{n+1}} G(s)Y^{(r+1)}(s)ds \qquad (2.25)$$

where, to avoid trouble at the upper limit, let[13]

$$\overline{(x_{n-i}-s)} = \begin{cases} x_{n-i} - s & \begin{cases} x_{n-i} \le s \le x_n & i \ne -1 \\ x_n \le s & i = -1 \end{cases} \\ 0 & \text{otherwise} \end{cases} \qquad (2.26)$$

The function G(s) is called the <u>influence</u> <u>function</u>. Where G(s) is of constant sign in the interval (x_{n-p}, x_{n+1}), the second law of the mean is applied to get the error in the form

$$T_n = \frac{Y^{(r+1)}(\eta)}{r!}\int_{x_{n-p}}^{x_{n+1}} G(s)ds \qquad (2.27)$$

where $x_{n-p} < \eta < x_{n+1}$.[13] The error term may not be of this form if G(s) is not of constant sign.

In making the substitutions $y_{n+1} = x+h, \cdots, y_{n-3} = x-3h$[13], two things must be noted before writing the influence function for equation (2.15).

1. Since h cancels out in the final result, there is no loss of generality in setting h = 1.

2. We set $x_n = 0$ since the coefficients are independent of the origin of the coordinate.

Now, G(s) for equation (2.15) is found by using equation (2.25) and is also given by Hamming[7] as

$$G(s) = \frac{1}{4!}\Big\{(h-s)^4 - a_o(h-s)^4 - a_1(-h-s)^4 - a_2(-2h-s)^4 - 4h\left[b_{-1}(h-s)^3 + b_o(-s)^3 + b_1(-h-s)^3 + b_2(-2h-s)^3\right]\Big\} \quad (2.28)$$

where r = 4.

Since the coefficients (2.18) are dependent upon a_1 and a_2, the remaining problem is to find those values of a_1, a_2 such that G(s) has a constant sign. Since it is impossible to find the zeros of G(s) for each pair (a_1, a_2), we set G(s) = 0 and find the region of the a_1, a_2 plane where G(s) is of constant sign. This linear function in a_1, a_2 is graphed for each interval of s, namely $h \geq s \geq 0$, $0 \geq s \geq -h$, and $-h \geq s \geq -2h$ in this example. From this graph, a_1 and a_2 are chosen such that G(s) is of constant sign, giving the desired error term.

A great deal of later algebra can be saved at this point if the stability analysis is performed on the "generalized corrector",

$$y_{n+1} = a_o y_n + a_1 y_{n-1} + a_2 y_{n-2} + h(b_{-1}y'_{n+1} + b_o y'_n + b_1 y'_{n-1} + b_2 y'_{n-2}) + E_5 \frac{h^5 y^{(5)}(\theta)}{5!}, \quad (2.29)$$

where $x_{n-2} < \theta < x_{n+1}$.

Let z be the true solution of the differential equation, then z satisfies

$$\frac{dz}{dx} = z' = f(x,z) \ . \tag{2.30}$$

The numerically calculated solution y satisfies

$$y_n' = f(x_n, y_n) + E_1(\eta) \tag{2.31}$$

where $E_1(\eta)$ is the error in the n^{th} value and is assumed to be small. The true solution z will approximate relation (2.29) and it therefore follows that

$$z_{n+1} = a_o z_n + a_1 z_{n-1} + a_2 z_{n-2} + h(b_{-1} z_{n+1}' + b_o z_n'$$

$$+ b_1 z_{n-1}' + b_2 z_{n-2}') + E_2(\eta) \tag{2.32}$$

where $E_2(\eta)$ is the truncation error at the point x_{n+1}.

Let the error between the true solution and the approximate solution be defined as

$$\epsilon_n = z_n - y_n \ . \tag{2.33}$$

Subtracting the two corrector equations (2.32) and (2.29) results in the relation

$$\epsilon_{n+1} = a_o \epsilon_n + a_1 \epsilon_{n-1} + a_2 \epsilon_{n-2} + h(b_{-1} \epsilon_{n+1}' + b_o \epsilon_n'$$

$$+ b_1 \epsilon_{n-1}' + b_2 \epsilon_{n-2}') + E_2(\eta) \ . \tag{2.34}$$

Assume that:

1. $f(x_n, y)$ is a continuous function of y for y contained in the closed interval whose end points are z_n and y_n.
2. $\partial f(x_n, y)/\partial y$ exists for y in the open interval whose end points are z_n and y_n.[11]

By applying the mean value theorem, there exists a θ contained in the open interval whose end points are z_n and y_n, for which

$$\epsilon'_n = z'_n - y'_n = f(x_n, z_n) - f(x_n - y_n) \left. \frac{\partial f(x,y)}{\partial y} \right|_{(x_n, \theta)}$$

$$= \epsilon_n \left. \frac{\partial f(x,y)}{\partial y} \right|_{(x_n, \theta)} \qquad (2.35)$$

The fact that $E_1(\eta)$, $E_2(\eta)$, and $\partial f/\partial y$ change slowly in practice, makes it reasonable to assume they are constants. Also, to simplify the equations, let

$$\frac{\partial f}{\partial y} = K, \text{ and } E_i(\eta) = E \ . \qquad (2.36)$$

It follows from equation (2.34) that

$$\epsilon_{n+1} = a_o \epsilon_n + a_1 \epsilon_{n-1} + a_2 \epsilon_{n-2}$$

$$+ H(b_{-1}\epsilon_{n+1} + b_o \epsilon_n + b_1 \epsilon_{n-1} + b_2 \epsilon_{n-2}) + E \qquad (2.37)$$

where $H = Kh$. This linear difference equation with constant coefficients may be solved by setting $\epsilon_n = C\lambda^n$, resulting in the following characteristic equation

$$C\lambda^{n+1} = a_o C\lambda^n + a_1 C\lambda^{n-1} + a_2 C\lambda^{n-2}$$

$$+ CH(b_{-1}\lambda^{n+1} + b_o\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2}) \qquad (2.38)$$

which may be rewritten as

$$(Hb_{-1} - 1)\lambda^3 + (Hb_o + a_o)\lambda^2 + (Hb_1 + a_1)\lambda + Hb_2 + a_2 = 0. \qquad (2.39)$$

For any given H, there are three roots.

Most of the assumptions made during this analysis will be carried over in subsequent analyses.

CHAPTER III

DISCUSSION OF WIDELY USED METHODS

This chapter presents a discussion of the stability properties of several well known predictor-corrector methods by Milne, Adams, and Hamming. It is felt that a discussion of these methods should be presented before the methods developed in this study are analyzed. Although this study is chiefly concerned with the use of predictor-corrector methods in an iterative fashion, Hamming's modified version of this technique will also be analyzed[6]. This method, which incorporates error estimates in the final result and uses the predictor and corrector equations only once, is of interest because it is generally regarded as one of the better methods. A detailed analysis of these and other methods is given in a paper by P. E. Chase[2].

Milne's Method

In spite of its well known stability problems, Milne's method is the classic predictor-corrector method for the numerical solution of ordinary differential equations. Since this is an iterative method, the stability properties are completely determined by the corrector equation and the predictor equation only influences the number of iterations required. The predictor and corrector equations for Milne's method are respectively:

$$p_{n+1} = y_{n-3} + \frac{4h}{3}\left(2y'_n - y'_{n-1} + 2y'_{n-2}\right) \qquad (3.1)$$

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(p'_{n+1} + 4y'_n + y'_{n-1}\right) \qquad (3.2)$$

where

$$p'_n = f(x_n, p_n)$$

$$y'_n = f(x_n, y_n) \; .$$

The characteristic equation for Milne's method as given by Hamming[6] and others is

$$(H - 3)\lambda^2 + 4H\lambda + H + 3 = 0. \qquad (3.3)$$

Equation (3.3) was solved for $-2 \leq H \leq 2$ to obtain twenty-one values for λ_1 and λ_2, using the Quadratic Formula. Figure 1 shows the magnitude of the roots as a function of H, where (◇) corresponds to a negative real root, (⊡) to a positive real root, and (⊡) signifies the magnitude of a complex root. The graphs of the dominant roots for each method are located in the appendix.

Adam's Method

Adam's method, also known as the modified Adam's or Moulton method, is characterized by the equations

$$p_{n+1} = y_n + \frac{h}{24}\,(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}) \qquad (3.4)$$

$$y_{n+1} = y_n + \frac{h}{24}\,(9p'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}) \qquad (3.5)$$

where (3.4) is used to predict and (3.5) to correct. To find the characteristic equation for Adam's corrector we use the results of equation (2.39) in Chapter II where

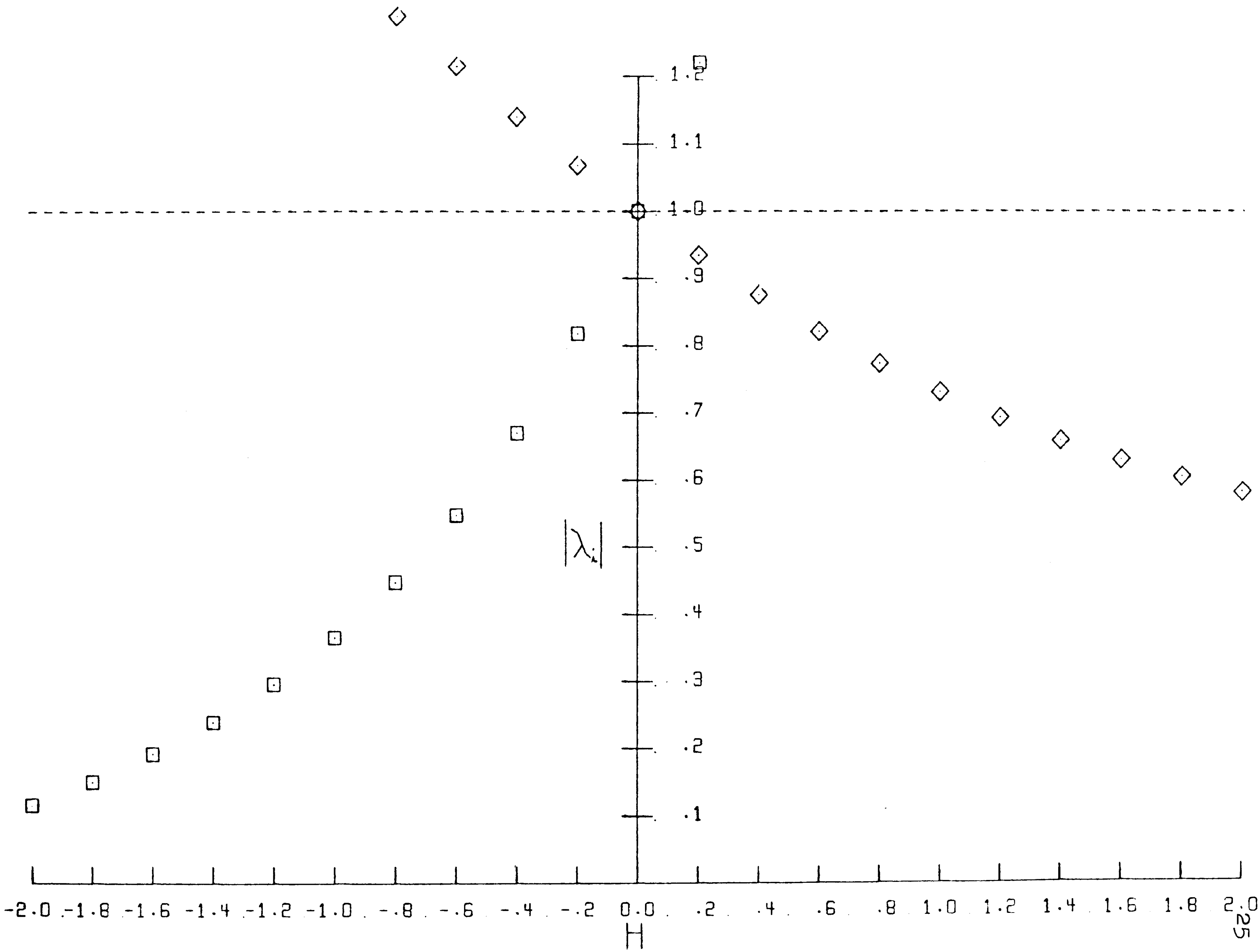

FIGURE 1. ROOT LOCI FOR MILNES CHARACTERISTIC EQUATION.

$$a_o = 1 \qquad b_{-1} = \frac{9}{24}$$

$$a_1 = 0 \qquad b_o = \frac{19}{24} \tag{3.6}$$

$$a_2 = 0 \qquad b_1 = \frac{-5}{24}$$

$$b_2 = \frac{1}{24}$$

which results in the characteristic equation

$$(\frac{9}{24} H - 1)\lambda^3 + (\frac{19}{24} H + 1)\lambda^2 - \frac{5}{24} H\lambda + \frac{1}{24} H = 0, \tag{3.7}$$

for Adam's method. The Fortran II subroutine ROTPOL, which uses the Bairstow method along with a Newton-Raphson iteration for greater accuracy, was used to find the roots of this characteristic equation in the interval $-2 \leq H \leq 2$ with $\Delta H = 0.2$. These roots are shown in Figure 2. The symbols showing the type and magnitude of each root as described earlier are used throughout this study.

Hamming's Method

The same type of analysis used in determining the characteristic equation associated with Adam's iterated method can be applied to Hamming's iterative method. The predictor and corrector for Hamming's iterative method are

$$p_{n+1} = y_{n-3} + \frac{4h}{3} (2y'_n - y'_{n-1} + 2y'_{n-2}) \tag{3.8}$$

$$y_{n+1} = \frac{1}{8} \left[9y_n - y_{n-2} + 3h (p'_{n+1} + 2y'_n - y'_{n-1}) \right] . \tag{3.9}$$

Again we may use the stability analysis performed in Chapter II, substituting the coefficients into equation (2.39).

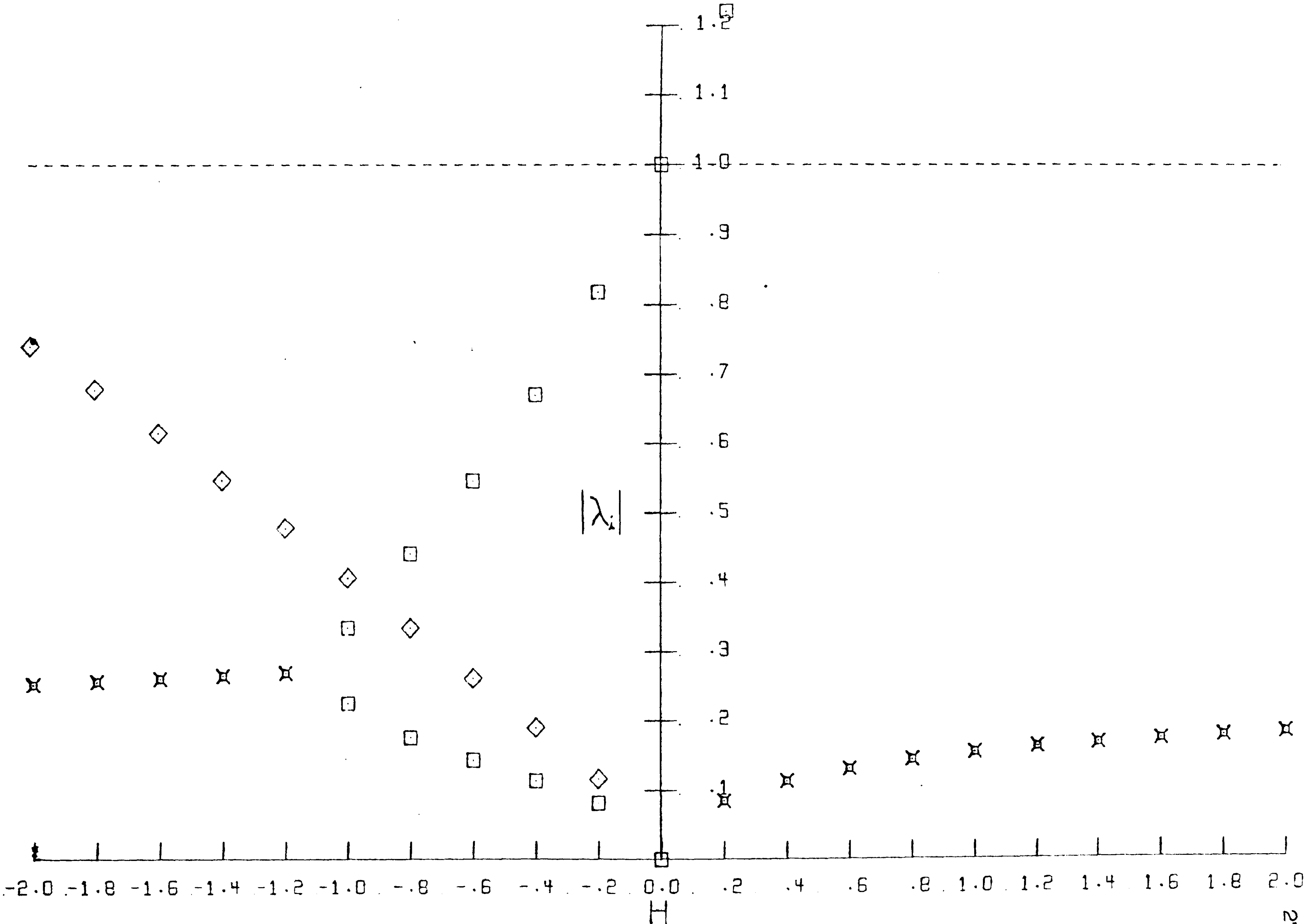

FIGURE 2. ROOT LOCI FOR ADAMS CHARACTERISTIC EQUATION.

They are:

$$a_o = \frac{9}{8} \qquad b_{-1} = \frac{3}{8}$$

$$a_1 = 0 \qquad b_o = \frac{3}{4} \tag{3.10}$$

$$a_2 = -\frac{1}{8} \qquad b_1 = -\frac{3}{8}$$

$$b_2 = 0 .$$

Performing this operation results in the characteristic equation

$$\left(\frac{3H}{8} - 1\right)\lambda^3 + \left(\frac{3H}{4} + \frac{9}{8}\right)\lambda^2 - \frac{3H}{8}\lambda - \frac{1}{8} = 0 \tag{3.11}$$

and again there are three roots for each H in the interval (-2,2). The roots of this third degree polynomial are plotted in Figure 3.

The modified method suggested by Hamming[6] is specified by the following relations:

$$\begin{aligned}
p_{n+1} &= y_{n-3} + \frac{4h}{3}\,(2y'_n - y'_{n-1} + 2y'_{n-2}) \\
m_{n+1} &= p_{n+1} - \frac{112}{121}\,(p_n - c_n) \\
m'_{n+1} &= f(x_{n+1}, m_{n+1}) \\
c_{n+1} &= \frac{1}{8}\left[9y_n - y_{n-2} + 3h\,(m'_{n+1} + 2y'_n - y'_{n-1})\right] \\
y_{n+1} &= c_{n+1} + \frac{9}{121}\,(p_{n+1} - c_{n+1})
\end{aligned} \tag{3.12}$$

Using the definitions

$$\nu_n = w_n - p_n, \quad \rho_n = w_n - m_n, \quad \sigma_n = w_n - c_n, \quad \epsilon_n = w_n - y_n, \tag{3.13}$$

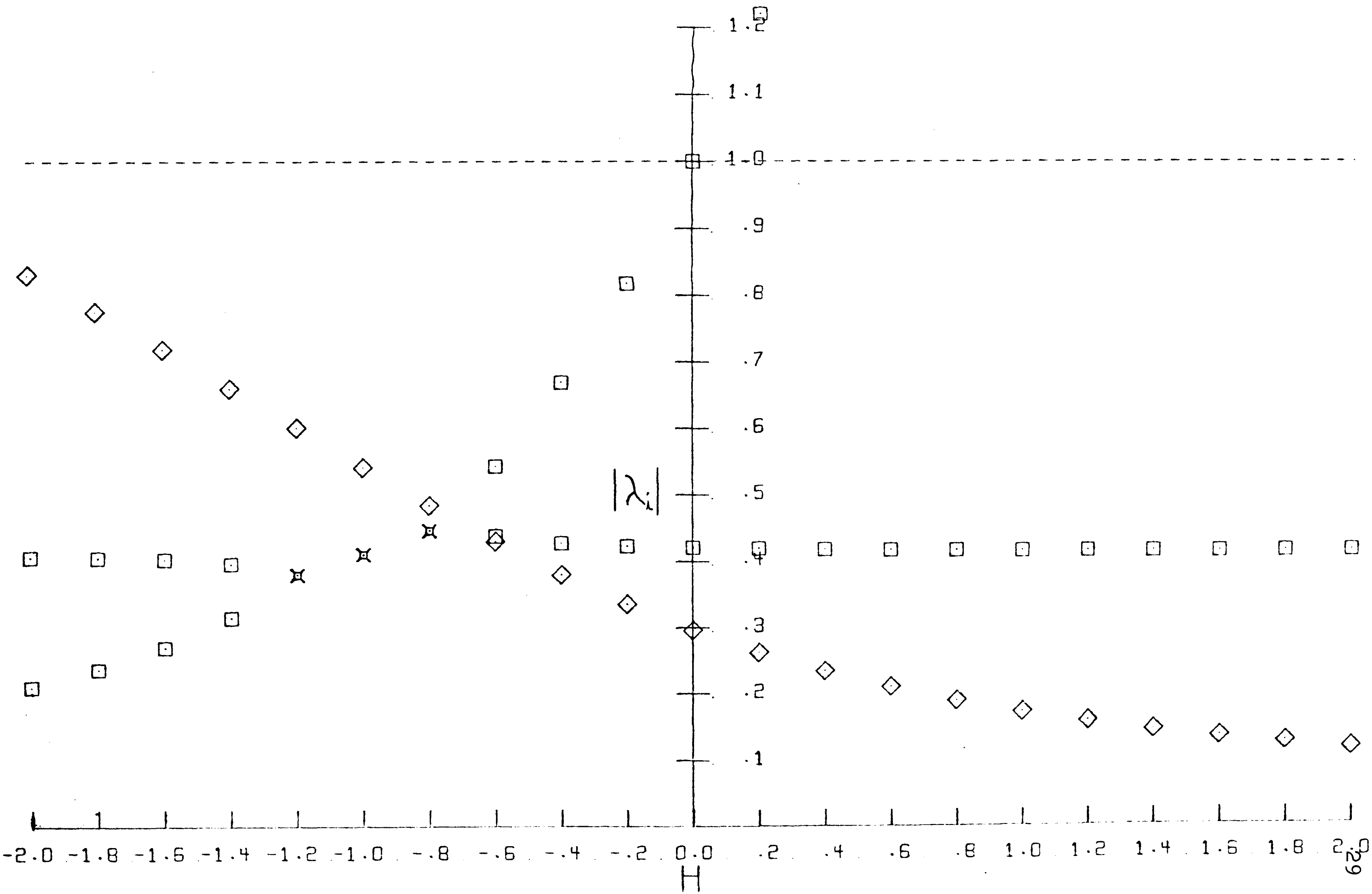

FIGURE 3. ROOT LOCI FOR THE ITERATED HAMMING CHARACTERISTIC EQUATION.

where w_n is the true solution, and following a similar procedure as in the analysis presented in Chapter II, the following difference equations are obtained[2]:

$$\nu_{n+1} = \epsilon_{n-3} + \frac{4H}{3} (2\epsilon_n - \epsilon_{n-1} + 2\epsilon_{n-2})$$

$$\rho_{n+1} = \nu_{n+1} - \frac{112}{121} (\nu_n - \sigma_n)$$

$$\sigma_{n+1} = \frac{9}{8} \epsilon_n - \frac{1}{8} \epsilon_{n-2} + \frac{3H}{8} (\rho_{n+1} + 2\epsilon_n - \epsilon_{n-1}) \qquad (3.14)$$

$$\epsilon_{n+1} = \sigma_{n+1} + \frac{9}{121} (\nu_{n+1} - \sigma_{n+1}) \quad .$$

The solution to this set of simultaneous difference equations is found by assuming

$$\nu_n = A\lambda^n, \quad \rho_n = B\lambda^n, \quad \sigma_n = C\lambda^n, \quad \epsilon_n = D\lambda^n \quad . \qquad (3.15)$$

Substituting these relations into the difference equations (3.14) results in a system of four simultaneous linear homogeneous equations in the constants A, B, C, and D. In order to obtain a nonzero solution, it is necessary and sufficient that the determinant of the coefficient matrix vanish. The characteristic equation, as given by Chase[2], for Hamming's modified predictor-corrector method is

$$121\lambda^5 + (-126 - 150H - 112H^2)\lambda^4 + (54H + 168H^2)\lambda^3$$

$$+ (14 - 24H - 168H^2)\lambda^2 + (-9 - 42H + 112H^2)\lambda + 42H = 0 . \qquad (3.16)$$

The magnitude of its roots as a function of H are shown in Figure 4.

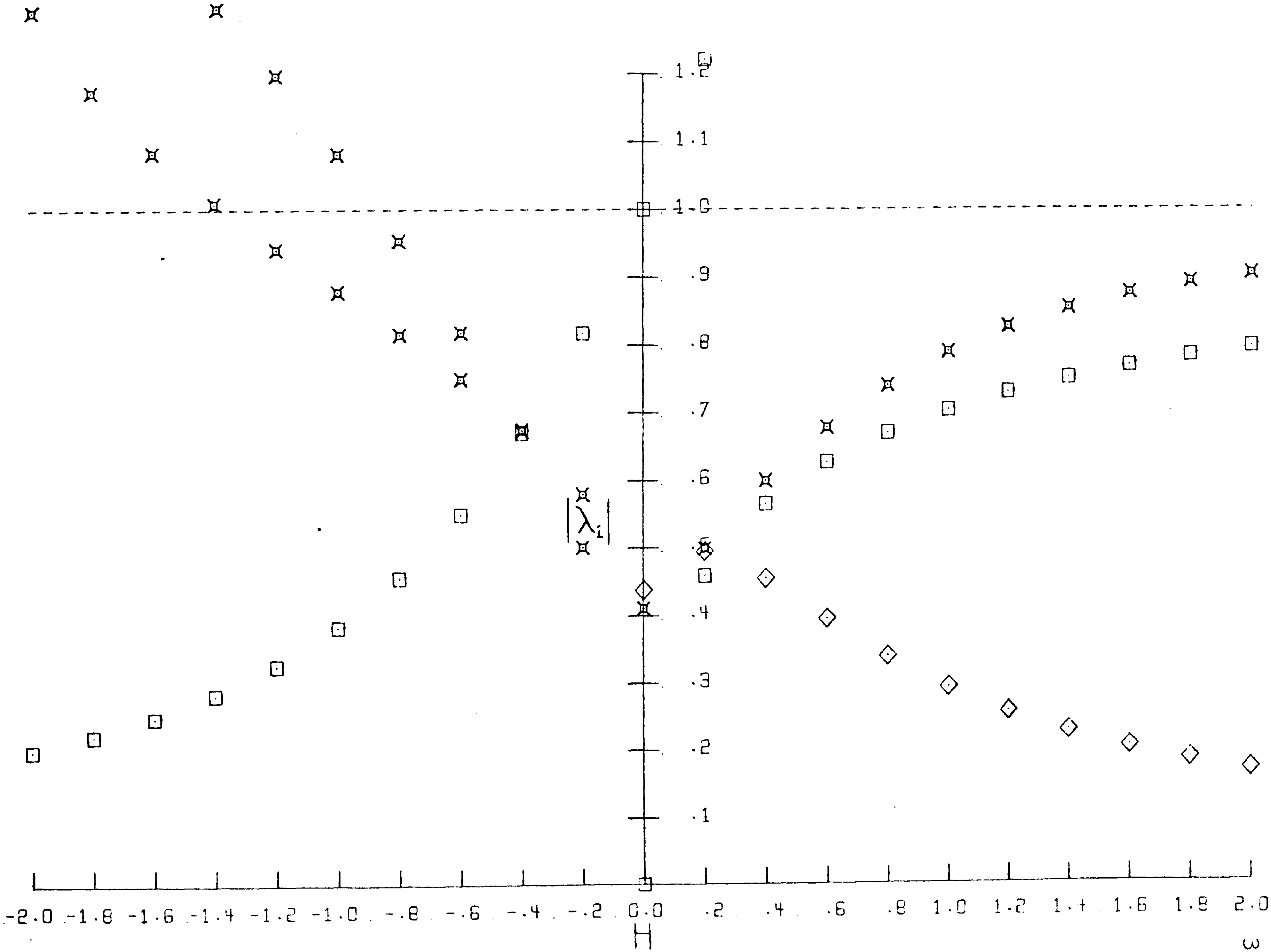

FIGURE 4. ROOT LOCI FOR THE MODIFIED HAMMING CHARACTERISTIC EQUATION.

CHAPTER IV

FIFTH-ORDER METHODS

This chapter presents a discussion of the commonly called four-point methods for the numerical solution of ordinary differential equations. All error terms for these four-point formulas are written in terms of the fifth power of the integration step length having the general form

$$\text{ERROR} = Ch^5 y^{(5)}(\theta)$$

where C denotes a constant and θ is somewhere between x_{n+1} and the smallest x value used in the particular formula.

The first stability analysis in this chapter is performed on the correctors derived from the difference equations in Appendix II. Differences higher than the third are truncated to obtain four-point formulas. The second analysis is performed on the corrector derived by making the formula exact for polynomials through degree four. The coefficients of this fifth order corrector are derived by Hamming[7] and are written as functions of two parameters a_1 and a_2. In choosing the values to assign these parameters, a compromise is made between excellent stability properties and a very small error term.

Writing the difference equations in terms of ordinates, results in the following predictor-corrector equations:

<u>Predictors</u>

$$y_{n+1} = y_n + \frac{h}{24}(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}) + \frac{251}{720}h^5y^{(5)}(\theta) \tag{4.1}$$

$$y_{n+1} = y_{n-1} + \frac{h}{3}(8y'_n - 5y'_{n-1} + 4y'_{n-2} - y'_{n-3}) + \frac{29}{90}h^5y^{(5)}(\theta) \tag{4.2}$$

$$y_{n+1} = y_{n-2} + \frac{3h}{8}(7y'_n - 3y'_{n-1} + 56'_{n-2} - y'_{n-3}) + \frac{27}{80}h^5y^{(5)}(\theta) \tag{4.3}$$

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_n - y'_{n-1} + 2y'_{n-2}) + \frac{14}{45}h^5y^{(5)}(\theta) \tag{4.4}$$

<u>Correctors</u>

$$y_{n+1} = y_n + \frac{h}{24}(9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}) - \frac{19}{720}h^5y^{(5)}(\theta) \tag{4.5}$$

$$y_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n+1} + 4y'_n + y'_{n-1}) - \frac{1}{90}h^5y^{(5)}(\theta) \tag{4.6}$$

$$y_{n+1} = y_{n-2} + \frac{3h}{8}(y'_{n+1} + 3y'_n + 3y'_{n-1} + y'_{n-2})$$

$$- \frac{3}{80}h^5 y^{(5)}(\theta) \ . \qquad (4.7)$$

There is no fourth corrector since the coefficient of y'_{n+1} is zero. Also, since corrector (4.5) is used in Adam's method and corrector (4.6) is used by Milne, only corrector (4.7) was not analyzed in Chapter III.

The stability analysis performed on the generalized corrector in Chapter II can again be utilized to find the characteristic equation of corrector (4.7). Substituting the coefficients of corrector (4.7) into equation (2.39) results in the equation

$$(\frac{3}{8}H - 1)\lambda^3 + \frac{9}{8}H\lambda^2 + \frac{9}{8}H\lambda + (\frac{3}{8}H + 1) = 0 \ . \qquad (4.8)$$

In Figure 5, the magnitude of the roots of this characteristic equation are shown as a function of H. The figure shows that corrector (4.7) has stability properties very similar to Milne's corrector, and is therefore unstable.

The generalized corrector of Chapter II is rewritten for the readers convenience.

$$y_{n+1} = a_o y_n + a_1 y_{n-1} + a_2 y_{n-2}$$

$$+ h(b_{-1}y'_{n+1} + b_o y'_n + b_1 y'_{n-1} + b_2 y'_{n-2}) + \frac{E_5 h^5 y^5(\theta)}{5!} \ . \qquad (4.9)$$

Making this corrector exact for polynomials through degree four, results in the coefficients given by Hamming[7]:

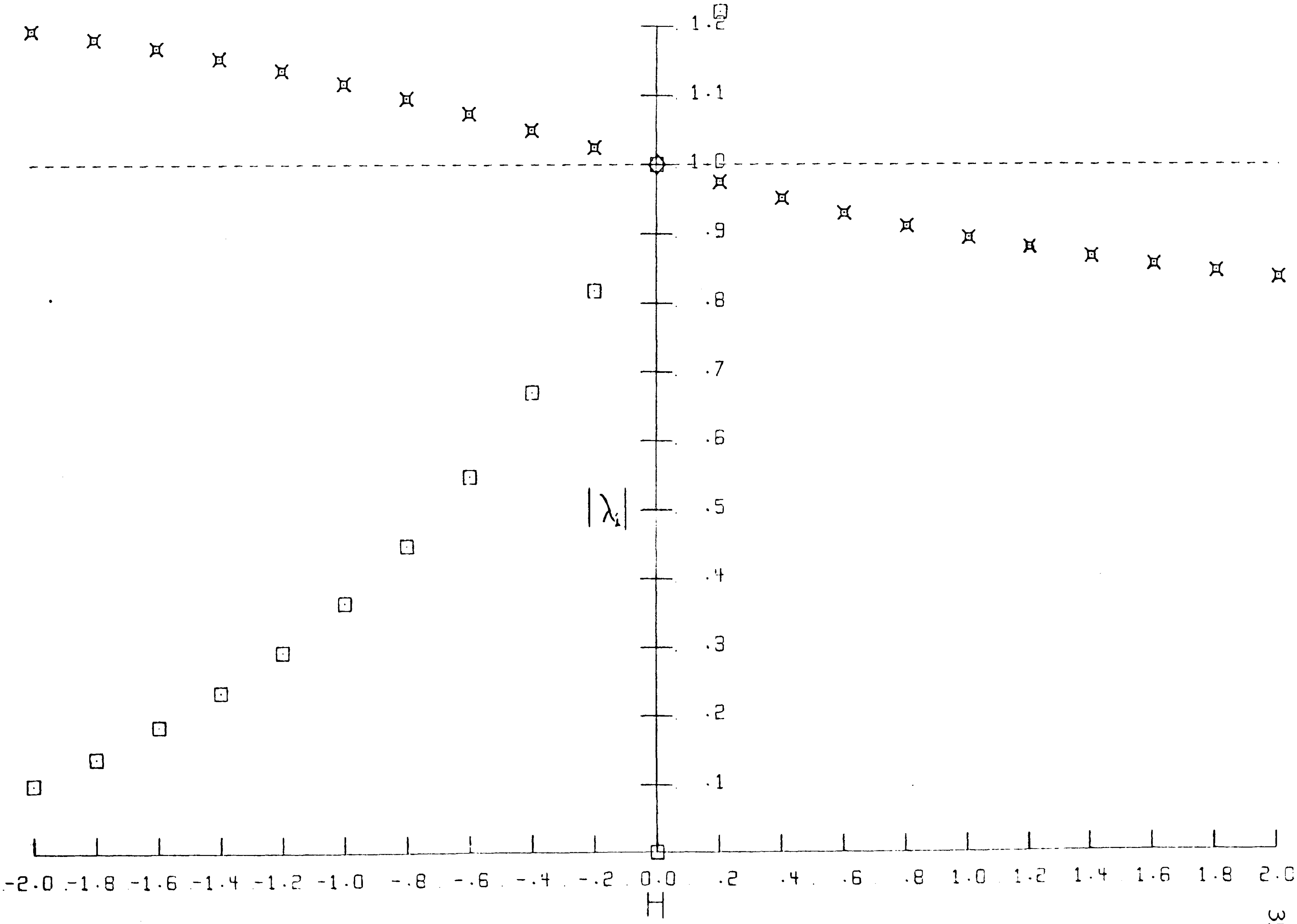

FIGURE 5. ROOT LOCI FOR THE CHARACTERISTIC EQUATION OF CORRECTOR (4.7).

$$a_o = 1 - a_1 - a_2 \qquad b_o = \frac{1}{24}(19 + 13a_1 + 8a_2)$$

$$a_1 = a_1 \qquad b_1 = \frac{1}{24}(-5 + 13a_1 + 32a_2)$$

$$a_2 = a_2 \qquad b_2 = \frac{1}{24}(1 - a_1 + 8a_2) \qquad (4.10)$$

$$b_{-1} = \frac{1}{24}(9 - a_1) \qquad E_5 = \frac{1}{6}(-19 + 11a_1 - 8a_2).$$

A suitable predictor to go with this corrector is also given by Hamming[7].

$$y_{n+1} = A_o y_n + A_1 y_{n-1} + A_2 y_{n-2} + h(B_o y'_n + B_1 y'_{n-1} + B_2 y'_{n-2}) + \frac{E_5 h^5 y^{(5)}(\theta)}{5!} \qquad (4.11)$$

where

$$A_o = -8 - A_2 \qquad B_o = \frac{17 + A_2}{3}$$

$$A_1 = 9 \qquad B_1 = \frac{14 + 4A_2}{3} \qquad (4.12)$$

$$A_2 = A_2 \qquad B_2 = \frac{-1 + A_2}{3}$$

$$E_5 = \frac{40 - 4A_2}{3}.$$

The coefficients for the predictor are derived by the same method as for the coefficients of the corrector equation. No attempt is made in this study to find the best predictor to go with any of the particular correctors developed in this paper.

The values chosen for the parameters a_1 and a_2 in equations (4.10) are $a_1 = .7$ and $a_2 = .53$. Other selections give slightly better accuracy but for stability H must be less than 0.2. The values of the coefficients for this equation are:

$$
\begin{aligned}
a_o &= -\frac{23}{100} & b_o &= \frac{539}{400} \\
a_1 &= \frac{7}{10} & b_1 &= \frac{351}{400} \\
a_2 &= \frac{53}{100} & b_2 &= \frac{227}{1200} \\
b_{-1} &= \frac{83}{240} & E_5 &= -\frac{259}{100} \quad .
\end{aligned}
\tag{4.13}
$$

The characteristic equation for this method is found by substituting these coefficients into equation (2.39), and is given by equation (4.14).

$$
\left(\frac{83}{240} H - 1\right)\lambda^3 + \left(\frac{539}{400} H - \frac{23}{100}\right)\lambda^2 + \left(\frac{351}{400} H + \frac{7}{10}\right)\lambda + \frac{217}{1200} H + \frac{53}{100} = 0 \quad . \tag{4.14}
$$

The roots of this equation are shown in Figure 6 as a function of H.

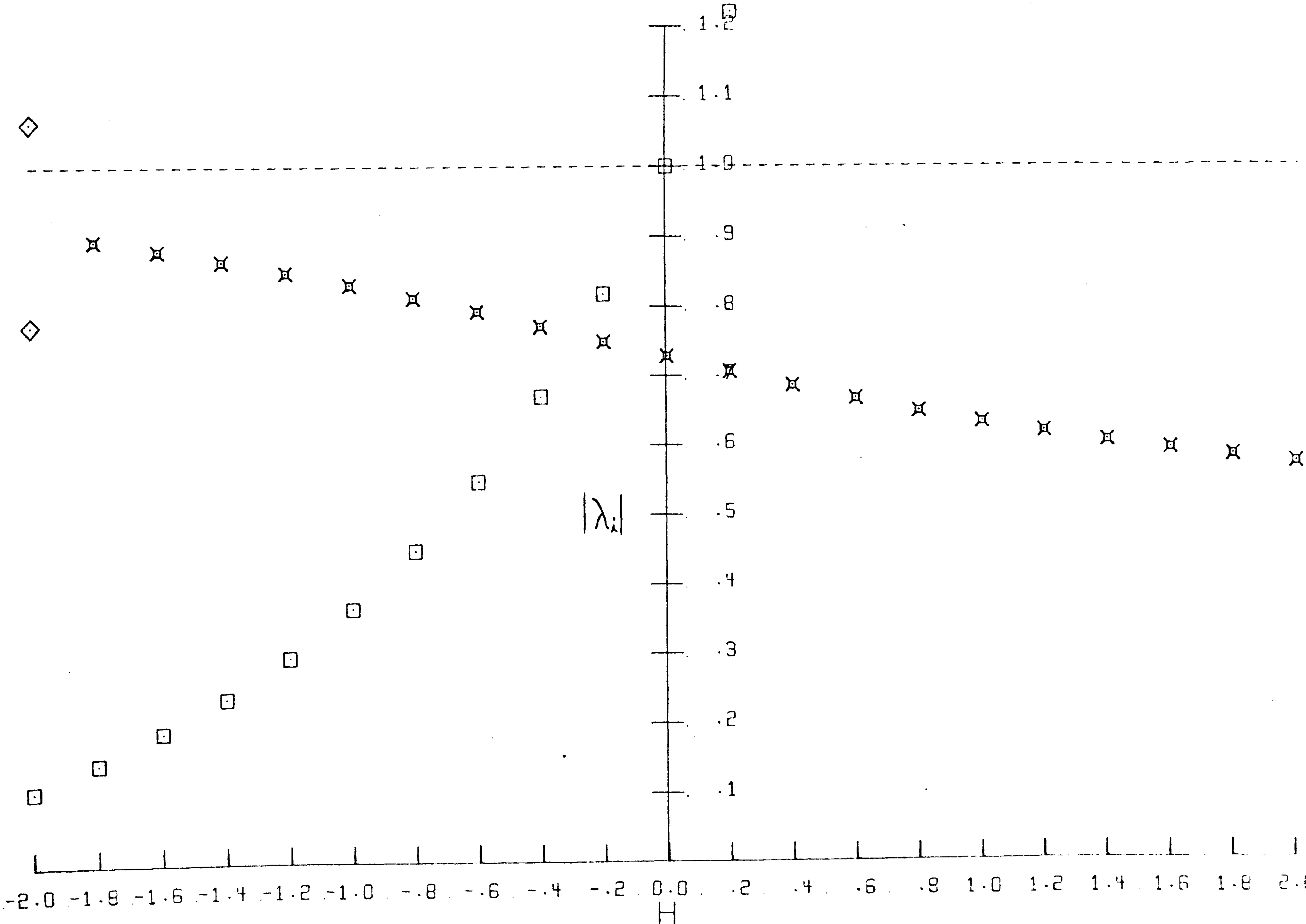

FIGURE 6. ROOT LOCI OF CHARACTERISTIC EQUATION (4.14).

CHAPTER V

SIXTH-ORDER METHODS

The sixth-order methods, or the commonly called five-point formulas, have a generalized corrector of the form

$$y_{n+1} = a_o y_n + a_1 y_{n-1} + a_2 y_{n-2}$$

$$+ h(b_{-1} y'_{n+1} + b_o y'_n + b_1 y'_{n-1} + b_2 y'_{n-2} + b_3 y'_{n-3})$$

$$+ \frac{E_6 h^6 y^{(6)}(\theta)}{6!} \quad . \qquad (5.1)$$

All of the predictor and corrector equations developed in this chapter have error terms of the same general form as equation (5.1).

Theoretically, the methods developed in this chapter should be more accurate than the four-point methods of Chapter IV. However, roundoff error becomes more of a problem as more points are utilized in the corrector equation. It also becomes increasingly more difficult to obtain stable formulas, since the characteristic polynomial is of a higher degree.

The following sixth-order predictors and correctors are derived from the difference equations in Appendix II:

<u>Predictors</u>

$$y_{n+1} = y_n + \frac{h}{720}(1901 y'_n - 2774 y'_{n-1} + 2616 y'_{n-2} - 1274 y'_{n-3}$$

$$+ 251 y'_{n-4}) + \frac{95}{288} h^6 y^{(6)}(\theta) \qquad (5.2)$$

$$y_{n+1} = y_{n-1} + \frac{h}{90}(269y'_n - 266y'_{n-1} + 294y'_{n-2} - 146y'_{n-3} + 29y'_{n-4}) + \frac{14}{45}h^6y^{(6)}(\theta) \quad (5.3)$$

$$y_{n+1} = y_{n-2} + \frac{3h}{80}(79y'_n - 66y'_{n-1} + 104y'_{n-2} - 46y'_{n-3} + 9y'_{n-4}) + \frac{51}{160}h^6y^{(6)}(\theta) \quad (5.4)$$

$$y_{n+1} = y_{n-3} + \frac{2h}{45}(67y'_n - 58y'_{n-1} + 102y'_{n-2} - 28y'_{n-3} + 7y'_{n-4}) + \frac{14}{45}h^6y^{(6)}(\theta) \quad (5.5)$$

<u>Correctors</u>

$$y_{n+1} = y_n + \frac{h}{720}(251y'_{n+1} + 646y'_n - 264y'_{n-1} + 106y'_{n-2} - 19y'_{n-3}) - \frac{3}{160}h^6y^{(6)}(\theta) \quad (5.6)$$

$$y_{n+1} = y_{n-1} + \frac{h}{90}(29y'_{n+1} + 124y'_n + 24y'_{n-1} + 4y'_{n-2} - y'_{n-3}) - \frac{1}{90}h^6y^{(6)}(\theta) \quad (5.7)$$

$$y_{n+1} = y_{n-2} + \frac{3h}{80}(9y'_{n+1} + 34y'_n + 24y'_{n-1} + 14y'_{n-2} - y'_{n-3}) - \frac{3}{160}h^6y^{(6)}(\theta) \quad (5.8)$$

$$y_{n+1} = y_{n-3} + \frac{2h}{45}(7y'_{n+1} + 32y'_n + 12y'_{n-1} + 32y'_{n-2} + 7y'_{n-3}) - \frac{8}{945}h^{(7)}y^{(7)}(\theta) \quad (5.9)$$

Equation (5.9) is a seventh-order equation, but will be analyzed in this chapter since the characteristic equation is of degree four.

To include all of the correctors developed in this chapter the equation,

$$y_{n+1} = a_o y_n + a_1 y_{n-1} + a_2 y_{n-2} + a_3 y_{n-3}$$

$$+ h(b_{-1} y'_{n+1} + b_o y'_n + b_1 y'_{n-1} + b_2 y'_{n-2} + b_3 y'_{n-3})$$

$$+ \frac{E_6 h^6 y^{(6)}(\theta)}{6!} \qquad (5.10)$$

is used in the stability analysis.

The differential equation to be solved is assumed to be of the form

$$y' = f(x,y). \qquad (5.11)$$

If z is the true solution of the differential equation, then z satisfies the relation

$$\frac{dz}{dx} = z' = f(x,z) \ . \qquad (5.12)$$

The calculated solution y satisfies

$$y'_n = f(x_n, y_n) + E_1(\eta) \qquad (5.13)$$

where $E_1(\eta)$ is the error in the n^{th} value and is assumed to be small. The true solution z approximates equation (5.10) and it follows that

$$z_{n+1} = a_o z_n + a_1 z_{n-1} + a_2 z_{n-2} + a_3 z_{b-3}$$

$$+ h(b_{-1} z'_{n+1} + b_o z'_n + b_1 z'_{n-1} + b_2 z'_{n-2} + b_3 z'_{n-3})$$

$$+ E_2(\eta) , \tag{5.14}$$

where $E_2(\eta)$ is the truncation error at the point x_{n+1}.

The error between the true solution and the approximate solution is defined as

$$\epsilon_n = z_n - y_n . \tag{5.15}$$

Subtracting equation (5.10) from (5.14) results in the relation

$$\epsilon_{n+1} = a_o \epsilon_n + a_1 \epsilon_{n-1} + a_2 \epsilon_{n-2} + a_3 \epsilon_{n-3}$$

$$+ h(b_{-1} \epsilon'_{n+1} + b_o \epsilon'_{n+1} + b_1 \epsilon'_{n-1} + b_2 \epsilon'_{n-2} + b_3 \epsilon'_{n-3})$$

$$+ E_2(\eta) . \tag{5.16}$$

By making the assumptions as in Chapter II, applying the mean value theorem, and letting

$$K = \frac{\partial f}{\partial y} \tag{5.17}$$

results in the equation

$$\epsilon_{n+1} = a_o\epsilon_n + a_1\epsilon_{n-1} + a_2\epsilon_{n-2} + a_3\epsilon_{n-3}$$
$$+ H(b_{-1}\epsilon_{n+1} + b_o\epsilon_n + b_1\epsilon_{n-1} + b_2\epsilon_{n-2} + b_3\epsilon_{n-3}) + E \quad (5.18)$$

where $H = Kh$. Solving this linear difference equation with constant coefficients by setting $\epsilon_n = C\lambda^n$, results in the characteristic equation

$$(Hb_{-1} - 1)\lambda^4 + (Hb_o + a_o)\lambda^3 + Hb_1 + a_1)\lambda^2$$
$$+ (Hb_2 + a_2)\lambda + (Hb_3 + a_3) = 0 \quad . \quad (5.19)$$

By substituting the coefficients of correctors (5.6 - 5.9) into equation (5.19), the characteristic equation of each can be found. The roots of each characteristic equation for twenty-one values of H, $-2 \leq H \leq 2$, are shown in Figure 7-10. The Bairstow method, subroutine ROTPOL, was again used to find each root.

Taking a_1 and a_2 as parameters and making equation (5.1) exact for polynomials through degree five, results in the coefficients:

$$a_o = 1 - a_1 - a_2 \qquad b_o = \frac{1}{720}(646 + 346a_1 - 1024a_2)$$
$$a_1 = a_1 \qquad b_1 = \frac{1}{720}(-264 + 456a_1 + 912a_2)$$
$$a_2 = a_2 \qquad b_2 = \frac{1}{720}(106 - 74a_1 + 272a_2) \quad (5.20)$$
$$b_{-1} = \frac{1}{720}(251 - 19a_1 - 8a_2) \qquad b_3 = \frac{1}{720}(-19 + 11a_1 - 8a_2)$$
$$E_6 = \frac{1}{2}(-27 + 11a_1) \ .$$

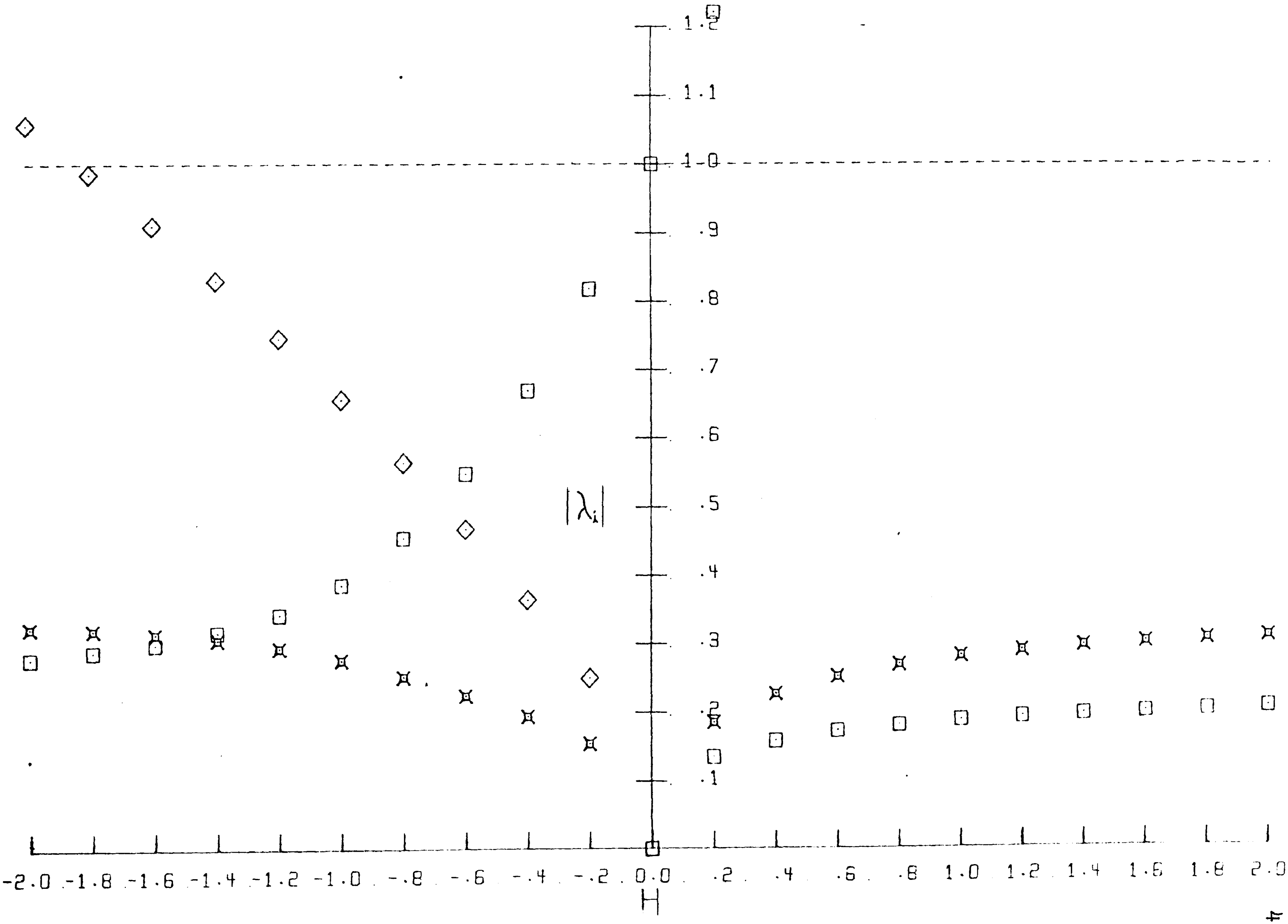

FIGURE 7. ROOT LOCI FOR THE CHARACTERISTIC EQUATION OF CORRECTOR (5.6).

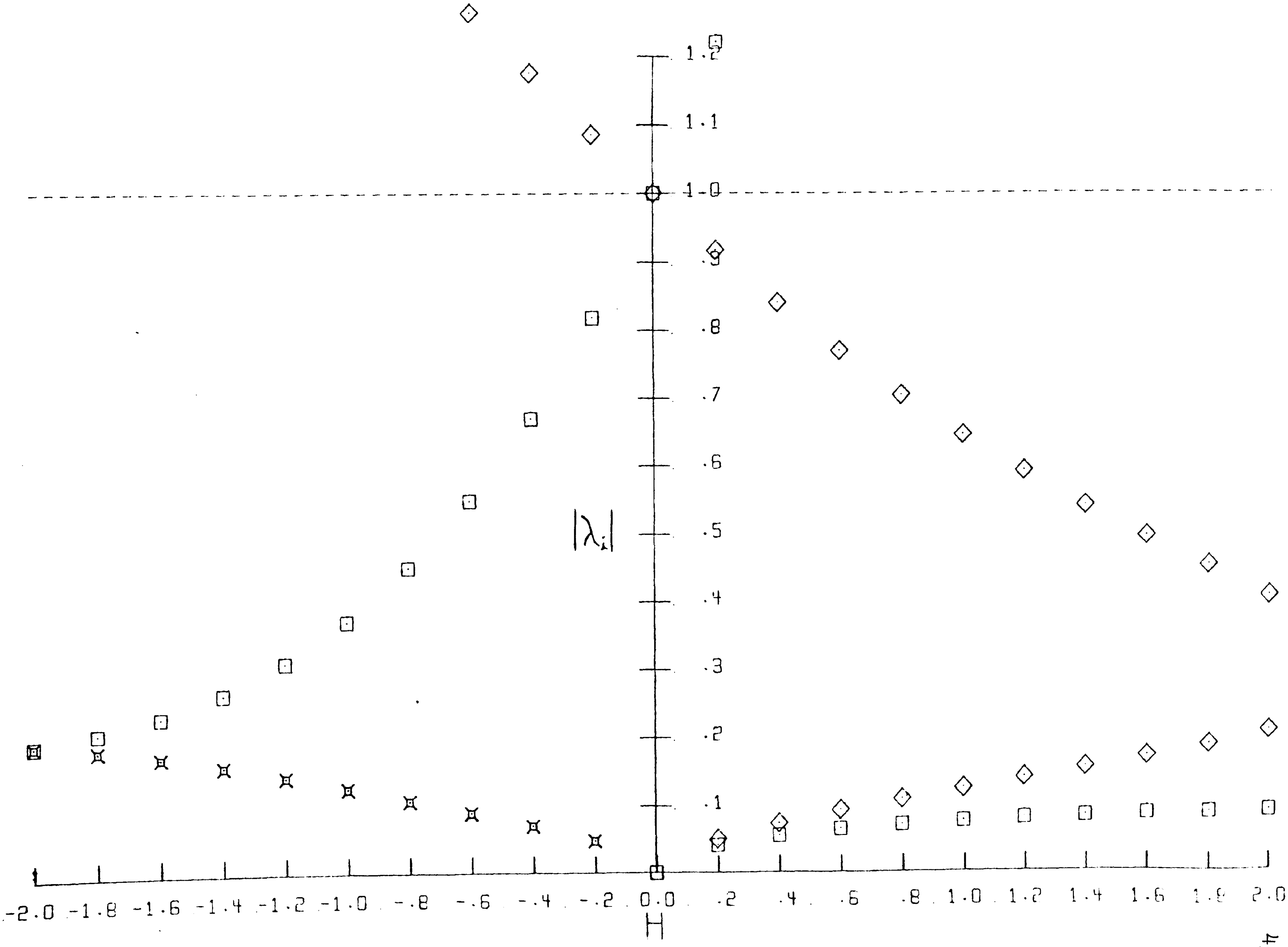

FIGURE 8. ROOT LOCI FOR THE CHARACTERISTIC EQUATION OF CORRECTOR (5.7).

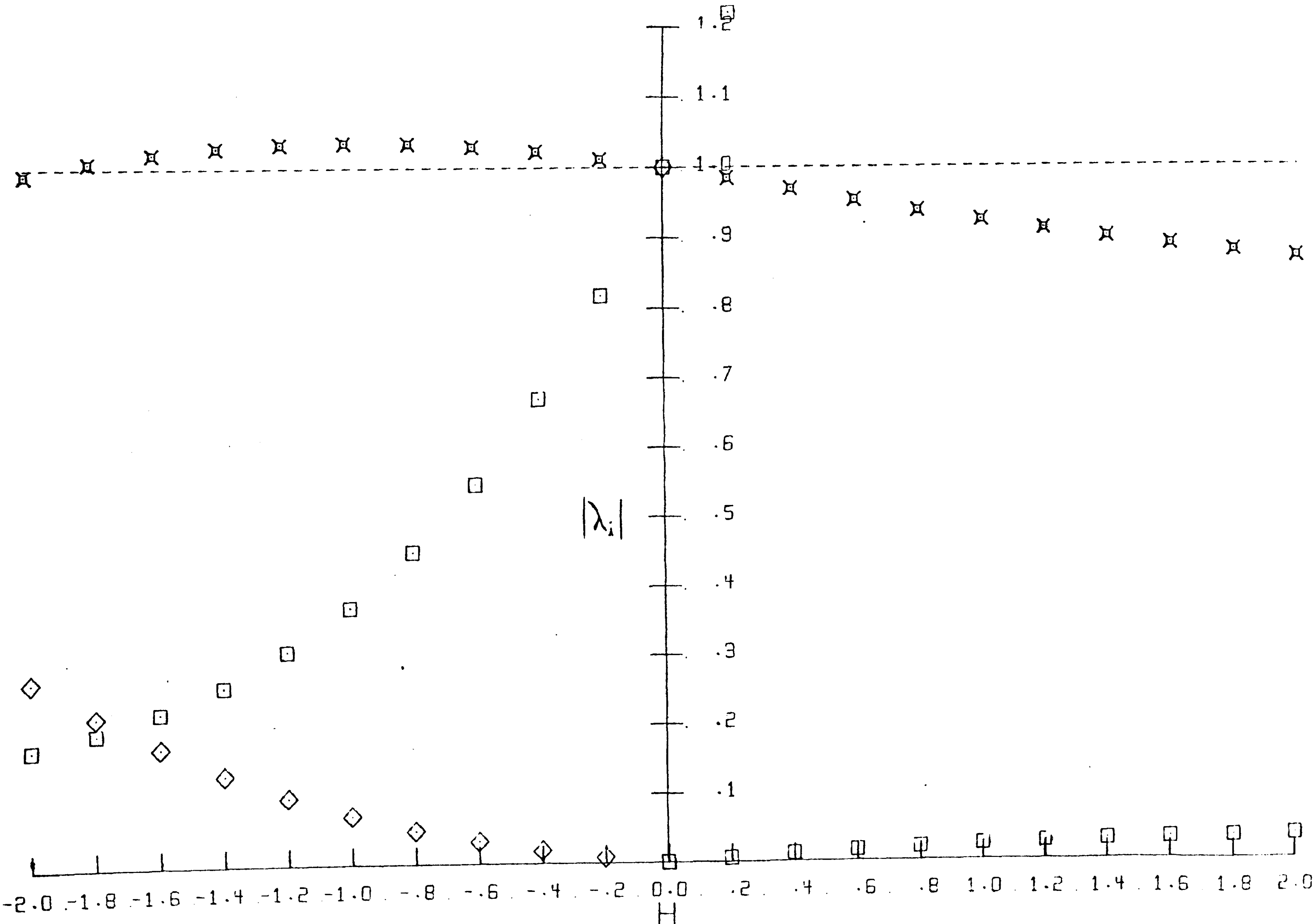

FIGURE 9. ROOT LOCI FOR THE CHARACTERISTIC EQUATION OF CORRECTOR (5.8).

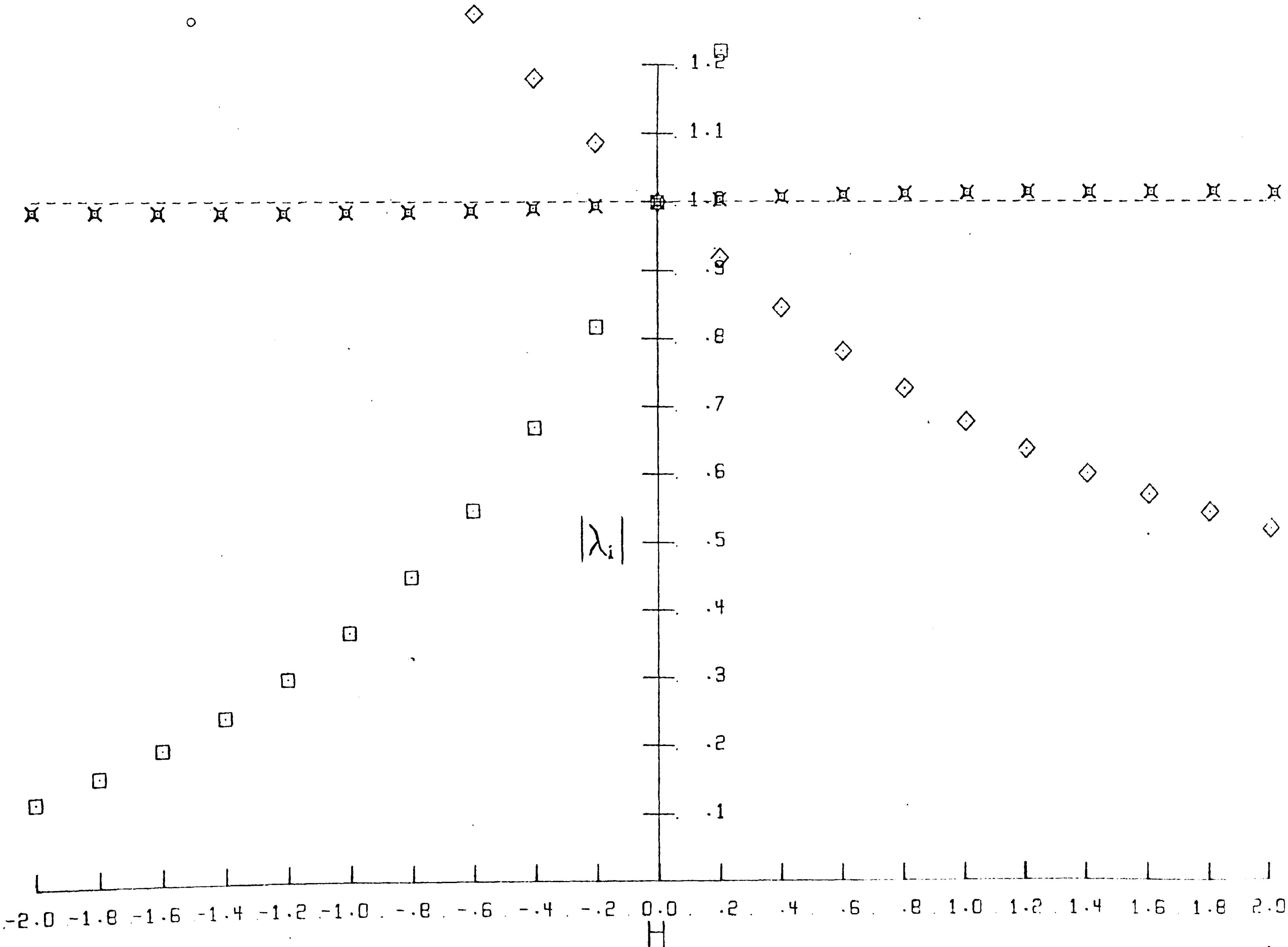

FIGURE 10. ROOT LOCI FOR THE CHARACTERISTIC EQUATION OF CORRECTOR (5.9).

A suggested sixth-order predictor to use with this corrector is

$$y_{n+1} = A_o y_n + A_1 y_{n-1} + A_2 y_{n-2}$$

$$+ h(B_o y'_n + B_1 y'_{n-1} + B_2 y'_{n-2} + B_3 y'_{n-3})$$

$$+ E_6 y^{(6)}(\theta) \qquad (5.21)$$

where

$$A_o = \frac{1}{19}(-232 - 11a_2) \qquad B_1 = \frac{1}{19}(152 + 19a_2)$$

$$A_1 = \frac{1}{19}(251 - 8a_2) \qquad B_2 = \frac{1}{19}(-23 + 8a_2)$$

$$A_2 = A_2 \qquad B_3 = \frac{1}{57}(10 - a_2) \qquad (5.22)$$

$$B_o = \frac{1}{57}(413 + 10a_2) \qquad E_6 = \frac{1}{570}(281 - 11a_2) .$$

These coefficients are found by making equation (5.21) exact for polynomials through degree five.

The values chosen for the parameters a_1 and a_2 for this sixth-order method are $a_1 = .7$ and $a_2 = 0$. The resulting coefficients of equation (5.1) are:

$$a_o = \frac{3}{10} \qquad b_o = \frac{4441}{3600}$$

$$a_1 = \frac{7}{10} \qquad b_1 = \frac{23}{300}$$

$$a_2 = 0 \qquad b_2 = \frac{271}{3600} \qquad (5.23)$$

$$b_{-1} = \frac{2377}{7200} \qquad b_3 = -\frac{113}{7200}$$

$$E_6 = -\frac{193}{20} .$$

The characteristic equation for this method, as given by equation (5.10), is

$$(\frac{2377}{7200}H - 1)\lambda^4 + (\frac{4441}{3600}H + \frac{3}{10})\lambda^3 + (\frac{23}{300}H + \frac{7}{10})\lambda^2$$

$$+ \frac{271}{3600}H\lambda - \frac{113}{7200}H = 0 \quad . \tag{5.24}$$

The roots of this equation were found by the usual method and are shown for the twenty-one values of H in Figure 11.

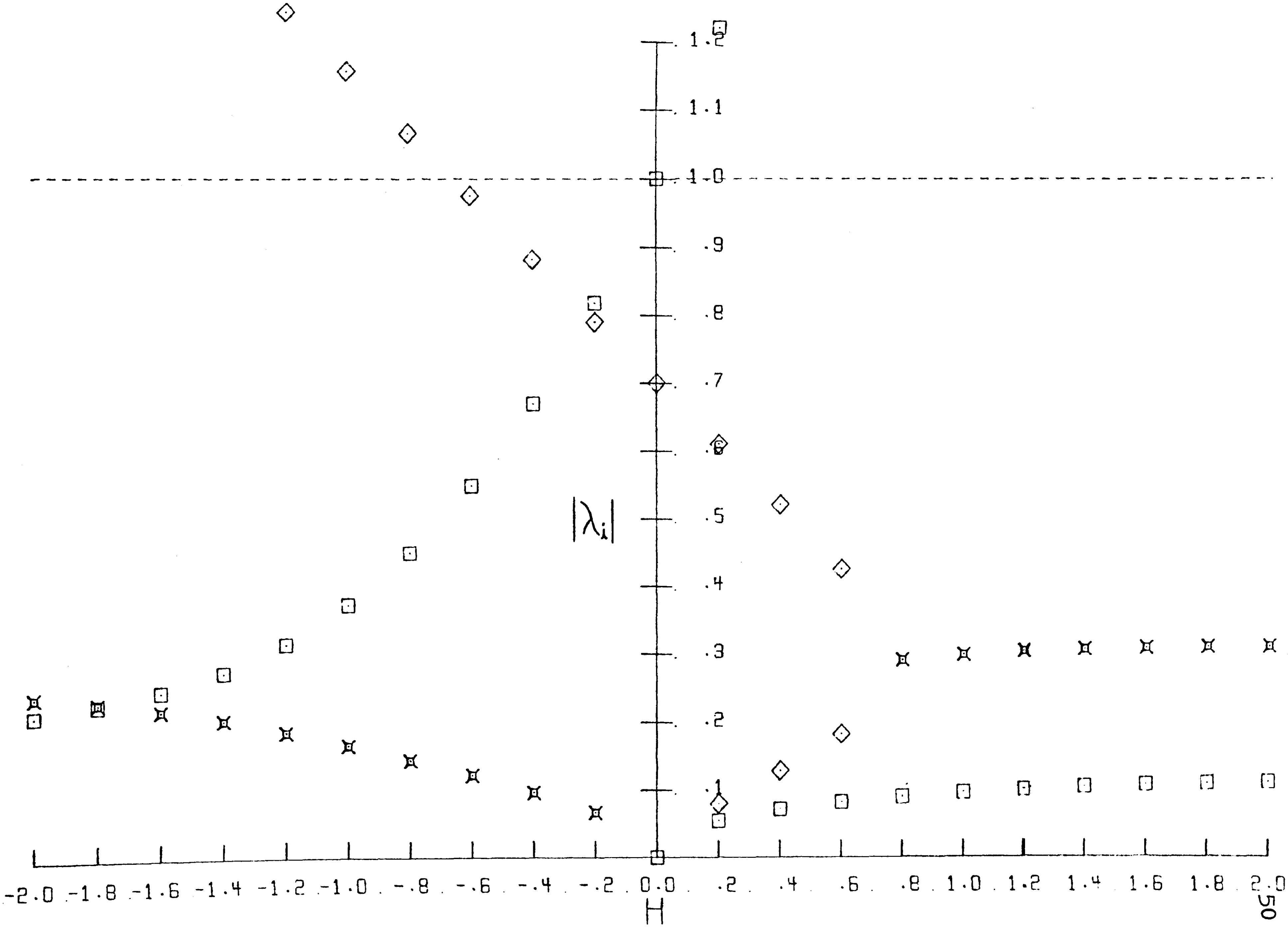

FIGURE 11. ROOT LOCI OF CHARACTERISTIC EQUATION (5.24).

CHAPTER VI

SEVENTH-ORDER METHODS

Stability becomes a definite problem in the derivation of the seventh-order methods. The correctors derived from the difference equations in Appendix II have fifth degree characteristic equations, while those derived by the method of undetermined coefficients have fourth degree characteristic equations.

The closed-type formulas of Appendix II which yield stable correctors are equations 1 and 3. The predictors listed are derived from open-type formulas 2 and 6, and were chosen because they have smaller error terms.

Predictors

$$y_{n+1} = y_{n-1} + \frac{h}{90}(297y'_n - 406y'_{n-1} + 574y'_{n-2} - 426y'_{n-3} + 169y'_{n-4} - 28y'_{n-5}) + \frac{13499}{60480}h^7 y^{(7)}(\theta) \quad (6.1)$$

$$y_{n+1} = y_{n-5} + \frac{3h}{10}(11y'_n - 14y'_{n-1} + 26y'_{n-2} - 14y'_{n-3} + 11y'_{n-4}) + \frac{481}{2240}h^7 y^{(7)}(\theta) \quad (6.2)$$

Correctors

$$y_{n+1} = y_n + \frac{h}{1440}(493y'_{n+1} + 1337y'_n - 618y'_{n-1} + 302y'_{n-2} - 83y'_{n-3} + 9y'_{n-4}) - \frac{863}{60480}h^7 y^{(7)}(\theta) \quad (6.3)$$

$$y_{n+1} = y_{n-2} + \frac{3h}{160}(17y'_{n+1} + 73y'_n + 38y'_{n-1} + 38y'_{n-2}$$

$$- 7y'_{n-3} + y'_{n-4}) - \frac{29}{2240}h^7 y^{(7)}(\theta) \quad . \qquad (6.4)$$

The generalized corrector

$$y_{n+1} = a_o y_n + a_1 y_{n-1} + a_2 y_{n-2} + a_3 y_{n-3} + a_4 y_{n-4}$$

$$+ h(b_{-1}y'_{n+1} + b_o y'_n + b_1 y'_{n-1} + b_2 y'_{n-2}$$

$$+ b_3 y'_{n-3} + b_4 y'_{n-4}) + \frac{E_7 h^7 y^{(7)}(\theta)}{7!} \qquad (6.5)$$

is used for the stability analysis of correctors (6.3) and (6.4). By a similar analysis, as presented in Chapter II and V, the characteristic equation of (6.5) is found to be

$$(b_{-1}H - 1)\lambda^5 + (b_o H + a_o)\lambda^4 + (b_1 H + a_1)\lambda^3 + (b_2 H + a_2)\lambda^2$$

$$+ (b_3 H + a_3)\lambda + (b_4 H + a_4) = 0 \quad . \qquad (6.6)$$

The magnitude of the roots for the characteristic equations of correctors (6.3) and (6.4) are shown in Figures 12 and 13.

The generalized seventh-order corrector is of the form

$$y_{n+1} = a_o y_n + a_1 y_{n+1} + a_2 y_{n-2} + a_3 y_{n-3}$$

$$+ h(b_{-1}y'_{n+1} + b_o y'_n + b_1 y'_{n+1} + b_2 y'_{n-2} + b_3 y'_{n-3})$$

$$+ \frac{E_7 h^7 y^{(7)}(\theta)}{7!} \quad . \qquad (6.7)$$

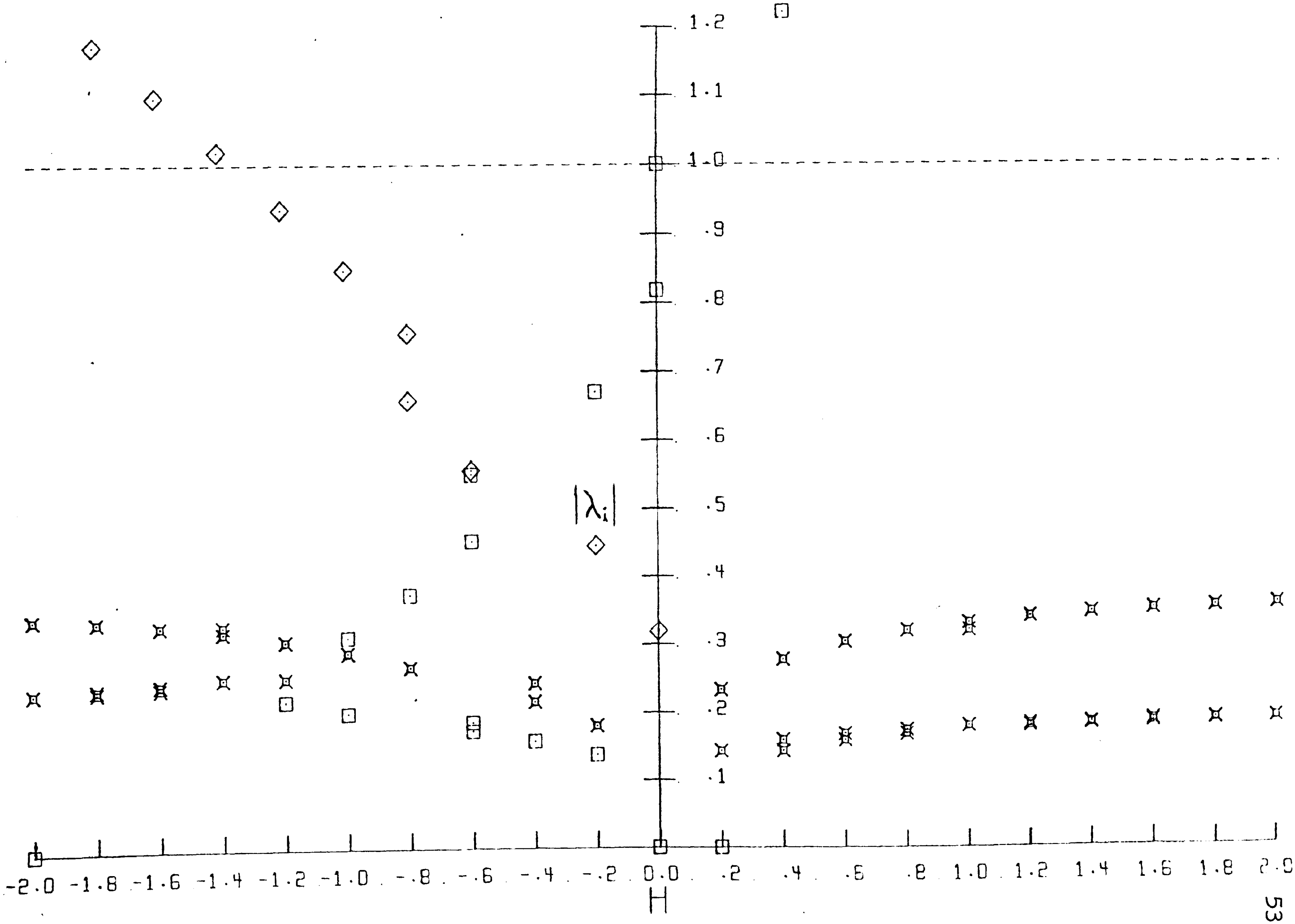

FIGURE 12. ROOT LOCI FOR THE CHARACTERISTIC EQUATION OF CORRECTOR (6.3).

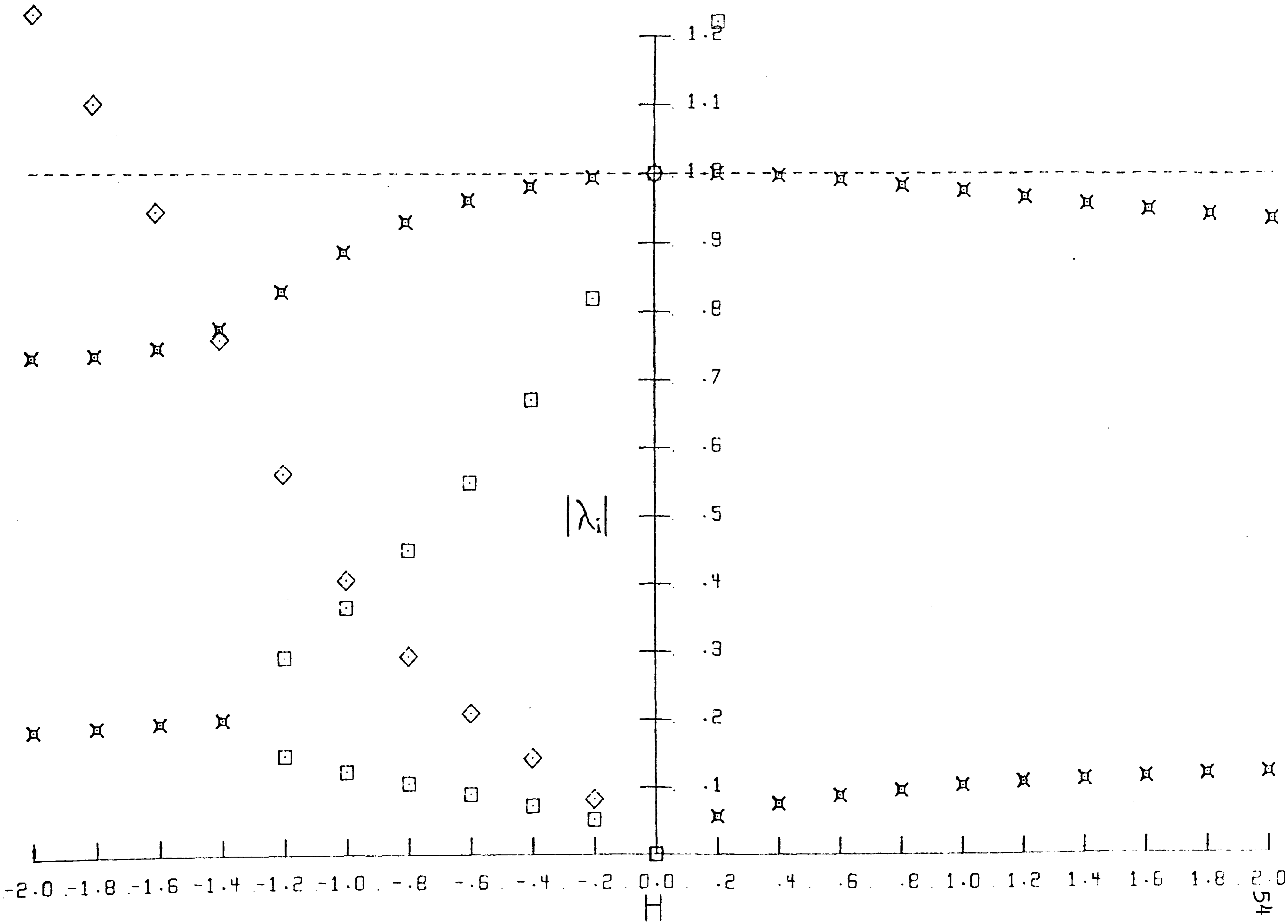

FIGURE 13. ROOT LOCI FOR THE CHARACTERISTIC EQUATION OF CORRECTOR (6.4).

Making this equation exact for polynomials through degree six, results in the coefficients

$$a_o = \frac{1}{11}(-16 - 11a_2 + 16a_3) \qquad b_o = \frac{1}{495}(1028 + 187a_2 - 324a_3)$$

$$a_1 = \frac{1}{11}(27 - 27a_3) \qquad b_1 = \frac{1}{165}(196 + 209a_2 - 108a_3)$$

$$a_2 = a_2 \qquad b_2 = \frac{1}{495}(-52 + 187a_2 + 756a_3) \quad (6.8)$$

$$a_3 = a_3 \qquad b_3 = \frac{1}{90}(1 - a_2 + 27a_3)$$

$$b_{-1} = \frac{1}{990}(281 - 11a_2 + 27a_3) \qquad E_7 = \frac{4}{693}(-408 - 11a_2 - 25653a_3) .$$

Equation (6.7) has a fourth degree characteristic polynomial and it can be found by using equation (5.19).

The suggested predictor to use with corrector (6.7) is

$$y_{n+1} = A_o y_n + A_1 y_{n-1} = A_2 y_{n-2} + A_3 y_{n-3}$$
$$+ h(B_o y'_n + B_1 y'_{n+1} + B_2 y'_{n-2} + B_3 y'_{n-3})$$
$$+ \frac{E_7 h^7 y^{(7)}(\theta)}{7!} \qquad (6.9)$$

with coefficients

$$A_o = \frac{1}{11}(-297 - 11A_3) \qquad B_o = \frac{1}{11}(129 + 3A_3)$$

$$A_1 = \frac{1}{11}(27 - 27A_3) \qquad B_1 = \frac{1}{11}(369 + 27A_3)$$

$$A_2 = \frac{1}{11}(281 + 27A_3) \qquad B_2 = \frac{1}{11}(105 + 27A3) \qquad (6.10)$$

$$A_3 = A_3 \qquad B_3 = \frac{1}{11}(-3 + 3A_3)$$

$$E_7 = \frac{1}{11}(1692 - 108A_3) .$$

For the coefficients (6.8), no values were found for the parameters a_2 and a_3 which make corrector (6.7) stable. Although stability was not achieved, for most differential equations tried, $a_2 = a_3 = 1$ worked very well.

The two extreme cases are shown in the examples given below. In the first example, the differential equation is

$$y' = -y, \quad y(0) = 1$$

which has the closed form solution

$$y = e^{-x} .$$

In the second example, the differential equation is

$$y' = -2xy, \quad y(0) = 1$$

with closed form solution

$$y = \frac{1}{1+x^2} .$$

Table I shows the error growth in the solution of each example. Predictor (6.9) with $A_3 = 0$, and corrector (6.7) with $a_2 = a_3 = 1$, is used to obtain the solution. A partial list of the results are shown.

TABLE I: ERROR GROWTH IN THE SOLUTION

OF $y' = -y$ AND $y' = -2xy$, $y(0) = 1$, AND $h = .1$.

x	$y' = -y$ ($x\ 10^{-10}$)	$y' = -2xy$ ($x\ 10^{-1}$)
0.5	1.33	0.17
1.0	7.76	1.37
1.5	0.75	1.95
2.0	5.64	1.98
2.5	2.72	0.73
3.0	10.61	3.94
3.5	11.59	16.39
4.0	16.69	126.04
4.5	19.14	1138.29
5.0	27.04	12658.81

CHAPTER VII

CONCLUSIONS

Several stable methods have been derived for the numerical solution of ordinary differential equations. For the readers convenience, they have been listed below with a brief description of each method.

METHOD I

The predictor and corrector for this method are derived from equations (4.11) and (4.9), respectively. They are derived by making the equations exact for polynomials through degree four. This method of derivation is usually referred to as the method of undetermined coefficients. This fifth-order method is stable for $-2 \leq H \leq 0$. For $H > 0$ the characteristic equation has points for which the magnitude of λ is greater than e^H, as do all the methods developed in this study and most other iterative methods. The dominant roots, in the negative H interval, for this method are shown in the Appendix, Figure 6A. This method is characterized by the following predictor and corrector:

$$P_{n+1} = -8y_n + 9y_{n-1} + \frac{h}{3}(17y'_n + 14y'_{n-1} - y'_{n-2}) + \frac{1}{9}h^5y^{(5)}(\theta) \qquad (7.1)$$

$$y_{n+1} = \frac{1}{10}\Big[-\frac{23}{10}y_n + 7y_{n-1} + \frac{53}{10}y_{n-2} + \frac{h}{120}(415p'_{n+1} + 1617y'_n + 1053y'_{n-1} + 227y'_{n-2})\Big] - \frac{193}{9000}h^5y^{(5)}(\theta) \;. \qquad (7.2)$$

METHOD II

This sixth-order iterative predictor-corrector method is developed from the difference equations located in Appendix II. The predictor is derived from open-type formula 4 and the corrector from closed-type formula 1. The dominant roots, in the negative H interval, for this method are shown in the Appendix, Figure 7A. As shown in this figure, $H \geq -1.9$ is required for stability. The predictor and corrector for this method are

$$p_{n+1} = y_{n-3} + \frac{2h}{45}(67y'_n - 58y'_{n-1} + 102y'_{n-2} - 28y'_{n-3} + 7y'_{n-4}) + \frac{14}{45}h^6 y^{(6)}(\theta) \qquad (7.3)$$

$$y_{n+1} = y_n + \frac{h}{720}(251p'_{n+1} + 646y'_n - 264y'_{n-1} + 106y'_{n-2} - 19y'_{n-3}) + \frac{3}{160}h^6 y^{(6)}(\theta) \qquad (7.4)$$

respectively.

METHOD III

This is the sixth-order iterative procedure developed by the method of undetermined coefficients. The predictor is developed by determining the coefficients of equation (5.21) and the corrector by determining the coefficients of equation (5.1). They are exact for polynomials through degree five. $H \geq -.66$ is required for the stability of this method, as is shown in Figure 11. The predictor and corrector equations are, respectively,

$$p_{n+1} = \frac{1}{19}\Big[-232y_n + 251y_{n-1} + \frac{h}{3}(413y'_n + 456y'_{n-1}$$

$$- 69y'_{n-2} + 10y'_{n-3})\Big] + \frac{281}{570}h^6 y^{(6)}(\theta) \qquad (7.5)$$

$$y_{n+1} = \frac{1}{10}\Big[3y_n + 7y_{n-1} + \frac{h}{720}(2377p'_{n+1} + 8882y'_n$$

$$+ 552y'_{n-1} + 542y'_{n-2} - 113y'_{n-3})\Big]$$

$$- \frac{193}{14400}h^6 y^{(6)}(\theta) \ . \qquad (7.6)$$

METHOD IV

This seventh-order method uses equation (6.2) to predict and (6.3) to correct. The predictor and corrector are derived from the difference equations in Appendix II. The dominant roots for this method are shown in Figure 12A, showing that this method is stable for $H \geq 1.36$. The predictor and corrector are

$$p_{n+1} = y_{n-5} + \frac{3h}{10}(11y'_n - 14y'_{n-1} + 26y'_{n-2} - 14y'_{n-3} + 11y'_{n-4})$$

$$+ \frac{481}{2240}h^7 y^{(7)}(\theta) \qquad (7.7)$$

$$y_{n+1} = y_n + \frac{h}{1440}(493p'_{n+1} + 1337y'_n - 618y'_{n-1} + 302y'_{n-2}$$

$$- 83y'_{n-3} + 9y'_{n-4}) - \frac{863}{60480}h^7 y^{(7)}(\theta) \ . \qquad (7.8)$$

METHOD V

This method is very similar to method IV. The same predictor is used with corrector (6.4). This method is stable for $H \leq -1.68$ as is shown in Figure 13A. The corrector for this method is

$$y_{n+1} = y_{n-2} + \frac{3h}{160}(17p'_{n+1} + 73y'_n + 38y'_{n-1} + 38y'_{n-2}$$

$$- 7y'_{n-3} + y'_{n-4}) - \frac{29}{2240}h^7y^{(7)}(\theta) \quad . \qquad (7.9)$$

To illustrate the theory discussed in this study, each of the various new predictor-corrector methods were tried on several first order differential equations. All computations were performed on the IBM 1620 Model II computer. An excellent correlation between the expected and actual rates of error growth is obtained. The differential equations solved are summarized in Table II.

TABLE II: NUMERICAL EXAMPLES

Example	Differential Equation	Initial Condition	Closed Form Solution
1	$y' = y$	$Y(0) = 1$	$y = e^x$
2	$y' = -y$	$y(0) = 1$	$y = e^{-x}$
3	$y' = x^2 - y$	$y(0) = 1$	$y = 2 - 2x + x^2 + e^{-x}$
4	$y' = 1/(1 + \tan^2 y)$	$y(0) = 0$	$y = \tan^{-1}x$

For all problems solved, the interval of integration is $h = .1$. Iteration is continued until the difference between the predicted and corrected value is less than 5×10^{-6}. The error growth for each method is measured by the difference between the closed form solution and the calculated values. To save valuable computer time the closed form values are used for the required starting values of each method. Each problem is also solved using the modified Hamming, Adams, Milne, and Runge-Kutta methods to provide a comparison with other well known and accurate methods. The computer results are summarized in Tables III through VI.

All of the new, stable methods are quite effective in providing an efficient and accurate solution to the first order differential equations tried. Methods I, II, and III are probably the best methods developed. Methods IV and V have the disadvantage of requiring extra starting values. Method II needs five starting values as written, but the number required can be reduced to four by using the predictor of Method III.

It is felt that either a fifth or sixth-order method is the optimum order, for both stability and convergence, in the numerical solution of ordinary differential equations.

TABLE III: ERROR GROWTH $(X10^{-6})$ IN THE SOLUTION OF $Y' = Y$, $Y(0) = 1$

BY USING VARIOUS METHODS

X	RUNGE-KUTTA	MILNE*	ADAMS	MOD. HAMMING	METHOD I	METHOD II	METHOD III	METHOD IV	METHOD V
0.5	0.8	4980.	0.6	0.0	0.2	0.1	0.1	**	**
1.0	2.9	29230.	3.8	0.2	1.1	0.4	0.7	0.2	0.3
1.5	6.8	78120.	11.4	0.4	3.2	0.8	1.6	0.6	0.5
2.0	14.6	180900.	27.3	0.5	7.5	1.4	2.8	1.0	1.5
2.5	30.0	381500.	59.0	0.0	17.0	2.0	6.0	2.0	3.0
3.0	61.0	766600.	119.0	4.0	32.0	6.0	14.0	6.0	6.0
3.5	116.0	1488000.	233.0	12.0	63.0	11.1	28.0	12.0	14.0
4.0	217.0	2822000.	446.0	19.0	121.0	18.0	52.0	21.0	24.0
4.5	398.0	5254000.	838.0	32.0	228.0	27.0	90.0	36.0	43.0
5.0	730.0	9648000.	1540.0	50.0	420.0	50.0	170.0	80.0	80.0

* ROUNDED TO FOUR SIGNIFICANT FIGURES
** STARTING VALUE

TABLE IV: ERROR GROWTH ($X10^{-7}$) IN THE SOLUTION OF $Y' = -Y$, $Y(0) = 1$

BY USING VARIOUS METHODS

X	RUNGE-KUTTA	MILNE*	ADAMS	MOD. HAMMING	METHOD I	METHOD II	METHOD III	METHOD IV	METHOD V
0.5	3.100	21550.	4.40	0.500	0.800	0.300	0.400	**	**
1.0	3.800	65730.	10.70	0.200	2.100	1.100	0.300	0.00	0.500
1.5	3.600	38650.	11.30	0.600	2.000	1.600	0.400	0.80	0.800
2.0	2.900	67840.	9.80	0.300	1.900	1.500	0.200	0.90	0.500
2.5	2.230	5807.	7.72	0.040	1.380	1.290	0.260	0.83	1.120
3.0	1.610	67490.	5.78	0.110	1.000	0.910	0.200	0.60	0.950
3.5	1.140	37690.	4.17	0.120	0.730	0.650	0.140	0.42	0.150
4.0	0.800	90720.	2.92	0.090	0.500	0.460	0.100	0.29	1.280
4.5	0.550	97690.	2.00	0.060	0.350	0.330	0.060	0.21	1.020
5.0	0.369	152600.	1.36	0.047	0.236	0.218	0.044	0.14	0.275

* ROUNDED TO FOUR SIGNIFICANT FIGURES

** STARTING VALUE

TABLE V: ERROR GROWTH ($X10^{-6}$) IN THE SOLUTION OF $Y' = X^2 - Y$, $Y(0) = 1$

BY USING VARIOUS METHODS

X	RUNGE-KUTTA	MILNE*	ADAMS	MOD. HAMMING	METHOD I	METHOD II	METHOD III	METHOD IV	METHOD V
0.5	0.61	3464.	0.41	0.02	0.05	0.02	0.07	**	**
1.0	1.06	13790.	1.08	0.08	0.18	0.08	0.10	0.02	0.02
1.5	1.30	16630.	1.10	0.10	0.10	0.20	0.20	0.10	0.20
2.0	1.30	25780.	0.80	0.30	0.10	0.40	0.50	0.40	0.20
2.5	1.40	22760.	0.40	0.80	0.30	0.50	1.00	0.60	0.30
3.0	1.50	33680.	0.20	1.00	0.30	0.70	1.00	0.80	0.40
3.5	1.60	23370.	0.00	1.00	0.20	0.70	0.90	0.70	0.20
4.0	1.20	40740.	0.80	1.20	0.80	1.10	1.30	0.90	0.20
4.5	0.00	19030.	2.00	4.00	2.00	3.00	3.00	2.90	2.90
5.0	2.00	50290.	4.00	7.00	2.00	5.00	5.00	4.00	2.00

* ROUNDED TO FOUR SIGNIFICANT FIGURES
** STARTING VALUE

TABLE VI: ERROR GROWTH ($X10^{-6}$) IN THE SOLUTION OF $Y' = 1/(1+\tan Y)$, $Y(0) = 0$ BY USING VARIOUS METHODS

X	RUNGE-KUTTA	MILNE*	ADAMS	MOD. HAMMING	METHOD I	METHOD II	METHOD III	METHOD IV	METHOD V
0.5	0.36	1348.	0.37	2.95	0.92	1.50	1.31	**	**
1.0	0.55	5357.	7.81	2.13	2.53	0.85	0.67	0.46	0.36
1.5	0.46	5778.	6.26	0.21	1.72	0.36	0.07	0.73	0.23
2.0	0.50	6099.	3.80	0.00	1.00	0.10	0.30	0.30	0.20
2.5	0.60	4351.	2.40	0.00	0.70	0.10	0.60	0.00	0.00
3.0	0.60	4839.	1.80	0.20	0.70	0.30	0.60	0.20	0.20
3.5	0.70	2723.	1.50	0.60	0.50	0.40	0.70	0.40	0.30
4.0	0.80	3984.	1.30	1.00	0.60	0.50	0.80	0.50	0.20
4.5	0.90	1504.	1.20	1.20	0.60	0.70	0.80	0.70	0.10
5.0	1.00	3542.	1.20	1.70	0.60	0.90	1.00	0.90	0.00

* ROUNDED TO FOUR SIGNIFICANT FIGURES

** STARTING VALUE

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APPENDIX I

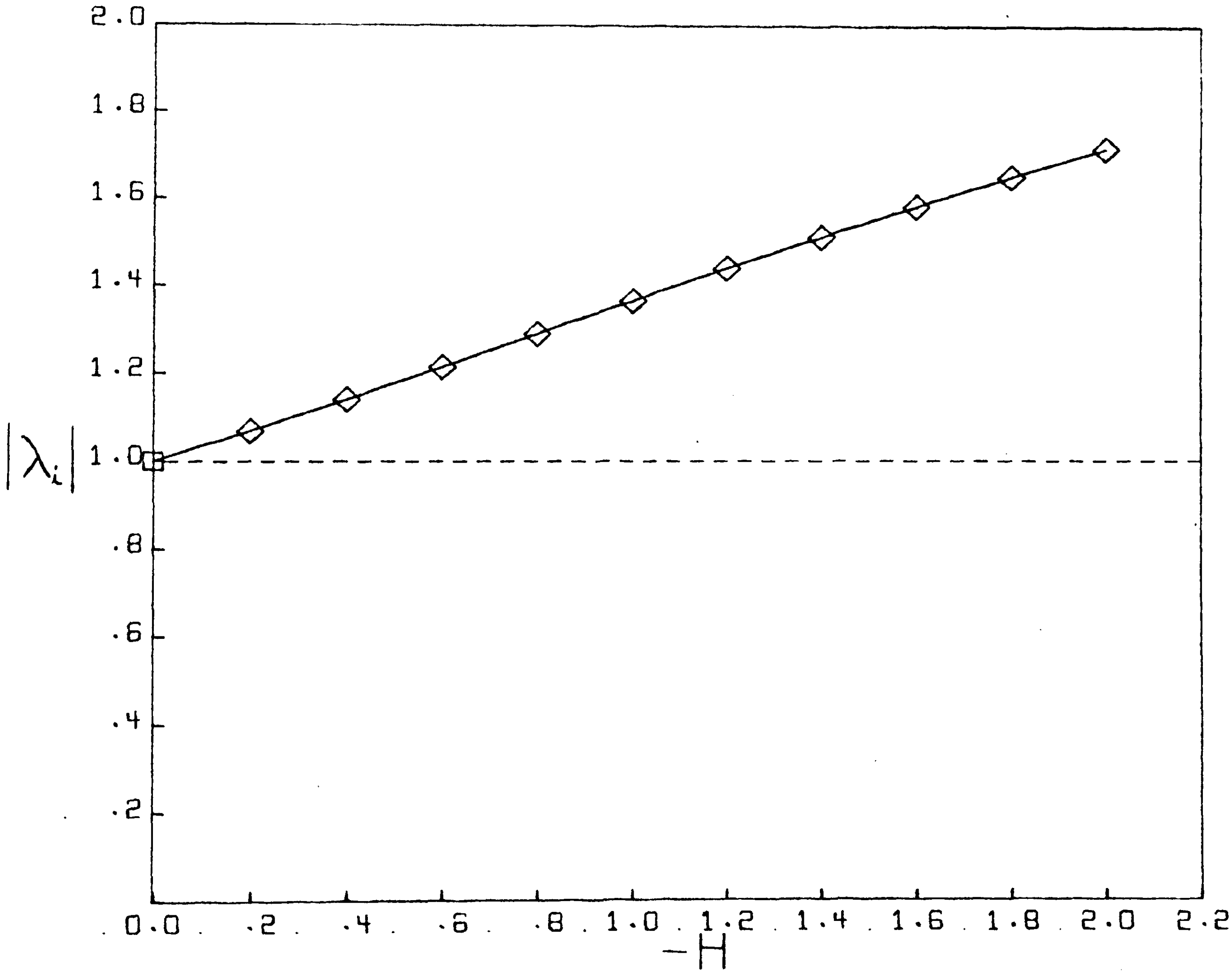

FIGURE 1A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 1.

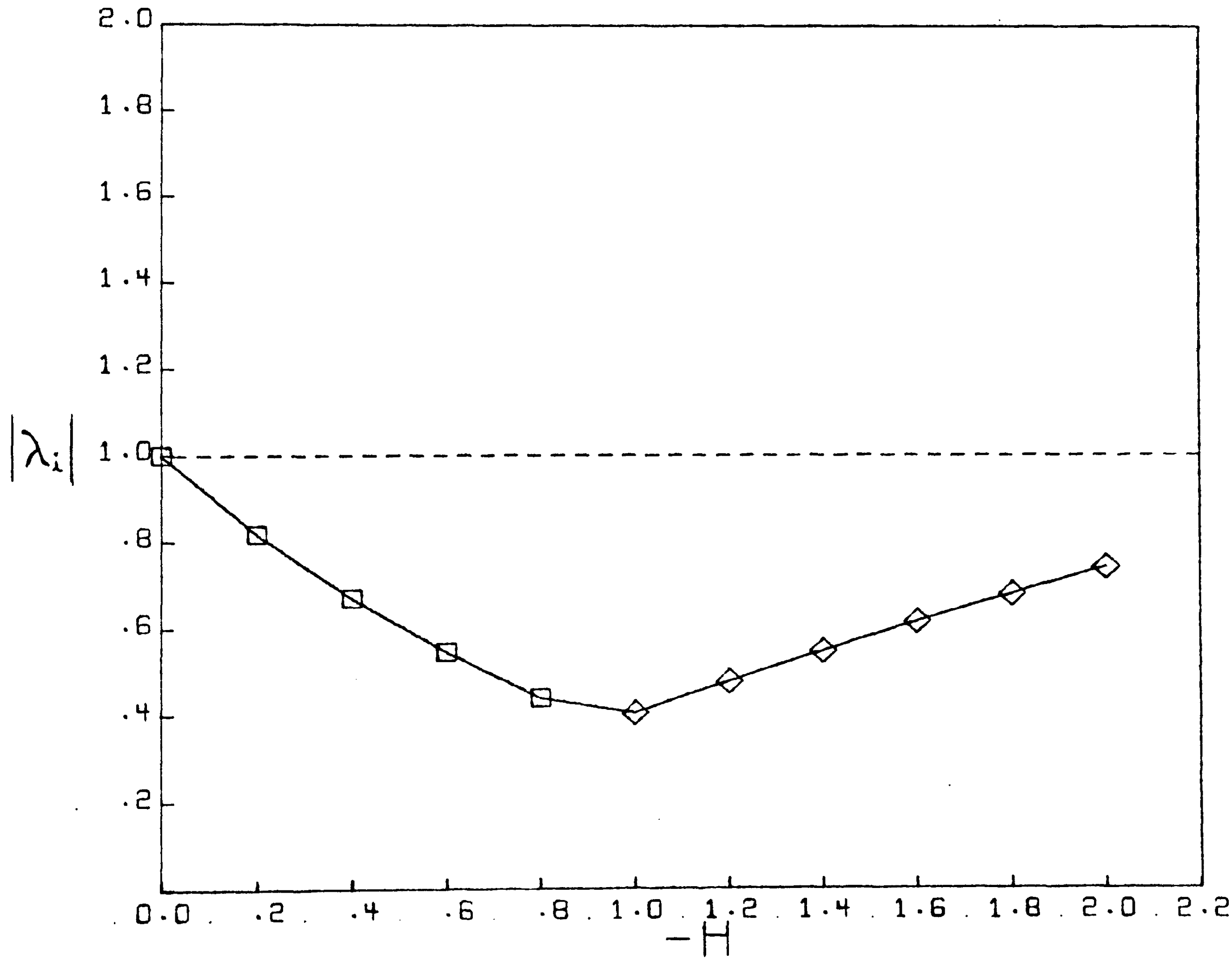

FIGURE 2A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 2.

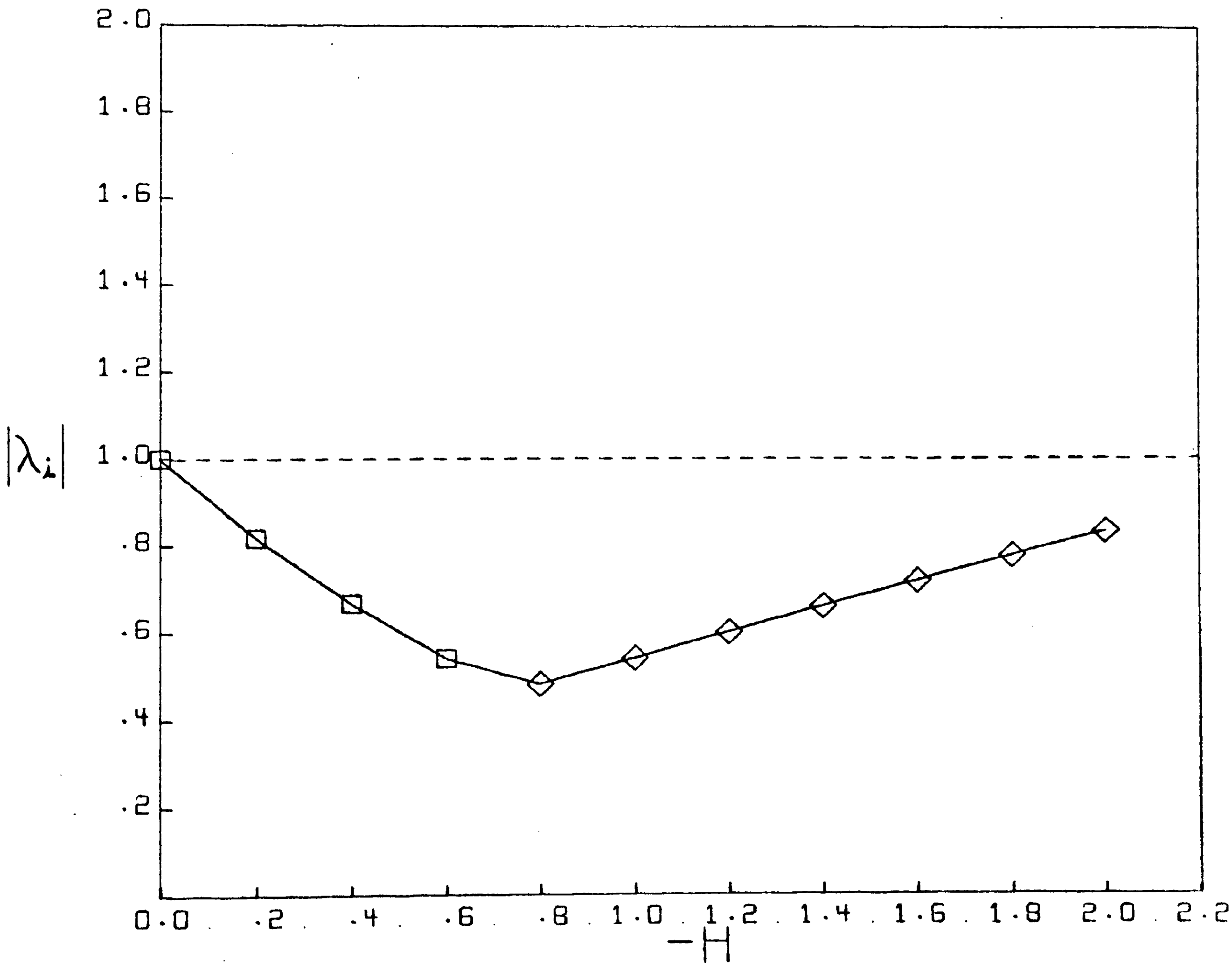

FIGURE 3A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 3.

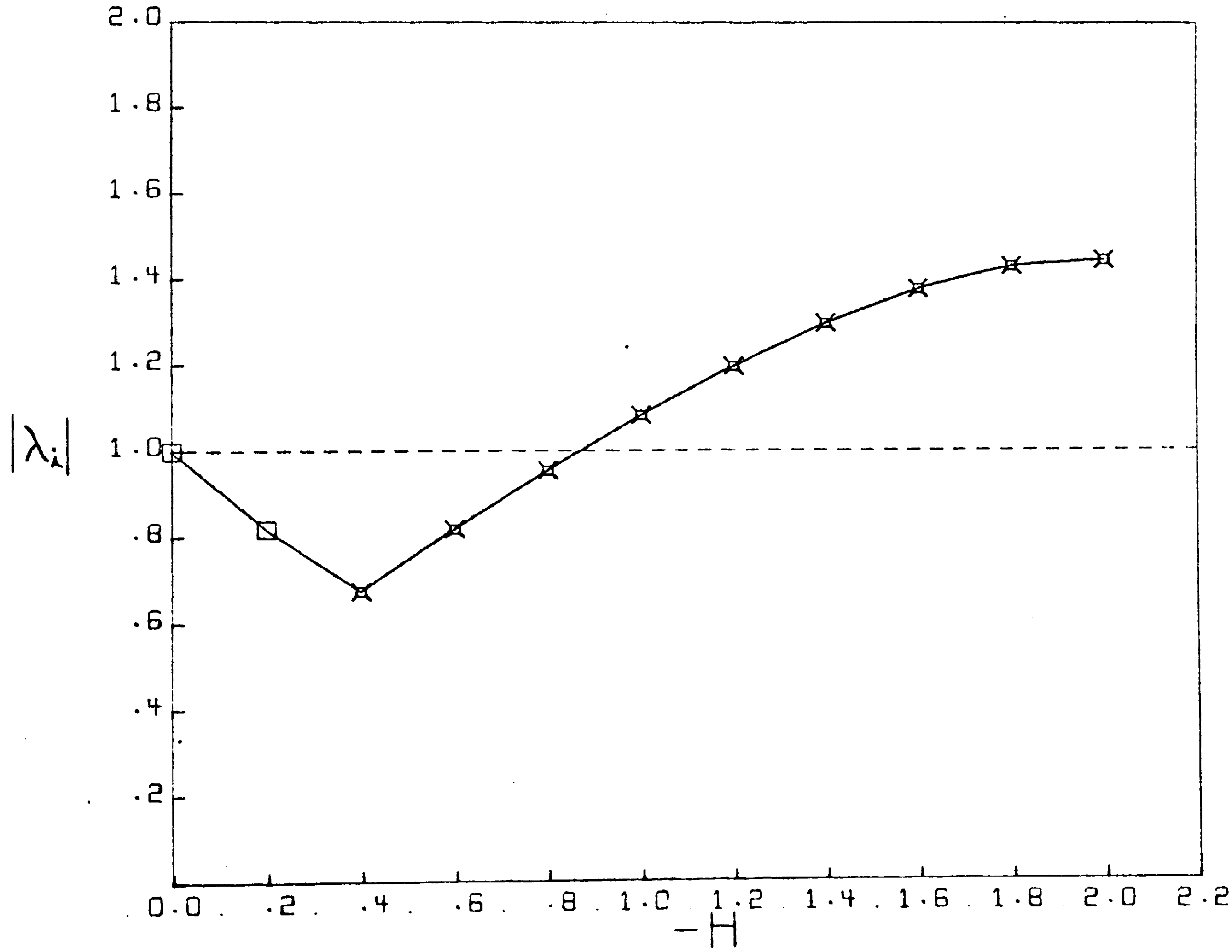

FIGURE 4A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 4.

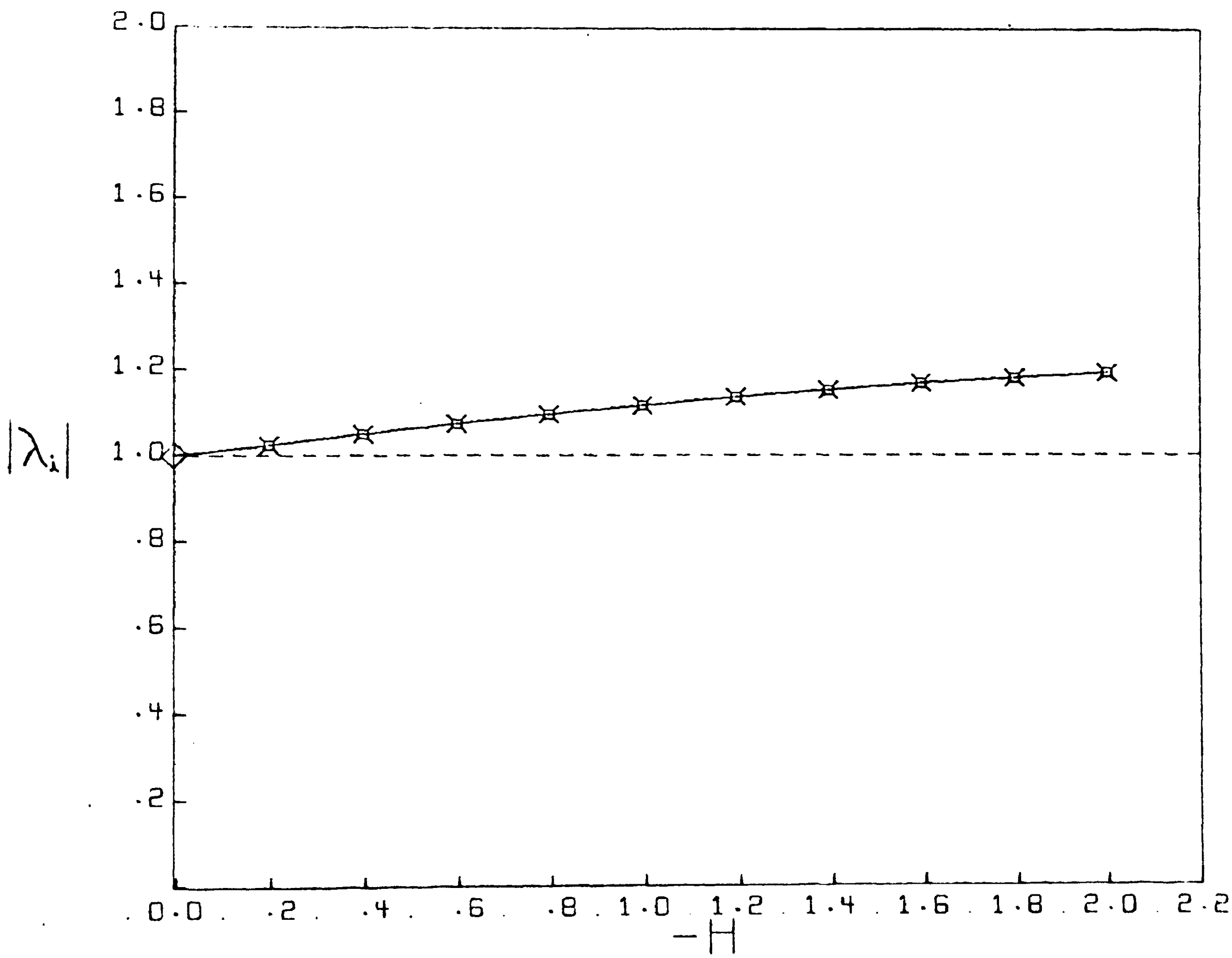

FIGURE 5A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 5.

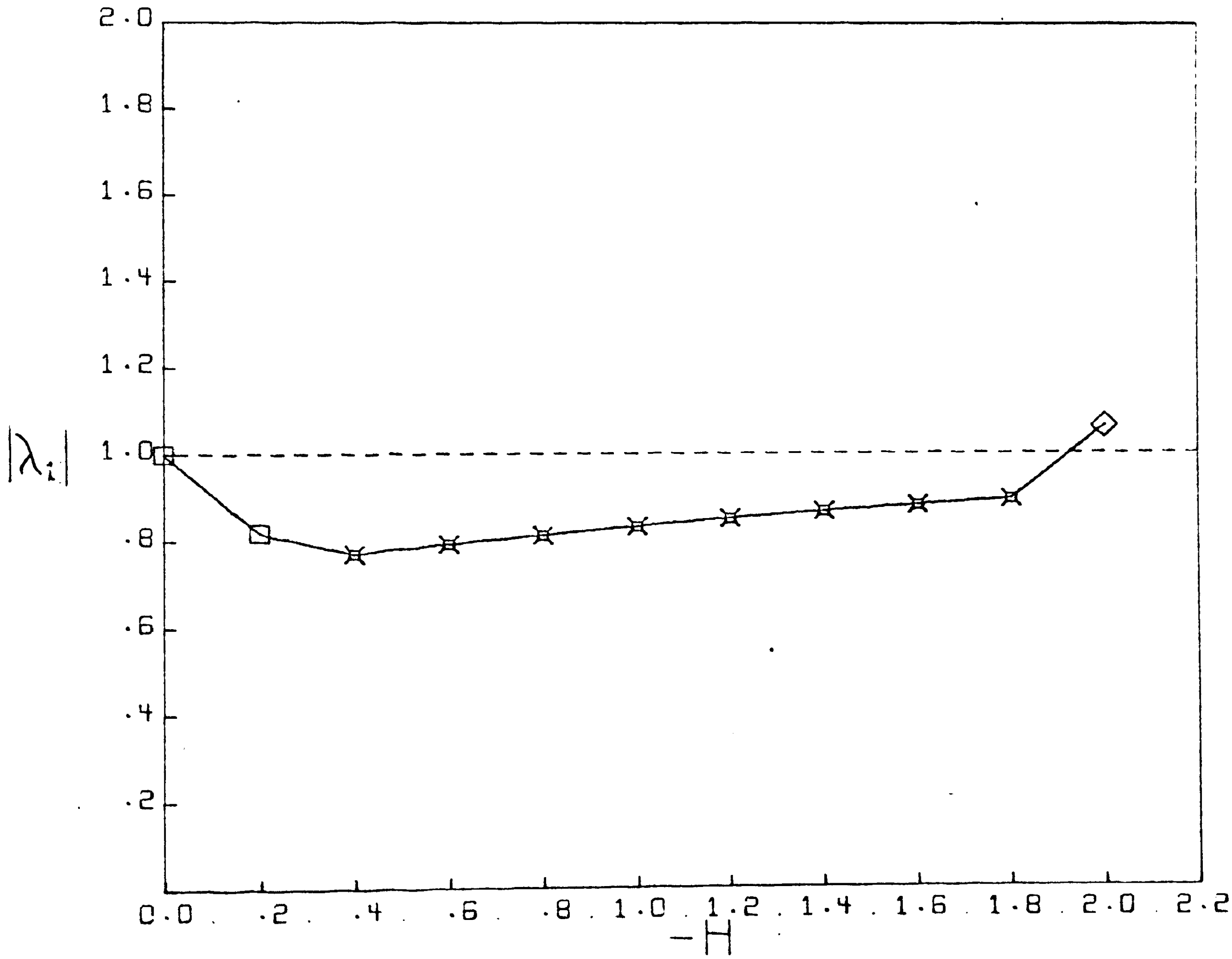

FIGURE 6A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 6.

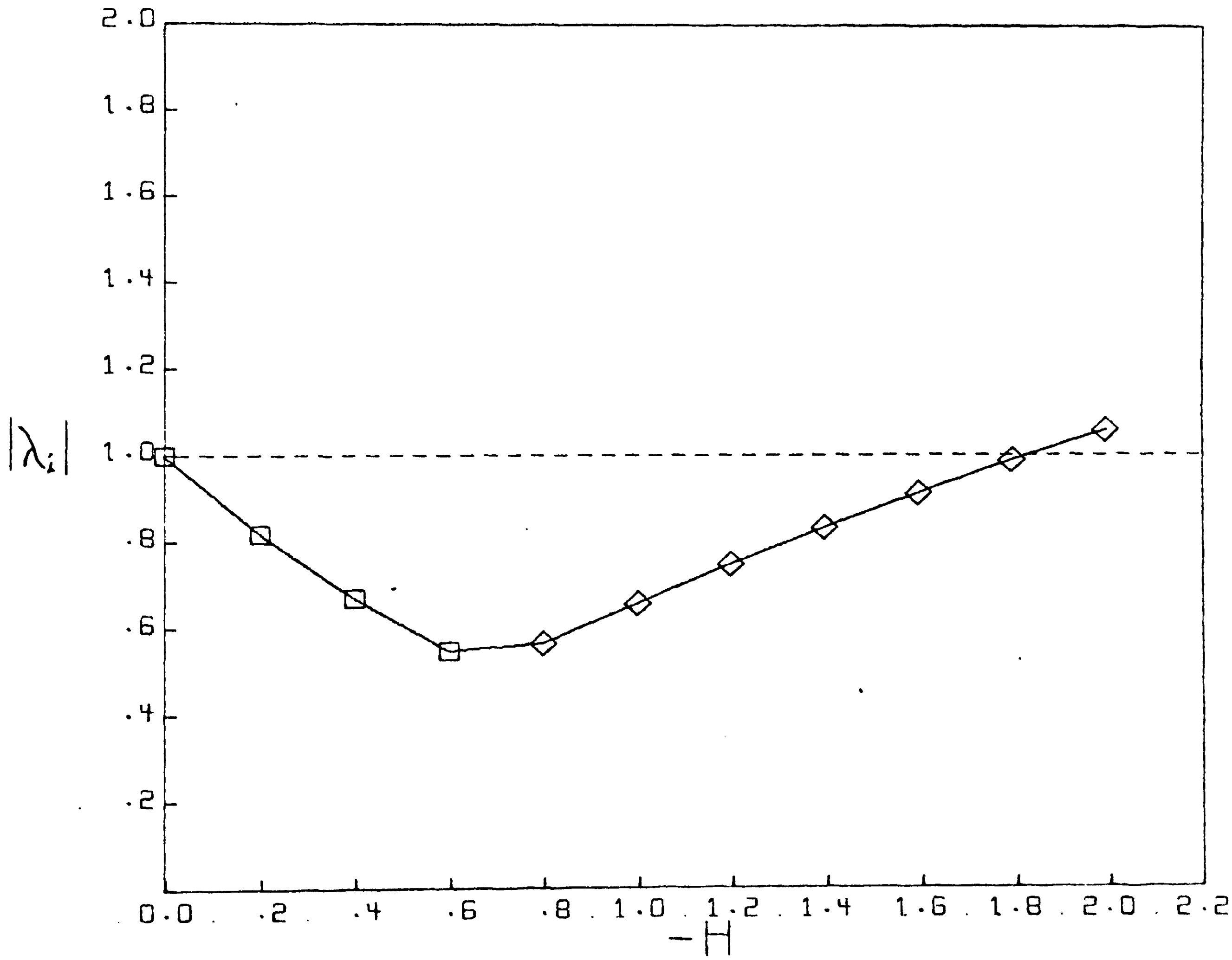

FIGURE 7A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 7.

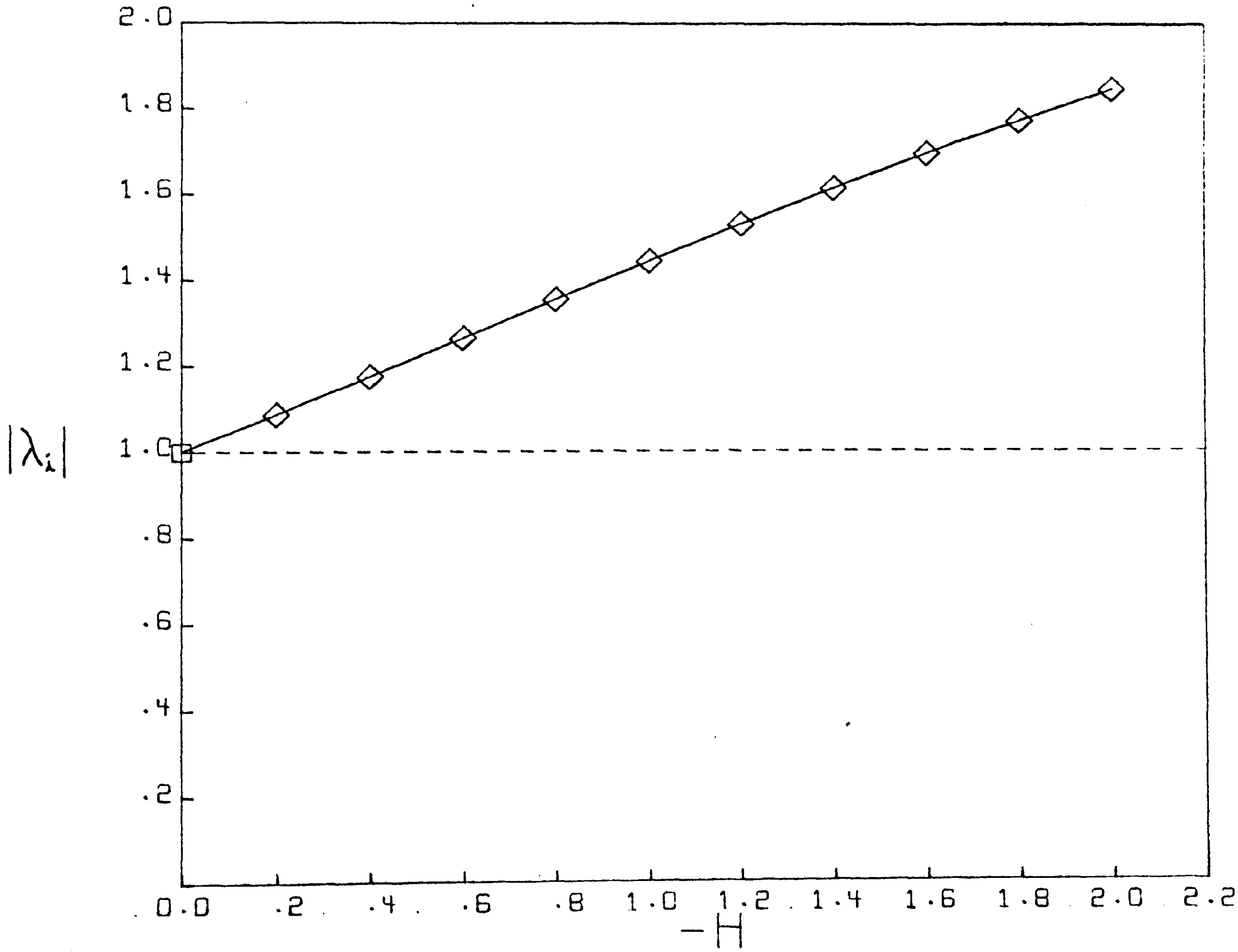

FIGURE 8A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 8.

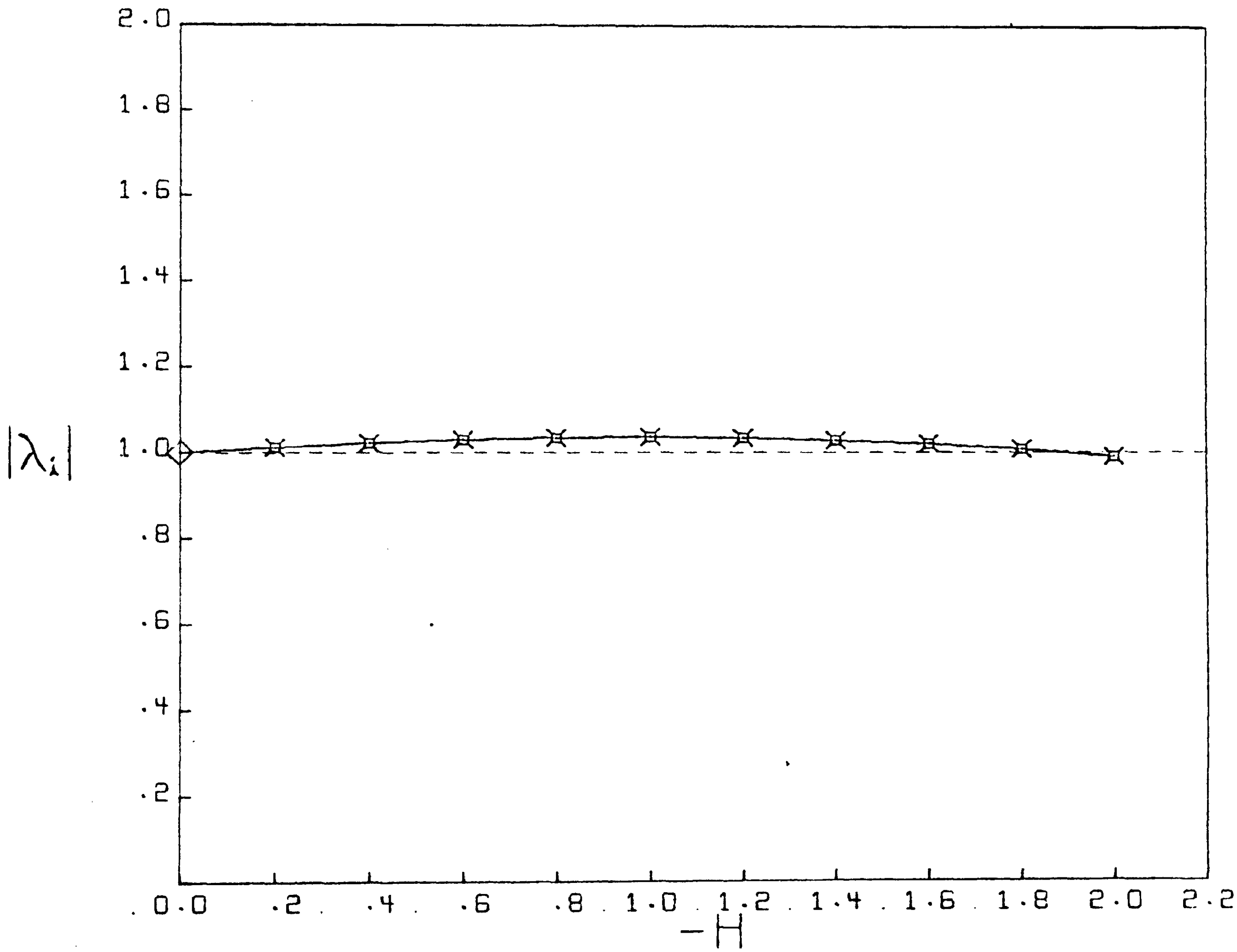

FIGURE 9A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 9.

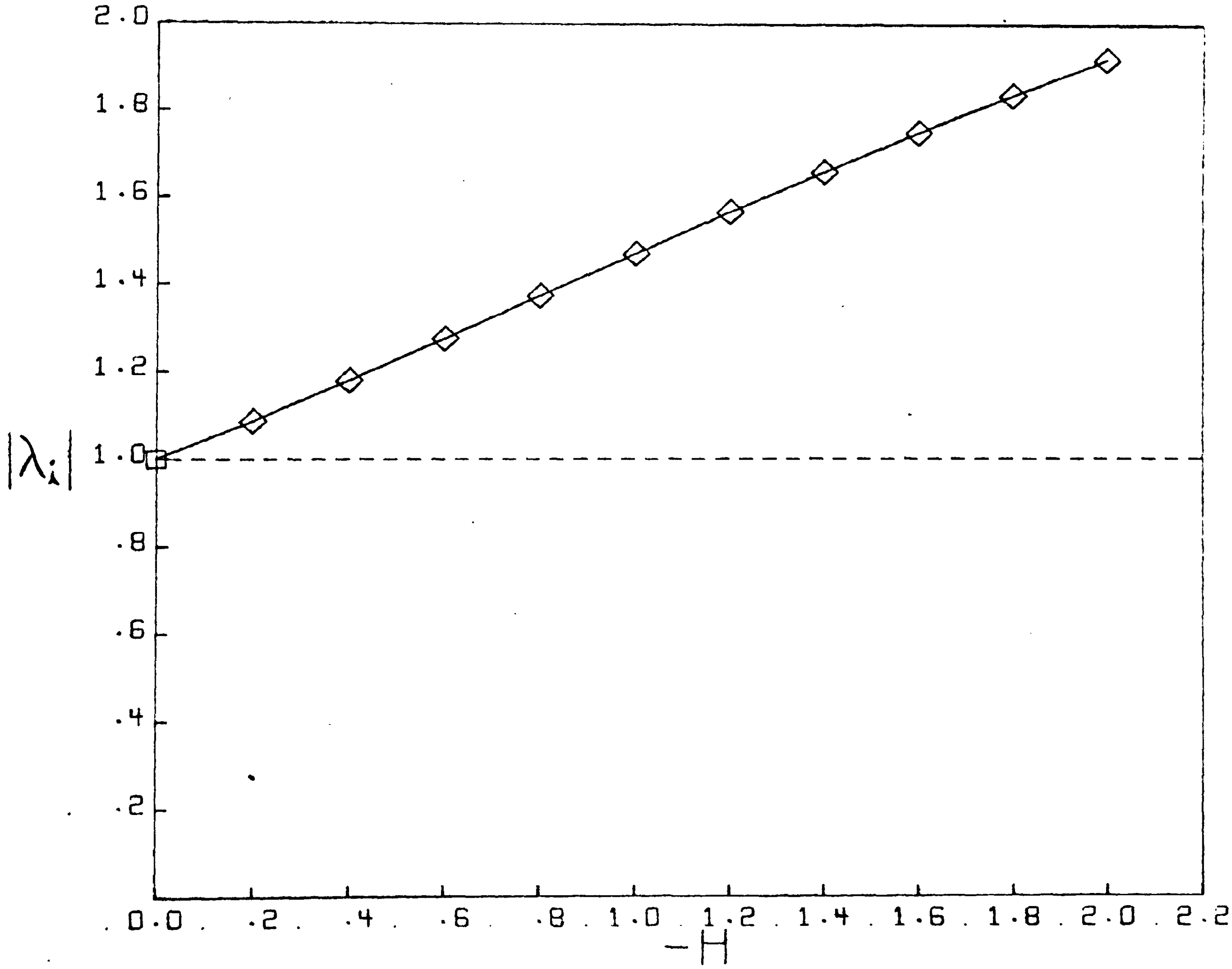

FIGURE 10A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 10.

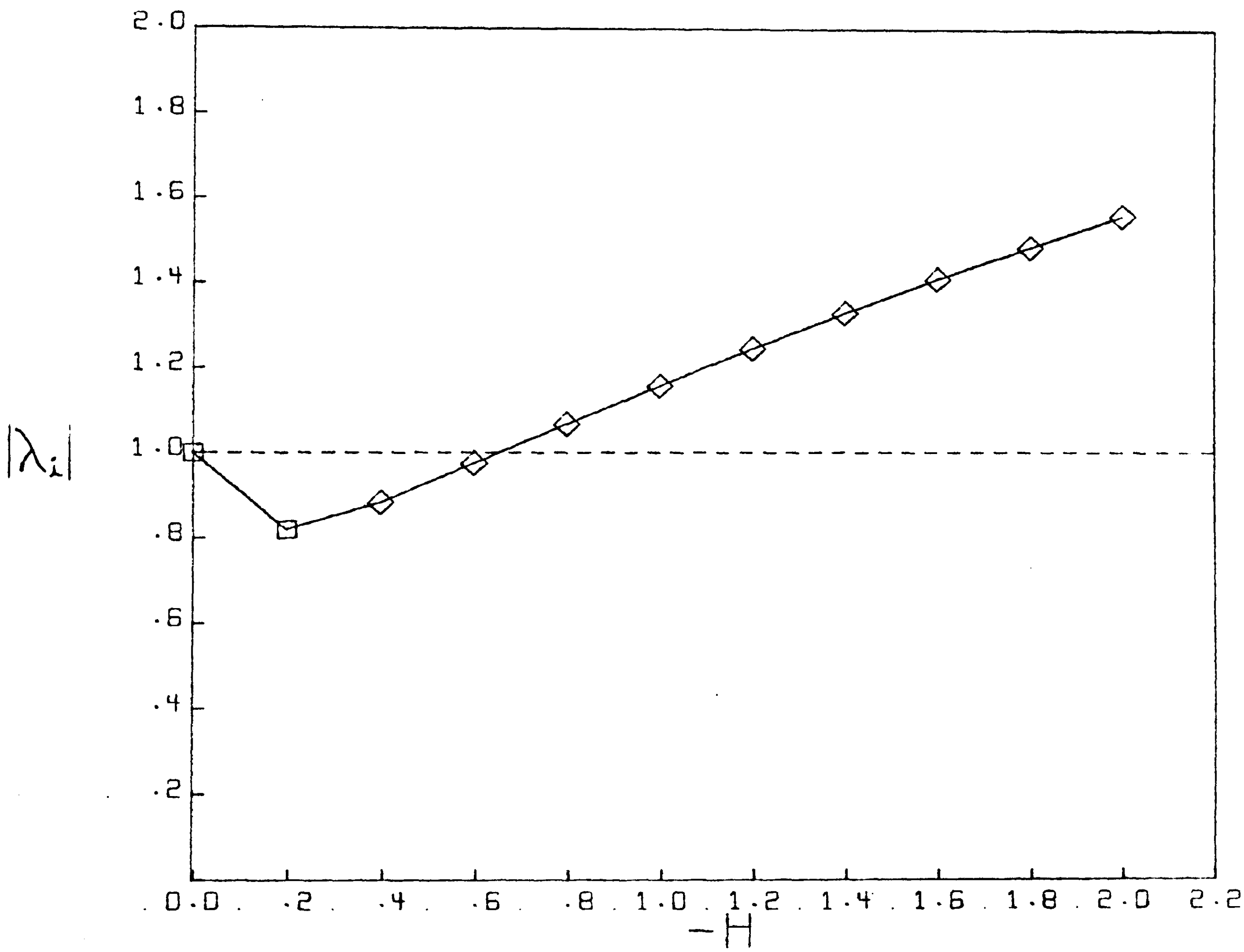

FIGURE 11A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 11.

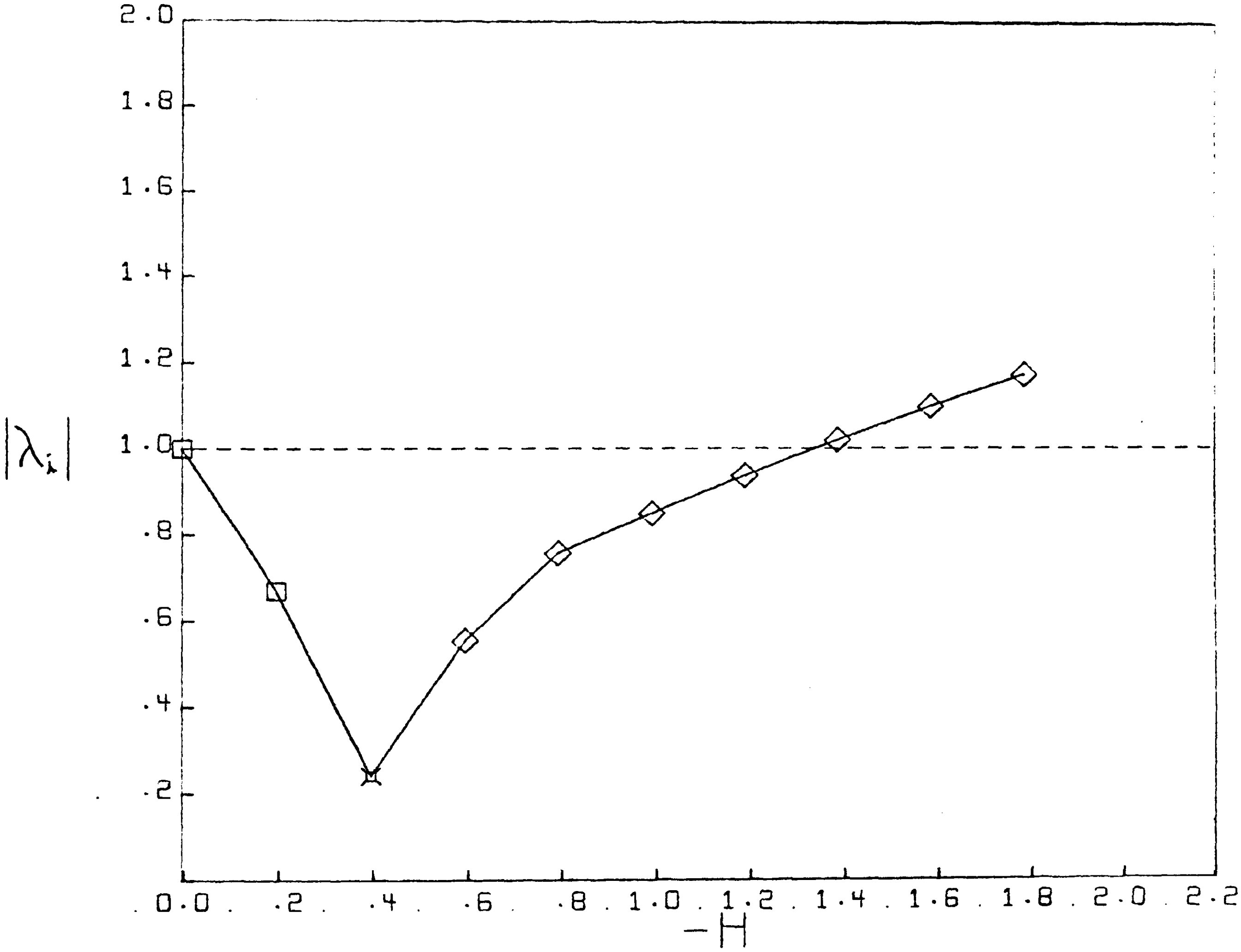

FIGURE 12A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 12.

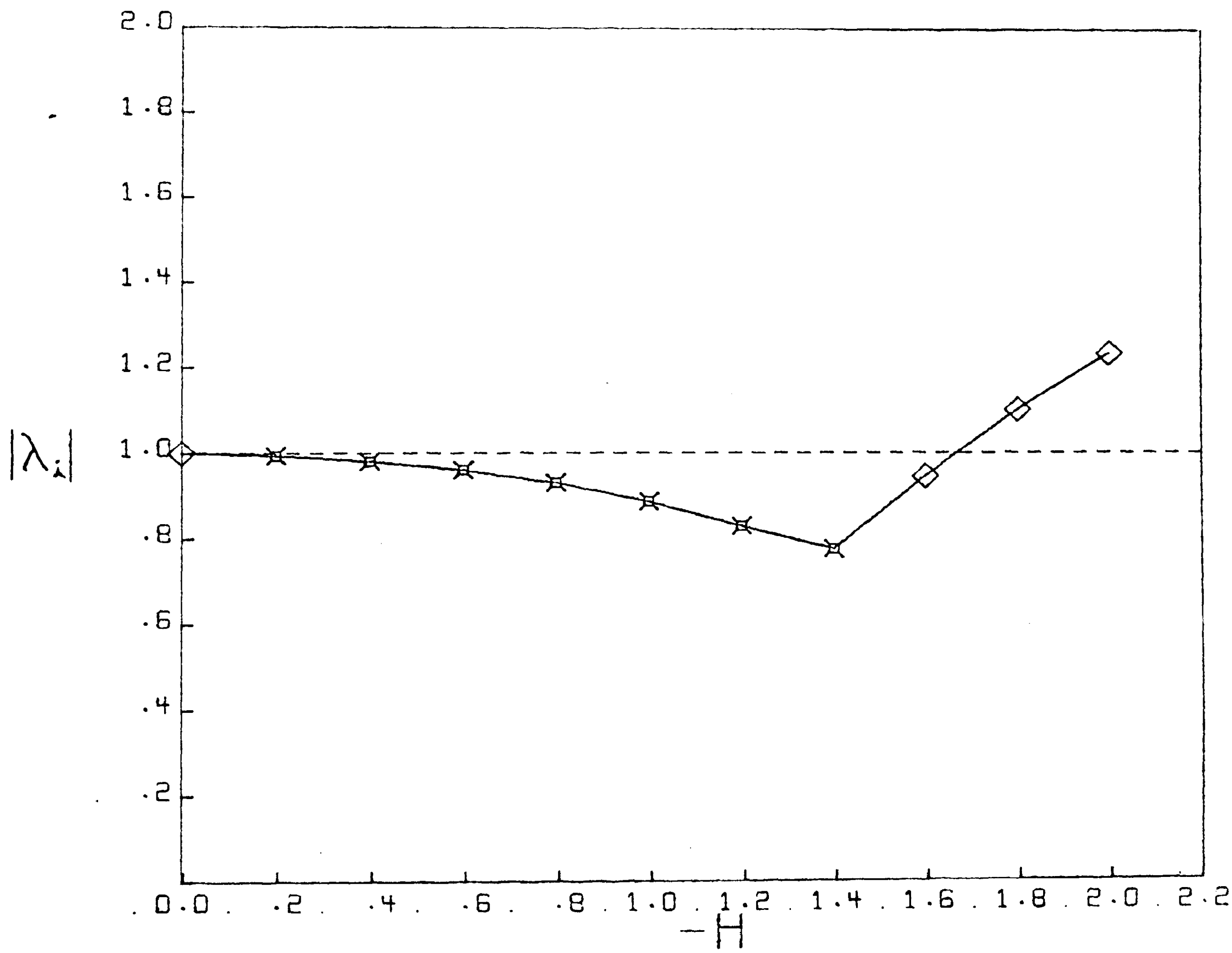

FIGURE 13A. DOMINANT ROOTS IN THE NEGATIVE H INTERVAL FOR FIGURE 13.

APPENDIX II

Predictors - Open Type Formulas

1. $$y_{n+1} = y_n + h(1 + \frac{1}{2}\nabla = \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^4 + \frac{251}{720}\nabla^4 + \frac{95}{288}\nabla^5 + \frac{7181}{30240}\nabla^6 + \cdots)y'_n$$

2. $$y_{n+1} = y_{n-1} + h(2 + \frac{1}{3}\nabla^2 + \frac{1}{3}\nabla^3 + \frac{29}{90}\nabla^4 + \frac{14}{45}\nabla^5 + \frac{13499}{60480}\nabla^6 + \cdots)y'_n$$

3. $$y_{n+1} = y_{n-2} + h(3 - \frac{3}{2}\nabla + \frac{3}{4}\nabla^2 + \frac{9}{24}\nabla^3 + \frac{27}{80}\nabla^4 + \frac{51}{160}\nabla^5 + \frac{479}{1120}\nabla^6 + \cdots)y'_n$$

4. $$y_{n+1} = y_{n-3} + h(4 - 4\nabla + \frac{8}{3}\nabla^2 + \frac{14}{45}\nabla^4 + \frac{14}{45}\nabla^5 + \frac{13579}{60480}\nabla^6 + \cdots)y'_n$$

5. $$y_{n+1} = y_{n-4} + h(5 - \frac{15}{2}\nabla + \frac{85}{12}\nabla^2 - \frac{55}{24}\nabla^3 + \frac{95}{144}\nabla^4 + \frac{95}{288}\nabla^5 + \frac{1385}{6048}\nabla^6 + \cdots)y'_n$$

6. $$y_{n+1} = y_{n-5} + h(6 - 12\nabla + 15\nabla^2 - 9\nabla^3 + \frac{33}{10}\nabla^4 + \frac{481}{2240}\nabla^6 + \cdots)y'_n$$

Correctors - Closed Type Formulas

1. $y_{n+1} = y_n + h(1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \frac{19}{720}\nabla^4 - \frac{3}{160}\nabla^5$

$- \frac{863}{60480}\nabla^6 - \cdots)y'_{n+1}$

2. $y_{n+1} = y_{n-1} + h(2 - 2\nabla + \frac{1}{3}\nabla^2 - \frac{1}{90}\nabla^4 - \frac{1}{90}\nabla^5$

$- \frac{37}{3780}\nabla^6 - \cdots)y'_{n+1}$

3. $y_{n+1} = y_{n-2} + h(3 - \frac{9}{2}\nabla + \frac{9}{4}\nabla^2 - \frac{3}{8}\nabla^3 - \frac{3}{80}\nabla^4 - \frac{3}{160}\nabla^5$

$- \frac{29}{2240}\nabla^6 - \cdots)y'_{n+1}$

4. $y_{n+1} = y_{n-3} + h(4 - 8\nabla + \frac{20}{3}\nabla^2 - \frac{8}{3}\nabla^3 + \frac{14}{45}\nabla^4$

$- \frac{8}{945}\nabla^6 - \cdots)y'_{n+1}$

5. $y_{n+1} = y_{n-4} + h(5 - \frac{25}{2}\nabla + \frac{175}{12}\nabla^2 - \frac{75}{8}\nabla^3 + \frac{425}{144}\nabla^4$

$- \frac{95}{288}\nabla^5 - \frac{275}{12096}\nabla^6 - \cdots)y'_{n+1}$

6. $y_{n+1} = y_{n-5} + h(6 - 18\nabla + 27\nabla^2 - 24\nabla^3 + \frac{123}{10}\nabla^4$

$- \frac{33}{10}\nabla^5 + \frac{41}{140}\nabla^6 - \cdots)y'_{n+1}$

VITA

The author was born March 2, 1942, at Princeton, Illinois. He received his primary education there and was graduated from Bureau Township High School in 1960. He received a Bachelor of Science Degree in Education, major in Mathematics, from Western Illinois University in Macomb, Illinois. He has been enrolled in the Graduate School of the University of Missouri at Rolla since September 1964.

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