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An Exploration of Doubly Transitive Designs in Affine Space

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AN EXPLORATION OF DOUBLY TRANSITIVE DESIGNS IN AFFINE SPACE

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I also am thankful for all of my friends who so often heard, "Not tonight, I'm working on my thesis." Thanks for all of your encouragement.

Lastly, I would like to thank my family for their love and support throughout this process.

Geometry has existed since the time of the Babylonians and Egyptians. It wasn't until about 300 B.C., however, that Euclid of Alexandria finally organized this geometric information and wrote his book The Elements. He formulated five specific postulates that led to almost 500 geometrical statements and theorems. For over 2000 years, mathematicians used Euclid's ideas to experiment with geometric designs.

In 1763, however, G.S. Klugel decided to evaluate the postulates Euclid proposed, and found that by eliminating some of the postulates, new types of geometry emerged (Ryan, 1-3). One of these emergences is called Affine Geometry which is the basis for my research of doubly-transitive designs in affine space.

Professor Tom Sibley's (SJU) intuitive knowledge as well as John Morrison's (class of '93, SJU) exploration of different 2-transitive designs have provided a foundation for my research. By expanding on their ideas, I have been able to extend the study of doubly-transitive designs to affine space.

We will begin with some primary definitions and theorems related to transitivity. Next, we will explore affine space and the properties that make it unique. Lastly, we will apply our definitions and theorems to find doubly-transitive designs in affine space.

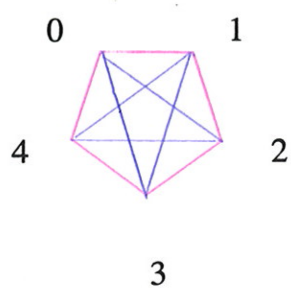
My research involves sets of points and a generalization of a distance relation on pairs of these points. A pair of distinct points A,B is called an edge and denoted AB (in n-dimensions this edge is denoted $\vec{a}\vec{b}$, where $\vec{a}$ and $\vec{b}$ are vectors). The generalized relation is called an equidistance relation, denoted $\equiv$.

Definition 1: Let $(S,\equiv)$ be a set S (whose elements will be called points), together with a relation $\equiv$ on pairs of distinct points in S. We say $(S,\equiv)$ is an **equidistance space** if and only if:

1) $\equiv$ is an equivalence relation.
2) For all A and B in S, $AB \equiv BA$.

*Note: An equidistance space will be denoted $(S, \equiv)$.

For example, let $S = \{0,1,2,3,4\}$ be the vertices in the following coloring of a pentagon, (K_5), and suppose that $AB \equiv CD$ if and only if AB and CD are the same color.



$01 \equiv 12 \equiv 23 \equiv 34 \equiv 40$
$02 \equiv 24 \equiv 41 \equiv 13 \equiv 30$

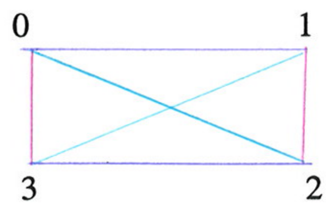
This figure is an equidistance space since:
1) $\equiv$ is an equivalence relation on pairs of points. When we examine the edges of K_5, we find the relation $\equiv$ is reflexive, symmetric, and transitive.
   a) Reflexive: For all AB in K_5, $AB \equiv AB$. e.g., $02 \equiv 02$.
   b) Symmetric: For all AB, CD in K_5, if $AB \equiv CD$, then $CD \equiv AB$. e.g., $02 \equiv 14$ and $14 \equiv 02$.
   c) Transitive: For all AB, CD, EF in K_5, if $AB \equiv CD$ and $CD \equiv EF$ then $AB \equiv EF$. e.g., $12 \equiv 23$, $23 \equiv 34$, and $12 \equiv 34$.
2) For all A and B in S, $AB \equiv BA$. e.g., $14 \equiv 41$.

The equidistance relation can be interpreted in several ways. Two edges can be equidistant if their lengths are the same. Yet others can be equidistant if they are parallel, perpendicular, etc. Equidistance does not necessarily hinge on distance. However, we call the relation an equidistance relation.

Definition 2: The figure $\mathbf{K_n}$ is the set of n vertices such that each pair of distinct vertices is connected by some edge.

All equidistance relations can be described as edge colorings of the complete graph on n vertices, K_n. The vertices of K_n are the points of the set S, and the colored edges of K_n represent the equidistance relation $\equiv$.

To demonstrate the interpretation of equidistance relation in terms of length, let us examine the following coloring of K_4.



The lengths of edge 01 and edge 32 are the same, therefore we color these edges the same color. We denote this relationship 01≡32. Similarly 12≡30 and 02≡13. The actual distance does not matter, only that the distances are equal.

Now that we have an understanding of the equidistance relation, we begin the explanation of transitivity, which hinges on the permutation of the points of an equidistance space.

Definition 3: A **similarity**, f, is a permutation on an equidistance space (S,≡) that preserves the equidistance relation. That is, f preserves ≡ if and only if for all AB, CD in S, if AB≡CD, then f(A)f(B)≡f(C)f(D).

For example, given the following coloring of K_4, the function f (x) = x+1 mod 4 permutes the points, sending 0→1, 1→2, 2→3, and 3→0. Note that horizontal edges are sent to vertical edges, vertical edges to horizontal edges, and diagonals to diagonals under this mapping.

Figure A

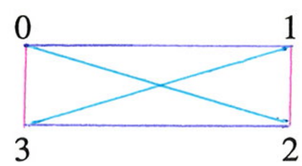
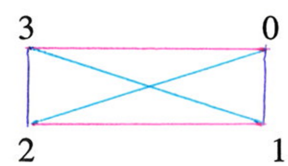


Figure B



In Figure A, edges that are horizontal are equidistant, as are vertical edges and diagonal edges. In Figure B, horizontal edges are still equidistant, as well as vertical and diagonal edges. The equidistance relation has been preserved. Therefore, f (x) = x+1 mod 4 is a similarity.

Definition 4: An equidistance space (S,≡) is said to be **1-transitive** if for any A and A' in (S,≡) there exists a similarity f such that f (A) = A'.

Definition 5: An equidistance space (S,≡) is said to be **2-transitive** if, given any four points A, B, A', and B' in (S,≡), where A≠B, A'≠B', there exists a similarity f such that f (A)=A' and f (B)=B'.

We will apply these definitions of transitivity to affine space. Theorems which provide us with ways to show a given equidistance relation on an affine space $(F^n, \equiv)$ is 1- or 2-transitive will follow an explanation of affine space and its properties.

Let F be any field, and let $\mathbf{F^n}$ be the n-dimensional vector space over F. Note that a 1-dimensional subspace of the space F^n is of the form $\{a\vec{v} : a \in F\}$ for some $\vec{v} \in F^n$, $\vec{v} \neq 0$.

Definition 6: A **line** is defined by $\{a\vec{v}+\vec{w} : a \in F\}$ for some $\vec{v},\vec{w} \in F^n$ and $\vec{v} \neq 0$. Thus, lines are cosets of 1-dimensional subspaces.

Definition 7: An **affine space** is defined as the vector space F^n , together with a set of lines.

Structures can be defined by distance between vertices, linearity, angles, or other geometric properties. Affine space is one example of a structure. The following are the axioms of an affine plane. The importance of these axioms is that they focus on linear structure.

Axioms of an Affine Plane:[Blumenthal, 55]

1) For each pair of points P,Q there exists a unique line L on P and Q.
2) For each point P and for each line L not on P, there exists a unique line M parallel to L such that P is on M. (See Appendix B for definition).
3) There exists a point P and a line L such that P is not on L.
4) For each line L there exist at least 3 points on L.

Let us examine the affine plane of 9 points to see how these axioms affect the structure. This plane contains 12 lines.

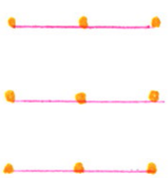


Figure 1


Figure 2

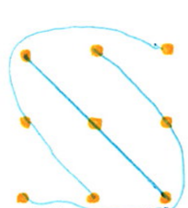


Figure 3

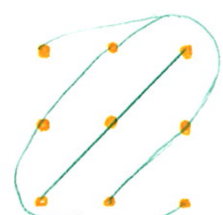


Figure 4

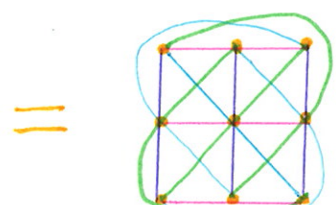
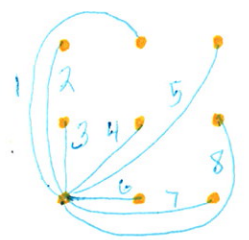


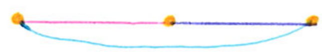
Figure 5

Figures 1,2,3 and 4 are simply a breakdown of figure 5 into groupings of parallel lines.

Each point has 8 incident edges.



Since each line contains three points, there are three edges on each line; one edge between each pair of points.



There are many different types of permutations of points in affine space. The following are examples.

Definition 8: A **translation** of F^n is a mapping of the form $\Omega_q(\vec{x}) = \vec{x}+\vec{q}$ for $\vec{q} \in F^n$.
*Note: Translations take lines to lines.

Definition 9: An **affine transformation** of F^n is a mapping of the form $\alpha(\vec{x}) = M\vec{x} + \vec{q}$ for $\vec{q} \in F$ and an invertible nxn matrix M.

Definition 10: AGL (F,n) is the group of all affine transformations on the n-dimensional affine space F^n.

Fact 1: AGL (F,n) is 2-transitive on F^n.
*Note: It is standard group theory to say that a group is 1-transitive on a set S if for all a, b in S, there exists a g in G such that g(a) = b. We have called the space itself 1-transitive. The same situation applies to 2-transitive.

Fact 2: The subgroup $H=\{\Omega_q: \vec{q} \in F^n\}$ is a one-transitive subgroup of AGL (F,n).

Fact 3: Every 2-transitive subgroup of AGL (F,n) has H as a 1-transitive (normal) subgroup.

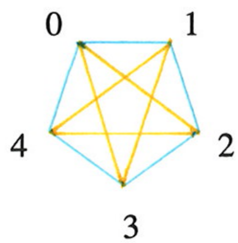
Fact 4: The set of similarities of a 1-transitive space forms a group under function composition.

Given the previous definitions and facts relating to affine space (Beth, 255), we are now able to prove two theorems which provide us with ways to show a given equidistance space is 1- or 2-transitive.

Theorem 1: Let $(F^n, \equiv)$ be an equidistance space where F^n is affine, and $\vec{a}$ a fixed point in F^n. Suppose if f is a similarity, then f^{-1} is a similarity. If for any point $\vec{a}'$ in F^n, there exists a similarity $f_{A'}$ such that $f_{A'}(\vec{a}) = \vec{a}'$, then $(F^n, \equiv)$ is 1-transitive.

Proof:
Let $\vec{b}, \vec{c} \in F^n$. Given a particular point $\vec{a}$ in F^n, let f_1 and f_2 be similarities such that $f_1(\vec{a}) = \vec{b}$ and $f_2(\vec{a}) = \vec{c}$. Show there exists a similarity g that takes $\vec{b}$ to $\vec{c}$. Pick $g = f_2(f_1^{-1})$. Then $g(\vec{b}) = f_2(f_1^{-1}(\vec{b})) = f_2(\vec{a}) = \vec{c}$. We know g is a similarity since f^{-1} is a similarity and the set of similarities is closed under function composition.
(See appendix C).
Therefore, $(F^n, \equiv)$ is 1-transitive.

For example, let us examine the following coloring of K_5.



$01 \equiv 12 \equiv 23 \equiv 34 \equiv 40$
$02 \equiv 24 \equiv 41 \equiv 13 \equiv 30$

Applying Theorem 1, if there are similarities f which move a particular point of K_5, say 0, to all points of K_5, then K_5 is 1-transitive.

If $f_0(x) = x+0 \bmod 5$	then $f_0(0) = 0 \bmod 5$
$f_1(x) = x+1 \bmod 5$	then $f_1(0) = 1 \bmod 5$
$f_2(x) = x+2 \bmod 5$	then $f_2(0) = 2 \bmod 5$
$f_3(x) = x+3 \bmod 5$	then $f_3(0) = 3 \bmod 5$
$f_4(x) = x+4 \bmod 5$	then $f_4(0) = 4 \bmod 5$

Therefore, since f_i is a similarity for all i, K_5 is 1-transitive.

*Note: Recall from linear algebra that all linear transformations can be represented as matrices with respect to a standard basis.

Theorem 2: Assume 1) The translations $\Omega_q(\vec{x}) = \vec{x}+\vec{q}$ preserve $\equiv$ for all q in F^n
and 2) There exist enough matrices which preserve $\equiv$ to map any particular non-zero vector to all other non-zero vectors.
Then the equidistance space $(F^n, \equiv)$ where F^n is an affine space, is 2-transitive.

Proof:
Assume 1) The translations $\Omega_q(\vec{x}) = \vec{x}+\vec{q}$ preserve $\equiv$ for all q in F^n.

2) There exist enough matrices which preserve $\equiv$ to map any particular non-zero vector to all other non-zero vectors.

1) Let $\vec{a}, \vec{b}, \vec{a}', \vec{b}' \in F^n$ with $\vec{a} \neq \vec{b}$ and $\vec{a}' \neq \vec{b}'$. Let f_1, f_2 be translations such that $f_1(\vec{0}) = \vec{a}'$ and $f_2(\vec{0}) = \vec{a}$.

Suppose $f_1(\vec{c}') = \vec{b}'$ and $f_2(\vec{c}) = \vec{b}$. Note that $\vec{c} \neq 0$, and $\vec{c}' \neq 0$ since $\vec{a}$ and $\vec{b}$ are distinct, and $\vec{a}'$ and $\vec{b}'$ are distinct.

By assumption 2, we know there exists a matrix M such that $M(\vec{c}) = (\vec{c}')$.

Then let $g = f_1 M f_2^{-1}$. Then $g(\vec{a}) = f_1 M f_2^{-1}(\vec{a}) = f_1 M(\vec{0}) = f_1(\vec{0}) = \vec{a}'$.

$$g(\vec{b}) = f_1 M f_2^{-1}(\vec{b}) = f_1 M(\vec{c}) = f_1(\vec{c}') = \vec{b}'$$

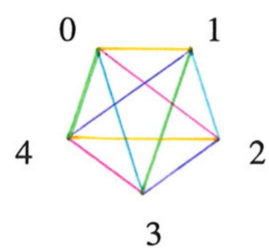
Therefore, given any four points $\vec{a}$, $\vec{b}$, $\vec{a}'$, and $\vec{b}'$ in $(F^n, \equiv)$ there exists a similarity g such that $g(\vec{a}) = \vec{a}'$ and $g(\vec{b}) = \vec{b}'$. Therefore $(F^n, \equiv)$ is 2-transitive.

***Note:** The following fact can be applied using Theorem 2 to prove that any equidistance space $(F^n, \equiv)$ where F^n is an affine space, is 2-transitive.

Fact 5: In an affine space F^n, for each non-zero vector $\vec{v}$, and each non-zero vector $\vec{a}$, there is a non-singular matrix M such that $M\vec{a} = \vec{v}$. (See Appendix A for Proof).

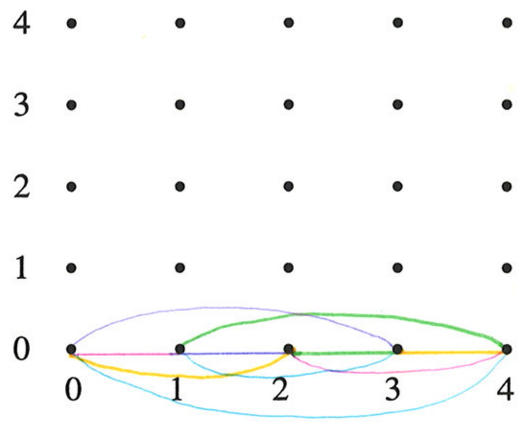
We know that AGL (F,n) is 2-transitive over F^n and that there are structures that are compatible with these affine space transformations. That is, the affine transformations preserve the equidistance relation on the structure. Using the theorems and properties of affine space, I have been able to find two families of compatible structures. I have also found a structure of the affine plane that is compatible with its affine transformations. The proofs that these spaces are 2-transitive follow.

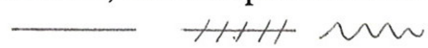
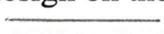
There are two specific structures that I have extended to affine space. The first is a coloring of K_5 such that parallel edges are equidistant.



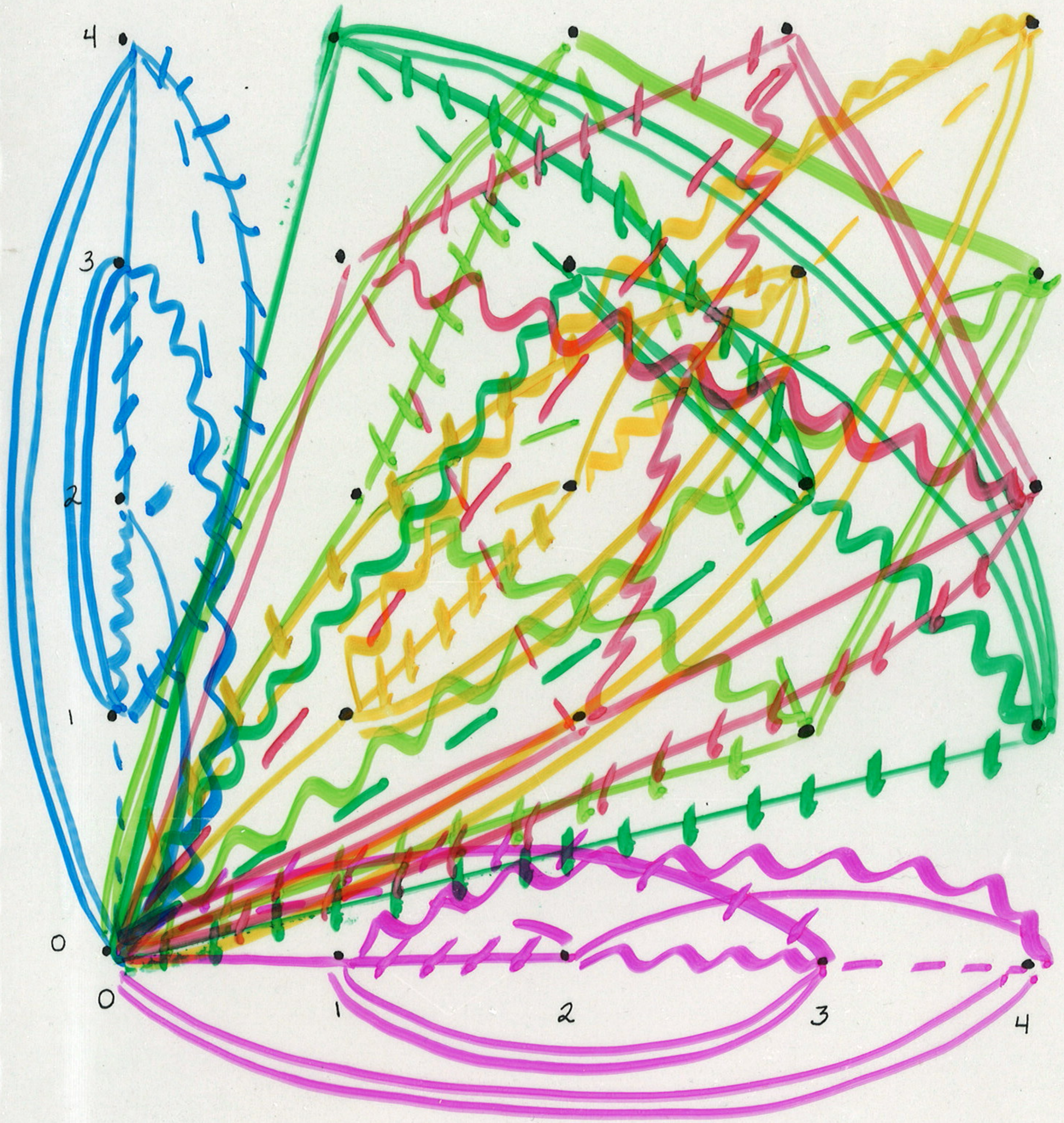
01 ≡ 42
12 ≡ 03
23 ≡ 14
34 ≡ 02
40 ≡ 31

I have extended this structure to a coloring of the lines of the affine plane. In this coloring of K_5, edge $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ if and only if $\vec{a}+\vec{b} = \vec{c}+\vec{d}$ mod 5. Thus when we look at the affine plane $(Z_5)^2$,



edges $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ when $(a_x + b_x, a_y + b_y) = (c_x + d_x, c_y + d_y)$. Every line of $(Z_5)^2$ is colored using the same criterion. There are 5 colors on each line, 6 different slopes, and 5 lines of each slope, which leaves us with 150 different colors. Because this design has so many colors, it is very difficult to represent. Therefore, I have represented 30 colors of this design on the following page. Note that  and 

represent different "colors".


4
3
2
1
0
0
1
2
3
4

I have generalized this idea to any affine space over any field. The following is a proof that this space is two-transitive.

Theorem 3: Define $\equiv$ on F^n by $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ iff $\vec{a} + \vec{b} = \vec{c} + \vec{d}$ and $\vec{a}\vec{b}$ and $\vec{c}\vec{d}$ are on the same line. Then $(F^n, \equiv)$ is 2-transitive.

We can prove this theorem by using Theorem 2.

Proof:

1) Show that the translations $\Omega_q(\vec{x}) = \vec{x} + \vec{q}$ preserve $\equiv$ for all $\vec{q}$ in F^n.

Given $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ and any translation Ω_q. Show $\Omega_q(\vec{a})\Omega_q(\vec{b}) \equiv \Omega_q(\vec{c})\Omega_q(\vec{d})$.

Since $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ we know that

$$\vec{a}+\vec{b} = \vec{c}+\vec{d}$$
$$\vec{a}+\vec{b}+2\vec{q} = \vec{c}+\vec{d}+2\vec{q}$$
$$\vec{a}+\vec{q}+\vec{b}+\vec{q} = \vec{c}+\vec{q}+\vec{d}+\vec{q}$$
$$\Omega_q(\vec{a})+ \Omega_q(\vec{b})= \Omega_q(\vec{c})+ \Omega_q(\vec{d})$$

and since translations take lines to lines, edges $\Omega_q(\vec{a})\Omega_q(\vec{b})$ and $\Omega_q(\vec{c})\Omega_q(\vec{d})$ are still on the same line. Therefore, the translations $\Omega_q(\vec{x}) = \vec{x} + \vec{q}$ preserve $\equiv$ for all $\vec{q}$ in F^n.

2) Show there exist enough matrices that preserve $\equiv$ to map any particular non-zero vector to all other non-zero vectors.

Define $\alpha(\vec{x}) = M\vec{x}$, where M is any non-singular nxn matrix.

Given $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$. Show $\alpha(\vec{a})\alpha(\vec{b}) \equiv \alpha(\vec{c})\alpha(\vec{d})$.

Since $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ we know that

$$\vec{a}+\vec{b} = \vec{c}+\vec{d}$$
$$M(\vec{a}+\vec{b}) = M(\vec{c}+\vec{d})$$
$$M\vec{a}+M\vec{b} = M\vec{c}+M\vec{d}$$
$$\alpha(\vec{a})+ \alpha(\vec{b})= \alpha(\vec{c})+ \alpha(\vec{d})$$

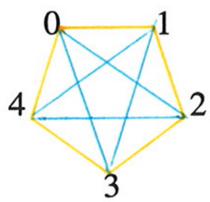
and since matrices take lines to lines, edges $M(\vec{a})M(\vec{b})$ and $M(\vec{c})M(\vec{d})$ are still on the same line.

Therefore, since all matrices preserve $\equiv$, and by application of Fact 5, there exist enough matrices that preserve $\equiv$ to map any particular non-zero vector to all other non-zero vectors. Hence, this design is two-transitive by Theorem 2.

The following definition is necessary for understanding the next design.

Definition 11: The **absolute value** of a vector $\vec{x}$, denoted $|\vec{x}|$, is the set $\{\vec{x}, -\vec{x}\}$. e.g., if $\vec{x}=(1,-2,3)$, then $|\vec{x}|=\{(1,-2,3), (-1,2,-3)\}$.

The next design that I have extended to affine space is the following.



01≡12≡23≡34≡40
02≡24≡41≡13≡30

Note that $|0-1| = |1-2| = |2-3|$, etc. for edges of the first color, and similarly $|0-2| = |1-3| = |2-4|$, etc. for edges of the second color.

One line of the corresponding affine plane would look like the following:

I have generalized this idea to affine spaces of any dimension over any field. The proof that this design is two-transitive follows.

Theorem 4: Define ≡ on F^n by $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ if and only if $|\vec{a} - \vec{b}| = |\vec{c} - \vec{d}|$. Then $(F^n, \equiv)$ is 2-transitive.

Proof:

1) Given $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ and any translation Ω_q, show that $\Omega_q(\vec{a})\Omega_q(\vec{b}) \equiv \Omega_q(\vec{c})\Omega_q(\vec{d})$.

Since $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ we know that

$$|\vec{a}-\vec{b}|=|\vec{c}-\vec{d}|$$
$$|\vec{a}-\vec{b}+\vec{q}-\vec{q}|=|\vec{c}-\vec{d}+\vec{q}-\vec{q}|$$
$$|(\vec{a}+\vec{q})-(\vec{b}+\vec{q})|=|(\vec{c}+\vec{q})-(\vec{d}+\vec{q})|$$
$$|\Omega_q(\vec{a})-\Omega_q(\vec{b})|=|\Omega_q(\vec{c})-\Omega_q(\vec{d})|$$

Therefore, the translations $\Omega_q(\vec{x}) = \vec{x} + \vec{q}$ preserve ≡ for all $\vec{q}$ in F^n.

2) Define $\alpha(\vec{x}) = M\vec{x}$, where M is any non-singular nxn matrix.
Given $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ show that $\alpha(\vec{a})\alpha(\vec{b}) \equiv \alpha(\vec{c})\alpha(\vec{d})$.

Since $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ we know that

$$|\vec{a}-\vec{b}|=|\vec{c}-\vec{d}|$$
$$|M(\vec{a}-\vec{b})|=|M(\vec{c}-\vec{d})|$$
$$|M(\vec{a})-M(\vec{b})|=|M(\vec{c})-M(\vec{d})|$$
$$|\alpha(\vec{a})-\alpha(\vec{b})|=|\alpha(\vec{c})-\alpha(\vec{d})|$$

Therefore, since all matrices preserve $\equiv$, and by application of Fact 5, there exist enough matrices that preserve $\equiv$ to map any particular non-zero vector to all other non-zero vectors.

Hence, $(F^n, \equiv)$ is two-transitive by Theorem 2.

This last design is my favorite because it has required the most thought and hard work. It originated by using the normal inner product to define perpendicular lines. Usually we define $(a_x, a_y)\bullet(b_x, b_y) = a_x b_x + a_y b_y$ and (a_x, a_y) is orthogonal to (b_x, b_y) if and only if their inner product is 0. For a number of fields, such as Z_3 and Z_7, this inner product works because distinct lines are perpendicular. For others, such as Z_5, this usual inner product doesn't work. For example, $(1,2)\bullet(1,2) = 0 \bmod 5$, but we don't want a vector to be perpendicular to itself. Therefore, it was necessary to change the definition of inner product.

Definition 12: For $p \in F$, $p \neq 0$ we define a **new inner product** $(a_x, a_y)\bullet_p(b_x, b_y) = p a_x b_x + a_y b_y$. Equivalently, we can represent the inner product $\bullet_p$ by a matrix $P = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$.

Then $\vec{a}\bullet_p\vec{b} = \vec{a}^T P \vec{b}$, where $\vec{a}, \vec{b}$ are column vectors.

Definition 13: For $\vec{a} \neq \vec{b}$, $\vec{c} \neq \vec{d}$, $\vec{a}\vec{b}$ is **parallel** to $\vec{c}\vec{d}$ if and only if there exists a non-zero $k \in F$ such that $(\vec{d}-\vec{c}) = k(\vec{b}-\vec{a})$. We write $\vec{a}\vec{b} \parallel \vec{c}\vec{d}$.

Definition 14: For $\vec{a} \neq \vec{b}$, $\vec{c} \neq \vec{d}$, $\vec{a}\vec{b}$ is **perpendicular** to $\vec{c}\vec{d}$ if and only if $(\vec{b}-\vec{a})\bullet_p(\vec{d}-\vec{c})=0$, where $\bullet_p$ is a new inner product. We write $\vec{a}\vec{b} \perp_p \vec{c}\vec{d}$.

Theorem 5: On F^2, for $\vec{a} \neq \vec{b}$, $\vec{c} \neq \vec{d}$, define $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ if and only if $\vec{a}\vec{b} \parallel \vec{c}\vec{d}$ or $\vec{a}\vec{b} \perp_p \vec{c}\vec{d}$. Then $(F^2, \equiv)$ is 2-transitive, provided p satisfies $px^2+y^2 \neq 0$ for all x, y ϵF, such that x,y are not both 0.

Proof:
1) Given: $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$ and any translation $\Omega_q(\vec{x}) = \vec{x}+\vec{q}$. Show $\Omega_q(\vec{a})\Omega_q(\vec{b}) \equiv \Omega_q(\vec{c})\Omega_q(\vec{d})$.

Let $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$.
Case 1) Assume that $\vec{a}\vec{b} \perp_p \vec{c}\vec{d}$. So $(\vec{b}-\vec{a}) \bullet_p (\vec{d}-\vec{c}) = 0$.

$$(\vec{b}-\vec{a}) \bullet_p (\vec{d}-\vec{c}) = 0$$
$$[(\vec{b}+\vec{q})-(\vec{a}+\vec{q})] \bullet_p [(\vec{d}+\vec{q})-(\vec{c}+\vec{q})] = 0$$
$$[\Omega_q(\vec{b})-\Omega_q(\vec{a})] \bullet_p [\Omega_q(\vec{d})-\Omega_q(\vec{c})] = 0$$
$$\Omega_q(\vec{a})\Omega_q(\vec{b}) \perp_p \Omega_q(\vec{c})\Omega_q(\vec{d})$$

Therefore, $\Omega_q(\vec{a})\Omega_q(\vec{b}) \equiv \Omega_q(\vec{c})\Omega_q(\vec{d})$.

Case 2) Assume that $\vec{a}\vec{b} \parallel \vec{c}\vec{d}$. So for some k$\epsilon$F, where $k \neq 0$, $(\vec{d}-\vec{c})=k(\vec{b}-\vec{a})$.

$$(\vec{d}-\vec{c}) = k(\vec{b}-\vec{a})$$
$$[(\vec{d}+\vec{q})-(\vec{c}+\vec{q})]= k[(\vec{b}+\vec{q})-(\vec{a}+\vec{q})]$$
$$[\Omega_q(\vec{d})-\Omega_q(\vec{c})] = k[\Omega_q(\vec{b})-\Omega_q(\vec{a})]$$
$$\Omega_q(\vec{a})\Omega_q(\vec{b}) \parallel \Omega_q(\vec{c})\Omega_q(\vec{d})$$

Therefore, $\Omega_q(\vec{a})\Omega_q(\vec{b}) \equiv \Omega_q(\vec{c})\Omega_q(\vec{d})$.

2) Show there exist enough matrices that preserve $\equiv$ to map any particular non-zero vector to all other non-zero vectors.

Define $\alpha(\vec{x}) = M\vec{x}$ where M is a non-singular 2x2 matrix $\begin{bmatrix} g & h \\ j & m \end{bmatrix}$ such that $p(g)(h)+(j)(m) = 0$ and $p(g^2)+(j^2) = p[p(h^2)+(m^2)]$.

Proof that these matrices exist:

Since M is non-singular, $(g,j) \epsilon F^2 \neq (0,0)$. Show there exists $(h,m) \epsilon F^2$ such that $pgh+jm = 0$ and $p(g^2)+(j^2) = p[p(h^2)+(m^2)]$.

Suppose $g \neq 0$, let m = g or -g, and set $h= (-jm)/(pg) = \pm j/p$. Note that $pgh+jm = 0$, and that h is well-defined unless p or g = 0. We know p≠0 because $\bullet_p$ is an inner product and we have stated that g≠0.

We show $p(g^2)+j^2 = p[p(h^2)+m^2]$.

$$\begin{aligned} p[p(h^2)+m^2] &= p[p\{(j^2m^2)/(p^2g^2)\}+g^2] \\ &= p[p\{(j^2g^2)/(p^2g^2)\}+g^2] \\ &= (p^2j^2)/p^2+pg^2 \\ &= pg^2+j^2. \end{aligned}$$

Now we consider the case when g = 0.
Given (0,j) find (h,m) such that
1) $pgh+jm = 0$ and
2) $p(g^2)+j^2 = p[p(h^2)+m^2]$.

Let m = 0 and $h = \pm (j/p)$
Then $pgh+jm = p(0)h+j(0) = 0$, and $p[p(h^2)+m^2] = p^2(j/p)^2+pm^2 = pg^2+j^2$.

Therefore, these matrices exist.

Now, show these matrices preserve ≡.

Suppose $\vec{a}\vec{b} \equiv \vec{c}\vec{d}$. Show $\alpha(\vec{a})\alpha(\vec{b}) \equiv \alpha(\vec{c})\alpha(\vec{d})$.
Case 1) Assume that $\vec{a}\vec{b} \perp_p \vec{c}\vec{d}$, so $(\vec{b}-\vec{a})^T P(\vec{d}-\vec{c}) = 0$. Let the matrix $A = (\vec{b}-\vec{a})$, $B = (\vec{d}-\vec{c})$. Then $A^TPB = 0$. To show $(MA)^TP(MB) = 0$, first note that $(MA)^TP(MB) = A^TM^TPMB$. Therefore it suffices to show that $M^TPM = \lambda P$ where $\lambda \in F$, $\lambda \neq 0$, since by substitution: $A^TM^TPMB = A^T\lambda PB = \lambda(A^TPB) = \lambda(0) = 0$.

Show that $M^TPM = \lambda P$. Let $\lambda = ph^2 + m^2$, and note that h,m are not both 0 since the matrix is non-singular. Then

$$\begin{bmatrix} g & j \\ h & m \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g & h \\ j & m \end{bmatrix} = \begin{bmatrix} pg & j \\ ph & m \end{bmatrix} \begin{bmatrix} g & h \\ j & m \end{bmatrix} = \begin{bmatrix} pg^2+j^2 & pgh+jm \\ pgh+jm & ph^2+m^2 \end{bmatrix} = \begin{bmatrix} p(ph^2+m^2) & 0 \\ 0 & (ph^2+m^2) \end{bmatrix} = (ph^2+m^2) \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}$$

Then $M^TPM = \lambda P$.

Case 2) Assume that $\vec{a}\vec{b} \parallel \vec{c}\vec{d}$, so for some scalar $k \in F$, $(\vec{d}-\vec{c}) = k(\vec{b}-\vec{a})$.
Let the matrix $A = (\vec{b}-\vec{a})$, $B = (\vec{d}-\vec{c})$. Then $B = kA$. Show $MB = kMA$.

$$\text{Since } B = kA, \text{ then } MB = M(kA) = kMA$$

Therefore, $\alpha(\vec{a})\alpha(\vec{b}) \equiv \alpha(\vec{c})\alpha(\vec{d})$.

Hence, the set of matrices M preserve $\equiv$, and we have shown there exist enough 2x2 matrices to take the vector (1,0) to all other non-zero vectors (g,j). Therefore there exist enough matrices M that preserve $\equiv$ to map any particular non-zero vector to all other non-zero vectors.

Therefore, $(F^2, \equiv)$ is two-transitive.

The exciting part of researching doubly-transitive design in affine space is the fact that I used many different disciplines of mathematics. It was terrific seeing how linear algebra, geometry, and algebraic structures could all fit together into one project. This subject area has not been widely explored and there are many different avenues one could take in continuing with this research. Since the last theorem has only been proven for 2 dimensions, it would be interesting to see if the inner product design could be generalized into all dimensions. I would also like to know why the new definition of inner product works.

Other designs to pursue include those which arise from combining designs. For example, the parallel and absolute value designs could be combined. My intuition tells me these are strong possibilities for doubly-transitive designs, however, time hasn't permitted me to explore them.

One area that I did examine was the projective plane. I found early on, however, that structures of this plane are not doubly transitive using just affine transformations. I challenge the next honors student to delve in.

Appendix A

Fact 5: In an affine space F^n, for each non-zero vector $\vec{v}$, and each non-zero vector $\vec{a}$, there is a non-singular matrix M such that $M\vec{a}=\vec{v}$.

Proof:

Let $\vec{b}= (1,0,0,0,....,0)$. Let $\vec{v} = (a_{11}, a_{12}, a_{13}, ...a_{1k}, ...a_{1n})$, where $a_{1k}\neq 0$.

Let M be the nxn matrix with the following stipulations: Let the first column vector of M $= \vec{v}$. Then let $a_{k1} = 1$, $a_{kk} = 0$, and $a_{ii} = 1$ for $i > 1$, $i \neq k$. Let all other terms be 0.

$$\begin{bmatrix} a_{11} & 0 & 0 & . & . & . & 1 & . & . & 0 \\ a_{12} & 1 & 0 & & & & & & & \\ a_{13} & 0 & 1 & & & & & & & \\ . & & & & & & & & & \\ a_{1k} & & & & & & 0 & & & \\ . & & & & & & & & & \\ a_{1n} & & & & & & & & & 1 \end{bmatrix}$$

Then $M(\vec{b})=\begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ . \\ a_{1k} \\ . \\ a_{1n} \end{bmatrix}$.

Show that M is non-singular:

The columns of M form a basis. Therefore, every vector can be written as a combination of these n column vectors.

Therefore: n = column rank of M (Fraleigh, 146) = rank M (Fraleigh, 149).

Since the rank of the nxn matrix M = n, M is non-singular (Fraleigh, 150).

Thus, for each non-zero vector $\vec{v}$, there is a non-singular matrix M such that $M(1,0,0,.....,0)=\vec{v}$.

Now, let M_a, M_v be non-singular nxn matrices such that $M_a(\vec{b}) = \vec{a}$ and $M_v(\vec{b}) = \vec{v}$ where $\vec{b} = (1,0,0,....,0)$.

Show that there exists a non-singular nxn matrix M such that $M(\vec{a}) = \vec{v}$.

Let $M = M_vM_a^{-1}$. Then $M(\vec{a}) = M_vM_a^{-1}(\vec{a}) = M_v(\vec{b}) = \vec{v}$.

Therefore, for each non-zero vector $\vec{v}$, and each non-zero vector $\vec{a}$, there is a non-singular matrix M such that $M\vec{a}=\vec{v}$.

Appendix B

Two distinct lines with a point in common are called *intersecting* and are said to *intersect*; if they do not intersect, they are mutually **parallel**. If two lines are mutually parallel, each is said to be parallel to the other, and vice versa (Blumenthal, 55).

Appendix C

Proof that the set of similarities is closed under function composition:

Let R be the set of all similarities on an equidistance space $(S, \equiv)$.
Let f_1 and $f_2 \in R$. Show $g = f_1f_2 \in R$.

Assume $f_1(E) = I$ $\quad f_2(A) = E$
$f_1(F) = J$ $\quad f_2(B) = F$
$f_1(G) = K$ $\quad f_2(C) = G$
$f_1(H) = L$ $\quad f_2(D) = H$

Then $g(A) = f_1f_2(A) = f_1(E) = I$
$g(B) = f_1f_2(B) = f_1(F) = J$
$g(C) = f_1f_2(C) = f_1(G) = K$
$g(D) = f_1f_2(D) = f_1(H) = L$

Let $AB \equiv CD$ then since $f_2 \in R$, $EF \equiv GH$, and since $f_1 \in R$, then $IJ \equiv KL$.
But $IJ = g(A)g(B)$ and $KL = g(C)g(D)$. Therefore $g(A)g(B) \equiv g(C)g(D)$. Thus $g \in R$.

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