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# Colored Independence of Cycle Graphs and Finite Grids

Michael Terhaar
*College of Saint Benedict/Saint John's University*

COLORED INDEPENDENCE ON CYCLE GRAPHS AND FINITE GRIDS

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Michael Terhaar

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## Section 1: Introduction

The earliest problem in graph theory is the Konigsberg Bridge Problem. The city of Konigsberg had seven bridges connecting four landmasses. As a game, the people in this city tried to walk around the city crossing each bridge exactly once. None of the residents seemed to be able to do this, but they could not prove that it was impossible. The Konigsberg Bridge Problem was solved by Leonard Euler who utilized what Gottfried Wilhelm Leibniz "referred to as *geometria situs*, or geometry of position. This so-called geometry of position is what is now called graph theory" [4]. Euler showed that it was impossible to walk around the city crossing each bridge exactly once. His reasoning in solving the Konigsberg Bridge Problem set the foundation for the concept of Eulerian circuits (to be defined in the next section) which is an integral part of much that has been proven in this paper.

Before delving too deeply into this subject, it is important to know just what is meant by a graph in the context of graph theory. When most people hear the word graph, they immediately think of a graph as it would be on a Cartesian coordinate system with the $x$ and $y$ axes and ordered pairs. Although this is certainly a graph in certain fields of math, this is not the type of graph we study in graph theory. In this paper, we will define a graph in the following way:

**Definition 1:** A graph is a set of vertices $V = \{v_1, v_2, v_3, ...\}$ with a set of edges $E = \{e_1, e_2, e_3, ...\}$ on those vertices. An edge is constructed by connecting two distinct vertices $v_i$ and $v_j$. We say $v_iv_j \in E$. The set of vertices of a graph $G$ will be denoted $V(G)$ and the set of edges $E(G)$. If two vertices form an edge, they are said to be adjacent.

Independence is a crucial topic within the scope of graph theory. We define independence in the following way:

**Definition 2:** Let $I$ be a set of vertices on a graph $G$. $I$ is said to be independent if for every $u, v \in I$, $uv$ is not an edge in $G$.

For an example of independent sets, consider the following scenario. There are nine computers connected on a network and three printers (printer 1, printer 2, and printer 3) that service this network. Three computers print to printer 1, three to printer 2, and three to printer 3. We say the three computers that print to printer 1 are independent of printer 2 and printer 3, the three computers that print to printer 2 are independent of printer 1 and printer 3, and the three computers that print to printer 3 are independent of printer 1 and printer 2. This example can be seen in Figure 1 below. The blue vertices represent computers, and the green vertices represent the three printers.

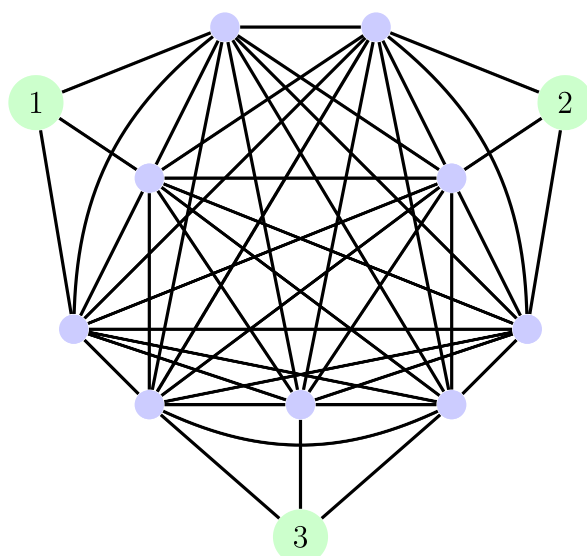


Figure 1: Graphical model of computer and printer network

This example illustrates that independence has to do with connections and lack thereof. Since the three computers that print to printer 1 have no connection to printer 2 and printer 3, they are independent of printer 2 and printer 3. In graph theory, independence works the same way. To describe events, we use vertices, and to describe connections, we use edges between vertices. If two vertices are connected by an edge, they are not independent, but if they are not connected by an edge, they are independent.

Vertex coloring is another concept used when looking into problems in graph theory.

**Definition 3:** We color a vertex $v$ by assigning it a color $i$. A set of vertices $S_i$ is a color class if every vertex in $S_i$ is colored $i$.

To gain a deeper understanding of vertex coloring, consider the following scheduling problem. Suppose a school is attempting to schedule times for the final test of each class the school provides. They have a total of 6 classes offered. Figure 2 below represents classes that cannot be scheduled at the same time as each other. The classes are represented by vertices and the conflicts are represented by edges.

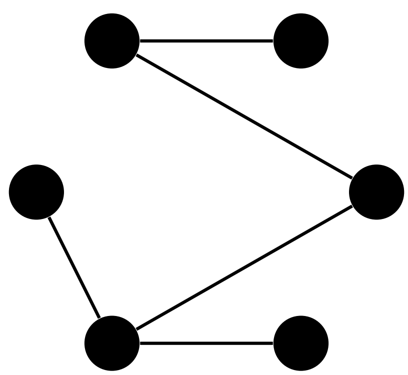

Figure 2: Graphical model of six classes and their conflicts

We can color the vertices of this graph in order to see how many testing times we will need to schedule all the final exams. Figure 3 below shows that we will need two time slots since all vertices can be colored with two color classes. All the classes colored red can be given during the same time slot and all the classes colored green can be given during a second time slot.

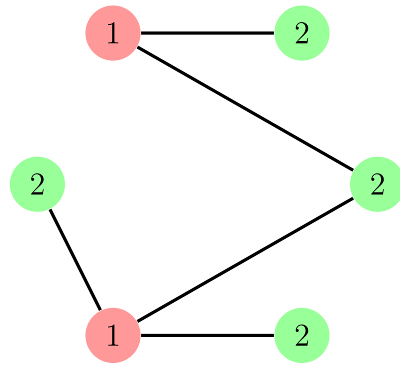


Figure 3: Test time scenario for six classes with conflicts modeled by vertex coloring

This example shows how vertex coloring can be a useful tool in solving scheduling problems.

In this paper, we will use coloring to signify vertices that must be placed together into independent sets. As described in Definition 3, the vertices that are all colored the same make up what is called a color class. This paper will focus on color classes that are also independent sets.

Thus far, we have considered independent vertex sets and vertex coloring separately. The edges of a graph represent connections between vertices of a graph, and these edges show us vertices which cannot exist together in independent sets. Vertex colorings show which vertices must exist together in independent sets. Independence and vertex coloring on graphs are two concepts that together will define colored independence.

Colored independence is a way in which we can understand scheduling/storage problems where events that cannot occur together are modeled by vertices connected by edges, and events that must occur together are modeled by vertices that have the same color. The dimension that is added by looking at colored independence versus just independence alone allows us to explore a broader set of problems. As an example, consider the graph in Figure 4 below with six different color classes.

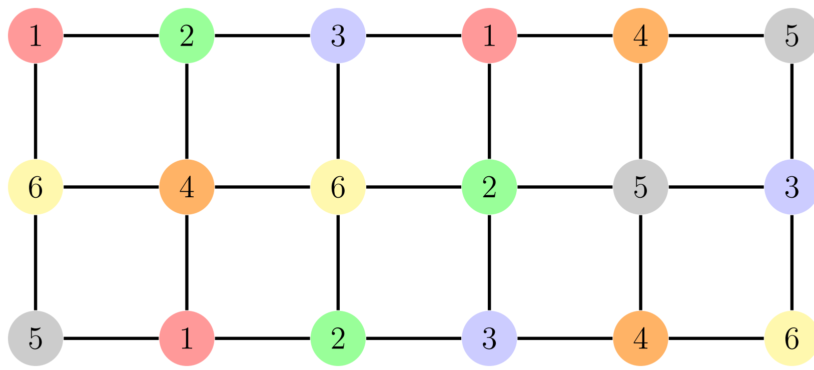


Figure 4: Coloring of a 3 by 6 grid

As it turns out, the way in which we partitioned the vertices of this graph into color classes yields an independent set of no more than three. This is the case because if a vertex from any color class is chosen, that vertex and the rest of the vertices in its color class are adjacent to a vertex in each of the other five color classes. So no other vertex could be chosen. Otherwise our set of vertices chosen would not be independent. Three is the size of the maximum independent set of vertices on this particular graph with this particular partition of vertices.

In the remainder of this paper, we will look more deeply into the idea of colored independence. In particular, we will apply this idea to cycle graphs as well as 2 by $n$ grids. By understanding this concept on more types of graphs, we can better model different situations and use this concept to solve scheduling problems.

## Section 2: Useful Definitions and Theorems

In this section, we will define several terms and state several theorems that are important to know in order to understand the remainder of this paper. Some of the definitions and theorems will be accompanied by examples in order to further enlighten the reader on the term or theorem. Many of the examples will use a type of graph called a path.

**Definition 4 [1]:** A path $P$ is a graph with vertex set $V = \{v_1, v_2, v_3, ..., v_n\}$ and edge set $E = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. If the path has $n$ vertices, we denote the path by $P_n$ and say $n$ is the order of $P_n$.

In Section 1, we formally defined a graph and how graphs are made of vertices and edges. The following definitions and theorem allow us to better understand graphs.

**Definition 5:** A vertex, $v$, and an edge, $e$, on a graph, $G$, are said to be incident if $v$ is one endpoint of $e$.

**Definition 6:** Let $v$ be a vertex in a graph $G$. The degree of $v$, denoted $\deg(v)$, is the number of edges incident with $v$.

**Theorem 1 [2]:** Let $G$ be a graph with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set $E = \{e_1, e_2, ..., e_m\}$. Then $\sum_{v \in V} \deg(v_i) = 2|E|$. In other words, the sum of the degrees of vertices in $G$ equals twice the number of edges.

Although we briefly discussed vertex colorings and independent vertex sets in the introduction, the following definitions will give further explaination of these ideas. First, we will expand on our knowledge of independent vertex sets by defining maximal and maximum independent vertex sets.

**Definition 7:** An independent vertex set $M$ is a maximal independent vertex set if for any vertex $v$ not in $M$, $M \cup \{v\}$ is not independent.

**Definition 8:** Let $L$ be a maximal independent vertex set on $V(G)$ where $G$ is any graph. $L$ is a maximum independent vertex set if for every maximal independent vertex set $M$ on $V(G)$, $|M| \leq |L|$. We denote the size of a maximum independent vertex set on a given graph $G$ as $\beta(G)$.

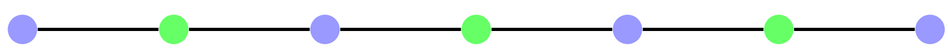

Figure 5: Maximal and maximum independent vertex sets on $P_7$

In Figure 5 above, the vertices colored green form a maximal independent vertex set, and the vertices colored blue form a maximum independent vertex set.

The next definitions will expand our knowledge of vertex colorings and color classes which were illustrated in the class scheduling example found in the introductory section.

**Definition 9:** Let $G$ be any graph. A partition of the vertices of $G$ into disjoint color classes, $S_1, S_2, S_3, ..., S_r$, is a proper vertex coloring if each color class $S_i$ is an independent set. Note that these sets need not be maximal.

**Definition 10:** Two color classes, $S_i$ and $S_j$, on the vertices of a graph $G$ are color class neighbors if there exists some $v_i \in S_i$ and some $v_j \in S_j$ such that $v_i v_j \in E(G)$.

**Definition 11:** Let $S_i$ and $S_j$ be color classes. $S_i$ and $S_j$ are said to be independent color classes if they are not color class neighbors.

In this paper, we will restrict the way in which we partition the vertices of a graph to proper vertex colorings. Recall that given a partition $\Phi = \{S_1, S_2, ..., S_r\}$ of the vertices of a graph $G$, $\bigcup_{i=1}^{r} S_i = V(G)$.

**Definition 12:** Given a graph $G$ and partition $\Phi = \{S_1, S_2, ..., S_r\}$ of the vertices of $G$, $D \subset V(G)$ is a colored independent set if for each $S_i \in \Phi$ either $S_i \cap D = S_i$ or $S_i \cap D = \emptyset$. Note, $D$ is an independent vertex set, but $D$ need not be maximal.

An example of a colored independent set on $P_7$ with partition $\Phi$ can be found in Figure 6 below. On this graph with this partition, $D = S_1 \cup S_3$ is a colored independent set since $S_1 \cap D = S_1$, $S_2 \cap D = \emptyset$, $S_3 \cap D = S_3$, and $S_4 \cap D = \emptyset$.

Next we will define the $\Phi$-independence number as well as the independence partition number. The independence partition number is the main focus of this paper.

**Definition 13 [3]:** Given a partition $\Phi$ of the vertices of a graph $G$, the $\Phi$-independence number is defined as the maximum size of a colored independent set. We denote the $\Phi$-independence number of a graph $G$ with partition $\Phi$ as $\beta(G; \Phi)$.

**Definition 14 [3]:** The $k$-independence partition number is defined as $\min\{\beta(G; \Phi) \mid \Phi \text{ is a partition of } V(G) \text{ into } r \text{ color classes and } |S_i| \leq k \text{ for } 1 \leq k \leq r\}$. We denote the $k$-independence partition number of a graph $G$ as $\beta_{PRT(k)}(G)$.

**Definition 15 [3]:** The independence partition number of a graph $G$ is $\beta_{PRT}(G) := \min\{\beta(G; \Phi) \mid \Phi \text{ is a partition of } V(G)\}$.

A more informal way to describe the definition of the independence partition number is as follows: For a graph $G$, there is a maximum independent vertex set associated with each partition $\Phi$ of the vertices of $G$. We call the size of these sets $\beta(G;\Phi)$ where $G$ is a graph and $\Phi$ is a partition of $V(G)$. The independence partition number is the smallest $\beta(G;\Phi)$ value we can get given every possible partition $\Phi$ of $V(G)$. The independence partition number is the number we strive to understand for different types of finite graphs.

Consider some partition $\Phi$. Two or more color classes in $\Phi$ can be in a colored independent set together as long as the color classes in the colored independent set are independent from each other. It is worth noting that the colored independent set formed in this way is not necessarily maximal. When looking for the independence partition number, we assume that every color class is a neighbor with every other color class. If not, then there exist independent color classes which could be combined into a single color class.

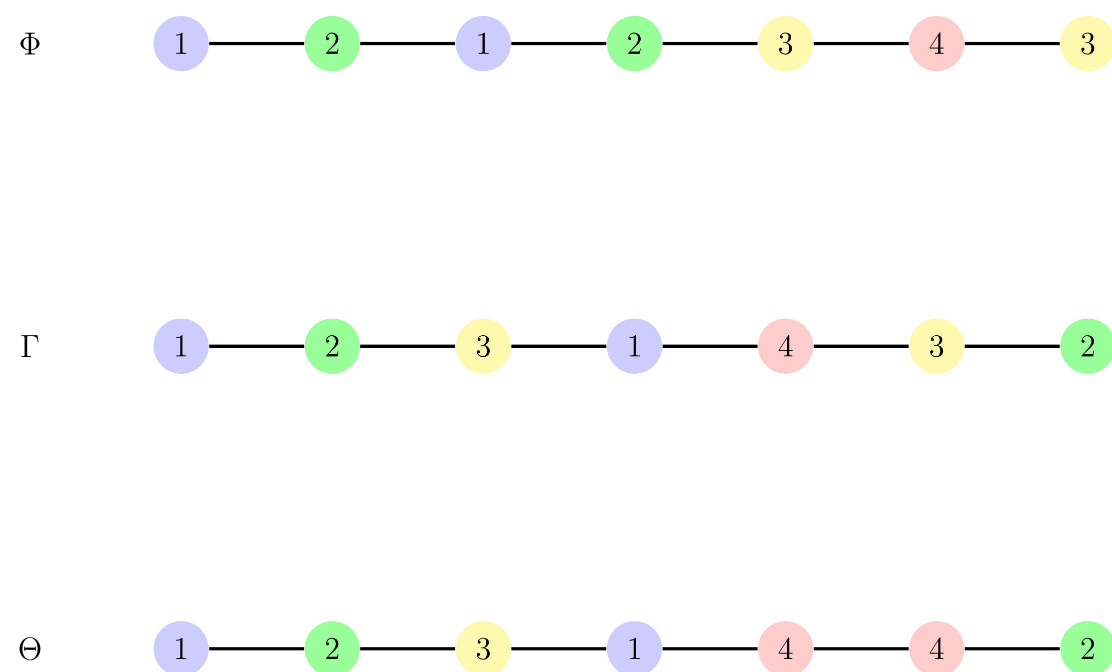


Figure 6: Three colorings of $P_7$

Figure 6 above shows examples for many of the previous definitions regarding coloring. First of all, the partitions of the vertices $\Phi$ and $\Gamma$ are proper vertex colorings, but the partition $\Theta$ is not a proper vertex coloring because the red color class includes two vertices which are adjacent. Next, notice there are many instances of color class neighbors. For example, in the graph with partition $\Phi$, the blue color class is a color class neighbor with the green color class. We also see that the blue color class is independent from the yellow color class. So, we could say the blue and yellow color class together form a colored independent set. In fact, this is the maximum independent vertex set that can be found on this graph with this particular partition. So, we say $\beta(P_7;\Phi) = 4$. Notice that $\beta(P_7;\Gamma) = 3$. It turns out that 3 is the minimum size of all the maximum independent sets found for all partitions of $P_7$. So $\beta_{PRT}(P_7) = 3$.

In order to prove claims for the independence partition number for different types of graphs in this paper, we will use the idea of colored independence on paths. The independence partition number on paths of order $n$ has already been described in the following theorem.

**Theorem 2 [1]:** For $t \geq 3$ and a path of order $n = 2(t-1)^2 + b$, we have

$$\beta_{PRT}(P_{2(t-1)^2+b}) = \begin{cases} t-1 & \text{if } b = 0 \\ t & \text{if } 1 \leq b \leq 3t-2 \\ t+1 & \text{if } 3t-1 \leq b \leq 4t-3 \end{cases}.$$

Note that Theorem 2 describes the independence partition number on all finite paths. In the proof of Theorem 2, another type of graph, called a linear forest, is used. We will also be using linear forests in this paper. In order to define a linear forest, we first need several other definitions.

**Definition 16 [2]:** A <u>walk</u> is a sequence of vertices and edges in a graph such that the sequence alternates between vertices and edges, starting and ending with vertices, and each edge in the sequence joins the vertices that occur immediately before and after it in the sequence.

**Definition 17 [2]:** Let $G$ be a graph. $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We say $G$ contains $H$. A graph is connected if every vertex is joined to every other vertex by a walk. A disconnected graph is a graph that is not connected.

**Definition 18 [2]:** Let $H$ be a subgraph of a graph $G$. The subgraph $H$ is a component of $G$ if $H$ is connected and $H$ is not contained in any connected subgraph $K$ of $G$ where $|V(H)| < |V(K)|$ or $|E(H)| < |E(K)|$.

**Definition 19 [2]:** A graph $LF_n$ is a linear forest if each component of $LF_n$ is a single vertex or a walk. We denote the number of vertices in the linear forest with the subscript $n$. Note that a path is a linear forest.

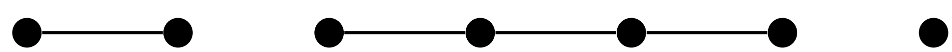

Figure 7: $LF_7$

An example of a linear forest and components of a graph can be seen in Figure 7 above. There are three components in Figure 7. In additon to the definition of a linear forest, we will use the following theorem concerning linear forests in the next section.

**Theorem 3 [1]:** $\beta_{PRT(k)}(LF_n) \geq \frac{n}{2k}$.

In Section 1, we mentioned the name Euler. In order to prove several results in the next section, we rely on a topic in graph theory attributed to Euler, namely, Eulerian circuits. We will first define a circuit and then use that definition to define an Eulerian circuit.

**Definition 20 [2]:** A circuit is a walk that starts and ends at the same vertex.

**Definition 21 [2]:** A circuit on a graph $G$ is an Eulerian circuit if the circuit uses each edge of $G$ exactly once.

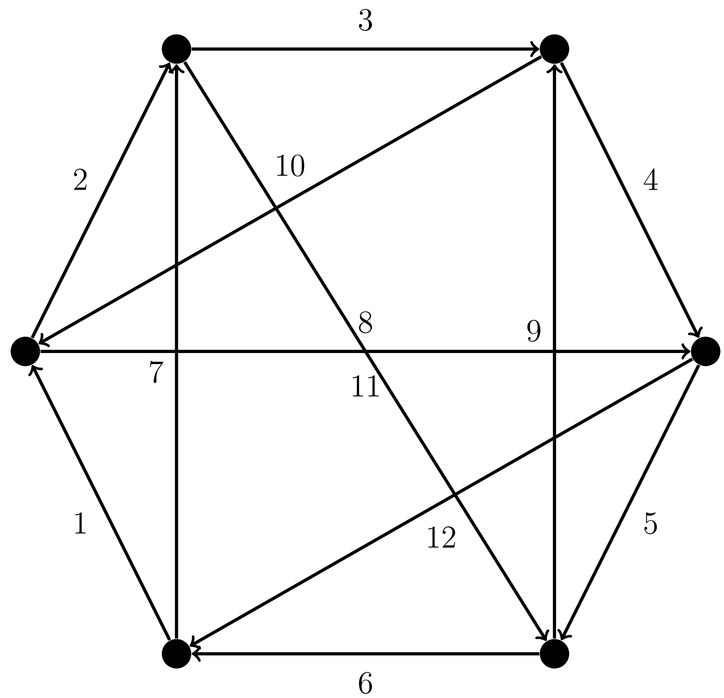


Figure 8: Eulerian circuit on a graph with 6 vertices of degree 4

In addition to the definition of an Eulerian circuit, we need the following theorem.

**Theorem 4 [2]:** A graph $G$ has an Eulerian circuit if and only if every vertex in $G$ has even degree.

Figure 8 above illustrates an Eulerian circuit as well as the theorem we have about Eulerian circuits. The vertices of the graph in Figure 8 all have even degree, and the Eulerian circuit is made evident by the arrows and numbers associated with each edge. If we follow the arrows starting at edge one, we will form an Eulerian circuit on this graph.

Before moving on to the next section, we need to define two other types of graphs.

**Definition 22 [2]:** A <u>complete graph</u> is a graph in which each pair of vertices of the graph is adjacent.

**Definition 23:** A <u>multigraph</u> is a graph in which multiple edges are allowed between vertices.

As we move into the next sections, we will apply colored independence to different types of graphs. Section 3 will look at colored independence on cycle graphs and Section 4 will concentrate on colored independence on finite grids. The definitions in this section will aid in the understanding of the lemmas, corollaries, and theorems found in the following sections.

## Section 3: Colored Independence on Cycle Graphs

With the definitions we covered in the last section, we can move on to describing the independence partition number for all cycle graphs. We will use the previous definitions and theorems to help us as we proceed. First of all, let us define a cycle.

**Definition 24:** A cycle $C_n$ is a graph with vertex set $V = \{v_1, v_2, v_3, ..., v_n\}$ and edge set $E = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n, v_nv_1\}$. If the cycle has $n$ vertices, we denote this with subscript $n$ and say $n$ is the order of the cycle.

Ultimately, in this section, we will show that the independence partition number for cycles $C_n$ can be described with a single formula. In order to prove this, we need to start with lemmas and theorems that will lead us to this end result. We will start with the following lemma.

**Lemma 1:** For a cycle $C_n$ of order $n$, $\beta_{PRT(k)}(C_n) \geq \frac{n-k}{2k}$.

**Proof.** Consider any cycle graph $C_n$, and suppose the vertices of $C_n$ have some partition $\Phi = \{S_1, S_2, S_3, ..., S_t\}$ where $\max_{1 \leq i \leq t} |S_i| \leq k$. We construct an independent set $A = S_I \subseteq V(C_n)$ where $S_I$ signifies one color class or the union of multiple color classes. The size of $A$ will then give us a lower bound for $C_n$.

Take any vertex $v$ in $C_n$. The vertex $v$ must be in some color class $S_i$. Put the elements of $S_i$ in $A$. Note that by definition of a proper vertex coloring $A$ is an independent set. Delete from $C_n$ all vertices of the set $S_i \cup \{S_j : u \in S_j, v \in S_i, \text{ and } uv \in E(C_n)\}$. In other words delete every vertex in $S_i$ and every vertex in a color class adjacent to $S_i$. So, $|S_i|$ elements have been added to $A$ and, at most, $|S_i| + 2k|S_i|$ total vertices have been deleted from $C_n$.

Since $\max_{1\leq i\leq t}|S_i| \leq k$, assume $|S_i| = k$. So, at most, $k+2k^2$ vertices are deleted. Note that deleting any edge or combination of edges in a cycle results in a linear forest. So, we are left with a linear forest with at least $n-k-2k^2$ vertices. We continue adding vertices to $A$ and deleting vertices from our original graph in the same way as before. From Theorem 3, we have $\beta_{PRT(k)}(LF_{n-k-2k^2}) \geq \frac{n-k-2k^2}{2k}$. Adding this to $|A|$, we have $k + \frac{n-k-2k^2}{2k} = \frac{n-k}{2k}$. Therefore, we can conclude $\beta_{PRT(k)}(C_n) \geq \frac{n-k}{2k}$.

□

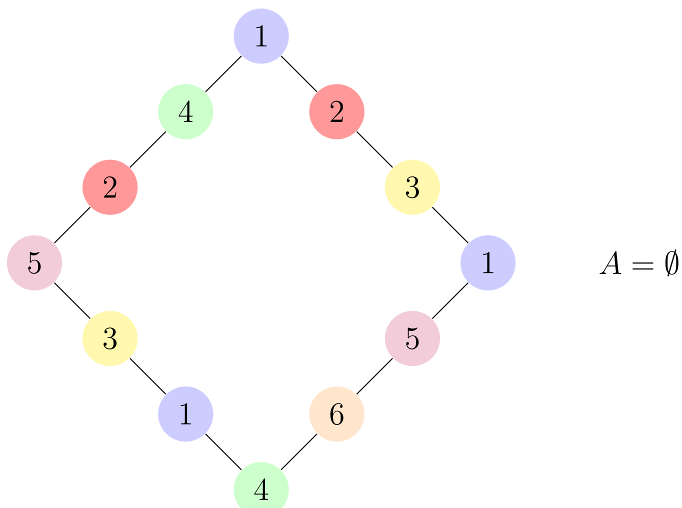


Figure 9: Coloring of $C_{12}$

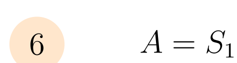


Figure 10: $C_{12}$ after removing $S_1$ and all color class neighbors of $S_1$

Figure 9 and Figure 10 above show the idea behind Lemma 1. Figure 9 is a colored cycle graph with 12 vertices and six color classes partitioning those vertices. Suppose we pick a vertex $v$ in $S_1$. By putting the vertices of color class $S_1$ into $A$, the resulting graph after the deletion steps found in Lemma 1 is shown in Figure 10. Thus far, $A = S_1$. Following

Lemma 1, we include the vertices of $S_6$ in $A$. So, $A = S_1 \cup S_6$. Therefore, $|A| = 4$ since $S_1$ has 3 vertices and $S_6$ has 1 vertex. And, indeed, $4 \geq \frac{n-k}{2k} = \frac{12-3}{6} = \frac{3}{2}$.

This result from Lemma 1 helps us to prove the next two theorems, namely, that $\beta_{PRT(k)}(C_{2k^2+k}) = k$ and $\beta_{PRT(k)}(C_{2k^2}) = k$.

**Theorem 5:** $\beta_{PRT(k)}(C_{2k^2+k}) = k$.

**Proof.** By Lemma 1, we know $\beta_{PRT(k)}(C_{2k^2+k}) \geq \frac{n-k}{2k} = \frac{2k^2+k-k}{2k} = k$. So, we need to show $\beta_{PRT(k)}(C_{2k^2+k}) \leq k$. Given $k$, we can use $2k+1$ color classes, each of order $k$, to construct a partition $\Phi$. We will construct $\Phi$ so that for $1 \leq i < j \leq 2k+1$, there exists elements $v_i \in S_i$ and $v_j \in S_j$ such that $v_iv_j \in E(C_{2k^2+k})$. This will ensure there will only be one color class in any independent set since all color classes will be color class neighbors.

Let $V(C_{2k^2+k}) = \{v_1, v_2, ..., v_{2k^2+k}\}$. Let $V(H) = \{u_1, u_2, ..., u_{2k+1}\}$ where $H$ is a complete graph on $2k+1$ vertices. Note that the vertices of $H$ all have degree $2k$. So, each vertex has even degree. This implies an Eulerian circuit exists on $H$ beginning and ending at $u_1$. The Eulerian circuit contains exactly $2k^2+k$ vertices. We get this number because there are $2k^2+k$ edges in $H$. Since we use each edge of $H$ exactly once in our Eulerian circuit, we have $2k^2+k$ vertices in the circuit. Note that the circuit formed in this way is not a subgraph of $H$. The Eulerian circuit found in $H$ is a tool for constructing a partition of the vertices of a cycle graph. In particular, this circuit will be used to construct the partition $\Phi$ on $C_{2k^2+k}$. Each vertex $v_i$ on $C_{2k^2+k}$ is placed into the color class $S_{u_j}$ where $u_j$ is the $i^{th}$ vertex in the Eulerian circuit on $H$.

For color classes $S_i$ and $S_j$ with $1 \leq i < j \leq 2k+1$, there are vertices $v_i \in S_i$ and $v_j \in S_j$ such that $v_iv_j \in E(C_{2k^2+k})$ since the edge $u_iu_j$ in $H$ was traversed by the Eulerian circuit. Therefore, any $\Phi$-independent set can contain no more than one color class. So, $\beta_{PRT(k)}(C_{2k^2+k}) \leq k$ which means $\beta_{PRT(k)}(C_{2k^2+k}) = k$.

$\square$

An instance of Theorem 5 can be seen in the following example. Consider the case $k = 3$. The cycle has the vertex set $V(C_{21}) = \{v_1, v_2\ , ..., v_{21}\}$ and the complete graph H has the vertex set $V(H) = \{u_1, u_2, ..., u_7\}$. Since each vertex of H is even, an Eulerian circuit exists on $H$. We will use this Eulerian circuit, $u_1u_2u_3u_4u_5u_6u_7u_1u_3u_5u_7u_2u_4u_6u_1u_4u_7u_3u_6u_2u_5$ (note that the first vertex $u_1$ and the last vertex $u_5$ connect to form the circuit), to construct the partition $\Phi$ on $C_{21}$. In fact, $\Phi = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$. Since $u_1$ is the first vertex in the Eulerian circuit, the first vertex in $C_{21}$, $v_1$, is placed into the color class $S_1$. Similarly, since $u_2$ is the second vertex in the Eulerian circuit, $v_2$ is placed into the color class $S_2$.

By placing every vertex $v_i$ in to the color class defined by the $i^{th}$ vertex in the Eulerian circuit, $S_1 = \{v_1, v_8, v_{15}\}$, $S_2 = \{v_2, v_{12}, v_{20}\}$, $S_3 = \{v_3, v_9, v_{18}\}$, $S_4 = \{v_4, v_{13}, v_{16}\}$, $S_5 = \{v_5, v_{10}, v_{21}\}$, $S_6 = \{v_6, v_{14}, v_{19}\}$, and $S_7 = \{v_7, v_{11}, v_{17}\}$. In this example, it is clear that $\beta_{PRT(k)}(C_{2k^2+k}) = k$. This is illustrated in Figure 11 and Figure 12 below. Figure 11 shows the Eulerian circuit on the complete graph with seven vertices. Figure 12 shows the coloring of the cycle $C_{21}$ based on the Eulerian circuit found in Figure 11.

In a similar way, we can show that $\beta_{PRT(k)}(C_{2k^2}) = k$.

**Theorem 6:** $\beta_{PRT(k)}(C_{2k^2}) = k$.

**Proof.** From Lemma 1, we know that $\beta_{PRT(k)}(C_{2k^2}) \geq \frac{n-k}{2k} = \frac{2k^2-k}{2k} = k - \frac{1}{2}$. Since $\beta_{PRT}(G)$ has to be a whole number, we can say that $\beta_{PRT(k)}(C_{2k^2}) \geq k$.

Now we need to show $\beta_{PRT(k)}(C_{2k^2}) \leq k$. Given $k$, we can use $2k$ color classes each of order $k$ to construct a partition $\Phi$. We will construct the partition $\Phi$ so that for $1 \leq i < j \leq 2k$, there exists some element $v_i \in S_i$ and $v_j \in S_j$ such that $v_iv_j \in E(C_{2k^2})$. This again will ensure there is only one color class in any independent set.

Let $V(C_{2k^2}) = \{v_1, v_2, ..., v_{2k^2}\}$ where $v_1v_2, v_2v_3, ..., v_{2k^2-1}v_{2k^2}, v_{2k^2}v_1 \in E(C_{2k^2})$. In order

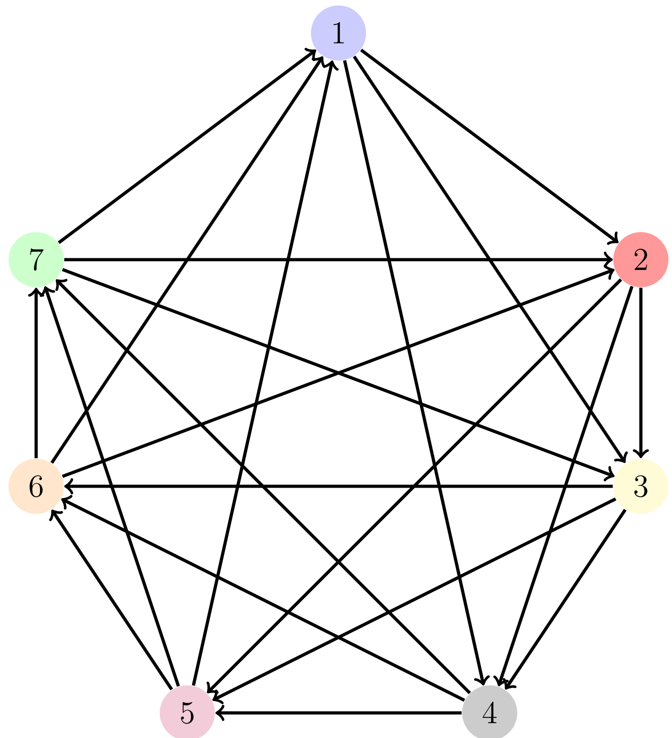


Figure 11: Eulerian circuit on the complete graph with six vertices

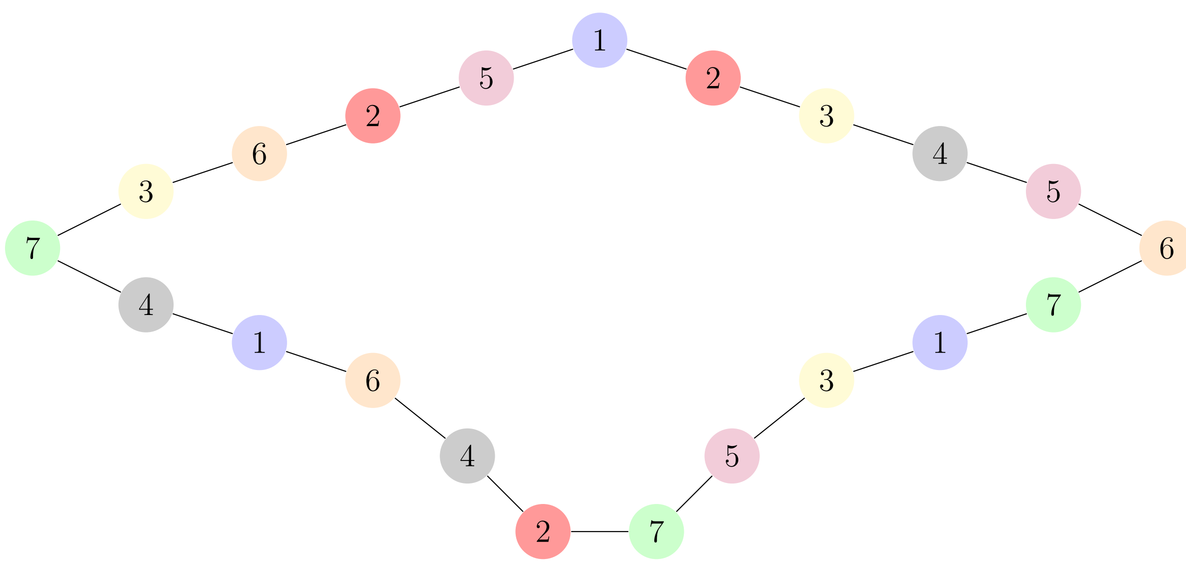


Figure 12: Coloring of $C_{21}$ based on the Eulerian circuit in Figure 11

to construct the partition on the cycle, consider a multigraph $H$ with $V(H) = \{u_1, u_2, ..., u_{2k}\}$ where $H$ is constructed by starting with a complete graph on $2k$ vertices and adding the $k$

edges $u_1u_2, u_3u_4, ..., u_{2k-1}u_{2k}$. Note that each vertex in $H$ has even degree. This implies that an Eulerian circuit exists on $H$ beginning and ending at $u_1$. The Eulerian circuit contains exactly $2k^2$ vertices. This circuit will be used to define the partition $\Phi$ on $C_{2k^2}$. Each vertex $v_i$ on $C_{2k^2}$ will be placed into the color class $S_{u_j}$ where $u_j$ is the $i^{th}$ vertex in the cycle on $H$.

For color classes $S_i$ and $S_j$, with $1 \leq i < j \leq 2k$, there are vertices $v_i \in S_i$ and $v_j \in S_j$ such that $v_iv_j \in E(C_{2k^2})$ since the edge $u_iu_j$ is traversed by the Eulerian circuit in the multigraph $H$. Therefore, any $\Phi$-independent set can contain no more than one color class, so $\beta_{PRT(k)}(C_{2k^2}) \leq k$, thus $\beta_{PRT(k)}(C_{2k^2}) = k$.

□

For an example of Theorem 6 for a specific value of $k$, consider the case $k = 3$. The cycle has the vertex set $V(C_{18}) = \{v_1, v_2, ..., v_{18}\}$ and the multigraph $H$ has the vertex set $V(H) = \{u_1, u_2, ..., u_6\}$. We construct the multigraph by starting with a complete graph on six vertices and add three additional edges, $u_1u_2$, $u_3u_4$, and $u_5u_6$. We use the Eulerian circuit $u_1u_3u_4u_2u_5u_6u_2u_3u_5u_4u_1u_6u_3u_4u_6u_5u_1u_2$ (Note that the first vertex $u_1$ and the last vertex $u_2$ connect to form the cycle.) of $H$ to construct the partition $\Phi$ on $C_{18}$. In fact, $\Phi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$. By placing every vertex $v_i$ into the color class defined by the $i^{th}$ vertex in the Eulerian circuit, $S_1 = \{v_1, v_{11}, v_{14}\}$, $S_2 = \{v_4, v_7, v_{18}\}$, $S_3 = \{v_2, v_8, v_{13}\}$, $S_4 = \{v_3, v_{10}, v_{16}\}$, $S_5 = \{v_5, v_9, v_{15}\}$, and $S_6 = \{v_6, v_{12}, v_{17}\}$. It is clear in this example that $\beta_{PRT(k)}(C_{2k^2}) = k$. This is illustrated in Figure 13 and Figure 14 below. Figure 13 shows the Eulerian circuit on the multigraph with six vertices. Figure 14 shows the coloring of the cycle $C_{18}$ based on the Eulerian circuit found in Figure 13.

Now that we have shown that $\beta_{PRT(k)}(C_{2k^2}) = k$ and $\beta_{PRT(k)}(C_{2k^2+k}) = k$ for some partition on their respective graphs, we need to show this is the case for any partition of the vertices of $C_{2k^2}$ and $C_{2k^2+k}$.

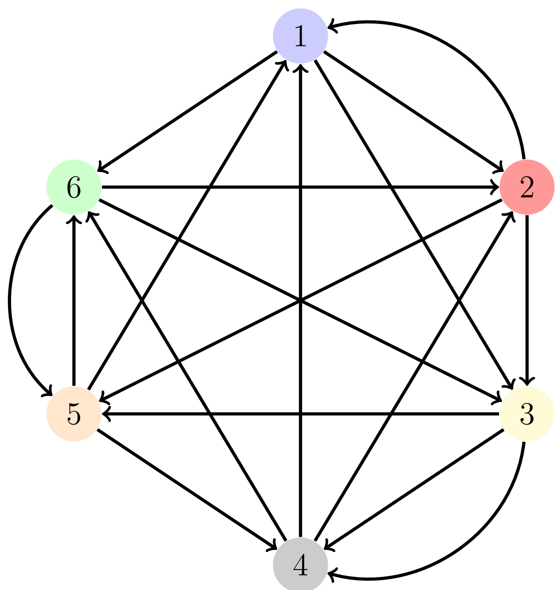

Figure 13: Eulerian circuit on a multigraph with six vertices.

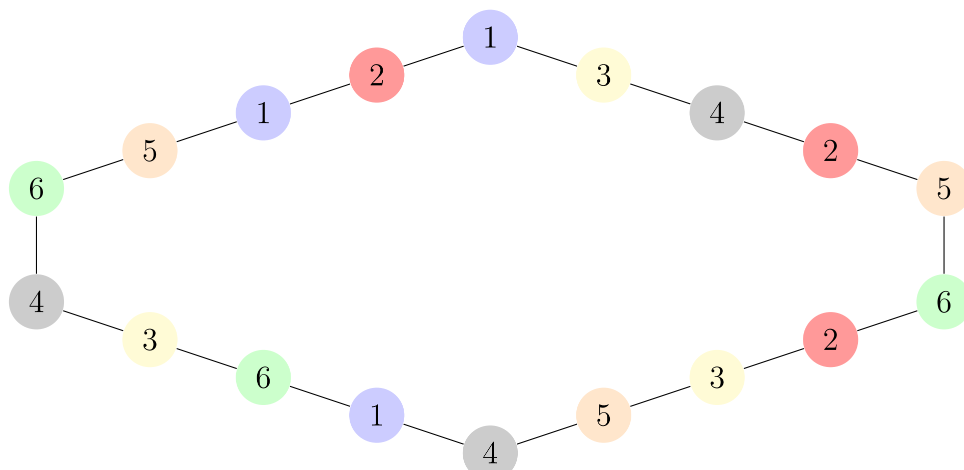

Figure 14: Coloring of $C_{18}$ based on the Eulerian circuit of Figure 13

**Theorem 7:** For a cycle of order $2t^2$ or $2t^2 + t$, $\beta_{PRT}(C_{2t^2}) = t$ and $\beta_{PRT}(C_{2t^2+t}) = t$.

**Proof.** From the previous two results, we know there are partitions $\Phi$ and $\Gamma$ such that $\beta(C_{2t^2}; \Phi) = t$ and $\beta(C_{2t^2+t}; \Gamma) = t$. Note that $\beta_{PRT}(C_{2t^2}) \leq \beta(C_{2t^2}; \Phi) = t$ and $\beta_{PRT}(C_{2t^2+t}) \leq \beta(C_{2t^2+t}; \Gamma) = t$. So, we need to show $\beta_{PRT}(C_{2t^2}) \geq t$ and $\beta_{PRT}(C_{2t^2+t}) \geq t$.

First, let us consider $C_{2t^2}$. Let $\Phi = \{S_1, S_2, ..., S_r\}$ be a partition of $V(C_{2t^2})$. Let $J = \max_{1\leq i\leq r}|S_i|$. If $J \geq t$, then it follows that $\beta(C_{2t^2}; \Phi) \geq t$. If, $J < t$, $\beta(C_{2t^2}; \Phi) \geq \beta_{PRT(J)}(C_{2t^2}) \geq \frac{2t^2-J}{2J} > \frac{2t^2-t}{2t} = t - \frac{1}{2}$. Since $\beta(G)$ must be a whole number, we can say $\beta(C_{2t^2}; \Phi) \geq t$. Since $\Phi$ was an arbitrary partition of $C_{2t^2}$, $\beta_{PRT}(C_{2t^2}) \geq t$. Thus, $\beta_{PRT}(C_{2t^2}) = t$.

Now, we want to show the same thing for $C_{2t^2+t}$, namely that $\beta_{PRT}(C_{2t^2+t}) \geq t$. Let $\Gamma = \{S_1, S_2, ..., S_r\}$ be a partition of $V(C_{2t^2+t})$. Let $Z = \max_{1\leq i\leq r}|S_i|$. If $Z \geq t$, then $\beta(C_{2t^2+t}; \Gamma) \geq t$. If, $Z < t$, $\beta(C_{2t^2+t}; \Gamma) \geq \beta_{PRT(Z)}(C_{2t^2+t}) \geq \frac{2t^2+t-Z}{2Z} > \frac{2t^2+t-t}{2t} = t$. Since $\Gamma$ was an arbitrary partition of $C_{2t^2+t}$, $\beta_{PRT}(C_{2t^2+t}) \geq t$. Thus $\beta_{PRT}(C_{2t^2+t}) = t$.

$\square$

In order to complete our description of the independence partition number on cycle graphs, we must prove the following lemma.

**Lemma 2:** For any cycle $C_n$ of order $n$ where $n > 2(t-1)^2 + (t-1)$, $\beta_{PRT}(C_n) \geq t$.

**Proof.** Let $\Phi = \{S_1, S_2, ..., S_r\}$ be a partition of $V(C_n)$ such that $\beta(C_n; \Phi) = \beta_{PRT}(C_n)$. Let $J = \max_{1\leq i\leq r}|S_i|$. If $J \geq t$, then it follows that $\beta(C_n; \Phi) \geq t$. If $J < t$, $\beta(C_n; \Phi) \geq \beta_{PRT(J)}(C_n) \geq \frac{n-J}{2J} \geq \frac{n-(t-1)}{2(t-1)} > \frac{2(t-1)^2+(t-1)-(t-1)}{2(t-1)} = t - 1$. It follows that $\beta_{PRT}(C_n) \geq t$.

$\square$

Now that we have the framework in place, we can prove the following formula describing the independence partition number on cycle graphs of order $n$.

**Theorem 8:** For $t \geq 3$, $t, b \in \mathbb{Z}$, and a cycle $C_n$ of order $n = 2(t-1)^2 + b$, we have

$$\beta_{PRT}(C_{2(t-1)^2+b}) = \begin{cases} t-1 & \text{if } b = 0 \text{ or } b = t-1 \\ t & \text{if } 1 \leq b \leq t-2 \text{ or } t \leq b \leq 3t-2 \cdot \\ t+1 & \text{if } 3t-1 \leq b \leq 4t-3 \end{cases}$$

**Proof.**

**Case 1.** Notice that this has already been proven in Theorem 7 for the cases when $b = 0$ and when $b = t - 1$.

**Case 2.** $1 \leq b \leq t - 2$

First, we must show $\beta_{PRT}(C_{2(t-1)^2+b}) \geq t$. We can assume that for each color class $S_i$, there exist vertices $v_i \in S_i$ and $v_j \in S_j$ such that $v_i v_j \in E(C_{2(t-1)^2+b})$ for $1 \leq i < j \leq r$ where $r$ is the total number of color classes. If not, there exist independent color classes which can be combined into a single color class.

Suppose there exists a partition $\Phi$ such that $\beta(C_n; \Phi) = t - 1$ where $2t^2 - 4t + 3 \leq n \leq 2t^2 - 3t$. Then, there must be at least $2t - 1$ color classes. Otherwise, $n \leq (2t-2)(t-1) = 2t^2 - 4t + 2$ where $t - 1$ is the maximum size of a color class. If the number of color classes is greater than $2t - 1$, then $n \geq \binom{2t}{2} = 2t^2 - t > 2t^2 - 3t$. Thus, if such a partition exists, it must contain exactly $2t - 1$ color classes. In order to prove $\beta_{PRT}(C_{2(t-1)^2+b}) \geq t$, it will suffice to show that no partition with exactly $2t - 1$ color classes exists.

Note that only one color class can be in a $\Phi$-independent set. In order to construct a cycle with such a partition, consider a complete graph $H$ on $2t-1$ vertices. There must exist an Eulerian circuit on $H$ because all the vertices of $H$ have degree $2t - 2$ which is even. It follows that $\sum_{v \in V(H)} deg(v_i) = (2t-1)(2t-2) = 4t^2 - 6t + 2$. From this fact and Theorem 1, we conclude that $|E(H)| = 2t^2 - 3t + 1$. Moreover, the cycle that is constructed from this graph will have $2t^2 - 3t + 1$ edges, but $2t^2 - 3t + 1 > 2t^2 - 3t$ which implies $b > t - 2$. Since $b \leq t - 2$, no such partition $\Phi$ exists, showing that $\beta_{PRT}(C_{2(t-1)^2+b}) \geq t$.

Now, all that is left to do for this case is to show $\beta_{PRT}(C_{2(t-1)^2+b}) \leq t$. From Theorem 2, we know there exists a path $P_n$ of order $n = 2(t-1)^2 + b$, $1 \leq b \leq t-2$ where $\beta_{PRT}(P_n) = t$. The way in which we color paths, outlined in the proof of Theorem 2 [1], ensures that the first and last vertices of the path are in independent color classes. So the same coloring found on a path yielding an independence partition number $t$ will work for partitioning the vericies of cycles of the same order. Therefore, we know $\beta_{PRT}(C_{2(t-1)^2+b}) \leq t$ showing $\beta_{PRT}(C_{2(t-1)^2+b}) = t$ for this case.

**Case 3.** $t \leq b \leq 3t - 2$

From Theorem 7, we know $\beta_{PRT}(C_{2(t-1)^2+b}) \geq t$. So all we must show for this case is that $\beta_{PRT}(C_{2(t-1)^2+b}) \leq t$.

We know from Theorem 2 there is a path, $P_n$ of order $n = 2(t-1)^2 + b$, $t \leq b \leq 3t - 2$ where $\beta_{PRT}(P_n) = t$. Using the same reasoning as in Case 2, the coloring found on paths of order $n$ will work for cycles of the same order. Note, we can do this since the way in which we color the vertices of the path, outlined in the proof of Theorem 2 [1], ensures the first and last vertices will not be in the same color class. From this we get $\beta_{PRT}(C_{2(t-1)^2+b}) \leq t$ which leads to the same conclusion as Case 2, namely, that $\beta_{PRT}(C_{2(t-1)^2+b}) = t$ for this case.

**Case 4.** $3t - 1 \leq b \leq 4t - 3$

As in the previous two cases, we know from paths that there exists a path $P_n$ of order $n = 2(t-1)^2 + b$, $3t - 1 \leq b \leq 4t - 3$ where $\beta_{PRT}(P_n) = t + 1$. It follows that a cycle $C_n$ of order $n = 2(t-1)^2 + b$, $3t - 1 \leq b \leq 4t - 3$ will have $\beta_{PRT}(C_{2(t-1)^2+b}) \leq t + 1$. Now, we must show that $\beta_{PRT}(C_{2(t-1)^2+b}) \geq t + 1$. Note that from Lemma 2, $\beta_{PRT}(C_{2(t-1)^2+b}) \geq t$. So, it will suffice to show that it is impossible to have a partition on cycles of this order in which $\beta_{PRT}(C_{2(t-1)^2+b}) = t$.

We can assume that for each color class, $S_i$, there are vertices $v_i \in S_i$ and $v_j \in S_j$ such that $v_i v_j \in E(C_{2(t-1)^2+b})$ for $1 \leq i < j \leq r$ where $r$ is the total number of color classes. If not, there exist independent color classes which can be combined into a single color class. If there exists a partition $\Phi$ such that $\beta(C_n;\Phi) = t$ where $2t^2 - t + 1 \leq n \leq 2t^2 - 1$, then there must be at least $2t$ color classes. Otherwise, $n \leq 2t^2 - t < 2t^2 - t + 1$. If the number of color classes is greater than $2t$, then $n \geq \binom{2t+1}{2} = 2t^2 + t > 2t^2 - 1$. Thus, if such a partition exists, it must have exactly $2t$ color classes.

Note that only one color class can be in a $\Phi$-independent set. In order to construct a cycle with such a partition, consider a multigraph $H$ constructed by adding $t$ edges to a complete graph with $2t$ vertices. The $t$ edges to be added to the complete graph are $u_1u_2, u_3u_4, ..., u_{2t-1}u_{2t}$. By adding $t$ edges in this way, note that all the vertices of $H$ are of degree $2t$ which is even. So, an Eulerian circuit must exist on $H$. We can derive that $|E(H)| = \frac{2t(2t-1)}{2} + t = 2t^2$ from Theorem 1. Moreover, the cycle that is constructed from this graph will have $2t^2$ edges, but $2t^2 \geq 2t^2 - 1$ which implies that $b > 4t - 3$. Since $b \leq 4t - 3$, no such partition $\Phi$ exists. Thus, $\beta_{PRT}(C_{2(t-1)^2+b}) \geq t+1$ showing that $\beta_{PRT}(C_{2(t-1)^2+b}) = t+1$.

Therefore, the theorem is proven.

□

Section 4: Colored Independence on 2 by $n$ grids.

Colored independence can be studied on other types of graphs besides cycle graphs and paths. In this section, we will look at colored independence on a type of graph called a grid. More specifically, we will be looking at grids with dimensions 2 by $n$. Unlike cycles, the conjecture for a formula that describes the independence partition number on 2 by $n$ grids has not been proven, but in this section, we will state the conjecture, understand the reasoning behind the conjecture, and identify reasons why proving the conjecture of the independence partition number on 2 by $n$ grids has posed problems. Let us first define a grid.

**Definition 25:** A grid, $G_{m,n}$ is a graph with vertex set $V = \{v_{1,1}, v_{1,2}, ..., v_{1,n}, v_{2,1}, v_{2,2}, ..., v_{2,n}, ..., v_{m,n}\}$ and edge set $E = \{v_{1,1}v_{1,2}, v_{1,2}v_{1,3}, ..., v_{1,n-1}v_{1,n}, v_{2,1}v_{2,2}, ..., v_{2,n-1} v_{2,n}, ..., v_{m,n-1}v_{m,n}\} \cup \{v_{1,1}v_{2,1}, v_{2,1}v_{3,1}, ..., v_{m-1,1}v_{m,1}, v_{1,2}v_{2,2}, ..., v_{m-1,2}v_{m,2}, ..., v_{m-1,n}v_{m,n}\}$. We denote an $m$ by $n$ grid with subscripts $m$ and $n$.

Figure 15 below gives an example of a 2 by 6 grid.

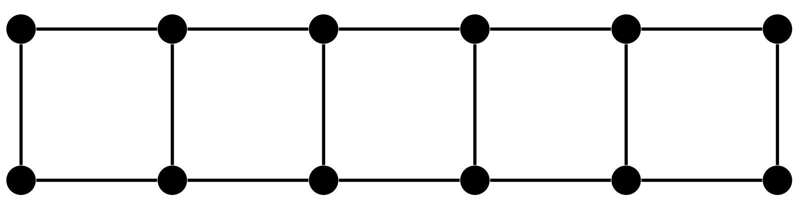

Figure 15: $G_{2,6}$

The following is the conjecture that we will be considering in this section.

**Conjecture 1.** For $t \geq 4$, $t, b \in \mathbb{Z}$, and a grid $G_{2,n}$ of order $n = \frac{3t^2-7t+4}{2} + b$, we have

$$\beta_{PRT}(G_{2,\frac{3t^2-7t+4}{2}+b}) = \begin{cases} t-1 & \text{if } b=0 \text{ or } b=t-1 \\ t & \text{if } 1 \le b \le \frac{5}{2}t-2 \text{ and } t \text{ is even} \\ t & \text{if } 1 \le b \le \frac{5}{2}t-\frac{5}{2} \text{ and } t \text{ is odd} \\ t+1 & \text{if } \frac{5}{2}t-1 \le b \le 3t-3 \text{ and } t\text{is even} \\ t+1 & \text{if } \frac{5}{2}t-\frac{3}{2} \le b \le 3t-3 \text{ and } t \text{ is odd} \end{cases}.$$

Much like the idea of the proof of Theorem 8, the first step in proving this claim is to determine some type of lower bound for 2 by $n$ grids with varying sizes of $n$ and varying sizes of color classes $k$.

**Lemma 3:** For a grid $G_{2,n}$ with order $n$, $\beta_{PRT(k)}(G_{2,n}) \geq \frac{2n}{3k}$.

**Proof.** Consider any grid $G_{2,n}$, and let $\Phi = \{S_1, S_2, S_3, ..., S_t\}$ be a partition of $V(G_{2,n})$ where $\max_{1\leq i\leq t}|S_i| \leq k$. We construct an independent set $A = S_I \subseteq V(G_{2,n})$ where $S_I$ signifies one color class or the union of several color classes.

Pick any vertex, $v$, in $G_{2,n}$ with the stipulation that $\deg(v) = 2$. The vertex, $v$, must be in some color class $S_i$. Put the vertices of $S_i$ in $A$. Next, delete from $G_{2,n}$ all vertices of the set $S_i \cup \{S_j : u \in S_j, v \in S_i, \text{ and } uv \in E(G_{2,n})\}$. In other words delete every vertex in $S_i$ and every vertex in a color class adjacent to $S_i$. So, at this point, we will have deleted, at most, $3k(|S_i| - 1) + 2k + |S_i|$ since $\deg(v) = 2$. The term $|S_i|$ accounts for the vertices in color class $S_i$, $2k$ accounts for the vertices in color classes adjacent to $v$, and $3k(|S_i| - 1)$ accounts for the vertices of color classes adjacent to the vertices in $S_i$ not including $v$. So, we have the ratio $\frac{|S_i|}{3k(|S_i|-1)+2k+|S_i|} = \frac{|S_i|}{3k|S_i|+|S_i|-k} \geq \frac{1}{3k}$.

Assuming there are vertices that have not been deleted, pick another vertex $q$ with $\deg(q) \leq 2$. The vertex $q$ must be in some color class $S_m$. Add the vertices of $S_m$ to $A$ and delete vertices from remaining vertices as above. Continue this process until no more vertices are remaining in $G_{2,n}$. Since we have the ratio $\frac{|S_i|}{3k|S_i|+|S_i|-k} \geq \frac{1}{3k}$, at most $3k$ vertices are deleted from $G_{2,n}$ for every vertex in $G_{2,n}$. So, for the given partition $\Phi$, $\beta_{PRT}(G_{2,n}; \Phi) \geq \frac{2n}{3k}$. This holds for any partition in which $\max_{1 \leq i \leq t} |S_i| \leq k$. Therefore, $\beta_{PRT(k)}(G_{2,n}) \geq \frac{2n}{3k}$.

□

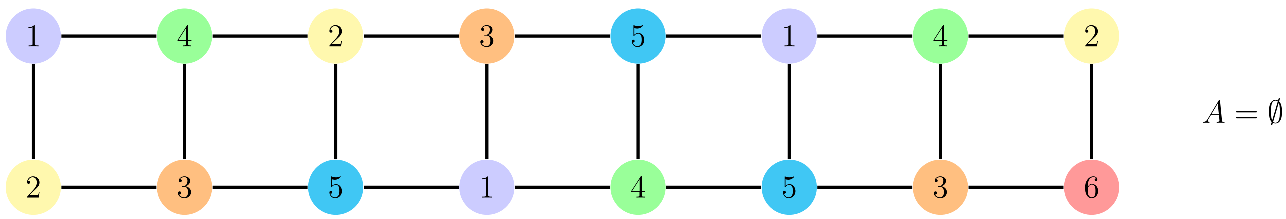


Figure 16: Coloring of $G_{2,8}$


Figure 17: $G_{2,8}$ after removing $S_1$ and all color class neighbors of $S_1$

Figure 16 and Figure 17 above show the idea behind Lemma 3. Figure 16 is a colored 2 by 8 grid with 18 vertices and 6 color classes partitioning those vertices. Suppose we first pick a vertex $v$ in $S_1$. By putting the color class $S_1$ into $A$, the resulting graph after the deletion steps found in Lemma 3 is shown in Figure 17. Thus far, $A = S_1$. Following Lemma 3, we include $S_6$ in $A$. So, $A = S_1 \cup S_6$. Therefore, $|A| = 4$ since $S_1$ has 3 vertices and $S_6$ has 1 vertex. And, indeed, $4 \geq \frac{2n}{3k} = \frac{2(8)}{3(3)} = \frac{16}{9}$.

From Lemma 3, we have the following corollary.

**Corollary 1:** For a grid $G_{2,n}$ with order $n = \frac{3t^2-7t+4}{2}$, $\beta_{PRT(t-1)}(G_{2,\frac{3t^2-7t+4}{2}}) \geq t-1$.

**Proof.** From Lemma 3, we know $\beta_{PRT(k)}(G_{2,n}) \geq \frac{2n}{3k}$. Using the fact that $n = \frac{3t^2-7t+4}{2}$

and $k = t-1$, we have $\beta_{PRT(t-1)}(G_{2,\frac{3t^2-7t+4}{2}}) \geq \frac{2\frac{3t^2-7t+4}{2}}{3(t-1)} = t - \frac{4}{3}$. Since the k-independence partition number of a graph $G$ must be a whole number, $\beta_{PRT(t-1)}(G_{2,\frac{3t^2-7t+4}{2}}) \geq t-1$ as desired.

$\square$

The fact that we have a lower bound for grids of order $n = \frac{3t^2-7t+4}{2}$ when all color classes are less than $t-1$ is reassuring, but even more reassuring is the fact that we can construct grids at certain values of $t$ that fit this lower bound. In fact, these graphs fit into the statement of Conjecture 1. The following graphs depicted below in Figure 18, Figure 19, and Figure 20 show a partition of vertices in each graph which produces the desired value for an upper bound of the independence partition number for the specified value of $n$. We will be looking at the specific values of $n$ using values $t = 4, 5,$ and 6. In other words, we will be looking at colorings of $G_{2,12}$, $G_{2,22}$, and $G_{2,35}$. In $G_{2,12}$, all color classes will be of size 3 or less; in $G_{2,22}$, all color classes will be of size 4 or less; and in $G_{2,35}$, all color classes will be of size 5 or less following the rule that all color classes have size no larger than $t-1$. Note, from Lemma 3, we have $\beta_{PRT(3)}(G_{2,12}) \geq \frac{2(12)}{3(3)} = \frac{8}{3}$, so $\beta_{PRT(3)}(G_{2,12}) \geq 3$. Similarly, $\beta_{PRT(4)}(G_{2,22}) \geq 4$ and $\beta_{PRT(5)}(G_{2,35}) \geq 5$. So, if we can show there are graphs that produce maximum independent vertex sets with sizes that match the lower bound for $\beta_{PRT(k)}(G_{2,n})$ when $n = 12, 22,$ and 35, this will give some credence to our conjecture. For the following figures, we will denote different color classes using numbers only.

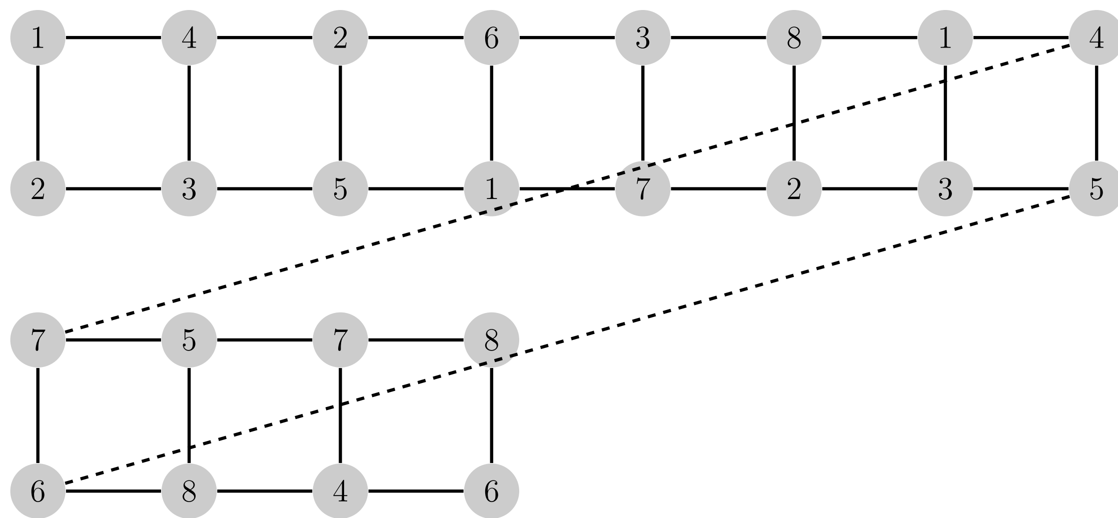


Figure 18: Coloring of $G_{2,12}$ showing $\beta_{PRT(3)}(G_{2,12}) \leq 3$

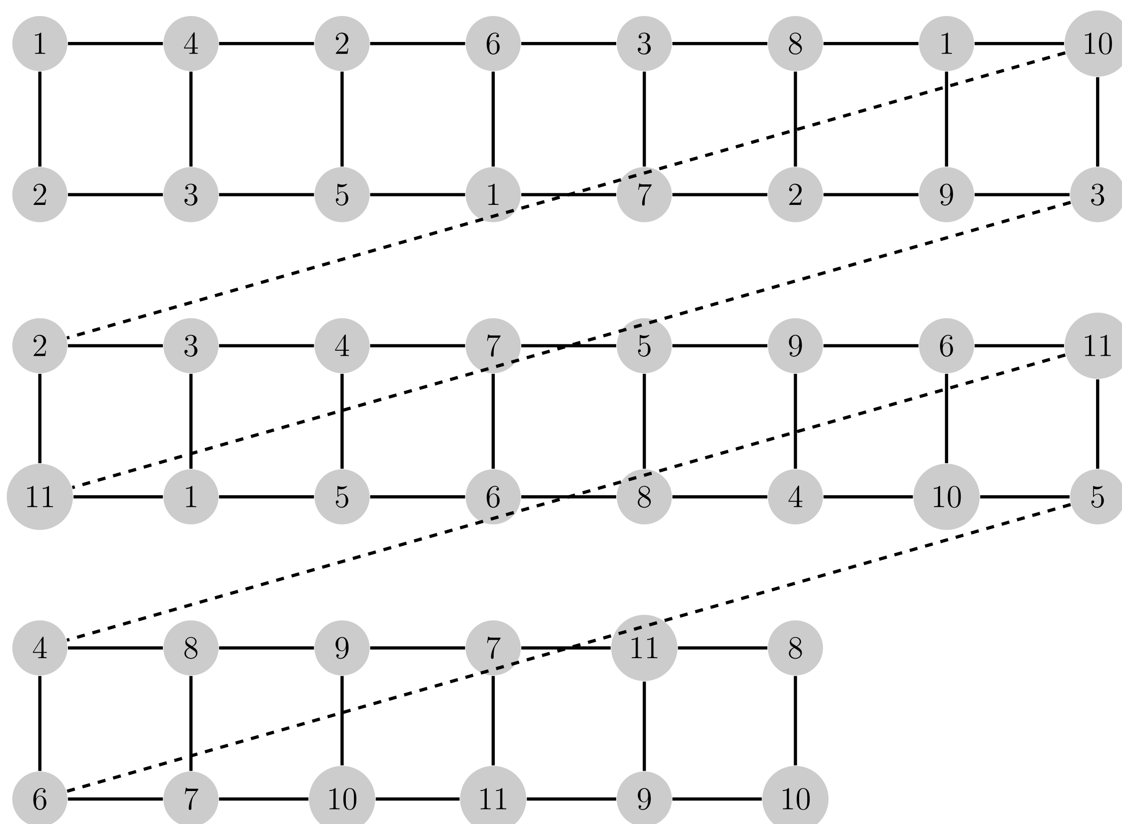


Figure 19: Coloring of $G_{2,22}$ showing $\beta_{PRT(4)}(G_{2,22}) \leq 4$

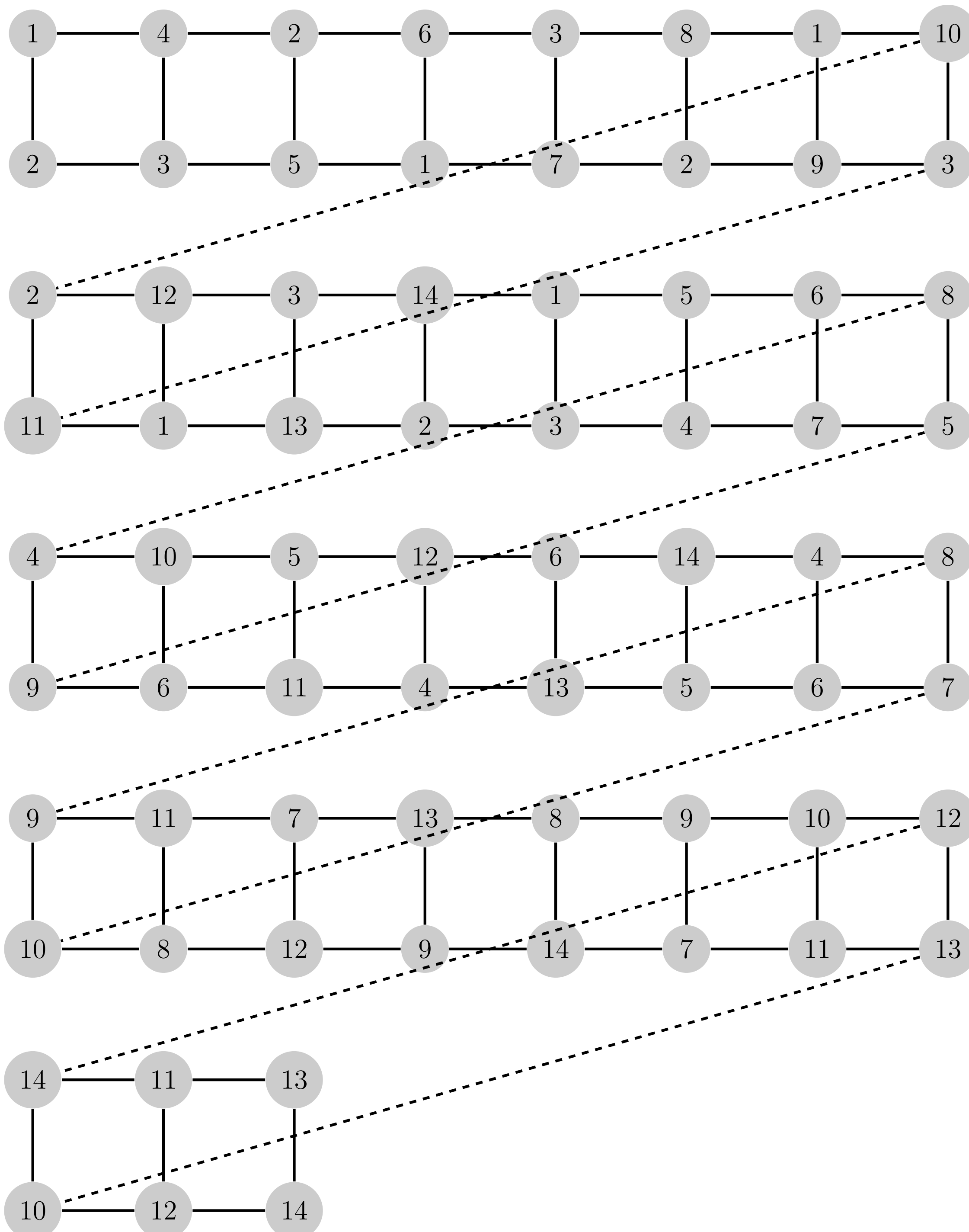


Figure 20: Coloring of $G_{2,35}$ showing $\beta_{PRT(5)}(G_{2,35}) \leq 5$

The fact that we can construct graphs to show $\beta_{PRT(t-1)}(G_{2,\frac{3t^2-7t+4}{2}}) = t-1$ for $t = 4, 5,$ and 6 gives reason to believe that the order chosen for $n$ in the conjecture is correct.

Although it is a little much to say these graphs give reason to believe the complete conjecture for 2 by $n$ grids to be true, it is encouraging to see a pattern hold for several examples.

With cycles, we had a way to construct the partition of the vertices of the graphs using Eulerian circuits. The real challenge in proving the conjecture for 2 by $n$ grids is to find a method of constructing partitions for the vertices of grids which yield the smallest maximum independent set. If we can find this construct, we are one step closer to describing the independence partition number on 2 by $n$ grids.

## Section 5: Conclusion and Further Direction in Research

Ideally, with further research, colored independence can be described on types of graphs beyond paths, cycles, and grids. Theoretically, we should be able to expand the idea of colored independence to any type of graph. In order to get to this point, several things have to be studied.

First and foremost, the conjecture stated in this paper regarding the description of the independence partition number for 2 by $n$ grids needs to be proven. As was said in Section 4, we need a way to construct the coloring of the vertices in these types of grids. Once the independence partition number has been established for 2 by $n$ grids, the next logical step would be to generalize to $m$ by $n$ grids. Once we have a firm grasp on the finite dimensional grid, one could look at infinite grids, and the implications of colored independence on them.

In the first paragraph of this section, I mentioned that colored independence could theoretically be studied on all types of graphs. This is not something that can be understood without first understanding colored independence on many more types of graphs, but there is a real chance that more research in this subject could reveal a more general idea of colored independence. Although I have only studied this subject specifically on cycle graphs and 2 by $n$ grids, I believe understanding colored independence on a more general level on any type of graph must consider the degrees of vertices. Further research on colored independence and its relation to the degree of vertices of a graph could give further insight into colored independence on any type of graph.

Colored independence is a relatively new subject to study in the area of graph theory. In Section 1, we discussed how the Konigsberg Bridge Problem was the tip of the iceberg in terms of different problems in graph theory. It just took that one result from Euler to spark the interest we have today in the field of graph theory. It is my hope that this paper on colored independence on cycle graphs and finite 2 by $n$ grids creates that same sort of spark in this area of graph theory and prompts researchers to make further strides in this relatively new area of mathematics. With continued research and findings, we can better understand colored independence and recognize how it can be helpful in solving problems in the future.

## Section 6: References

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