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Analysis of a substitute for the impact damper to damp near-resonant mechanical vibrations

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ANALYSIS OF A SUBSTITUTE FOR THE IMPACT DAMPER TO DAMP NEAR-RESONANT MECHANICAL VIBRATIONS

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ABSTRACT

This report investigates a substitute for the impact damper which, although highly effective in reducing vibration amplitudes of near-resonant mechanical systems, in operation causes often unacceptable intensive noise.

The present damper consists of a piston free to move in a cylinder, at either end of which is a ball valve set to open at a preset pressure, and in the middle along its length an intake port.

An approximate analytical study is made to determine the conditions for the existence of "noiseless periodic operation" of the damper, periodic operation without the occurence of impacts. This approach is based on the describing function method which harmonically linearizes the nonlinear damping force involved in the equations of motion of the system. The excitation force to produce this periodic operation and the response results from this operation may be predicted by this analytical approach.

Digital simulation of the system is used to verify the predictions by analytical approach.

A study on a given system indicates that appropriate design parameters may be selected for a damper of this type to obtain a reduction in response amplitude of the primary system at resonance to 1/5 of its value without the damper. This is essentially the reduction that might be obtained with a properly designed impact damper. Unlike the impact

damper, the new damper is expected to operate relatively noiselessly.

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NOMENCLATURE

A = cross-sectional area of the damper piston (in.^{2})

A_f = F_o/M (in./sec.^{2})

B = length of the damper piston (in.)

C = damping coefficient of primary system (lb. sec./in.)

C_r = critical damping coefficient of primary system (lb. sec./in.)

D = total piston stroke (in.)

e = coefficient of restitution

F_o = maximum force of excitation (lb.)

$F(y,\dot{y})$ = damping force, $(P_2 - P_1)A$ (lb.)

f = excitation function, $\frac{F_o}{M} \sin (\Omega t - \phi)$

K = spring stiffness (lb./in.)

M = mass of primary system (lb./in. per sec.^{2})

m = mass of damper-piston (lb./in. per sec.^{2})

N = describing function for damping force, $N_1 + jN_2$

P_a = atmospheric pressure (p.s.i.a.)

P_c = preset absolute pressure of the ball valves (p.s.i.a.)

P_1 = absolute pressure of the air in cylinder on the left hand side of the piston (p.s.i.a.)

P_2 = absolute pressure of the air in cylinder on the right hand side of the piston (p.s.i.a.)

r = ratio of forcing frequency to natural frequency, $\frac{\Omega}{\omega}$

t = time (sec.)

x = displacement of M (in.)

X = maximum amplitude of $x(t)$ (in.)

X_o = maximum amplitude of $x(t)$ for system at resonance without damper (in.)

x_- = displacement of M immediately before impact (in.)

x_+ = displacement of M immediately after impact (in.)

y = relative displacement of damper piston m with respect to M (in.)

Y = maximum amplitude of $y(t)$ (in.)

Y_B = absolute value of amplitude of relative displacement of damper piston when air-discharge begins (in.)

W = length of the intake port (in.)

ζ = ratio of critical damping, $\frac{C}{C_r}$, of primary system

μ = ratio of damper mass to primary system mass, $\frac{m}{M}$

Φ = phase angle

ω = natural frequency of undamped oscillation, $\sqrt{K/M}$ (rad./sec.

Ω = forcing frequency (rad./sec.)

I. INTRODUCTION

In many vibratory systems, the system response to an excitation characterized by a frequency near a resonance may be excessive and may require reduction. A successful technique for reducing the near-resonant response amplitude of a vibratory system is to use an impact damper. An impact damper consists of a small mass constrained to move in a container attached to the primary system. The periodic impacts between the impacting mass and the ends of the container may often cause subtantial reductions in the vibration amplitude of the primary system through momentum transfer.

The impact damper was first investigated by Lieber and Jensen (1945) [1]. The theory of the impact damper was developed by the contributions of Grubin [2], Arnold [3], Warburton [4] and Masri [5], [15].

The application of impact dampers to reduce the vibrations of such systems as ship hulls, cantilever beams, turbine buckets, and antenna structures was investigated by McGoldrick [6], Lieber and Tripp [7], Duckwald [8], Rocke and Masri [9], respectively. Recent communication indicates that Masri is attempting to extend the application of impact dampers to the reduction of earthquake-induced vibrations in buildings.

Although impact dampers appear to be effective in reducing the response amplitude of vibratory systems near resonance, the excessive noise caused by impacts of the mass and its container may be of such an intensity that it is unacceptable to the human ear, according to private communications with Rocke and Masri.

The objective of the present study is to investigate the feasibility of a similar bolt-on damper -- a free piston gas compression, throttling, expansion device -- as a possible replacement for the impact damper. Feasibility is assessed in terms of a reasonably-sized damper producing amplitude reductions in the response of the primary system for steady-state near-resonant operation like those produced by the impact damper. The new damper, at least in its conceptual state, should have the important advantage that its operation should be relatively noiseless.

The present damper consists of a piston free to move in a cylinder as shown in Fig. 1. At the mid-point along the length of the cylinder is an intake port, and at either end of the cylinder is a ball valve which is set to open at a preset pressure. In operation the piston compresses air at one end of the cylinder and forces it past the ball valve out of the cylinder; the piston then expands the remaining air, takes in more air at mid-stroke and the process repeats at the other end of the cylinder. If the ball valves are set to open at a preset pressure such that no impact can occur when the damper is in operation, the noise which would be caused by the impacts is eliminated and the new damper should be relatively noiseless. This "noiseless" periodic operation is of primary interest in the present study. However, a relatively quiet operation which allows occasional impacts, impacts at starting or impacts with reduced force may not be undesirable and

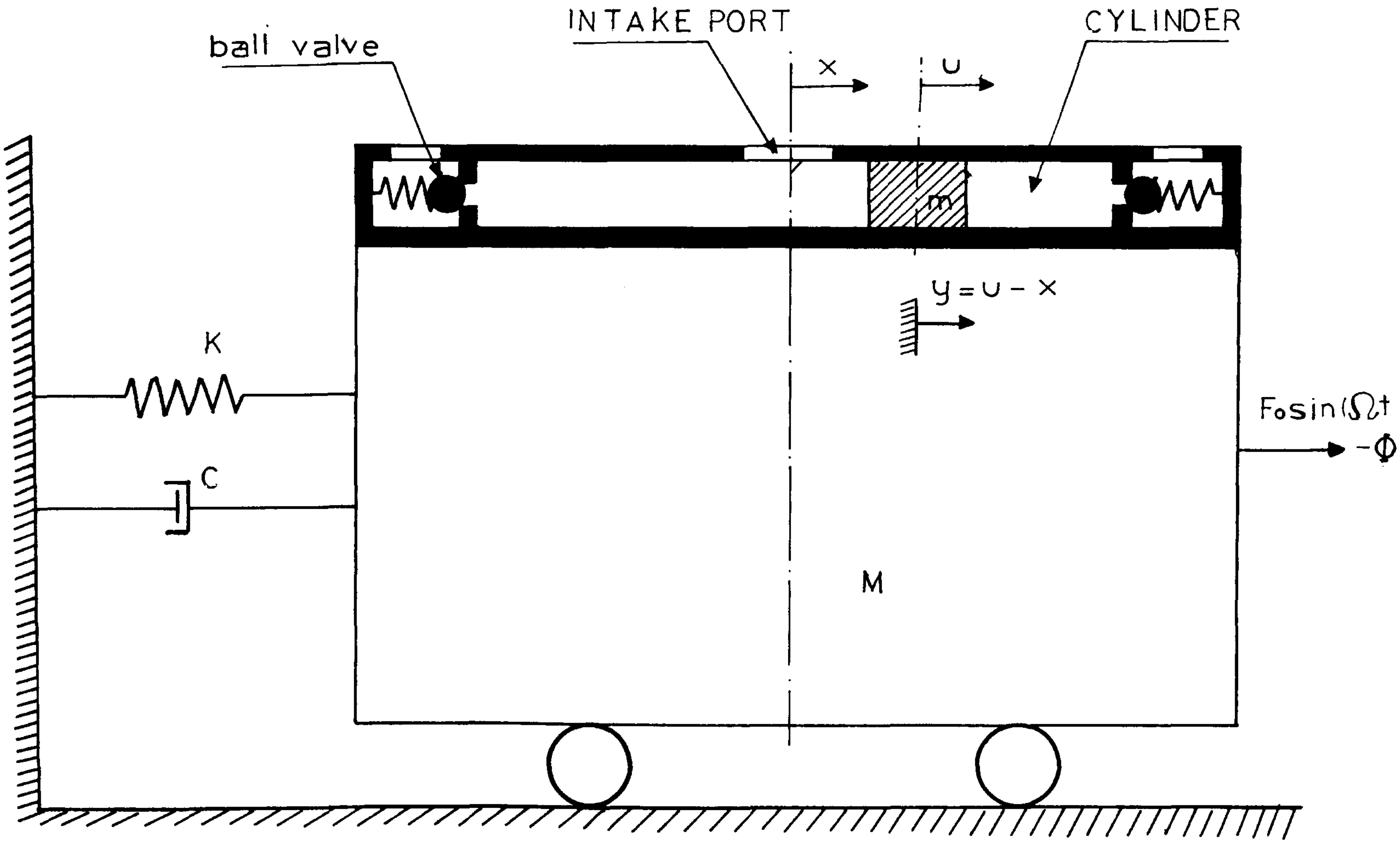


Figure 1. Model of the System.

should perhaps be an objective of future work.

Since the new damper is a nonlinear device, its performance, as is the case for impact dampers, is dependent on the amplitude of the exciting force. In the present study, a general analytical approach leading to a description of when noiseless periodic operation of the damper will occur has been developed. Using the analytical approach, the performance of particular systems, and their behavior as functions of the input and design parameters have been studied. Numerical studies have been used to examine the effects of the various approximations made in the foregoing analytical study.

The primary goals of the present study can be summarized as follows:

1) To find a general method which will permit an approximate analysis of noiseless periodic operation of the damper applied to a general system for near-resonant operation.

By applying this method to a particular system described by

$\mu = \frac{m}{M}$ = ratio of damper mass to primary system mass = 0.1

ζ = damping ratio of primary system = 0.02

e = coefficient of restitution = 0.8

ω = natural frequency of primary system = 188.5 rad./sec. (or 30 c.p.s.)

Ω = forcing frequency $\doteq \omega$ (operation near-resonance)

W_1 = weight of primary system = 10 lbs.

F_o = maximum force of excitation = 20 to 50 lbs., the following goals are also addressed:

2) To select the remaining parameters to obtain noiseless periodic operation of the damper.

3) To obtain a reduction in the system response amplitude comparable to those obtained by impact dampers, e.g. this amplitude ratio should be

$$\frac{X}{X_o} \leqslant \frac{1}{3}$$

When the damper is operating noiselessly and periodically, $\frac{X}{X_o}$ is the ratio of maximum amplitude of the system with damper attached to maximum amplitude of the system at resonance without the damper.

4) To work effectively when the excitation force varies within the range of 20 to 50 lbs.

5) To work effectively when the system is at resonance, as well as in the vicinity of this resonance.

6) To study the effects of the design parameters, especially the adjustable ones, P_c and D, on the performance of the damper.

A brief description of the contents of the remaining chapters follows.

The mathematical description of the system with the damper attached is given in Chapter II. First the system is described by the equations of motion. Then the assumptions are discussed, and the mathematical description of the damping force $F(y,\dot{y})$ is given, where y is the relative displacement of the damper piston m with respect to the primary system M. The impact conditions are also discussed in this chapter.

In chapter III a digital computer model is described that simulates the motion of the system with the damper attached. In this model the motion of the piston is not constrained to be periodic and the possibility of impacts is included.

An approximate analytical method to determine the noiseless periodic operation of the new damper, is outlined in Chapter IV. The governing nonlinear differential equation relating the damping force $F(y,\dot{y})$ to the input variable $y(t)$ is first derived from the equations of motion. The damping force $F(y,\dot{y})$ is approximated by harmonic linearization using the describing function method. This method was developed by many authors, among them Siljak [10], Shridhar [11] and Minorsky [13]. The conditions for the existence of periodic operation of the damper are established. Then the steady-state response of the system is derived, and the stability of the system is discussed using the Hurwitz criteria.

Chapter V presents a comparison of the results obtained by the digital computer simulation with those from the analytical approach described in Chapter IV.

Summary and conclusions drawn from this study are stated in Chapter VI.

Appendix A contains a description of the computer program used to simulate the motion of the system.

Appendix B contains details of the computer program verifications.

II. MATHEMATICAL DESCRIPTION OF THE SYSTEM

A. Equations of Motion

The equations of motion of the system with damper shown in Figure 1 are:

$$M\ddot{x} + C\dot{x} + Kx = (P_2 - P_1)A + F_o \sin(\Omega t - \Phi) \tag{2.1}$$

$$m(\ddot{x} + \ddot{y}) = (P_1 - P_2)A \tag{2.2}$$

Equations (2.1) and (2.2) may be rewritten as:

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = \frac{F(y,\dot{y})}{M} + f(t) \tag{2.3}$$

$$\ddot{x} + \ddot{y} = -\frac{F(y,\dot{y})}{m} \tag{2.4}$$

where:

x = displacement of primary system

y = relative displacement of the damper mass m with respect to the primary system M

$\zeta = \frac{C}{Cr}$ = damping ratio

C = damping coefficient of primary system

C_r = critical damping coefficient, $2\sqrt{KM}$

$\omega = \sqrt{K/M}$, undamped natural frequency of primary system

P_1 = Pressure of the air in cylinder on the left hand side of the piston (ref. Figure 1)

P_2 = Pressure of the air in cylinder on the right hand side of the piston (ref. Figure 1)

$\mu = \frac{m}{M}$ = ratio of damper mass to primary system mass

$f(t) = \frac{F_o}{M} \sin(\Omega t - \phi)$ = excitation function

$F(y,\dot{y}) = (P_2 - P_1)A$ = damping force

A = cross-sectional area of the piston

Φ = phase factor on the input

B. Dimensions of the Damper

For the present investigation the damper, as shown in Fig. 2, is designed such that:

1. The length B of the piston is equal to the length W of the intake port.

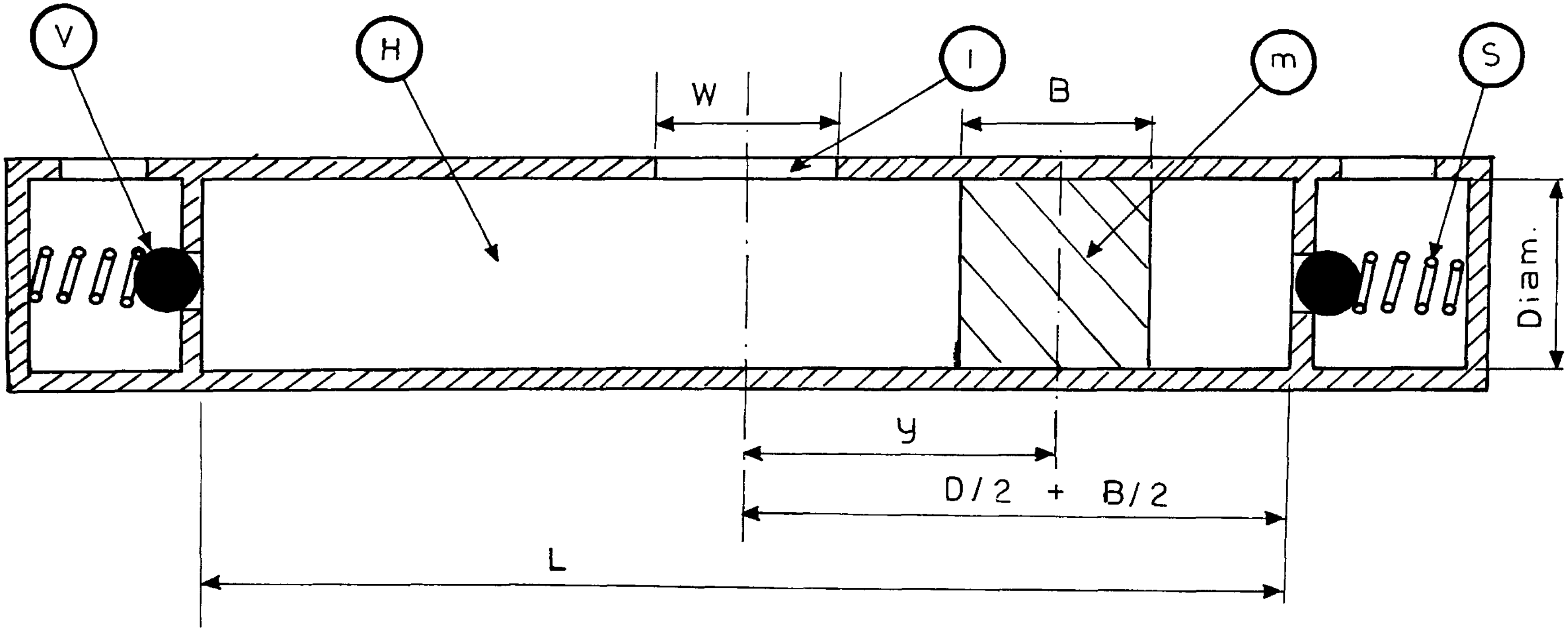
$$W = B$$

As a consequence, when the piston moves past the mid-stroke position (y = 0), compression begins on one side of the cylinder, and the expansion of the air in the other side terminates as air exhausts through the port.

2. The length L of the cylinder is equal to the sum of the stroke D and the length B of the piston:

$$L = D + B$$

When the piston contacts an end of the cylinder, its relative



B = Length of the damper piston

W = Length of the intake port

D = Total Piston Stroke

L = Length of the cylinder

S Ball Valve Springs

H Cylinder or container

m Damper piston

I Intake port

V Ball valves

Figure 2. Schematic of the Damper

coordinate y(t) with respect to the cylinder is $\pm D/2$.

C. Assumptions

The following assumptions are made for the purpose of analysis:

1. The piston moves frictionlessly in the cylinder

2. Whenever the ball valves are closed (no air-discharge) the compression or expansion of the air in the cylinder is polytropic*

$$P_i V_i^n = \text{constant}$$

where:

P_i, V_i = pressure and volume of the air under compression or expansion, and

$n = 1.3$

3. The ball valve mass, and valve spring are selected so that ball valve dynamics, e.g., chatter, etc., are negligible.

4. After the piston has reached the mid-stroke position ($y = 0$), as soon as the compression of air begins on one side of the cylinder, the expansion of the remaining air on the other side terminates immediately, and its pressure regains the value of atmospheric pressure.

5. The impacts, if they occur, take place in an incrementally

---

* The factor $n = 1.3$ describes an observed value for the operation of real machines. It represents a process somewhere between isothermal ($n = 1$) and adiabatic ($n = 1.4$).

small time. During this time the positions of the primary system mass M and damper piston m remain the same while their velocities change discontinuously.

D. Damping Force

The damping force $F(y,\dot{y})$ is first described for the case when the damper operates periodically with 2 compressions per cycle, then it is determined for the general case when the piston is not constrained to periodic operation and can move back and forth as shown in Figure 5.

1. Periodic 2 compressions/cycle operation.

The damping force is:

$$F(y,\dot{y}) = (P_2 - P_1)A \tag{2.5}$$

The pressure P_i of the air under compression or expansion in the cylinder is determined by the equation:

$$P_i V_i^{\,n} = \text{constant} \tag{2.6}$$

For the portion AB in Figure 3:

$$P_i V_i^{\,n} = P_a V_a^{\,n}$$

$$P_i = P_2 = P_a \left[\frac{D/2}{D/2 - y}\right]^n$$

The portion BC of the curve in Figure 3 is defined by:

$$P_i = P_2 = P_c = \text{constant} \tag{2.6b}$$

By applying equation (2.6) to portions AB, CD, AB', C'D', and

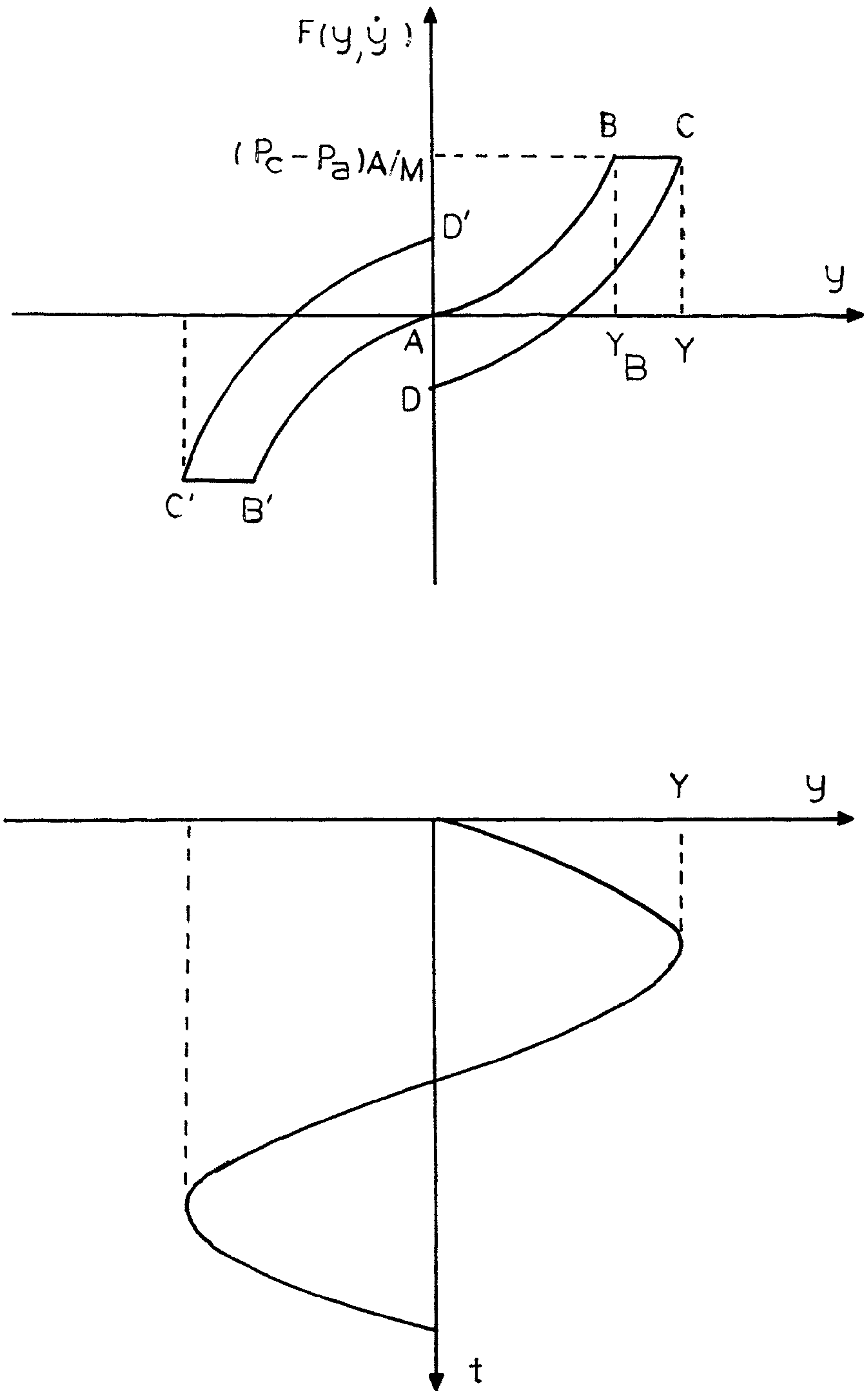


Figure 3. Damping Force in Periodic 2 Compressions/Cycle Operation

equation (2.6b) to portions BC, B'C', and noting that the pressure of the air under compression or expansion in the cylinder at points A, C and C' are known, the damping force $F(y,\dot{y})$ may be obtained for these portions:

$$\text{Portion AB:} \quad F(y,\dot{y}) = \left(P_a \left[\frac{D/2}{D/2 - y}\right]^n - P_a \right) A \quad \begin{cases} 0 \leqslant y \leqslant y_B \\ \dot{y} \geqslant 0 \end{cases}$$

$$\text{Portion BC:} \quad F(y,\dot{y}) = \left(P_c - P_a \right) A \quad \left[\begin{array}{l} Y_B \leqslant y \leqslant Y < D/2 \\ \dot{y} \geqslant 0 \end{array} \right.$$

$$\text{Portion CD:} \quad F(y,\dot{y}) = \left(P_c \left[\frac{D/2 - Y}{D/2 - y}\right]^n - P_a \right) A \quad \left[\begin{array}{l} 0 \leqslant y < D/2 \\ \dot{y} \leqslant 0 \end{array} \right.$$

$$\text{Portion AB':} \quad F(y,\dot{y}) = \left(P_a - P_a \left[\frac{D/2}{D/2 + y}\right]^n \right) A \quad \left[\begin{array}{l} -Y_B \leqslant y \leqslant 0 \\ \dot{y} \leqslant 0 \end{array} \right.$$

$$\text{Portion B'C':} \quad F(y,\dot{y}) = \left(P_a - P_c \right) A \quad \begin{cases} -D/2 < -Y \leqslant y \leqslant -Y_B \\ \dot{y} \leqslant 0 \end{cases}$$

$$\text{Portion C'D':} \quad F(y,\dot{y}) = \left(P_a - P_c \left[\frac{D/2 - Y}{D/2 + y}\right]^n \right) A \quad \begin{cases} -D/2 < y \leqslant 0 \\ \dot{y} \geqslant 0 \end{cases} \tag{2.7}$$

where:

Y = maximum amplitude of the relative displacement of the damper piston $< D/2$

Y_B = amplitude of relative displacement of damper piston when air discharge begins

If $Y \leqslant Y_B$ the ball valves will never open, and the device will behave as a mass on a nonlinear spring (See Figure 4.) In this case the damping force is

$$F(y,\dot{y}) = \pm \left(P_a \left[\frac{D/2}{D/2 \mp y}\right]^n - P_a \right) A \qquad (2.8)$$

where:

The upper sign corresponds to $y > 0$

The lower sign corresponds to $y < 0$

2. <u>General case</u>

In the general case when the damper is not constrained to periodic operation, one possible movement back-and-forth of the piston is shown in Fig. 5.

In the program for digital simulation, as described in Chapter III, a control flag named VALVE is used to indicate the state of the air in the cylinder. This flag is introduced here to aid the present discussion.

VALVE = -1	if the piston having passed the mid-stroke position (y = 0) compresses the air, but the compression has not yet opened the ball valve.
VALVE = 0	When the ball valve is forced to open
VALVE = 1	if during the same half-cycle period the ball

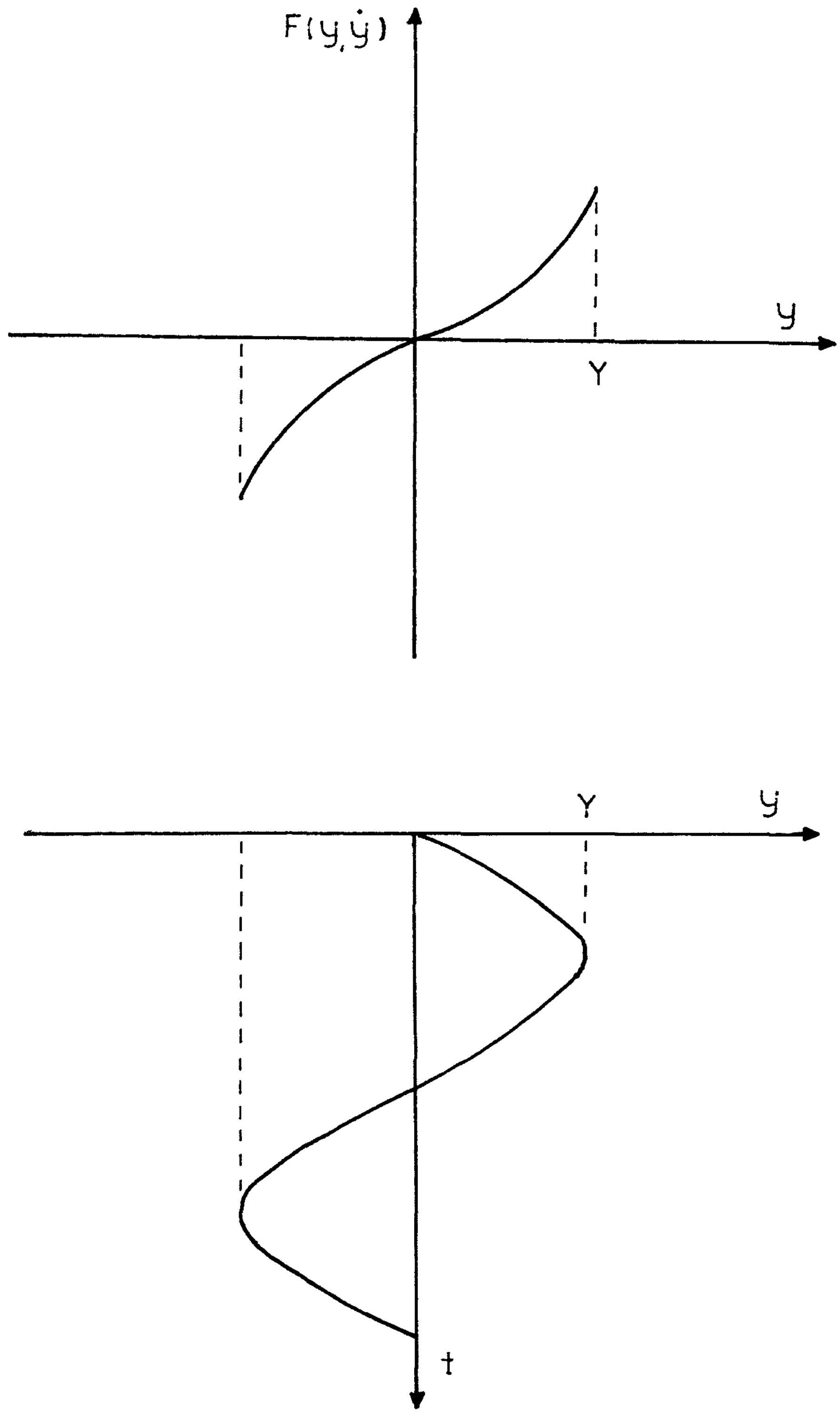


Figure 4. Damping Force in Periodic 2 Compressions/Cycle Operation Case of no Air-Discharge $(Y \leqslant Y_B)$.

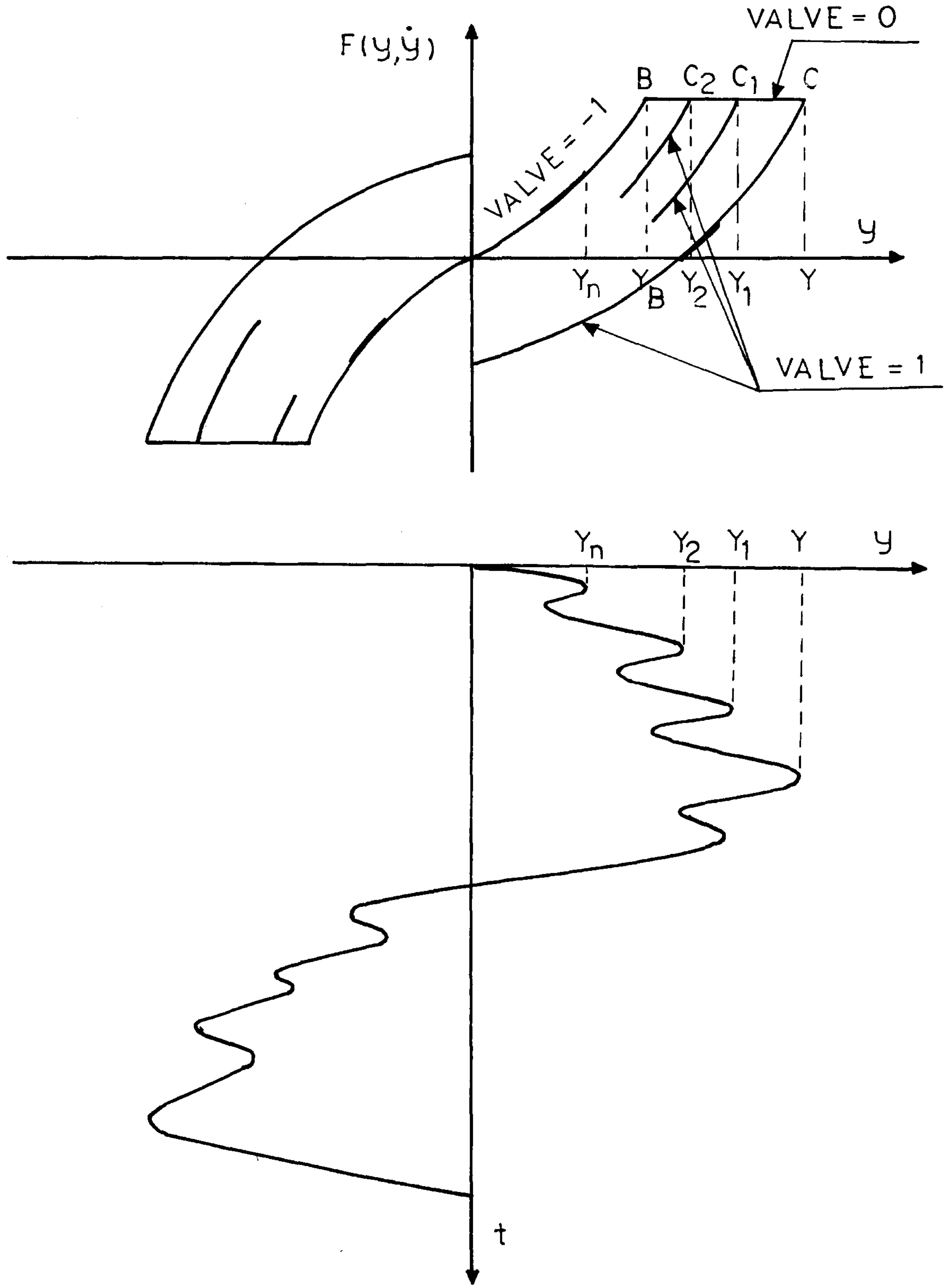


Figure 5. Damping Force - General Case.

valve which has been open at least one time during the period is closed again.

The damping force can be derived from equations (2.5), (2.9), (2.10), and (2.11):

$$F(y,\dot{y}) = \text{Sign }(y) \left(P_a \left[\frac{D/2}{D/2 - |y|}\right]^n - P_a \right) A \qquad \text{if VALVE} = -1$$

$$F(y,\dot{y}) = \text{Sign }(y) \left(P_c - P_a \right) A \qquad \text{if VALVE} = 0$$

$$F(y,\dot{y}) = \text{Sign }(y) \left(P_c \left[\frac{D/2 - Y_n}{D/2 - |y|}\right]^n - P_a \right) A \qquad \text{if Valve} = 1 \qquad (2.12)$$

where:

$Y_n = Y, Y_1, Y_2, \ldots\ldots$

= displacement of the piston at the moment when the ball valve which has been open during the same half cycle closes, as shown in Fig. 5.

E. Impact Conditions

For the purpose of the present study, impact is considered as potentially superposing a force upon the system in addition to $F(y,\dot{y})$. In general, motion, for example, in the positive y direction may be terminated by impact which occurs, depending on the choice of damper parameters, during the air compression phase (line AB on Fig. 3) or during the air expulsion phase (line BC on Fig. 3). Impacts are assumed to occur instantly such that during the impacts, the displacements of the primary system mass M and the piston m remain the same, while their absolute velocties are discontinuously changed.

The momentum equation of the system during impact is:

$$M\dot{x}_- + mV_{m_-} = M\dot{x}_+ + mV_{m_+} \tag{2.13}$$

where:

$V_m = \dot{y} + \dot{x}$ = absolute velocity of the damper piston

The - and + subscripts designate times immediately preceding and following the impact.

By the definition of the coefficient of restitution e:

$$\dot{x}_+ - V_{m_+} = -e(\dot{x}_- - V_{m_-}) \tag{2.14}$$

The impact conditions can then be summarized as follows:

$$x_+ = x_-$$

$$y_+ = y_-$$

$$\dot{x}_+ = \dot{x}_- + \mu\frac{(1 + e)}{(1 + \mu)}\ \dot{y}_- \tag{2.15}$$

$$\dot{y}_+ = -e\dot{y}_-$$

where:

$\mu = \frac{m}{M}$ = ratio of damper mass to primary system mass

e = coefficient of restitution

Conceptually impacting forces may be superimposed upon the damping force as illustrated in Fig. 6.

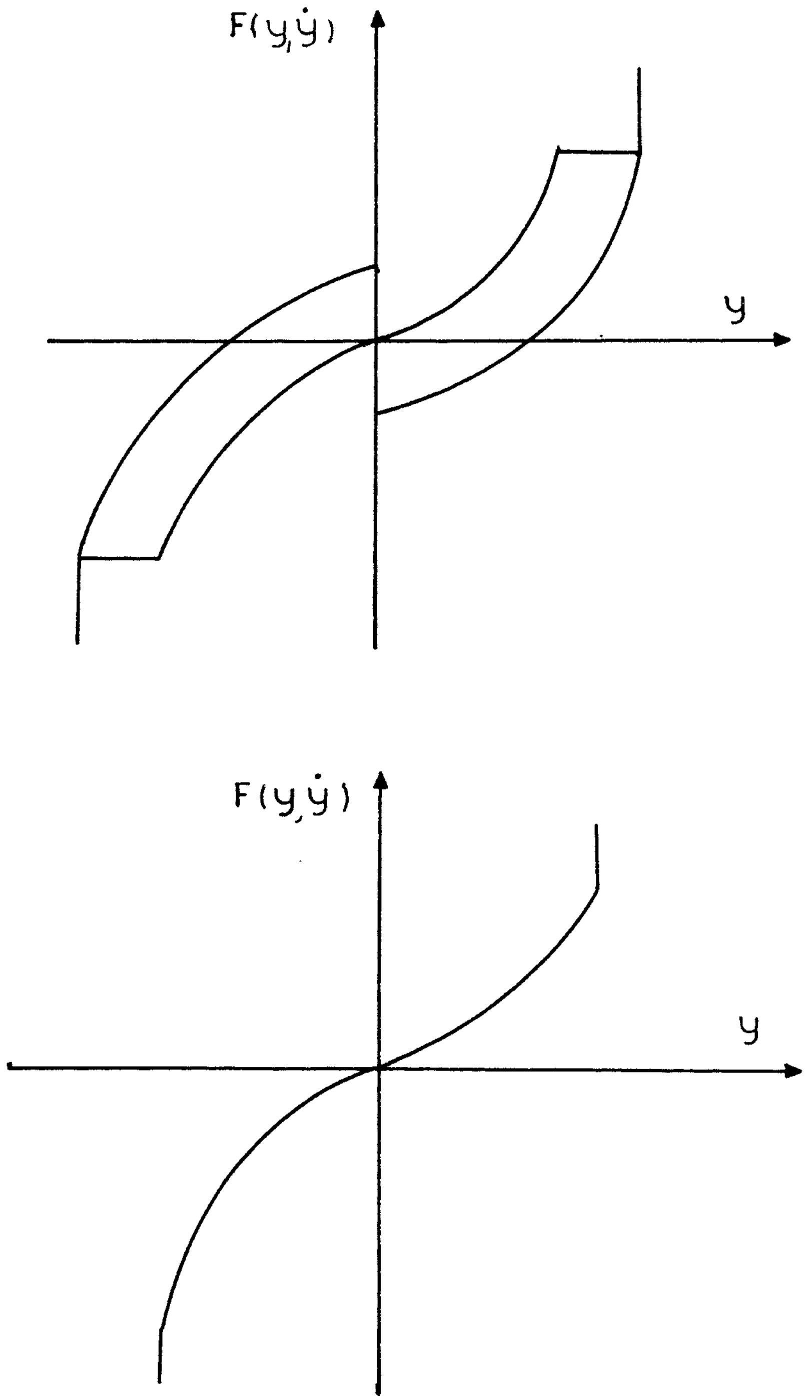


Figure 6. Damping Force Conceptually Superposed by Impacts

III. DIGITAL COMPUTER MODEL SIMULATING THE MOTION OF THE SYSTEM

A. Solution of the Differential Equations by Kutta-Merson Method

The solution of the system of two second-order nonlinear differential equations:

$$M\ddot{x} + C\dot{x} + Kx = (P_2 - P_1)A + F_o \sin (\Omega t - \phi) \tag{2.1}$$

$$m(\ddot{x} + \ddot{y}) = - (P_2 - P_1)A \tag{2.2}$$

is obtained by treating these equations as four first-order differential equations and by using the Kutta-Merson [14] method.

The system of equations (2.1) and (2.2) are the equations of motion of the system when there is no impact. If impacts occur, these are the equations of motion of the system between impacts.

When new variables are defined:

$$\begin{aligned} Z_1 &= x \\ Z_2 &= \dot{x} \\ Z_3 &= y \\ Z_4 &= \dot{y} \end{aligned} \tag{3.1}$$

Then equations (2.1) and (2.2) may be rewritten as:

$$\dot{Z}_1 = f_1 (t, Z_1, Z_2, Z_3, Z_4) = Z_2 \tag{3.2}$$

$$\dot{Z}_2 = f_2 (t, Z_1, Z_2, Z_3, Z_4) = G_2 \tag{3.3}$$

$$\dot{Z}_2 = (P_2 - P_1)\frac{A}{M} + \frac{F_o}{M}\sin(\Omega t - \phi) - \frac{C}{M}Z_2 - \frac{K}{M}Z_1$$

$$= (P_2 - P_1)\frac{A}{M} + A_f \sin(\Omega t - \phi) - 2\zeta\omega Z_2 - \omega^2 Z_1$$

$$\dot{Z}_3 = f_3(t, Z_1, Z_2, Z_3, Z_4) = Z_4 \qquad (3.4)$$

$$\dot{Z}_4 = f_4(t, Z_1, Z_2, Z_3, Z_4) = G_4 \qquad (3.5)$$

$$= -(P_2 - P_1)\frac{A}{M} - \dot{Z}_2$$

The initial conditions throughout are taken to be:

$$\begin{aligned} Z_1(0) &= x(0) \\ Z_2(0) &= \dot{x}(0) \\ Z_3(0) &= y(0) \\ Z_4(0) &= \dot{y}(0) \end{aligned} \qquad (3.6)$$

In terms of vector notations:

$$\{Z\} = \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{Bmatrix} \qquad (3.7)$$

$$\{F_i\} = \begin{Bmatrix} f_1(t,Z) \\ f_2(t,Z) \\ f_3(t,Z) \\ f_4(t,Z) \end{Bmatrix} = \begin{Bmatrix} Z_2 \\ G_2 \\ Z_4 \\ G_4 \end{Bmatrix} \qquad (3.8)$$

$$\{Z(0)\} = \begin{Bmatrix} Z_1(0) \\ Z_2(0) \\ Z_3(0) \\ Z_4(0) \end{Bmatrix} = \begin{Bmatrix} x(0) \\ \dot{x}(0) \\ y(0) \\ \dot{y}(0) \end{Bmatrix} \qquad (3.9)$$

and the system of equations (2.1) and (2.2) may be rewritten as follows:

$$\{\dot{Z}\} = \begin{Bmatrix} F_i \end{Bmatrix} \qquad (3.10)$$

with initial conditions:

$$\{Z(0)\} = \begin{Bmatrix} Z_1(0) \\ Z_2(0) \\ Z_3(0) \\ Z_4(0) \end{Bmatrix} \qquad (3.9)$$

The Kutta-Merson method functions in the following way.

If $\{Z(t)\}$ is known, $\{Z(t + h)\}$ may be determined by:

$$\{Z(t + h)\} = \{Z(t)\} + h \left[\frac{1}{6} \begin{Bmatrix} K_1 \end{Bmatrix} + \frac{2}{3} \begin{Bmatrix} K_4 \end{Bmatrix} + \frac{1}{6} \begin{Bmatrix} K_5 \end{Bmatrix}\right] \qquad (3.11)$$

where

$$\{Z(t + h)\} = \begin{Bmatrix} Z_1(t + h) \\ Z_2(t + h) \\ Z_3(t + h) \\ Z_4(t + h) \end{Bmatrix} = \begin{Bmatrix} x(t + h) \\ \dot{x}(t + h) \\ y(t + h) \\ \dot{y}(t + h) \end{Bmatrix} \qquad (3.12)$$

h = time increment

$$\{K_1\} = \begin{Bmatrix} K(1,1) \\ K(1,2) \\ K(1,3) \\ K(1,4) \end{Bmatrix} = \begin{Bmatrix} f_1(t, \{Z(t)\}) \\ f_2(t, \{Z(t)\}) \\ f_3(t, \{Z(t)\}) \\ f_4(t, \{Z(t)\}) \end{Bmatrix} \quad (3.13)$$

$$\{K_2\} = \begin{Bmatrix} K(2,1) \\ K(2,2) \\ K(2,3) \\ K(2,4) \end{Bmatrix} = \begin{Bmatrix} f_1\ (t + h/3,\ \{Z(t)\} + \frac{1}{3}\, h\ \{K_1\}) \\ f_2\ (t + h/3,\ \{Z(t)\} + \frac{1}{3}\, h\ \{K_1\}) \\ f_3\ (t + h/3,\ \{Z(t)\} + \frac{1}{3}\, h\ \{K_1\}) \\ f_4\ (t + h/3,\ \{Z(t)\} + \frac{1}{3}\, h\ \{K_1\}) \end{Bmatrix} \quad (3.14)$$

$$\{K_3\} = \begin{Bmatrix} K(3,1) \\ K(3,2) \\ K(3,3) \\ K(3,4) \end{Bmatrix} = \begin{Bmatrix} f_1\ (t + h/3,\ \{Z(t)\} + h[\frac{1}{6}\ \{K_1\} + \frac{1}{6}\ \{K_2\}]) \\ f_2\ (t + h/3,\ \{Z(t)\} + h[\frac{1}{6}\ \{K_1\} + \frac{1}{6}\ \{K_2\}]) \\ f_3\ (t + h/3,\ \{Z(t)\} + h[\frac{1}{6}\ \{K_1\} + \frac{1}{6}\ \{K_2\}]) \\ f_4\ (t + h/3,\ \{Z(t)\} + h[\frac{1}{6}\ \{K_1\} + \frac{1}{6}\ \{K_2\}]) \end{Bmatrix} \quad (3.15)$$

The vector $\{K_3\}$ is determined by using equations (3.13), (3.14), (3.15) and will be used in equations (3.16) and (3.17).

$$\{K_4\} = \begin{Bmatrix} K(4,1) \\ K(4,2) \\ K(4,3) \\ K(4,4) \end{Bmatrix} = \begin{Bmatrix} f_1(t + h/2,\ \{Z(t)\} + h[\frac{1}{8}\ \{K_1\} + \frac{3}{8}\ \{K_3\}]) \\ f_2(t + h/2,\ \{Z(t)\} + h[\frac{1}{8}\ \{K_1\} + \frac{3}{8}\ \{K_3\}]) \\ f_3(t + h/2,\ \{Z(t)\} + h[\frac{1}{8}\ \{K_1\} + \frac{3}{8}\ \{K_3\}]) \\ f_4(t + h/2,\ \{Z(t)\} + h[\frac{1}{8}\ \{K_1\} + \frac{3}{8}\ \{K_3\}]) \end{Bmatrix}$$

(3.16)

$$\{K_5\} = \begin{Bmatrix} K(5,1) \\ K(5,2) \\ K(5,3) \\ K(5,4) \end{Bmatrix} = \begin{Bmatrix} f_1(t + h, \{Z(t)\} + h[\frac{1}{2}\{K_1\} - \frac{3}{2}\{K_3\} + 2\{K_4\}]) \\ f_2(t + h, \{Z(t)\} + h[\frac{1}{2}\{K_1\} - \frac{3}{2}\{K_3\} + 2\{K_4\}]) \\ f_3(t + h, \{Z(t)\} + h[\frac{1}{2}\{K_1\} - \frac{3}{2}\{K_3\} + 2\{K_4\}]) \\ f_4(t + h, \{Z(t)\} + h[\frac{1}{2}\{K_1\} - \frac{3}{2}\{K_3\} + 2\{K_4\}]) \end{Bmatrix} \tag{3.17}$$

K(i,j) define an approximation to the "average" derivatives of the dependent variables in the internal (t, t+h), in the Kutta-Merson formulae.

The Kutta-Merson method has been used for this problem because it is self-starting and has a correction capability such that the time increment h may be halved or doubled to hold a specified level of accuracy as will be discussed in section III-B. This capability was thought to be of value because the velocities of the primary mass M and the damper mass m change discontinuously at impacts, and using other multiple-step methods of numerical integration may lead to large errors in such situations.

B. Tests for Time Increment Halving and Doubling

By using the Kutta-Merson Method the time increment h may be halved or doubled to hold a specified level of accuracy and minimize computation time.

If x_{pr}, y_{pr} are defined as the values of x and y used as arguments in the vector K_5, as given by equation (3.17), and x_{co}, y_{co} are defined as the values of x(t + h) and y(t + h) computed from x(t) and y(t), as given by equation (3.11), then:

$$x_{pr} = x(t) + [\frac{1}{2} K(1,1) - \frac{3}{2} K(3,1) + 2K(4,1]h$$

$$y_{pr} = y(t) + [\frac{1}{2} K(1,3) - \frac{3}{2} K(3,3) + 2K(4,3)]h$$

$$x_{co} = x(t + h) = x(t) + [\frac{1}{6} K(1,1) + \frac{2}{3} K(4,1) + \frac{1}{6} K(5,1)]h$$

$$y_{co} = y(t + h) = y(t) + [\frac{1}{6} K(1,3) + \frac{2}{3} K(4,3) + \frac{1}{6} K(5,3)]h$$

If:

$$0.2 \left| x_{pr} - x_{co} \right| > \epsilon \left| x_{co} \right|$$

and/or $$0.2 \left| y_{pr} - y_{co} \right| > \epsilon \left| y_{co} \right|$$

Then the time increment h will be reduced to h/2, otherwise h will remain the same.

If:

$$12.8 \left| x_{pr} - x_{co} \right| < \epsilon \left| x_{co} \right|$$

and $$12.8 \left| y_{pr} - y_{co} \right| < \epsilon \left| y_{co} \right|$$

then the time increment will be doubled, otherwise h will remain the same, where ϵ is an error criterion chosen primarily by experiment.

C. Flow Charts

1. Main Program

The main program, which is shown in Fig. 7, consists of the integration routine, the integration step halving, the impact and the integration step doubling tests.

If U, V, UU and VV are defined as:

$$U = 0.2 \left| x_{pr} - x_{co} \right| - \epsilon \left| x_{co} \right|$$

$$V = 0.2 \left| y_{pr} - y_{co} \right| - \epsilon \left| y_{co} \right|$$

$$UU = 12.8 \left| x_{pr} - x_{co} \right| - \epsilon \left| x_{co} \right|$$

$$VV = 12.8 \left| y_{pr} - y_{co} \right| - \epsilon \left| y_{co} \right|$$

then the condition for step-halving is positive U and/or V, the condition for step-doubling is negative or zero UU and VV. The time increment is also halved whenever a ball valve begins to open or to close, and when the piston approaches the mid-stroke position (y = 0). In such a case FLAG is set equal to 1 in the subroutine FORCE as a signal to the main program to halve step.

Impacts are detected when:

$$YYY = \left| y(t) \right| > D/2$$

at this time the time increment h is halved until:

$$\frac{D}{2} < \left| y(t) \right| \leqslant C = \frac{D}{2} (1 + EEF)$$

EEF being an error criterion.

The occurence of impacts is determined by:

a) $\frac{D}{2} < \left| y(t) \right| \leqslant \frac{D}{2} (1 + EEF)$

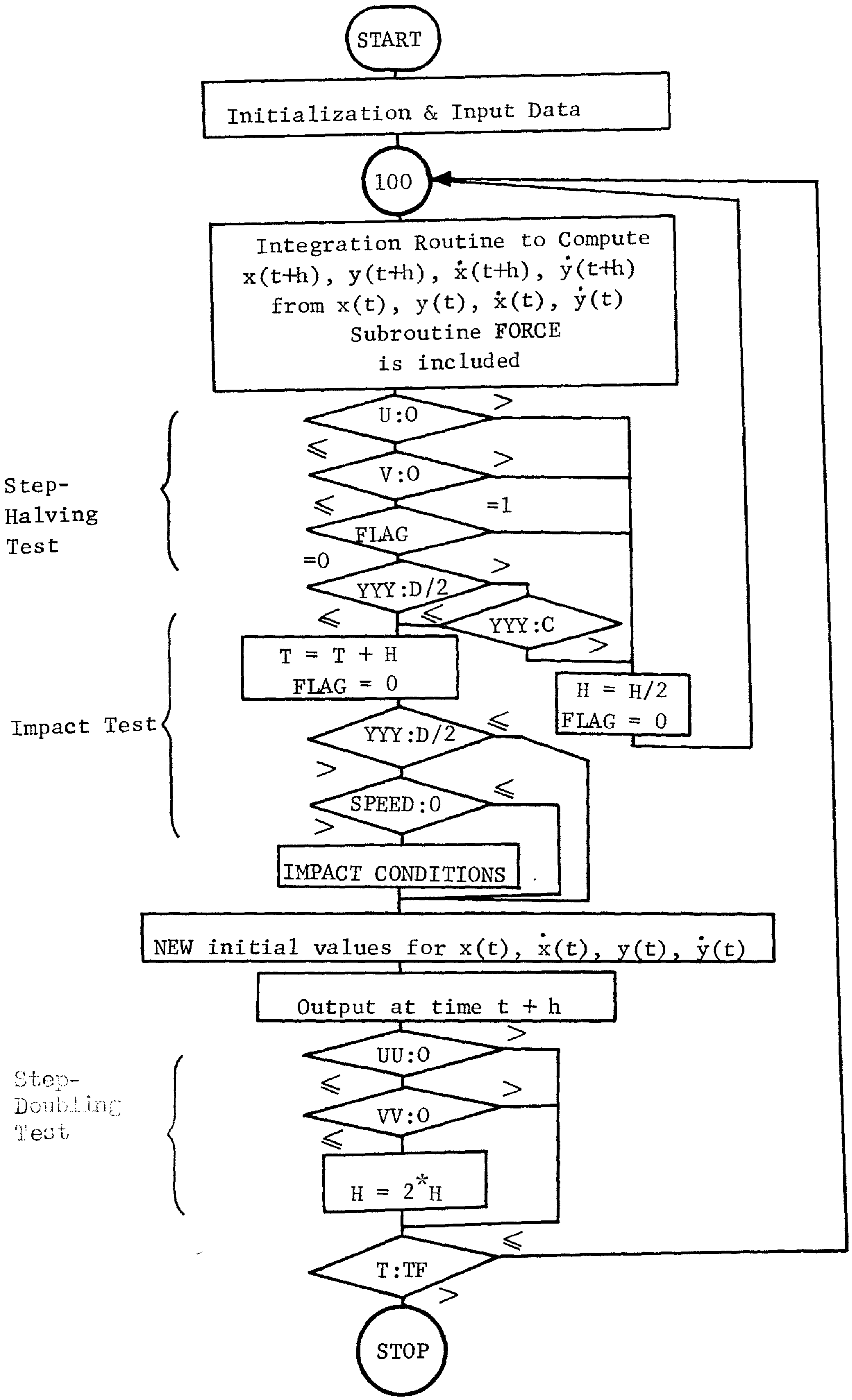


Figure 7. Main Program

b) SPEED = $y.\dot{y} > 0$

Once the impact is determined, the impact conditions given by equation (2.15) are applied:

$$x_+ = x_-$$

$$y_+ = y_-$$

$$\dot{x}_+ = \dot{x}_- + \mu \frac{(1 + e)}{(1 + \mu)} \dot{y}_-$$

$$\dot{y}_+ = - e\dot{y}_-$$

Since impacts are determined for

$$\frac{D}{2} < \left| y(t) \right| \leqslant \frac{D}{2} (1 + EEF)$$

the arguments of the expressions $[\frac{D/2}{D/2 - |y|}]^n$ and $[\frac{D/2 - Y_n}{D/2 - |y|}]^n$ may become negative or zero.

Since the computation of a fractional power of a negative number is not permissible, the damper is assumed to have a small dead chamber V_o at each end of the cylinder.

Equations (2.9) and (2.11) becomes:

$$P_i = P_a \left[\frac{D/2 + S_o}{D/2 - |y| + S_o}\right]^n \qquad \text{if VALVE} = -1$$

$$P_i = P_c \left[\frac{D/2 - Y_n + S_o}{D/2 - |y| + S_o}\right]^n \qquad \text{if VALVE} = 1$$

where

$$S_o = \frac{V_o}{A}$$

However, the dead chambers are designed such that the volume V_o of which is small:

$$V_o = \frac{(\frac{D}{2} \times A)}{1000}$$

and can be ignored in the approximate analytical approach outlined in Chapter IV.

2. Integration Routine

The purpose of the integration routine, which is shown in Fig. 8, is to compute $\{z(t + h)\}$ from the values $\{z(t)\}$.

The vector $\{z(t + h)\}$ is computed through the intermediate steps of approximation using the "average" derivatives K(i,j) defined by eq. (3.13), (3.14), (3.15), (3.16) and (3.17).

These equations may be summarized as follows:

$$\{K_i\} = \{K(i,j)\} = \{f_j(TT, \; ZZ)\}$$

where the dummy arguments TT and ZZ may be written as follows:

If $\{K_i\} = \{K_1\}$ then: $TT = t$, $\{ZZ\} = \{Z(t)\} = (1)^*$

If $\{K_i\} = \{K_2\}$ then: $TT = t + h/3$, $\{ZZ\} = \{Z(t)\} + \frac{1}{3} h \{K_1\} = (2)^*$

If $\{K_i\} = \{K_3\}$ then: $TT = t + h/3$, $\{ZZ\} = \{Z(t)\} + h[\frac{1}{6}\{K_1\} + \frac{1}{6}\{K_2\}] = (3)^*$

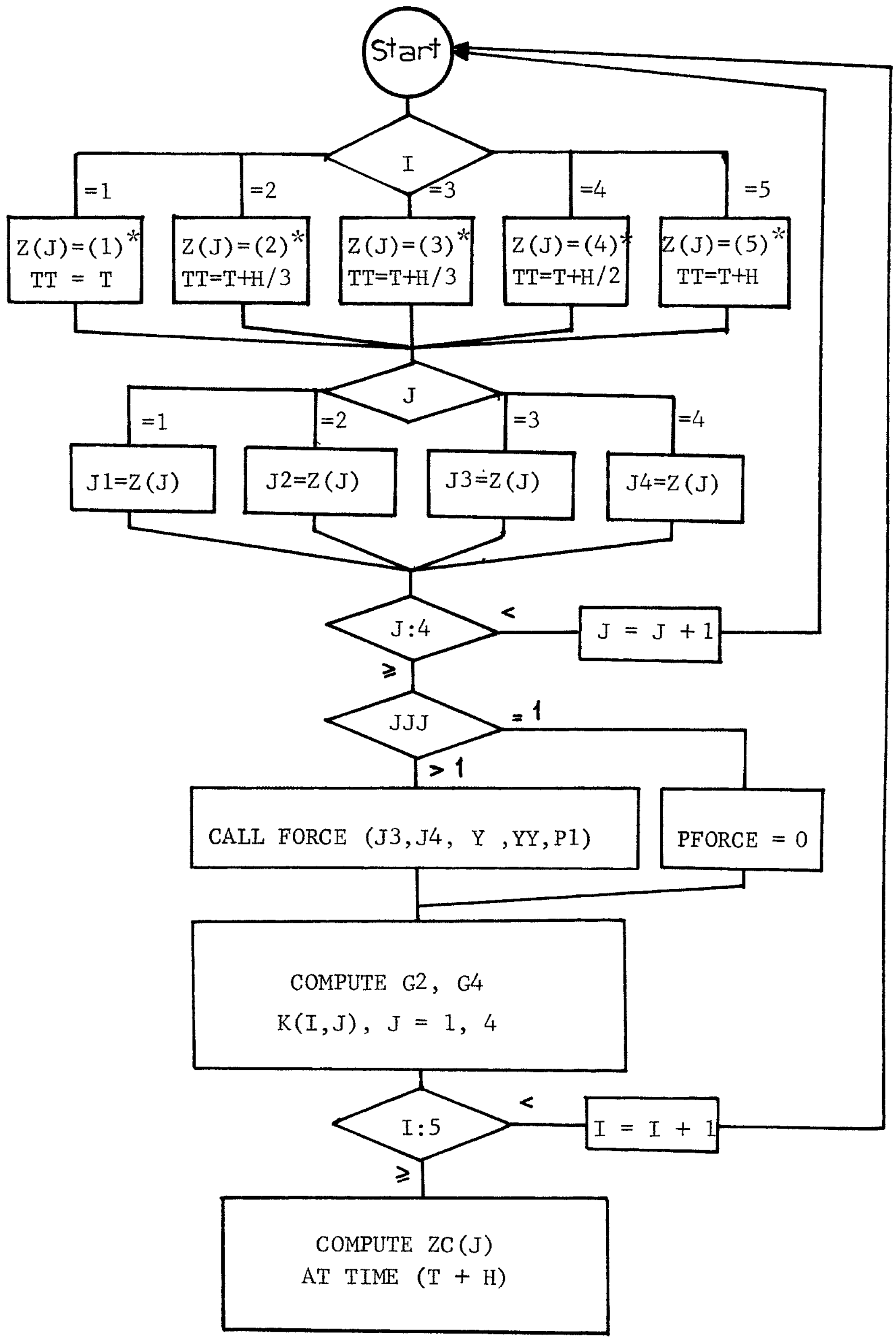


Figure 8. Integration Routine

If $\{K_1\} = \{K_4\}$ then: $TT = t + h/2$, $\{ZZ\} = \{Z(t)\} + h[\frac{1}{8}\{K_1\} + \frac{3}{8}\{K_3\}] = (4)^*$

If $\{K_1\} = \{K_5\}$ then: $TT = t + h$, $\{ZZ\} = \{Z(t)\} + h[\frac{1}{2}\{K_1\} - \frac{3}{2}\{K_2\} + 2\{K_4\}] = (5)^*$

The motion of the system with the damper attached is computed by calling subroutine FORCE (setting JJJ > 1). The motion of the system without the damper is computed by setting PFORCE = $(P_2 - P_1)A = 0$ (JJJ = 1).

3. Subroutine FORCE

The subroutine FORCE is shown in Fig. 9.

A control flag named VALVE, used to indicate the state of the air in the cylinder, has been described previously in section II-D.

In general the pressure P_i of the air in the cylinder under compression, expansion, or discharge is given by:

$$P_i = P_a \left[\frac{D/2}{D/2 - |y|}\right]^n = P^* \qquad \text{if VALVE} = -1$$

$$P_i = P_c \qquad \text{if VALVE} = 0$$

$$P_i = P_c \left[\frac{D/2 - Y_n}{D/2 - |y|}\right]^n = P^{**} \qquad \text{if VALVE} = 1$$

where:

$$Y_n = Y, Y_1, Y_2, \ldots\ldots$$

= displacement of the piston at the moment

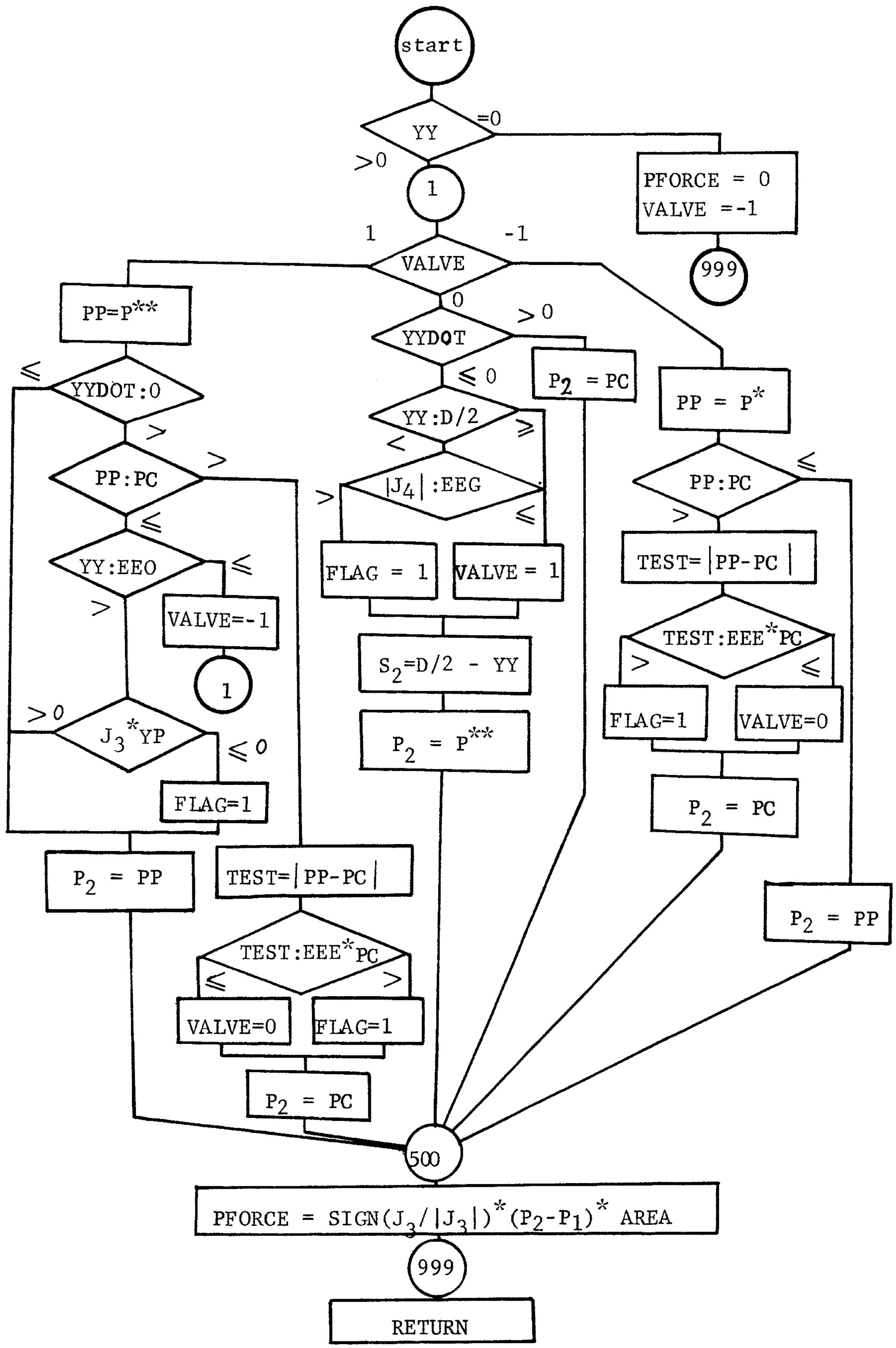


Figure 9. Subroutine FORCE

when the ball valve which has been open during the same half cycle closes, as shown in Fig. 5.

The flag VALVE, as shown in Fig. 9, switches from -1 to 0, when the pressure P_i becomes greater than P_c. It switches from 0 to 1 when the piston, which has been approaching the ball valve during the same half cycle, changes the direction and moves away from it. VALVE switches from 1 to 0 again if the pressure P_i regains a value greater than P_c. VALVE switches from 1 to -1 when the piston passes the mid-stroke position (y = 0).

Another flag, named FLAG, is used as a signal to the main program to halve step whenever a ball valve begins to open or to close, or when the piston approaches the mid-stroke position (y = 0).

Since the damper is symmetrical through its center, the damping force is computed from equation (2.12) for the right-hand half of the damper, and then generalized to the other half, using the equation:

$$F(y,\dot{y}) = \text{sign}\,(y) \quad (P_2 - P_1)A$$

The absolute value $YY = \left| y(t) \right|$ is used in computation instead of y(t).

If VALVE = 0, the closing of the ball valve is determined by

$$YYDOT = y\dot{y} \leqslant 0$$

If VALVE = 1, the reopening of the ball valve is determined by

$$YYDOT = y\dot{y} > 0$$

and $$P_i > P_c$$

All the other variables which appear in the flow charts are listed in Appendix A.

IV. APPROXIMATE ANALYTICAL APPROACH FOR THE DETERMINATION OF NOISELESS PERIODIC OPERATION

This chapter outlines an approximate analytical approach for the determination of noiseless periodic operation of the damper. This analysis leads to an equation relating the maximum amplitude Y of the function y(t) to the amplitude of the excitation, F_o, which results in periodic operation for a given damper. Periodic operation of the damper is defined by the existence of a periodic solution y(t). The operation is relatively noiseless if there is no impact. In order to force the ball valves to open at a preset pressure giving rise to dissipation and to prevent impacting, Y is constrained to lie within the range (Y_B, D/2). Hence, for a given damper the excitation force F_o corresponding to a certain maximum amplitude Y, which should produce periodic operation of the damper, can be predicted. The value of Y is selected such that the operation is noiseless.

An approach to the determination of the noiseless periodic operation of the damper may be summarized as follows:

First a governing nonlinear differential equation relating the damping force $F(y,\dot{y})$ and its imput variable y(t) is derived from the equations of motion (2.3) and (2.4). The nonlinear damping force $F(y,\dot{y})$ is approximated using the harmonic linearization method, or describing function method [10, 13]. The linearized damping force $F(y,\dot{y})$ is then substituted into the governing nonlinear differential equation. After this substitution, the conditions for the existence of a periodic quasi-sinusoidal solution, y(t), are established. Once y(t) has been so constrained, the damping force $F(y,\dot{y})$ is periodic. The steady-state response X of the system is derived from the equations of motion approximately, using the basic assumption that y(t) is quasi-sinusoidal. The stability of the system is discussed using the Hurwitz criteria.

A. Governing Nonlinear Differential Equation

In the system of equations:

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = \frac{F(y,\dot{y})}{M} + f(t) \tag{2.3}$$

$$\ddot{x} + \ddot{y} = -\frac{F(y,\dot{y})}{m} \tag{2.4}$$

where:

$$F(y,\dot{y}) = (P_2 - P_1)A$$

$$f(t) = \frac{F_o}{M}\sin(\Omega t - \phi) = A_f \sin(\Omega t - \phi)$$

the damping force $F(y,\dot{y})$ is a nonlinear function of y and $\dot{y}$.

By uncoupling the equations (2.3) and (2.4), a single equation relating $F(y,\dot{y})$ and y may be obtained.

If the following change of variables is defined:

$$Z_1 = x$$

$$Z_2 = \dot{x}$$

$$Z_3 = y$$

$$Z_4 = \dot{y} \tag{4.1}$$

and:

$$s = \frac{d}{dt} = \text{operator} \tag{4.2}$$

then the following equations result:

$$\omega^2 Z_1 + 2\zeta\omega Z_2 + \dot{Z}_2 = f(t) + \frac{F(y,\dot{y})}{M}$$

$$\dot{Z}_1 - Z_2 = 0$$

$$\dot{Z}_2 + \dot{Z}_4 = -\frac{F(y,\dot{y})}{m}$$

$$\dot{Z}_3 - Z_4 = 0$$

or in matrix form:

$$\begin{bmatrix} \omega^2 & 2\zeta\omega+s & 0 & 0 \\ s & -1 & 0 & 0 \\ 0 & s & 0 & s \\ 0 & 0 & s & -1 \end{bmatrix} \begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{Bmatrix} = \begin{Bmatrix} f(t) + \frac{F(y,\dot{y})}{M} \\ 0 \\ -\frac{F(y,\dot{y})}{m} \\ 0 \end{Bmatrix} \tag{4.3}$$

or:

$$[D] \quad \{Z\} = \{f\} \tag{4.4}$$

If Cramer's rule is applied to equation (4.4), then:

$$Z_3 = \frac{\Delta 3}{\Delta} \tag{4.5}$$

where

$\triangle$ = Determinant $[D]$

$\triangle_3$ = Determinant of the matrix obtained from $[D]$ by replacing its third column with the column vector $\{f\}$.

$$\triangle = \begin{vmatrix} \omega^2 & 2\zeta\omega+s & 0 & 0 \\ s & -1 & 0 & 0 \\ 0 & s & 0 & s \\ 0 & 0 & s & -1 \end{vmatrix} \tag{4.6}$$

$$\triangle_3 = \begin{vmatrix} \omega^2 & 2\zeta\omega+s & f(t) + \frac{F(y,\dot{y})}{M} & 0 \\ s & -1 & 0 & 0 \\ 0 & s & \frac{-F(y,\dot{y})}{m} & s \\ 0 & 0 & 0 & -1 \end{vmatrix} \tag{4.7}$$

Equation (4.5) may be rewritten as:

$$D(s)y = \triangle_3 = \triangle_{13} f(t) + \triangle_{13} \frac{F(y,\dot{y})}{M} - \triangle_{33} \frac{F(y,\dot{y})}{m}$$

or

$$D(s)y + \left[\frac{B(s)}{m} - \frac{C(s)}{M}\right] F(y,\dot{y}) = C(s)f(t) \tag{4.8}$$

where:

$D(s) = \triangle$ = determinant of $[D]$

$B(s) = \triangle_{33}$ = cofactor of the third row third column element of $\triangle$

$C(s) = \Delta_{13}$ = cofactor of the first row third column element of Δ

$$B(s) = \begin{vmatrix} \omega^2 & 2\zeta\omega+s & 0 \\ s & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} \qquad (4.9)$$

$$C(s) = \begin{vmatrix} s & -1 & 0 \\ 0 & s & s \\ 0 & 0 & -1 \end{vmatrix} \qquad (4.10)$$

$$D(s) = s^3 \ (2\zeta\omega+s) + \omega^2 s^2$$

$$B(s) = \omega^2 + s \ (2\zeta\omega+s)$$

$$C(s) = -s^2$$

Equation (4.8) may be rewritten as:

$$[s^4 + 2\zeta\omega s^3 + \omega^2 s^2] \ y = - [(\frac{1}{m} + \frac{1}{M})s^2 + \frac{2\zeta\omega s}{m} + \frac{\omega^2}{m}]F(y,\dot{y}) - s^2 f(t) \qquad (4.11)$$

An alternative method to obtain equation (4.11) is to write:

$$\ddot{x} = - \frac{F(y,\dot{y})}{m} - \ddot{y} \qquad (4.12)$$

By substituting (4.12) into equation (2.3), taking derivatives of both sides, and repeating the procedure until x and its derivatives disappear from the equation the same result may be obtained.

B. Basic Assumption

The goal of this analysis is to obtain the conditions for periodic operation of the damper. Such operation can be obtained only if the damping force $F(y,\dot{y})$ and its input variable y are periodic functions of time. The basic assumption is that the periodic solution y(t) may be modeled by a simple sinusoid:

$$y \doteq Y \sin \Omega t \tag{4.13}$$

where

Y = maximum amplitude of y(t)

Ω = frequency of the excitation force f(t) = $A_f \sin (\Omega t - \phi)$

The existence of such a sinusoidal function y(t) will satisfy the fundamental requirement for application of the describing function method [10]. Using this method the damping force $F(y,\dot{y})$ can be linearized by expanding it in a Fourier series and taking account of only its fundamental harmonic.

C. Describing Function or Harmonic Linearization Method

The nonlinear damping force is defined as:

$$g = F(y,\dot{y}) = (P_2 - P_1)A \tag{4.14}$$

Since y varies harmonically by the basic assumption, the nonlinear function $g = F(y,\dot{y})$ may be expanded in a Fourier series:

$$g(t) = A_o + A_1 \cos \Omega t + B_1 \sin \Omega t + A_2 \cos 2 \Omega t + B_2 \sin 2 \Omega t$$

$$+ \; . \; . \; . \; . + A_n \cos n \Omega t + B_n \sin n \Omega t \tag{4.15}$$

where the coefficients of this Fourier series are given by

$$A_o = \frac{1}{2\pi}\int_0^{2\pi} F(y,\dot{y})\, d\alpha$$

$$A_n = \frac{1}{\pi}\int_0^{2\pi} F(y,\dot{y}) \cos n\alpha d\alpha \tag{4.16}$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} F(y,\dot{y}) \sin n\alpha d\alpha$$

with $\alpha = \Omega t$

The first harmonic of $F(y,\dot{y})$ is defined by:

$$g_1(t) = A_1 \cos \Omega t + B_1 \sin \Omega t \tag{4.17}$$

Harmonic linearization in this method is defined by the approximation of the nonlinear function $F(y,\dot{y})$ by the first harmonic of its Fourier series:

$$g = F(y,\dot{y}) \doteq g_1 = A_1 \cos \Omega t + B_1 \sin \Omega t \tag{4.18}$$

The first constant A_o is equal to zero (i.e., the damping force is symmetric), and the higher harmonics ($A_n \cos n\,\Omega t$, $B_n \sin n\,\Omega t$ for $n \geqslant 2$) are ignored.

Equation (4.18) may be rewritten as:

$$g = F(y,\dot{y}) = \frac{A_1 Y\Omega}{Y\Omega} \cos \Omega t + \frac{B_1 Y}{Y} \sin \Omega t \tag{4.19}$$

or $$g = N_1 y + \frac{N_2}{\Omega}\dot{y} \tag{4.20}$$

where:

$$N_1 = \frac{B_1}{Y} = \frac{1}{\pi Y}\int_0^{2\pi} F(y,\dot{y}) \sin \alpha \, d\alpha$$

$$N_2 = \frac{A_1}{Y} = \frac{1}{\pi Y}\int_0^{2\pi} F(y,\dot{y}) \cos \alpha \, d\alpha \tag{4.21}$$

with:

$$y = Y \sin \alpha$$

$$\alpha = \Omega t$$

Equation (4.20) may also be rewritten as:

$$g = F(y,\dot{y}) = A_1 \cos \Omega t + B_1 \sin \Omega t$$

$$= \sqrt{A_1^2 + B_1^2} \;\; \sin (\Omega t + \tan^{-1} \frac{A_1}{B_1})$$

$$= Y\sqrt{N_1^2 + N_2^2} \quad \sin\left(\Omega t + \tan^{-1}\frac{N_2}{N_1}\right) \tag{4.22}$$

The describing function for the damping force $F(y,\dot{y})$ is defined to be:

$$\begin{aligned} \text{Describing function} = N &= \left|\frac{F(y,\dot{y})}{Y}\right| \angle \tan^{-1}\frac{N_2}{N_1} \\ &= \sqrt{N_1^2 + N_2^2} \;\angle \tan^{-1}\frac{N_2}{N_1} \end{aligned} \tag{4.23}$$

or in complex form:

$$\text{Describing function} = N = N_1 + jN_2 \tag{4.24}$$

The magnitude of N is the ratio of damping force $F(y,\dot{y})$ amplitude to imput variable y amplitude.

D. Harmonic Linearization of the Damping Force

From section IV-C it is shown that if a quasi-sinusoidal solution $y \doteq Y \sin \Omega t$ exists, the damping force $F(y,\dot{y})$ may be approximated by:

$$F(y,\dot{y}) \doteq N_1 y + \frac{N_2}{\Omega}\dot{y}$$

where

$$\begin{aligned} N_1 &= \frac{1}{\pi Y}\int_0^{2\pi} F(Y\sin\alpha,\ Y\Omega\cos\alpha)\ \sin\alpha\ d\alpha \\ N_2 &= \frac{1}{\pi Y}\int_0^{2\pi} F(Y\sin\alpha,\ Y\Omega\cos\alpha)\ \cos\alpha\ d\alpha \end{aligned} \tag{4.21}$$

$$\alpha = \Omega t$$

If Y_B is defined as:

Y_B = relative coordinate of the piston, at which the air-discharge begins,

then:

$$P_c \left(A\ [D/2 - Y_B] \right)^n = P_a \left(A \frac{D}{2} \right)^n$$

It follows that:

$$Y_B = Y \sin \alpha_B = \frac{D}{2} - \left(\frac{P_a}{P_c}\right)^{\frac{1}{n}} \left[\frac{D}{2}\right]$$

and:

$$\alpha_B = \sin^{-1} \frac{Y_B}{Y} = \sin^{-1} \left\{ [D/2 - \left(\frac{P_a}{P_c}\right)^{\frac{1}{n}} (D/2)] \,/\, Y \right\} \qquad (4.25)$$

The value of α when the piston reaches the maximum amplitude Y is $\frac{\pi}{2} + 2n\pi$, $n = 0, 1, 2 \ldots$,

$$y(t) = Y \sin \alpha = Y.$$

The expressions N_1 and N_2 will be determined by summing the integrals corresponding to the potions AB, BC, CD, AB', B'C' and C'D' of the curve $F(y,\dot{y})$ shown in Fig. 3.

Since y_i, y_f, α_i, α_f are defined as:

y_i = lower limit of the integration interval

y_f = upper limit of the integration interval

α_i = value of α corresponding to y_i

α_f = value of α corresponding to y_f,

the limits of the integration and appropriate integrands may be summarized in Table I.

$$N_1 = \sum_{j=1}^{6} N_{1j}$$

$$N_2 = \sum_{j=1}^{6} N_{2j}$$

where:

$$N_{1j} = \frac{1}{\pi Y} \int_{\alpha_i}^{\alpha_f} F(Y \sin \alpha,\ Y\Omega \cos \alpha) \sin \alpha \, d\alpha$$

$$N_{2j} = \frac{1}{\pi Y} \int_{\alpha_i}^{\alpha_f} F(Y \sin \alpha,\ Y\Omega \cos \alpha) \cos \alpha \, d\alpha$$

The limits α_i, α_f for each integration interval have been given previously.

Since:

$$\left[\frac{D/2}{D/2 - y}\right]^n = \left[\frac{1}{1 - \frac{Y}{D/2} \sin \alpha}\right]^n = (1 - a \sin \alpha)^{-n} \qquad (4.26)$$

where:

$$a = \frac{Y}{D/2}$$

Portion #	Portion	y_i	y_f	α_i	α_f	$F(y,\dot{y})$	Corresponding Integrals
1	AB	0	Y_B	0	α_B	$\left(P_a[\frac{D/2}{D/2 - y}]^n - P_a\right) A$	N_{11}, N_{21}
2	BC	Y_B	Y	α_B	$\frac{\pi}{2}$	$\left(P_c - P_a\right) A$	N_{12}, N_{22}
3	CD	Y	0	$\frac{\pi}{2}$	π	$\left(P_c[\frac{D/2 - Y}{D/2 - y}]^n - P_a\right) A$	N_{13}, N_{23}
4	AB'	0	$-Y_B$	π	$\pi + \alpha_B$	$\left(P_a - [\frac{D/2}{D/2 + y}]^n P_a\right) A$	N_{14}, N_{24}
5	B'C'	$-Y_B$	-Y	$\pi + \alpha_B$	$\frac{3\pi}{2}$	$\left(P_a - P_c\right) A$	N_{15}, N_{25}
6	C'D'	-Y	0	$\frac{3\pi}{2}$	2π	$\left(P_a - P_c[\frac{D/2 - Y}{D/2 + y}]^n\right) A$	N_{16}, N_{26}

Table I. Limits of Integration and Appropriate Integrands in the Describing Function N.

$[\frac{D/2}{D/2 - y}]^n$ may be expanded in a binomial series:

$$[\frac{D/2}{D/2-y}]^n = (1 - a \sin \alpha)^{-n} = 1 + na \sin \alpha + \frac{n(n+1)a^2 \sin^2 \alpha}{2!}$$

$$+ \frac{n(n+1)(n+2)a^3 \sin^3 \alpha}{3!} + \ldots .$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(n+r)a^r \sin^r \alpha}{r!\ \Gamma(n)}$$

If $[\frac{D/2}{D/2 - y}]^n$ is approximated by the first four terms of its binomial series, then the approximation of $G(\alpha, \alpha_i, \alpha_f)$ will follow:

$$G(\alpha, \alpha_i, \alpha_f) = \int_{\alpha_i}^{\alpha_f} [\frac{D/2}{D/2 - y}]^n \sin \alpha \, d\alpha = \int_{\alpha_i}^{\alpha_f} \sum_{r=0}^{\infty} \frac{\Gamma(n+r)a^r \sin^{r+1} \alpha \, d\alpha}{r!\ \Gamma(n)}$$

$$\doteq \int_{\alpha_i}^{\alpha_f} \sum_{r=0}^{3} \frac{\Gamma(n+r)a^r \sin^{r+1} \alpha d\alpha}{r!\,\Gamma(n)}$$

Hence

$$G(\alpha, 0, \alpha_B) \doteq 1 - \cos \alpha_B + \frac{na}{2}(\alpha_B - \frac{\sin 2\alpha_B}{2})$$

$$- \frac{n(n+1)a^2}{6} [\cos \alpha_B (\sin^2 \alpha_B + 2) - 2]$$

$$+ \frac{n(n+1)(n+2)a^3}{48} (3\alpha_B - 2 \sin 2\alpha_B + \frac{\sin 4\alpha_B}{4}) \qquad (4.27)$$

$$G(\alpha, \frac{\pi}{2}, \pi) \doteq 1 + \frac{na\pi}{4} + \frac{n(n+1)a^2}{3} + \frac{n(n+1)(n+2)a^3\pi}{32} \qquad (4.28)$$

Using equations (4.27) and (4.28) one may obtain the expressions N_{11}, N_{12}, N_{13}:

$$N_{11} = \frac{1}{\pi Y}\int_{0}^{\alpha_B} F(Y \sin \alpha,\ Y\Omega \cos \alpha) \sin \alpha d\alpha = \frac{P_a A}{\pi Y}\left[G(\alpha,\ 0,\ \alpha_B) - 1 + \cos \alpha_B \right]$$

$$N_{12} = \frac{1}{\pi Y}\int_{\alpha_B}^{\pi/2} \left(P_c - P_a \right) A \sin \alpha d\alpha = \frac{(P_c - P_a)A}{\pi Y} \cos \alpha_B \tag{4.29}$$

$$N_{13} = \frac{1}{\pi Y}\int_{\pi/2}^{\pi} \left(P_c [\frac{D/2 - Y}{D/2 - Y \sin \alpha}]^n - P_a \right) A \sin \alpha\, d\, \alpha$$

$$= \frac{P_c A(1 - a)^n}{\pi Y} G(\alpha,\ \pi/2,\ \pi) - \frac{P_a A}{\pi Y} \tag{4.30}$$

By expanding $[\frac{D/2}{D/2 + y}]^n$ in binomial series, and by applying the same procedure, one may obtain the following equalities:

$$N_{14} = N_{11}$$

$$N_{15} = N_{12}$$

$$N_{16} = N_{13}$$

Finally the expression N_1 may be written as:

$$N_1 = \sum_{j=1}^{6} N_{1j} = \frac{2\ P_a A}{\pi Y}\left[G(\alpha,\ 0,\ \alpha_B) - 2 \right] + \frac{2\ P_c A}{\pi Y}\left[\cos \alpha_B + (1 - a)^n G(\alpha,\ \frac{\pi}{2},\ \pi) \right] \tag{4.31}$$

Since the denominator of the expression $[\frac{D/2 - Y}{D/2 - Y \sin \alpha}]^n$ tends to zero when α approaches $\frac{\pi}{2}$ and Y becomes close to D/2, one may expect a degeneration of approximation to N_1.

Unlike N_1, the expression N_2 can be obtained very simply.

If a new variable u is defined:

$$u = 1 - a \sin \alpha$$

$$du = -a \cos \alpha \, d\alpha$$

then the following integrations result:

$$I = \int (\frac{D/2}{D/2 - Y \sin \alpha})^n \cos \alpha \, d\alpha = \int (1 - a \sin \alpha)^{-n} \cos \alpha \, d\alpha$$

$$- \frac{1}{a} \int u^{-n} \, du = -\frac{u^{-n+1}}{a(1 - n)} = \frac{(1 - a \sin \alpha)^{-n+1}}{a(n - 1)} \tag{4.32}$$

Substituting equation (4.32) into the expressions N_{21}, N_{23}, these expressions may be rewritten as follows:

$$N_{21} = \frac{1}{\pi Y} \int_0^{\alpha_B} \left(P_a \, [\frac{D/2}{D/2 - y}]^n - P_a \right) A \cos \alpha \, d\alpha$$

$$= \frac{P_a A}{\pi Y} \left(\frac{(1- a \sin \alpha_B)^{-n+1} - 1}{a(n - 1)} - \sin \alpha_B \right) \tag{4.33}$$

$$N_{23} = \frac{1}{\pi Y} \int_{\pi/2}^{\pi} \left(P_c \, [\frac{D/2 - Y}{D/2 - y}]^n - P_a \right) A \cos d\alpha$$

$$= \frac{P_c A \ (1 - a)^n}{\pi Y} \int_{\pi/2}^{\pi} (1 - a \sin \alpha)^{-n} \cos \alpha \, d\alpha + \frac{P_a A}{\pi Y}$$

$$N_{23} = \frac{P_c A(1 - a)^n}{\pi Y} \left[\frac{(1 - a \sin \alpha)^{-n+1}}{a(n - 1)}\right]_{\pi/2}^{\pi} + \frac{P_a A}{\pi Y}$$

$$= \frac{P_c A(1 - a)^n}{\pi Y} \left(\frac{1 - (1 - a)^{-n+1}}{a(n - 1)}\right) + \frac{P_a A}{\pi Y}$$

$$= \frac{P_c A}{\pi Y} \left(\frac{(1 - a)^n - 1 + a}{a(n - 1)}\right) + \frac{P_a A}{\pi Y} \qquad (4.34)$$

$$N_{22} = \frac{1}{\pi Y}\int_{\alpha_B}^{\pi/2} \left(P_c - P_a\right) A \cos d\alpha = \frac{(P_c - P_a)A}{\pi Y} (1 - \sin \alpha_B) \qquad (4.35)$$

By applying the same procedure to the expressions N_{24}, N_{25}, N_{26} the following equalities can be obtained:

$$N_{21} = N_{24}$$

$$N_{22} = N_{25}$$

$$N_{23} = N_{26}$$

The expression N_2 can be obtained by summing all the expressions N_{2j}, $j = 1,6$.

$$N_2 = \sum_{j=1}^{6} N_{2j}$$

$$N_2 = \frac{2 P_a A}{\pi Y} \left(\frac{(1 - a \sin \alpha_B)^{-n+1} - 1}{a(n - 1)}\right)$$

$$+ \frac{2P_c A}{\pi Y} \left(1 - \sin \alpha_B + \left[\frac{(1 - a)^n - 1 + a}{a(n - 1)}\right] \right) \qquad (4.36)$$

Finally the harmonically linearized function of the damping force $F(y,\dot{y})$ and the describing function N for $F(y,\dot{y})$ can be obtained:

$$F(y,\dot{y}) = N_1 y + \frac{N_2}{\Omega}\dot{y} \qquad (4.20)$$

$$N = N_1 + j\,N_2 \qquad (4.24)$$

where

N_1 and N_2 are given by the equations (4.31) and (4.36).

E. Conditions for the Existence of the Periodic Solution $y = Y \sin \Omega t$

The governing nonlinear differential equation representing the system is:

$$[s^4 + 2\zeta\omega s^3 + \omega^2 s^2]y =$$

$$- [(\frac{1}{m} + \frac{1}{M})s^2 + \frac{2\zeta\omega s}{m} + \frac{\omega^2}{m}]\, F(y,\dot{y}) - s^2 f(t) \qquad (4.11)$$

The excitation force f(t) may be rewritten as:

$$f(t) = \frac{F_o}{M} \sin(\Omega t - \phi) = A_f \sin(\Omega t - \phi)$$

$$= A_f \cos\phi \sin\Omega t - A_f \sin\phi \cos\Omega t$$

$$= \frac{A_f}{Y}(\cos\phi\, y - \frac{\sin\phi}{\Omega}\dot{y})$$

$$= \frac{A_f}{Y}(\cos\phi - \frac{\sin\phi}{\Omega}s)y \qquad (4.37)$$

By substituting equations (4.20) and (4.37) into equation (4.11), this equation may be rewritten as follows:

$$\left[s^4 + \left(2\zeta\omega - \frac{A_f \sin\phi}{Y\Omega}\right)s^3 + \left(\omega^2 + \frac{A_f \cos\phi}{Y}\right)s^2\right]y =$$

$$- \left\{ \left(\frac{1}{m} + \frac{1}{M}\right)\frac{N_2}{\Omega} s^3 + \left[\left(\frac{1}{m} + \frac{1}{M}\right) N_1 + \frac{2\zeta\omega}{m\Omega} N_2\right]s^2 + \left(\frac{2\zeta\omega N_1}{m} + \frac{\omega^2 N_2}{m\Omega}\right)s + \frac{\omega^2 N_1}{m} \right\} y$$

or

$$\left\{ s^4 + \left[2\zeta\omega - \frac{A_f \sin\phi}{Y\Omega} + \left(\frac{1}{m} + \frac{1}{M}\right)\frac{N_2}{\Omega}\right] s^3 + \left[\omega^2 + \frac{A_f \cos\phi}{Y} + \left(\frac{1}{m} + \frac{1}{M}\right)N_1 + \frac{2\zeta\omega N_2}{m\Omega}\right]s^2 + \left(\frac{2\zeta\omega N_1}{m} + \frac{\omega^2 N_2}{m\Omega}\right)s + \frac{\omega^2 N_1}{m} \right\} y = 0 \tag{4.38}$$

This equation may be written in the form:

$$g(s)y = 0 \tag{4.39}$$

where g(s) is a linearized differential operator.

The solutions of equation (4.38) can be determined from the four roots, λ_i, i = 1, 2, 3, 4, of the auxiliary equation

$$g(\lambda) = 0 \tag{4.40}$$

For each real root λ_i there corresponds a solution $y = C_i e^{\lambda_i t}$

For each pair of complex roots:

$$\lambda_K = u + jv$$

$$\lambda_{K'} = u - jv$$

there corresponds a solution $y = C_K e^{ut} \cos vt + C_{K'} e^{ut} \sin vt$.

Therefore the condition for equation (4.38) to yield a steady-state solution close to $y = Y \sin \Omega t$ is that the equation

$$g(s) = s^4 + [2\zeta\omega - \frac{A_f \sin \Phi}{Y\Omega} + (\frac{1}{m} + \frac{1}{M}) \frac{N2}{\Omega}] s^3$$

$$+ [\omega^2 + \frac{A_f \cos \Phi}{Y} + (\frac{1}{m} + \frac{1}{M}) N_1 + \frac{2\zeta\omega N_2}{m\Omega}] s^2$$

$$+ (\frac{2\zeta\omega N_1}{m} + \frac{\omega^2 N_2}{m\Omega}) s + \frac{\omega^2 N_1}{m} = 0 \qquad (4.41)$$

yields a pair of pure imaginary roots $\pm jv = \pm j\Omega$ and the real part of the other roots must be negative.

There should be no repeated roots, since in this case the equation (4.38) will yield solutions of the forms: either Cte^{ut} or $Ct(e^{ut} \cos vt + e^{ut} \sin vt)$ which grow as t increases.

If equation (4.41) has a pair of pure imaginary roots $\pm j\Omega$, it can be rewritten as:

$$g(s) = (s^2 + \Omega^2)(C_o + C_1 s + C_2 s^2) = 0$$

or

$$C_2 s^4 + C_1 s^3 + (C_o + C_2 \Omega^2) s^2 + \Omega^2 C_1 s + \Omega^2 C_o = 0 \qquad (4.42)$$

Equating the coefficients of the same power of s in the equations

(4.41) and (4.42), we obtain:

$$C_2 = 1$$

$$C_1 = 2\zeta\omega - \frac{A_f \sin\Phi}{Y\Omega} + \left(\frac{1}{m} + \frac{1}{M}\right)\frac{N_2}{\Omega} \tag{4.43}$$

$$= \frac{\omega^2 N_2}{m\Omega^3} + \frac{2\zeta\omega N_1}{m\Omega^2}$$

$$C_o = \frac{\omega^2 N_1}{m\Omega^2} = \omega^2 + \frac{A_f \cos\Phi}{Y} + \left(\frac{1}{m} + \frac{1}{M}\right)N_1 + \frac{2\zeta\omega N_1}{m\Omega} - \Omega^2 \tag{4.44}$$

From equation (4.43) and (4.44) one obtains:

$$2\zeta\omega\Omega + \left(\frac{1}{m} + \frac{1}{M}\right)N_2 - \frac{A_f \sin\Phi}{Y} - \frac{2\zeta\omega N_1}{m\Omega} - \frac{\omega^2 N_2}{m\Omega^2} = 0 \tag{4.45}$$

and

$$\frac{2\zeta\omega N_2}{m\Omega} + \left(\frac{1}{m} + \frac{1}{M}\right)N_1 + \frac{A_f \cos\Phi}{Y} + \omega^2 - \Omega^2 - \frac{\omega^2 N_1}{m\Omega^2} = 0 \tag{4.46}$$

The equations (4.45) and (4.46) represent the necessary condition for the existence of the periodic solution $y = Y \sin \Omega t$.

If the system is at resonance, these equations reduce to

$$2\zeta\Omega^2 + \frac{N_2}{M} - \frac{A_f \sin\Phi}{Y} - \frac{2\zeta N_1}{m} = 0 \tag{4.47}$$

$$\frac{2\zeta N_2}{m} + \frac{N_1}{M} + \frac{A_f \cos\Phi}{Y} = 0 \tag{4.48}$$

or

$$A_f \sin\Phi = \left[2\zeta\Omega^2 + \frac{N_2}{M} - \frac{2\zeta N_1}{m}\right] Y$$

$$A_f \cos \phi = -\left[\frac{2\zeta\Omega N_2}{m} + \frac{N_1}{M}\right]Y$$

$$F_o = A_f M = \left([2\zeta\Omega^2 M + N_2 - \frac{2\zeta\Omega N_1}{\mu}]^2 + [\frac{2\zeta\Omega N_2}{\mu} + N_1]^2 \right)^{1/2} Y$$

or

$$\frac{Y}{F_o} = \left([2\zeta\Omega^2 M + N_2 - \frac{2\zeta\Omega N_1}{\mu}]^2 + [\frac{2\zeta\Omega N_2}{\mu} + N_1]^2 \right)^{-1/2} \quad (4.49)$$

For the system at resonance the equation (4.49) represents the necessary condition for the existence of the periodic solution $y \doteq Y \sin \Omega t$. It is also the necessary condition for the existence of a periodic operation of the damper. Since Y is constrained to be within the range $(Y_B, D/2)$ so that the operation of the damper is noiseless, the necessary condition for the existence of a noiseless periodic operation of the damper can be stated as follows.

For a given system and damper described by ζ, μ, Ω, M, N_1 and N_2, and for each amplitude Y within the range $(Y_B, D/2)$, an excitation force F_o satisfying the equation (4.49) will produce a noiseless periodic operation of the damper, when the system is at resonance.

Equations (4.45) and (4.46) representing the conditions for the existence of the periodic solution $y(t) = Y \sin \Omega t$ can also be obtained by substituting $s = j\Omega$ in equation (4.41) and separating the real and the imaginary parts.

Equation (4.41) may be rewritten as:

$$R(\Omega, Y) + jI(\Omega, Y) = 0$$

By separating the real and the imaginary parts, two equations may be obtained:

$$R(\Omega, Y) = 0$$

$$I(\Omega, Y) = 0$$

which correspond to equations (4.46) and (4.45), respectively.

Equations (4.45) and (4.46) may also be rewritten as:

$$A_f \sin \phi = [2\zeta\omega\Omega + (\frac{1}{m} + \frac{1}{M})N_2 - \frac{2\zeta\omega N_1}{m\Omega} - \frac{\omega^2 N_2}{m\Omega^2}]Y$$

$$A_f \cos \phi = -[\frac{2\zeta\omega N_2}{m\Omega} + (\frac{1}{m} + \frac{1}{M})N_1 + \omega^2 - \Omega^2 - \frac{\omega^2 N_1}{m\Omega^2}]Y$$

One obtains an equation similar to equation (4.49) for the general case:

$$\frac{Y}{F_o} = \left([2\zeta\omega\Omega M + (\frac{1}{\mu} + 1)\, N_2 - \frac{2\zeta\omega N_1}{\mu\Omega} - \frac{\omega^2 N_2}{\mu\Omega^2}]^2 + [\frac{2\zeta\omega N_2}{\mu\Omega} + (\frac{1}{\mu} + 1)N_1 + \omega^2 M - \Omega^2 M - \frac{\omega^2 N_1}{\mu\Omega^2}]^2 \right)^{-1/2} \tag{4.49b}$$

The phase angle Φ is given by:

$$\phi = \sin^{-1} \left\{ [2\zeta\omega\Omega M + (\frac{1}{\mu} + 1)N_2 - \frac{2\zeta\omega N_1}{\mu\Omega} - \frac{\omega^2 N_2}{\mu\Omega^2}] \frac{Y}{F_o} \right\}$$

F. Steady-state Response

Once the periodic operation of the damper has been established, the function $y(t)$ and $F(y,\dot{y})$ are periodic and can be approximated by:

$$y \doteq Y \sin \Omega t$$

$$F(y,\dot{y}) \doteq N_1 \ y + \frac{N_2}{\Omega} \ \dot{y} = N_1 Y \sin \Omega t + N_2 Y \cos \Omega t$$

By substituting these approximations into equation (2.4), one obtains:

$$\ddot{x} = \Omega^2 Y \sin \Omega t - \frac{1}{m} [N_1 Y \sin \Omega t + N_2 Y \cos \Omega t] \qquad (4.50)$$

If the effect of starting conditions is ignored one may integrate equation (4.50) twice to obtain:

$$\dot{x} = - \Omega Y \cos \Omega t + \frac{N_1 Y}{m\Omega} \cos \Omega t - \frac{N_2 Y}{m\Omega^2} \sin \Omega t$$

$$x = - Y \sin \Omega t + \frac{N_1 Y}{m\Omega^2} \sin \Omega t + \frac{N_2 Y}{m\Omega^2} \cos \Omega t \qquad (4.51)$$

$$x = X \sin (\Omega t - \theta)$$

$$= X \cos \theta \sin \Omega t - X \sin \theta \cos \Omega t \qquad (4.52)$$

By equating the corresponding coefficients of sin Ωt and cos Ωt in equations (4.51) and (4.52) one obtains:

$$X \cos \theta = [\frac{N_1}{m\Omega^2} - 1]Y = [\frac{N_1}{m} - \Omega^2] \frac{Y}{\Omega^2}$$

$$X \sin \theta = - \frac{N_2 Y}{m\Omega^2}$$

$$\therefore \qquad X = \left([\frac{N_1}{m} - \Omega^2]^2 + \frac{N_2^{\ 2}}{m^2} \right)^{1/2} \frac{Y}{\Omega^2} \qquad (4.53)$$

By combining equations (4.53) and (4.49b) one obtains:

$$\frac{X}{F_o} = \frac{1}{\Omega^2}\left([\frac{N_1}{m} - \Omega^2]^2 + \frac{N_2^{\,2}}{m^2}\right)^{1/2}\left\{[2\zeta\omega\Omega M + (\frac{1}{\mu} + 1)N_2 - \frac{2\zeta\omega N_1}{\mu\Omega} - \frac{\omega^2 N_2}{\mu\Omega^2}]^2 + [\frac{2\zeta\omega N_2}{\mu\Omega} + (\frac{1}{\mu} + 1)N_1 + \omega^2 M - \Omega^2 M - \frac{\omega^2 N_1}{\mu\Omega^2}]^2\right\}^{-1/2}$$

Equation (4.54) leads to the prediction of the steady-state response X when periodic operation of the damper has been established.

The response amplitude X of the primary system with damper attached will be compared with the response amplitude X_A of the same primary system operating without a damper:

$$X_A = \frac{F_o/K}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

where

$K = \omega^2 M$ = stiffness of the primary system

$r = \frac{\Omega}{\omega}$ = ratio of forcing frequency to natural frequency

At resonance the response amplitude of the system without the damper becomes:

$$X_o = \frac{F_o}{2\zeta K} = \frac{F_o}{2\omega^2 M\zeta}$$

The magnitude of the amplitude ratio $\frac{X}{X_o}$ indicates the degree of reduction in response amplitude of the primary system as a result of the noiseless damper.

Alternative methods may be used to obtain an approximation of the steady-state response amplitude X.

By substituting equation (4.50) into equation (2.3), the trial solution (4.52) into the new equation, and by equating the corresponding coefficients of sin Ωt and cos Ωt on two sides of the equation, one may obtain:

$$X = \sqrt{\frac{S_1^2 + S_2^2}{4\ \zeta^2 \omega \Omega^2 + \omega^4}} \qquad (4.54b)$$

where:

$$S_1 = A_f \cos \phi - \Omega^2 Y + (\frac{1}{m} + \frac{1}{M}) N_1 Y$$

$$S_2 = (\frac{1}{m} + \frac{1}{M}) N_2 Y - A_f \sin \phi$$

G. Stability Analysis

Once the damping force $F(y,\dot{y})$ has been linearized, the stability of the system with damper attached can be treated using the Hurwitz's criteria applicable to linear differential equations.

The stability of the system depends on the roots of the auxiliary equation (4.41) of the governing nonlinear differential equation.

Equation (4.41) has a pair of pure imaginary roots $\pm j\Omega$. The two other roots are the roots of equation

$$C_o + C_1 s + C_2 s^2 = 0 \qquad (4.55)$$

where:

$$C_2 = 1$$

C_1 and C_o are given by equations (4.43) and (4.44).

The system will be stable if all the roots of equation (4.55) have negative real parts.

If the Hurwitz's theorem is applied to eq. (4.55), then this equation has all roots with negative real part if, and only if, the determinants

$$\triangle_1 = C_1 \qquad \triangle_2 = \begin{vmatrix} C_1 & C_3 \\ C_o & C_2 \end{vmatrix}$$

are positive, provided $C_o > 0$.

By applying the Hurwitz criteria to the present analysis, the stability condition of the system with damper attached may be summarized as follows:

a) $N_1 > 0$

b) $N_2 + 2\,\zeta r N_1 > 0$

c) $\dfrac{(1 + \mu)N_1 r}{2\zeta} + N_2 > \dfrac{mr\Omega^2}{2\zeta}\left[1 - \dfrac{1}{r^2} - \dfrac{A_f \cos\Phi}{\Omega^2 Y}\right]$

d) $N_2 > m\,\dfrac{[A_f \sin\Phi - 2\zeta\omega\Omega Y]}{Y(1 + \mu)}$ (4.56)

V. COMPARISON OF SIMULATION AND THE APPROXIMATE ANALYTICAL APPROACH

The objectives of the study are:

1. To investigate the feasibility of a possible substitute for the impact damper, i.e., the feasibility of a damper which has the effectiveness in reducing mechanical vibrations comparable to that obtained by the impact damper, and can work noiselessly.

2. To determine the damper parameters necessary for reducing the vibration amplitude near resonance for the given system described in Chapter I to value of less than 1/3 of its value when the system vibrates without the damper.

3. To show that the general approximate method described in Chapter IV may be usefully applied to the analysis of noiseless periodic operation of the damper attached to a general system for near-resonant operation, and how in particular it applied to the system described in Chapter I.

These studies are presented with comparisons of the results obtained by digital computer simulation to those obtained by application of the approximate analytical approach described in Chapter IV.

Some details on noiseless periodic operation are also discussed at the end of this chapter.

A. Determination of Damper Parameters

In general problems, the damper parameters are to be selected to obtain a response amplitude ratio $\frac{X}{X_o}$ below a specified limit, when a given primary system is excited by a specified excitation force F_o.

Since the damper should operate periodically and noiselessly, the excitation force to produce such an operation is a function of the maximum amplitude Y, and may be predicted for a given primary system and damper, using eq. (4.49b), as described in Chapter IV. It is easier to predict the force F_o to produce periodic operation of a given damper attached to a given primary system (direct problem) than to select appropriate design parameters for a damper, which, attached to a given primary system, and excited by a specified excitation force F_o, should operate noiselessly and periodically (indirect problem). The indirect problems, which are discussed in this part of the study, may therefore be solved more conveniently by first considering the direct problems concerning various trial dampers.

A description of the given primary system mentioned in Chapter I is shown in Table II. The ratio μ of damper mass to primary system mass, and the coefficient of restitution e were selected to correspond to values used in a study of Masri [5] for comparative purposes. The remaining design parameters P_c, D and A are to be selected such that the vibration amplitude of the given system is reduced to less than 1/3 of its initial value without a damper, when the excitation force is specified to lie within the limits (20 to 50 lbs.).

W_1 = weight of primary system = 10. lbs.

ζ = damping ratio of primary system = 0.02

ω = undamped natural frequency of primary system

= 188.5 rad./sec. (or 30. c.p.s.)

$\mu = \frac{m}{M}$ = ratio of damper mass to primary system mass = 0.1

e = coefficient of restitution = 0.8

Table II. Description of the Primary System Under Discussion

This indirect problem was solved by considering the direct problem concerning various trial dampers. Trial sets of damper parameters with values of preset pressure P_c ranging from 30 to 200 p.s.i.a., those of total piston stroke D from 1 to 10 inches and those of piston diameter from 1 to 5 inches were taken in the present study. The force F_o to produce periodic operation of these dampers and the ratio of response amplitude $\frac{X}{X_o}$ were predicted, using eq. (4.49b) and (4.54b). Among these trial sets of damper parameters those satisfying the goals of the indirect problem were selected for a damper designed to work in the prescribed conditions.

<u>1. Prediction of noiseless periodic operation for trial dampers.</u>

The periodic operation was first determined for systems at resonance. The following successive steps of prediction were taken:

a) Predictions of the excitation force F_o which would produce noiseless periodic operation of the damper for the following trial sets of parameters:

P_c = 30., 50., 100., 200. p.s.i.a

D = 1., 1.5, 5., 10. inches

Diam. = 1., 1.5, 5. inches

where: Diam. = diameter of the damper piston.

This excitation force F_o was predicted by using eq. (4.49b). By using eq. (4.54b), one may predict the amplitude ratio $\frac{X}{X_o}$ which indicates the degree of reduction in response amplitude of the primary system.

The predictions were summarized and illustrated in Figures 10 through 18.

It can be shown from these predictions that for each set of parameters P_c, D, Diam, noiseless periodic operation was obtained for a certain operation range of excitation force F_o. For example, the operation range of F_o which would result in noiseless periodic operation for the set of parameters:

$$P_c = 30.\ \text{p.s.i.a.}$$

$$D = 1.\ \text{inch}$$

$$\text{Diam.} = 1.\ \text{inch}$$

is:

$$11.6\ \text{lbs.} \leqslant F_o \leqslant 31.8\ \text{lbs.}$$

as shown in Figure 10.

This operation range of F_o is limited on one end by the value of F_o corresponding to $Y = Y_B$, where Y_B is the absolute value of y(t) where air-discharge begins, and on the other by that corresponding to $Y = D/2$.

b) By analyzing the predictions illustrated in Figures 10 through 18, one may eliminate the sets of parameters P_c, D and Diam. which do not meet the required condition:

$$\frac{X}{X_o} \leqslant 1/3$$

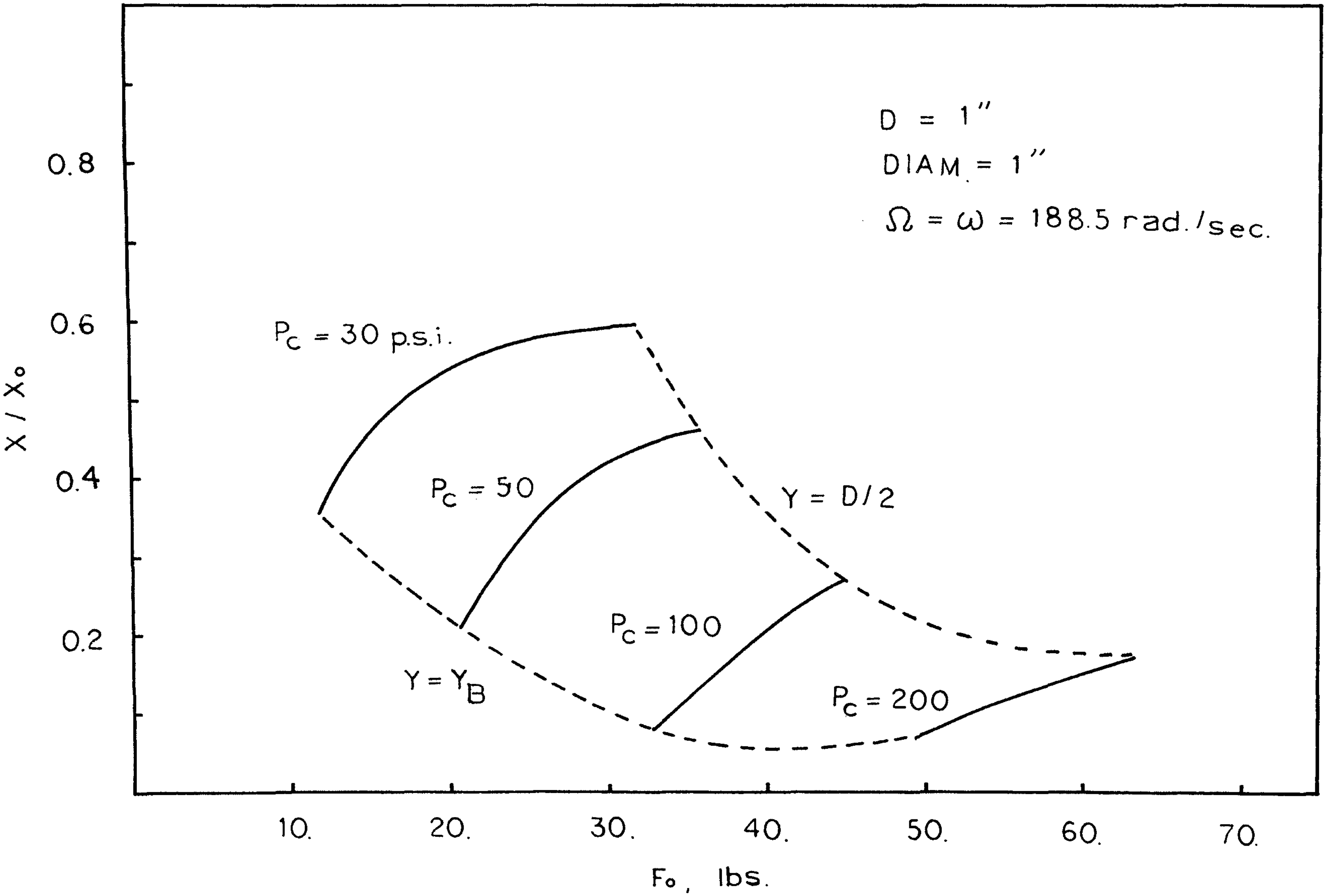


Figure 10. Solution Curve of Predicted Noiseless Periodic Operation.

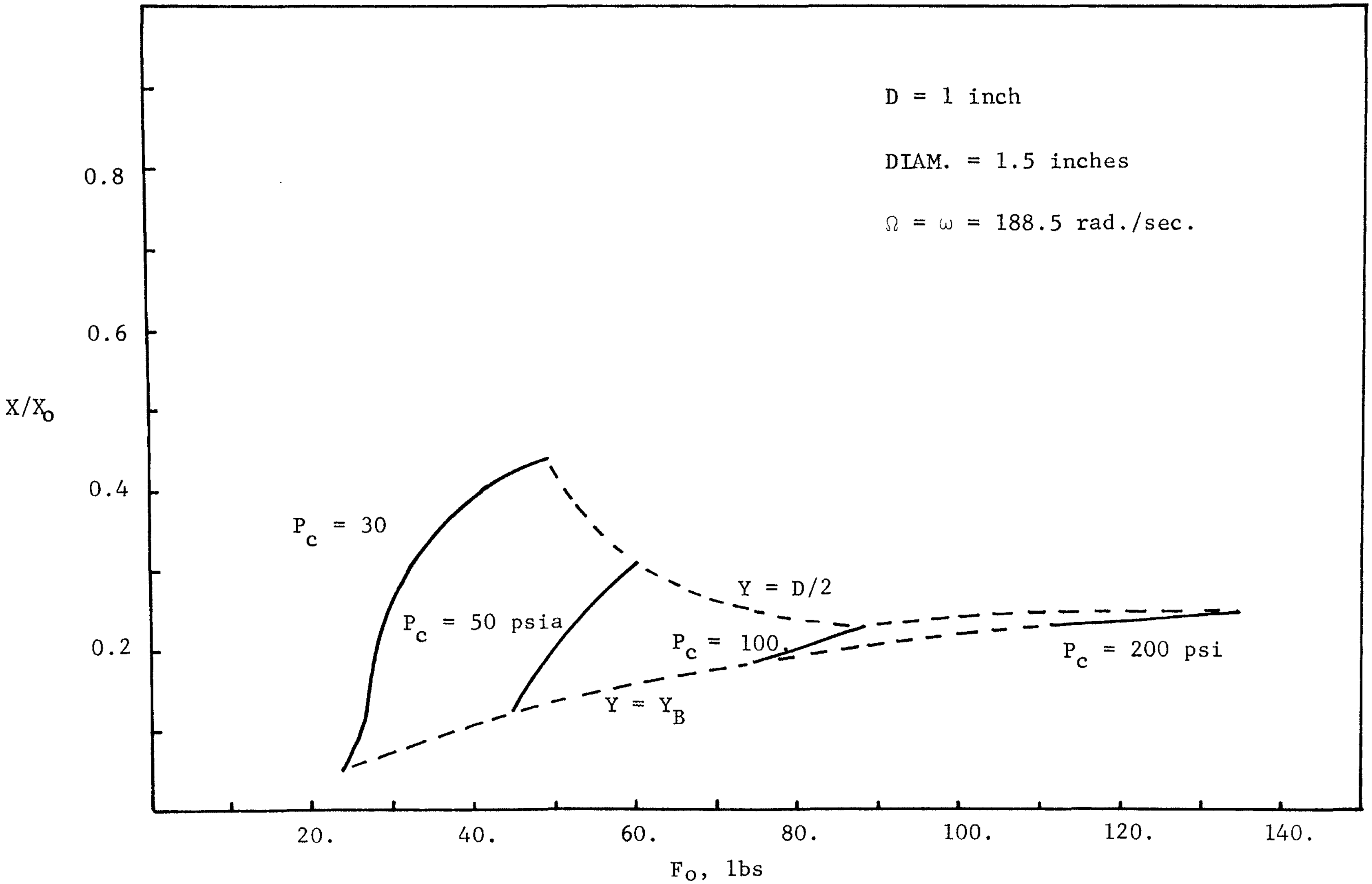


Figure 11. Solution Curve of Predicted Noiseless Periodic Operation.

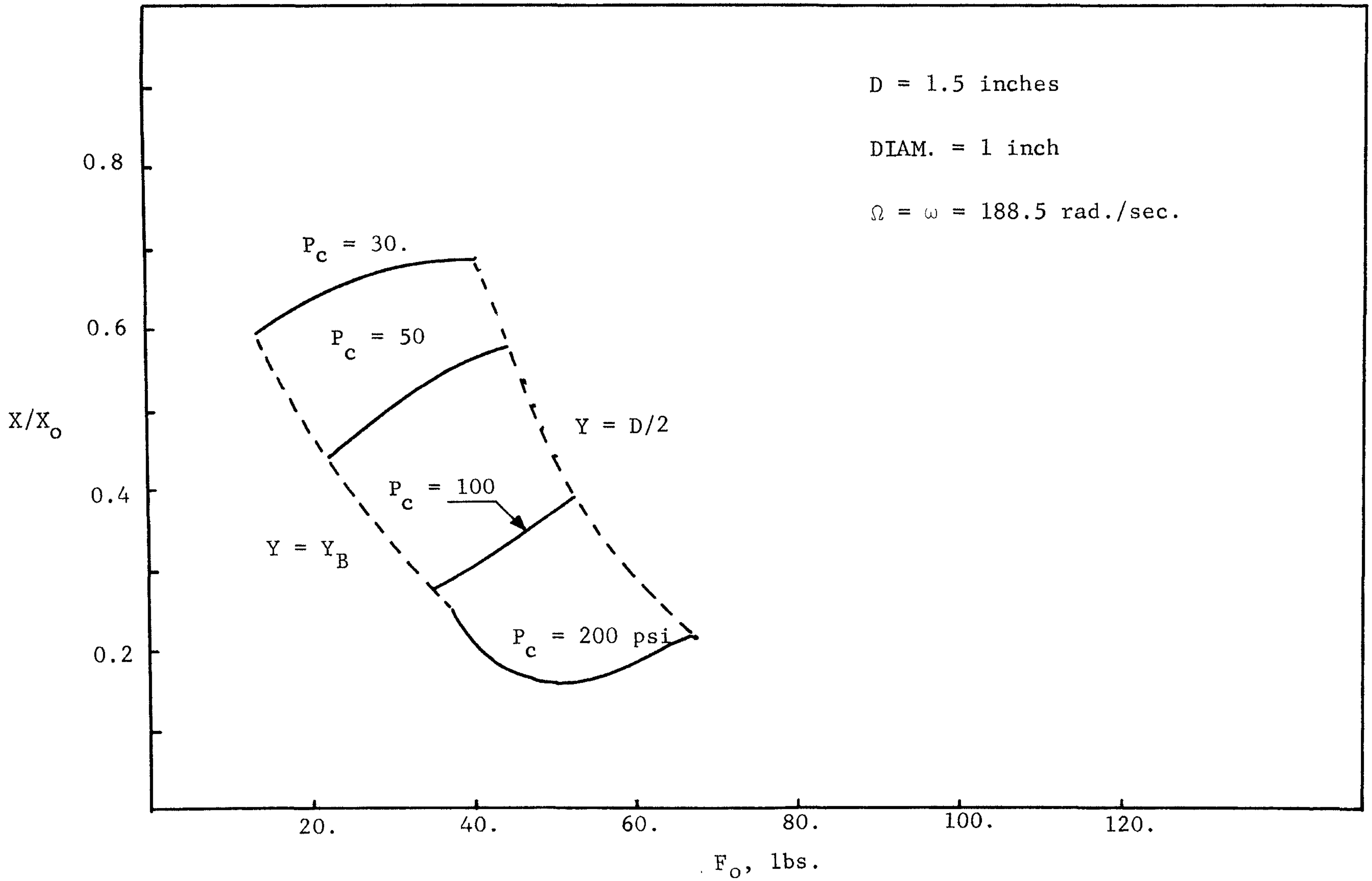


Figure 12. Solution curve of predicted noiseless periodic operation.

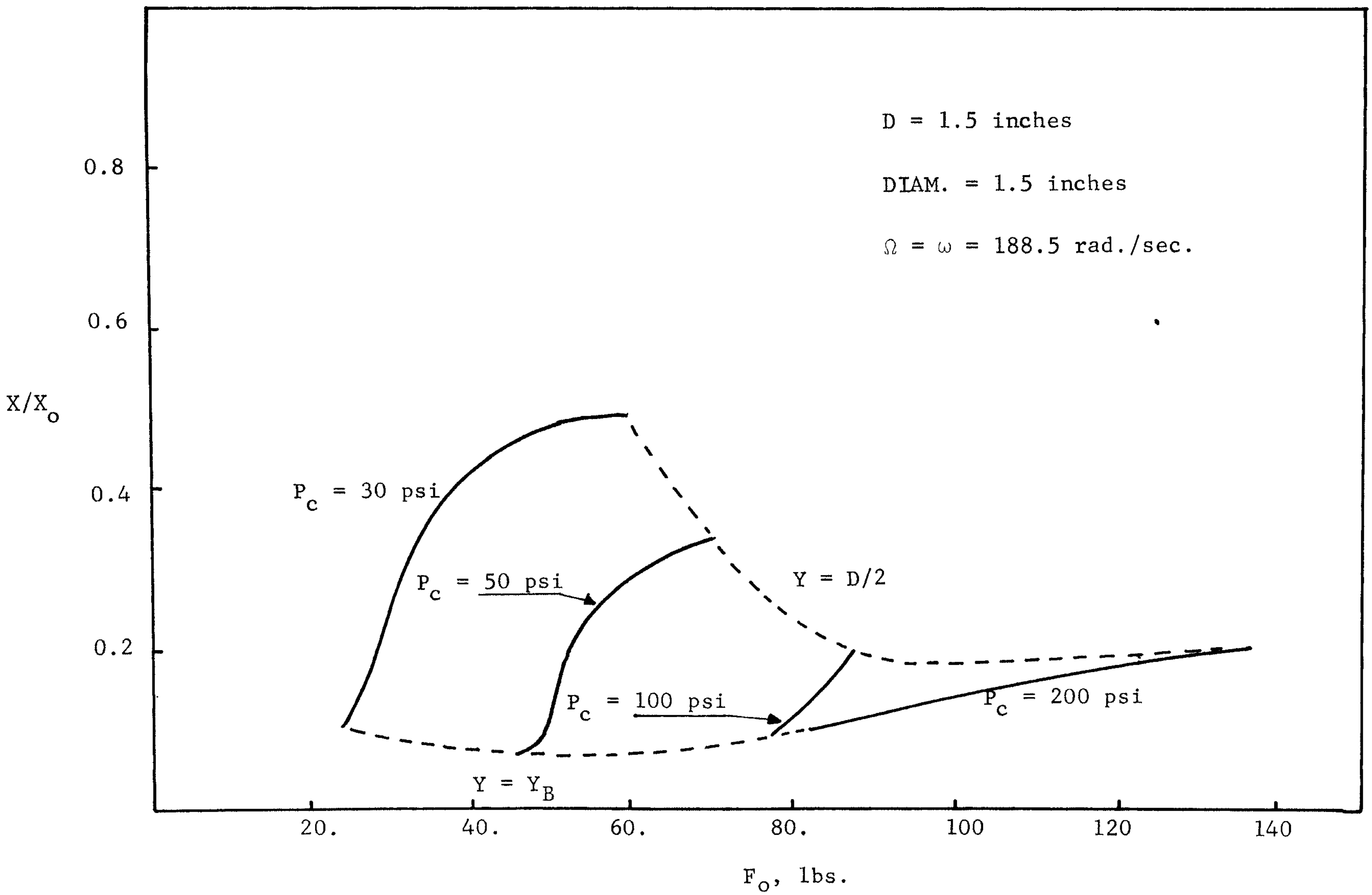


Figure 13. Solution Curve of Predicted Noiseless Periodic Operation

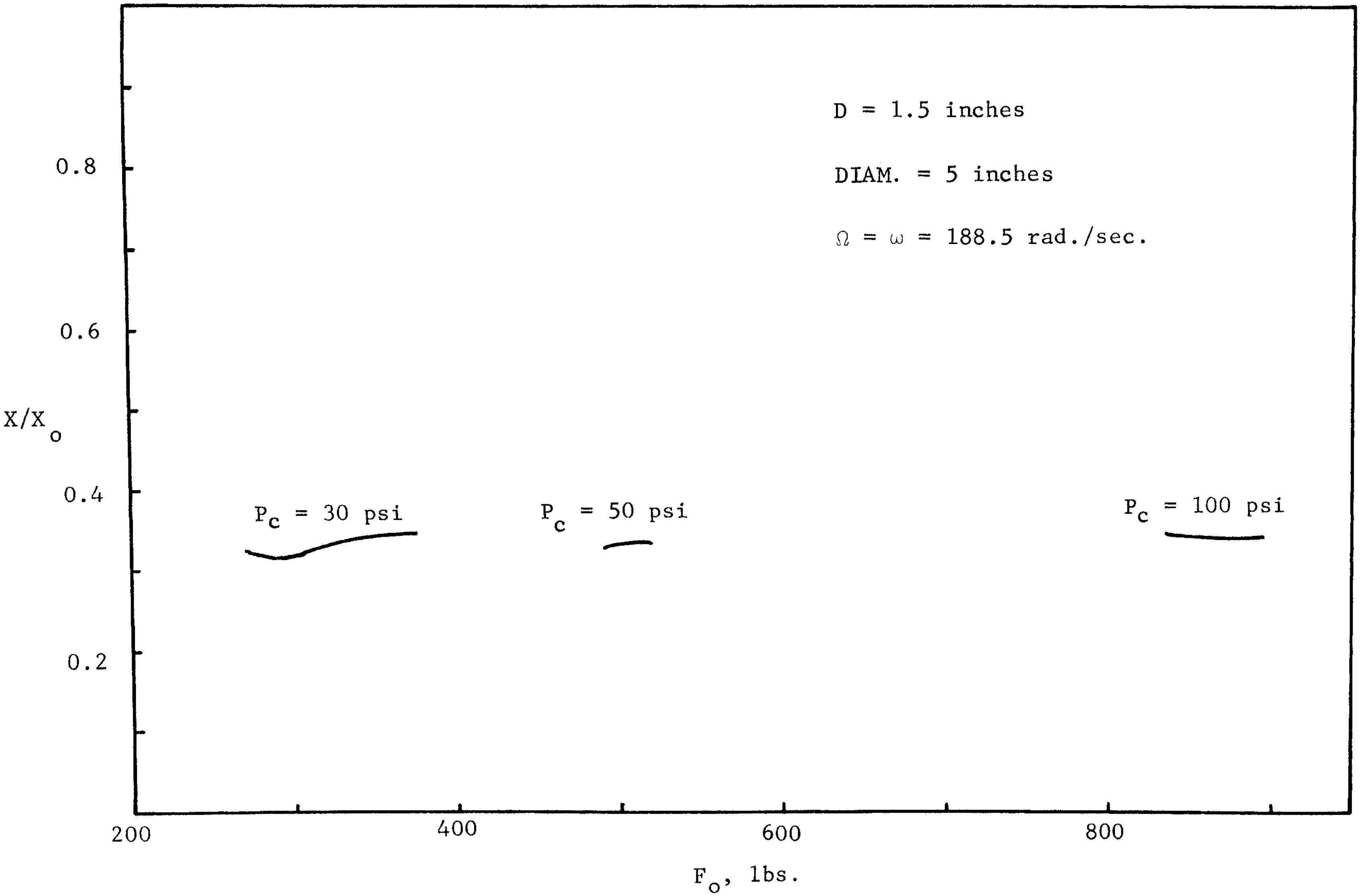


Figure 14. Solution Curve of Predicted Noiseless Periodic Operation.

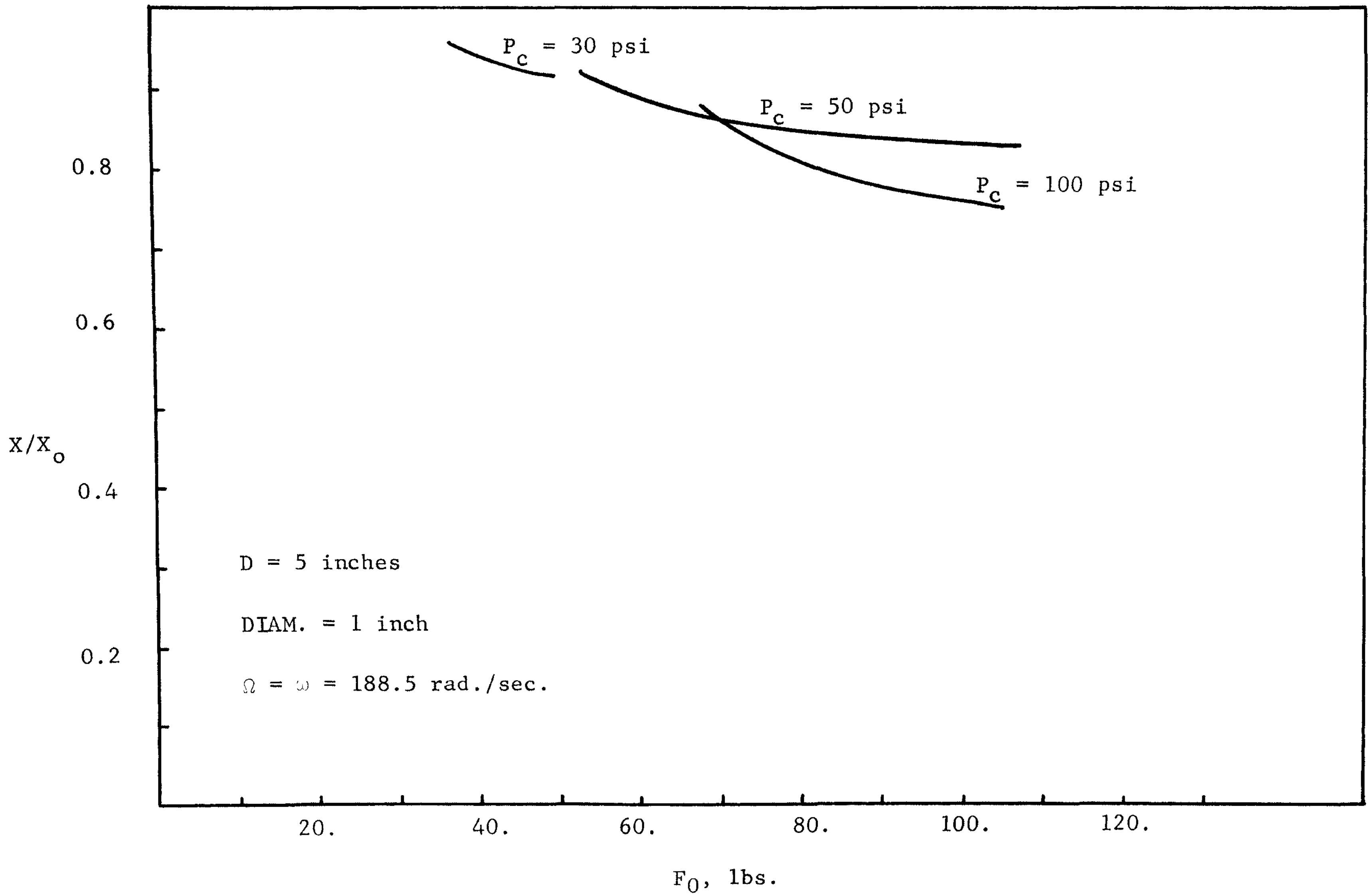


Figure 15. Solution Curve of Predicted Noiseless Periodic Operation

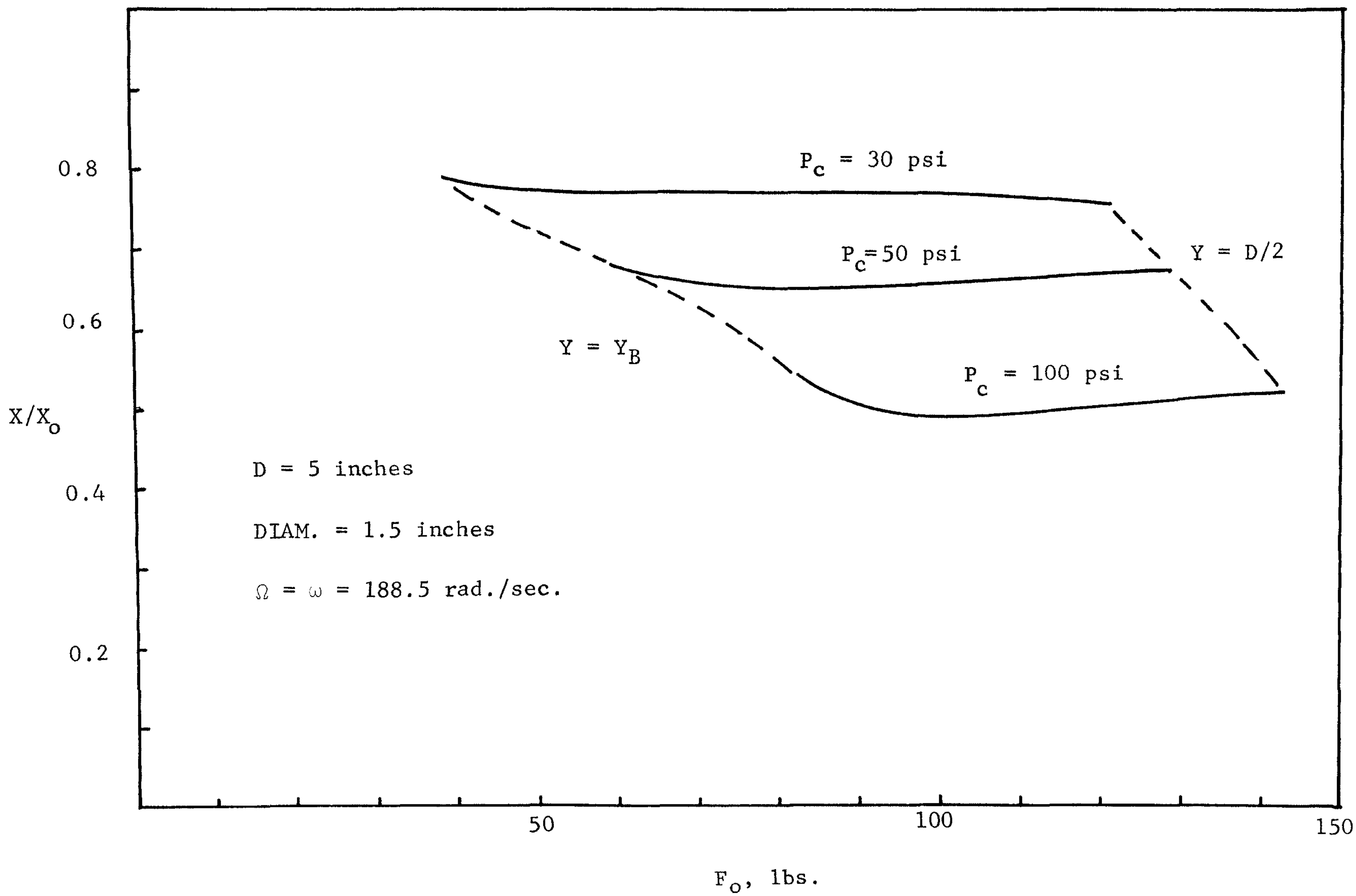


Figure 16. Solution Curve of Predicted Noiseless Periodic Operation.

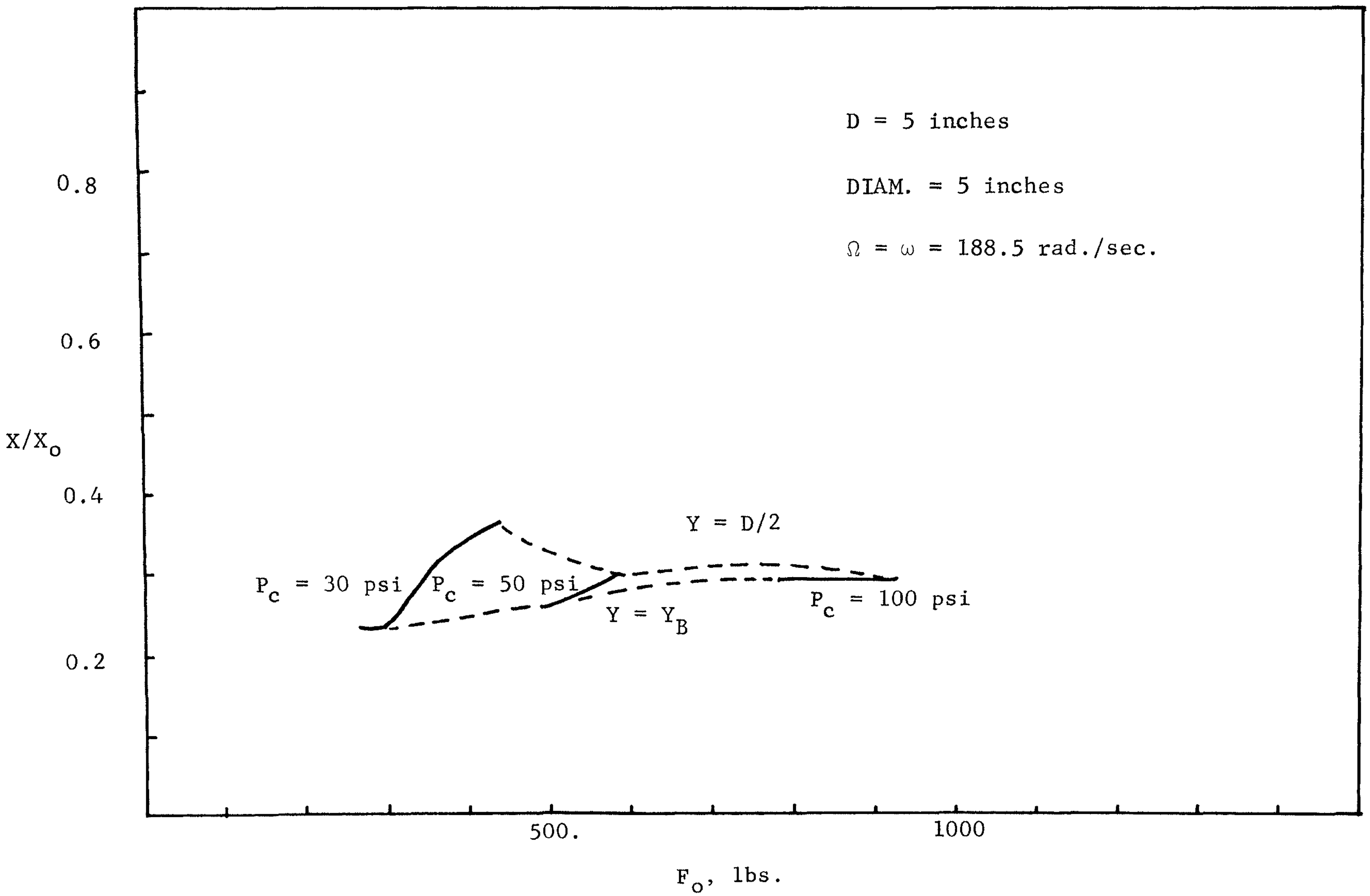


Figure 17. Solution Curve of Predicted Noiseless Periodic Operation.

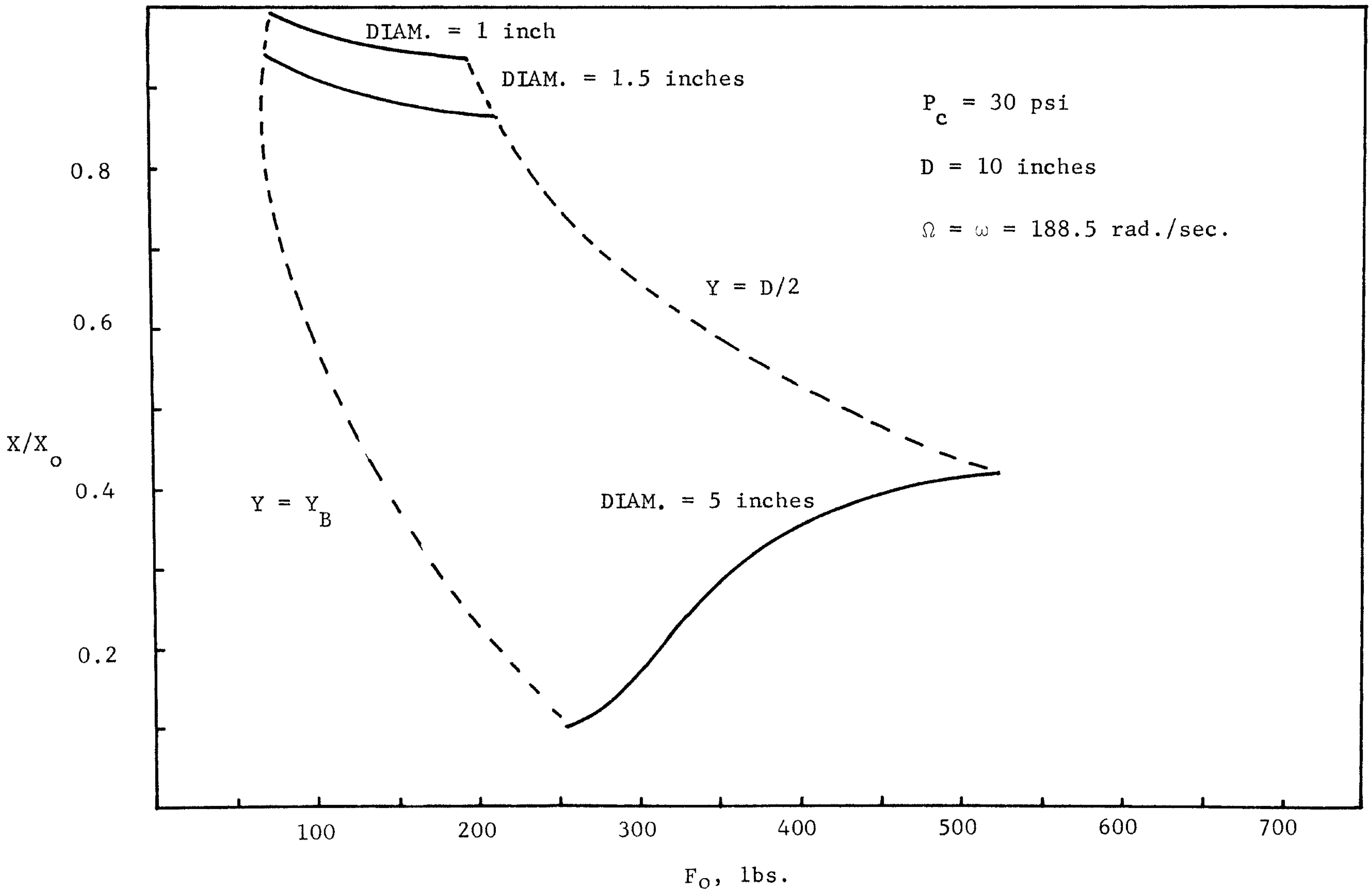


Figure 18. Solution Curve of Predicted Noiseless Periodic Operation.

The sets of parameters shown in Figures 14, 15, 16, for examples, do not meet this condition.

c) By analyzing the remaining sets of parameters one may eliminate those which do not meet the required condition:

$$F_o = 20. \text{ to } 50. \text{ lbs.}$$

The sets of parameters:

D = 1. inch

Diam. = 1.5 inches

P_c = 200. p.s.i.a. (Fig. 10)

and:

D = 10. inches

Diam. = 5. inches

P_c = 30. p.s.i.a. (Fig. 18)

for examples, do not meet this condition.

d. After successive steps of elimination, one may list the remaining sets of parameters which were predicted to meet all the required conditions as follows:

Set 1: D = 1.5 inches

Diam. = 1.5 inches

P_c = 30., 50. p.s.i.a. (Fig. 13)

Set 2: D = 1. inch

Diam. = 1.5 inches

P_c = 30., 50. p.s.i.a. (Fig. 11)

Set 3: D = 1.5 inches

Diam. = 1. inch

P_c = 200. p.s.i.a. (Fig. 12)

Set 4: D = 1. inch

Diam. = 1. inch

P_c = 100. p.s.i.a. (Fig. 10)

Digital simulation was used to verify the results predicted by the approximate analytical approach and to make decision on the selection of the best among these four sets of parameters.

2. Digital simulation of noiseless periodic operation of the trial dampers.

For the system described by the data given in Table II, with a trial damper having the following set of parameters:

P_c = 30., 50., 100. p.s.i.a.

D = 1.5 inches

Diam. = 1.5 inches,

The excitation force F_o to produce noiseless periodic operation was predicted by using eq. (4.49b). The results, as shown in Fig. 13 and 19, have been tabulated in Table III. Once predicted, this force was used as an input in the digital simulation of the system.

The prediction and the simulation of noiseless periodic operation of the given system and damper are shown in Fig. 19.

The following observations based on Figures 13 and 19 can be made:

a) Although noiseless periodic operation has been predicted for

$$Y_B < Y < D/2,$$

results obtained through digital simulation show that impacts may occur for Y within these limits.

For example, noiseless periodic operation for the system under discussion, when P_c = 30. p.s.i.a., has been predicted for:

$$Y_B = 0.317 \text{ inch} < Y < 0.75 \text{ inch} = D/2$$

$$\text{and} \quad 23.4 \text{ lbs.} \leqslant F_o \leqslant 58.2 \text{ lbs.}$$

as shown in Table III.

Results through digital simulation indicate that for the same system and damper, noiseless periodic operation has been obtained only for:

$$0.317 \text{ inch} < Y < 0.487 \text{ inch}$$

$$23.4 \text{ lbs.} \leqslant F_o \leqslant 31.6 \text{ lbs.},$$

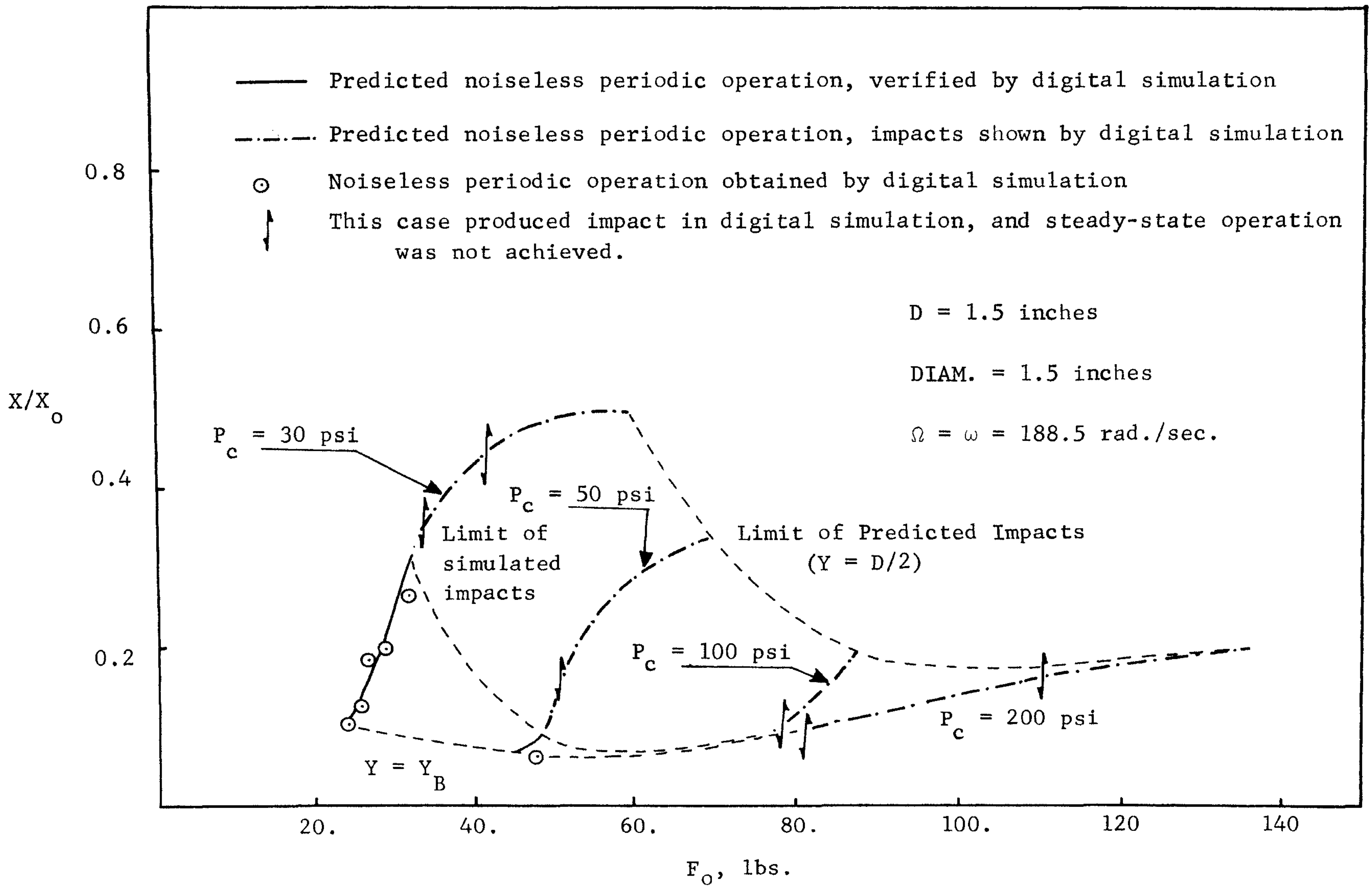


Figure 19. Solution Curves of Noiseless Periodic Operation

Y in.	F_o lbs.	Y in.	F_o lbs.	Y in.	F_o lbs.
0.325	23.494	0.475	43.476	0.600	72.883
0.350	24.830	0.500	45.207	0.650	80.091
0.400	27.034	0.550	47.417	0.700	86.089
0.450	29.346	0.600	49.875	0.750	93.564
0.500	32.086	0.650	53.715		
0.550	35.451	0.700	59.764		
0.600	39.599	0.750	69.288		
0.650	44.657				
0.700	50.736				
0.750	58.178				

P_c = 30. p.s.i.a.	P_c = 50. p.s.i.a.	P_c = 100 p.s.i.a.
D = 1.5 inches	D = 1.5 inches	D = 1.5 inches
Diam. = 1.5 inches	Diam. = 1.5 inches	Diam. = 1.5 inches
Y_B = 0.317 inch	Y_B = 0.457 inch	Y_B = 0.578 inch

Table III. Predicted Excitation Force to Produce Periodic Operation of the Trial Dampers.

as shown in Fig. 19.

The same system and damper, when excited by a force F_o of 33.3 lbs., produced impacts, and steady-state operation was not achieved.

b) There is, however, good agreement between the results obtained by digital computer simulation and those obtained by application of the approximate analytical approach described in Chapter IV, if the excitation force F_o falls within the operation range:

$$23.4 \text{ lbs.} \leqslant F_o \leqslant 31.6 \text{ lbs.},$$

as can be seen in Fig. 19 and as will be discussed with details in section V-B.

It was predicted by using eq. (4.49b) that an excitation force F_o of 27.385 lbs., for example, would produce periodic operation for the system and damper under discussion, the maximum amplitude Y would be 0.408 inch, and the response amplitude ratio $\frac{X}{X_o}$ would be 0.172. When the same system and damper was simulated by a digital computer, with an excitation force F_o = 27.385 lbs. as an input, the motions of the primary system and that of the damper were periodic, in agreement with the predictions, the maximum amplitude Y was 0.391 inch, and the response amplitude ratio $\frac{X}{X_o}$ was 0.179.

c) Impacts have occured and noiseless periodic operation has not been established, whenever Y_B is close to D/2.

Since:

$$Y_B = D/2 \left[1 - \left(\frac{P_a}{P_c}\right)^{\frac{1}{n}}\right],$$

the larger P_c is, the closer to D/2 the amplitude Y_B will be.

Therefore noiseless periodic operation for the given system with damper has not been obtained for high values of preset pressure, P_c, of the ball valves, as can be seen in Fig. 19.

The system and damper under discussion, when P_c = 100. p.s.i.a., was predicted to operate periodically and noiselessly for:

$$72.883 \text{ lbs.} \leq F_o < 93.564 \text{ lbs.},$$

as shown in Table III.

When the same system was simulated by a digital computer, using an excitation force of 77.126 lbs. as an input, the motion of the primary system was disturbed by impacts and was not periodic.

For this case,

$$Y_B = 0.578 \text{ inch}$$

$$\text{and} \qquad D/2 = 0.750 \text{ inch}$$

From this observation, one may come to the conclusion that among the four trial sets of parameters predicted for noiseless periodic operation and selected previously, set 3 and set 4 may result in undesired impacts.

Finally after all these considerations, the following set of damper parameters:

$$\mu = \frac{m}{M} = 0.1$$

$$e = 0.8$$

$$P_c = 30. \text{ to } 50. \text{ p.s.i.a.}$$

$$D = 1.5 \text{ inches}$$

$$\text{Diam.} = 1.5 \text{ inches}$$

has been selected for a damper designed to work with the given primary system, which may be described by the data given in Table II, such that a response amplitude ratio $\frac{X}{X_o}$ of less than 1/3 was obtained for an excitation force of 20 to 50 lbs. The performance of the damper has been illustrated in Fig. 19.

B. Determination of the Motion of the Given System with the Selected Damper Attached

The behavior of the selected damper, as illustrated in Fig. 19, was investigated with details in this part of the present study.

1. Reduction in response amplitude at resonance.

The solution curve of the selected damper attached to the given primary system has been shown in Fig. 19 for:

$$P_c = 30 \text{ and } 50 \text{ p.s.i.a.}$$

Figure 20 shows the solution curve of the same damper attached to the given primary system when:

$$P_c = 24, 27, 30, 33, 36 \text{ and } 50 \text{ p.s.i.a.}$$

Eleven cases of noiseless periodic operation of the selected damper attached to the given primary system have been simulated by a digital computer for different values of P_c ranging from 24 to 50 p.s.i.a. and illustrated in Fig. 20. Details of these cases are summarized in Table IV.

The mean response amplitude ratio $\frac{X}{X_o}$ of these eleven cases is:

$$\text{mean } \frac{X}{X_o} = 0.173$$

A two impacts/cycle solution curve of an impact damper attached to a primary system is reproduced from a study of Masri [5] for comparative purpose, and shown in Fig. 21. The ratio μ of damper mass, m, to primary system mass, M, and the coefficient of restitution e of this system have the same values with those of the system under discussion in the present study, e.g.:

$$\mu = \frac{m}{M} = 0.10$$

$$\text{and} \qquad e = 0.8$$

The mean value of $\frac{X}{X_o}$ for eleven cases obtained by a digital computer

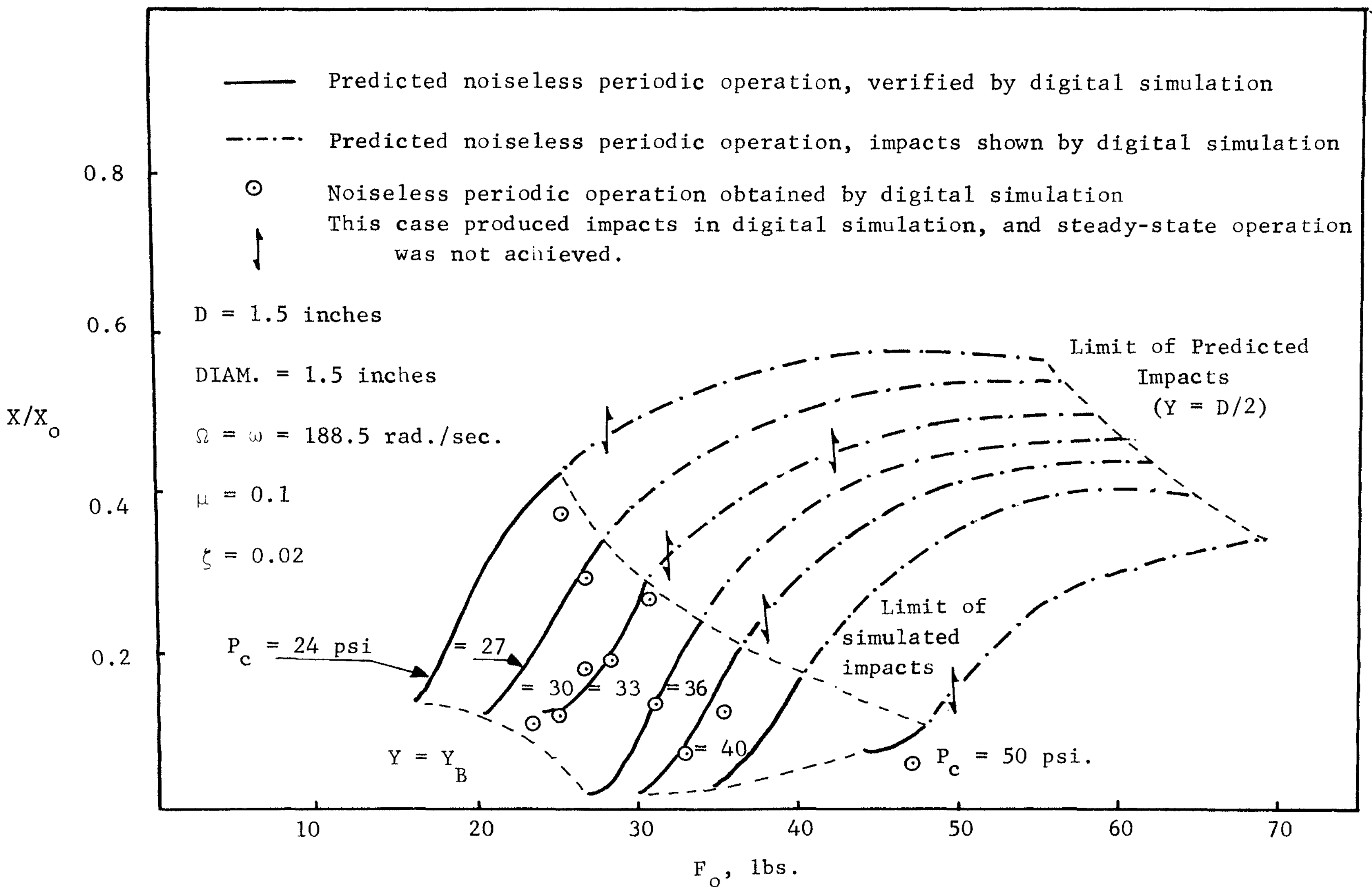


Figure 20. Solution Curves of Noiseless Periodic Operation

P_c p.s.i.	F_o lb.	$\frac{X}{X_o}$ Predicted	$\frac{X}{X_o}$ Simulated	% Difference	Y Predicted	Y Simulated	% Difference
30.	23.494	0.118	0.103	14.	0.325	0.331	-2.
"	24.928	0.122	0.125	-2.4	0.352	0.350	0.5
"	27.385	0.172	0.179	-4.	0.408	0.391	4.
"	28.676	0.220	0.199	10.	0.436	0.408	7.
"	31.609	0.307	0.266	15.	0.492	0.487	1.
24.	25.318	0.423	0.370	14.	0.435	0.518	-16.
27.	26.793	0.316	0.290	9.	0.436	0.415	5.
33.	30.761	0.138	0.131	5.	0.435	0.412	5.
36.	32.854	0.072	0.068	6.	0.431	0.411	5.
36.	35.428	0.168	0.121	38.	0.491	0.457	7.
50	47.118	0.088	0.055	60.	0.543	0.506	7.

Table IV. Cases of Noiseless Periodic Operation at Resonance Simulated by a Digital Computer.

D = 1.5 inches, Diam = 1.5 inches, $\Omega = \omega = 188.5$ rad./sec.

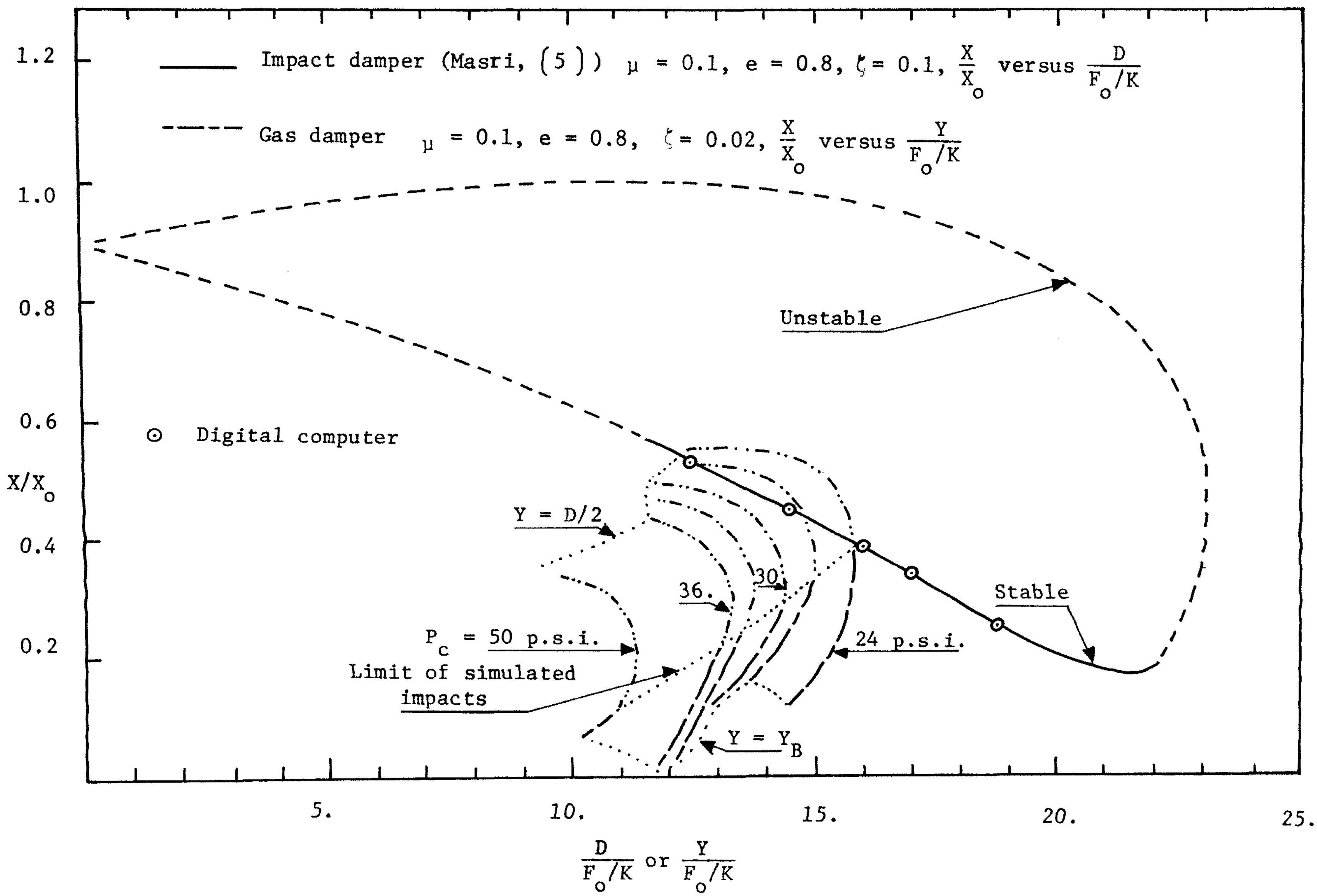


Figure 21. Solution Curve of Impact Damper in Masri's Study (5) Compared with those of the Damper in Figure 20.

in the present study is comparable to minimum values obtained by Masri [5] equal to 0.2 for impact dampers, as shown in Fig. 21. The operation of the new damper is expected to be relatively noiseless, since in steady-state operation the maximum amplitude Y is clearly less than D/2, as can be seen in Table IV.

2. Reduction in response amplitude of systems near resonance.

The selected damper which has been designed with:

$$D = 1.5 \text{ inch}$$

$$\text{Diam.} = 1.5 \text{ inch}$$

and P_c set at 30 p.s.i.a. is considered again.

Figure 22 and Table V show that the operation of the damper remained noiselessly periodic and the reduction in response amplitude of the system comparable to those obtained by impact dampers, when the system operated in the vicinity of the resonance. The excitation force F_o which produced noiseless periodic operation was predicted by using eq. (4.49b), and the response amplitude ratio $\frac{X}{X_o}$ was predicted by eq. (4.54b).

3. Widening the operation range of the excitation force F_o by adjustment of the preset pressure P_c

It was found that for the selected damper, if

$$P_c = 24 \text{ p.s.i.a.},$$

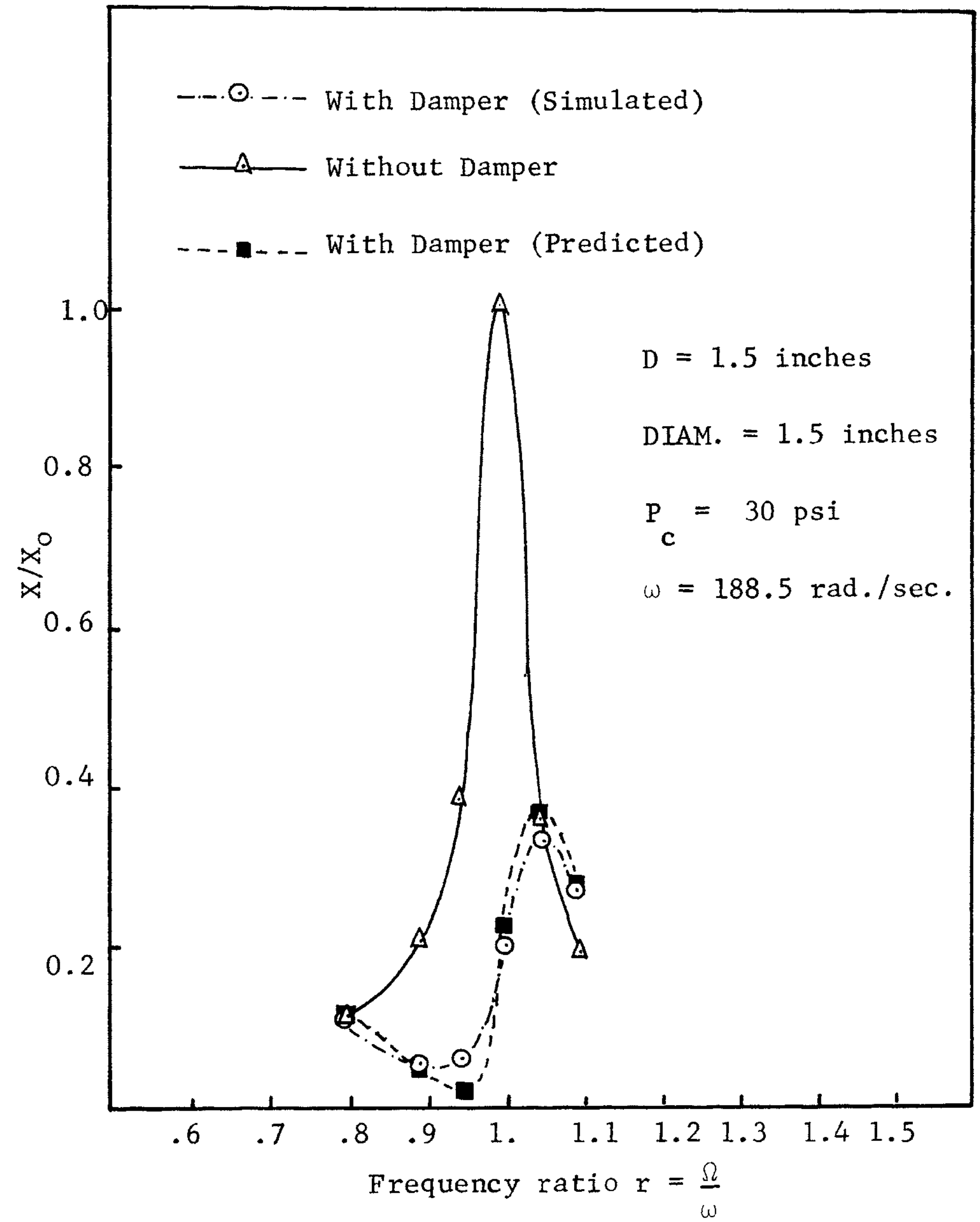


Figure 22. Operation Near Resonance.

P_c p.s.i.	$r = \frac{\Omega}{\omega}$	F_o lb.	$\frac{X}{X_o}$ Predicted	$\frac{X}{X_o}$ Simulated	% Difference
30	0.8	28.676	0.114	0.104	9.
30	0.9	"	0.050	0.054	-7.
30	0.95	"	0.013	0.056	-76.
30	1.05	"	0.367	0.322	13.
30	1.1	"	0.273	0.263	3.

Table V. Cases of Noiseless Periodic Operation Near the Resonant Frequency ω = 188.5 rad./sec. Simulated by a Digital computer.

D = 1.5 inches, Diam. = 1.5 inches, P_c = 30 p.s.i.

then noiseless periodic operation will be obtained for an excitation force F_o within the range:

$$15.9 \text{ lbs.} \leqslant F_o \leqslant 25.5 \text{ lbs}$$

This operation range will be shifted to:

$$43.5 \text{ lbs} \leqslant F_o \leqslant 47 \text{ lbs.}$$

when:

$$P_c = 50 \text{ p.s.i.a.},$$

as can be seen in Table IV and Fig. 20.

Therefore if the preset pressure P_c is adjustable within the range of 24 to 50 p.s.i.a., then the operation range of F_o will be widened to:

$$15.9 \text{ lbs.} \leqslant F_o \leqslant 47 \text{ lbs.}$$

as shown in Fig. 20 and Table IV.

4. Stability

The predicted envelope shown in Figures 19 and 20 encloses those configurations characterized by roots of the auxillary polynomial equal to $\pm j\Omega$ The complete picture of stability must, however, rely also on an examination of the other roots of the characteristic polynomial. By applying the Hurwitz criteria, as given in equations (4.56), to the system shown in Fig. 20, one may determine the nature of these remaining roots. The results of such a determination are shown in Figure 23 and Table VI. Fig. 23 illustrates that the predicted stability envelope is smaller than the predicted envelope where roots $\pm j\Omega$ exist as shown in Figures 19 and 20. Computed results do predict an even smaller stability envelope as a comparison of Figures 20 to Figure 23 illustrates. The discrepency here is believed to be due in part to the approximation of the integral used in the determination of N_1 and in part due to the linearization process.

Based on the results shown in Figures 19, 20, 21, 22, 23, one may conclude that a damper of the type under discussion in the present study may produce a reduction in response amplitude of the primary system at resonance comparable to that obtained by the impact damper. The operation of the new damper is expected to be relatively noiseless. A substitute for the impact damper to reduce mechanical vibrations, as conceived in the present study, is therefore feasible.

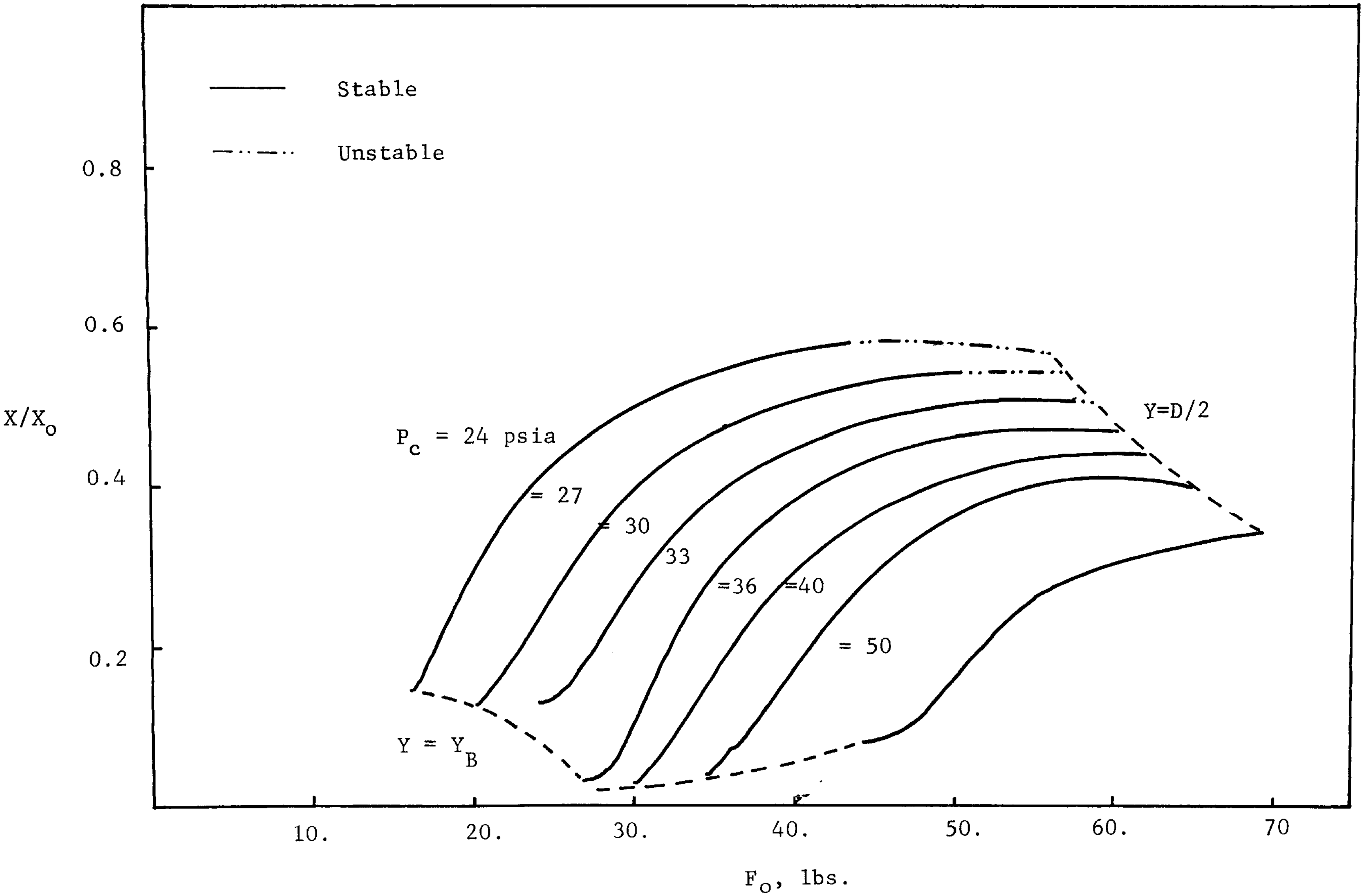


Figure 23. Predicted Stability of the System Shown in Figure 20.

P_c p.s.i.a.	F_o lbs.	ϕ radian	Y in.	N_1	N_2	Hurwitz's Criteria
24	15.941	0.245	0.250	60.7	2.9	Stable
"	23.209	0.817	0.400	32.3	18.4	Stable
"	33.583	1.208	0.550	11.7	24.9	Stable
"	43.648	1.404	0.655	-0.4	28.7	Unstable
"	48.822	1.465	0.700	-4.9	30.6	Unstable
"	55.371	1.503	0.750	-8.4	33.5	Unstable
27	20.341	0.222	0.300	64.6	3.9	Stable
"	27.550	0.770	0.450	35.9	20.2	Stable
"	34.334	1.064	0.550	20.0	25.8	Stable
"	43.783	1.296	0.650	6.1	30.4	Stable
"	50.357	1.390	0.705	-0.5	33.3	Unstable
"	56.731	1.433	0.750	-4.2	36.4	Unstable
30	23.475	0.142	0.325	70.8	1.7	Stable
"	29.346	0.603	0.450	46.2	18.7	Stable
"	35.451	0.932	0.550	28.0	26.1	Stable
"	44.657	1.203	0.650	11.9	32.0	Stable
"	50.736	1.306	0.700	4.9	35.1	Stable
"	58.178	1.369	0.750	-0.1	39.1	Unstable
33	26.441	0.098	0.350	75.0	00.5	Stable
"	31.429	0.461	0.450	55.9	16.6	Stable
"	36.866	0.813	0.550	35.6	26.1	Stable
"	45.704	1.117	0.650	17.5	33.4	Stable
"	51.874	1.235	0.700	9.6	37.0	Stable
"	59.703	1.310	0.750	3.9	41.6	Stable
36	29.412	0.075	0.375	78.1	0.3	Stable
"	33.631	0.341	0.450	64.8	14.1	Stable
"	38.507	0.706	0.550	43.0	25.8	Stable
"	46.898	1.038	0.650	22.9	34.5	Stable
"	53.111	1.170	0.700	14.1	38.7	Stable
"	61.294	1.255	0.750	7.8	44.0	Stable
50	43.476	0.063	0.475	89.5	4.7	Stable
"	47.417	0.342	0.550	72.7	21.2	Stable
"	53.715	0.742	0.650	45.9	37.4	Stable
"	59.764	0.923	0.700	33.6	44.7	Stable
"	69.288	1.053	0.750	24.4	53.2	Stable

Table VI. Predicted Stability of the System Shown in Figure 20.

C. Details on the Noiseless Periodic Operation

1. Basic assumption.

The basic assumption that:

$$y(t) = Y \sin \Omega t,$$

once periodic operation of the system has been established, e.g. if the excitation force F_o is related to the maximum amplitude Y by eq. (4.49b), appears justified for the range of variables examined by the results obtained through digital simulation.

Noiseless periodic operation of a damper with the following design-parameters:

$$P_c = 30.\ \text{p.s.i.a.}$$

$$D = 1.5\ \text{inches}$$

$$\text{Diam.} = 1.5\ \text{inches}$$

$$\mu = 0.1$$

$$e = 0.8,$$

attached to the following primary system:

$$W_1 = 10.\ \text{lbs}$$

$$\zeta = 0.02$$

$$\omega = 314.16\ \text{rad./sec.}$$

was sought.

It was predicted by eq. (4.49b) that if the system at resonance was excited by a force of 49 lbs., the function y(t) would be periodic and close to 0.425 sin Ωt. A curve of the predicted function 0.425 sin Ωt was drawn with continuous line in Fig. 24.

When the motion of the same system was simulated by a digital computer, with the same excitation force as an input, it was found that the function y(t) was a quasi-sinusoidal function of time, with maximum amplitude of Y = 0.427 inch. This simulated function y(t) was shown in Fig. 24, with circled points. The curve of this simulated function was redrawn from a computer plot.

2. Damping force.

The damping force has been defined as:

$$F(y,\dot{y}) = (P_2 - P_1)A$$

The damping force of the system which has just been discussed was redrawn from a computer splot and shown in Fig. 25 in continuous line.

Curve 1 represents this force in function of y and y, curve 2 represents the same force in function of time.

The damping force was harmonically linearized by

$$F(y,\dot{y}) = N_1 Y \sin \Omega t + N_2 Y \cos \Omega t$$

For the system which has just been discussed, the expressions N_1 and N_2 were derived from equations (4.31) and (4.36)

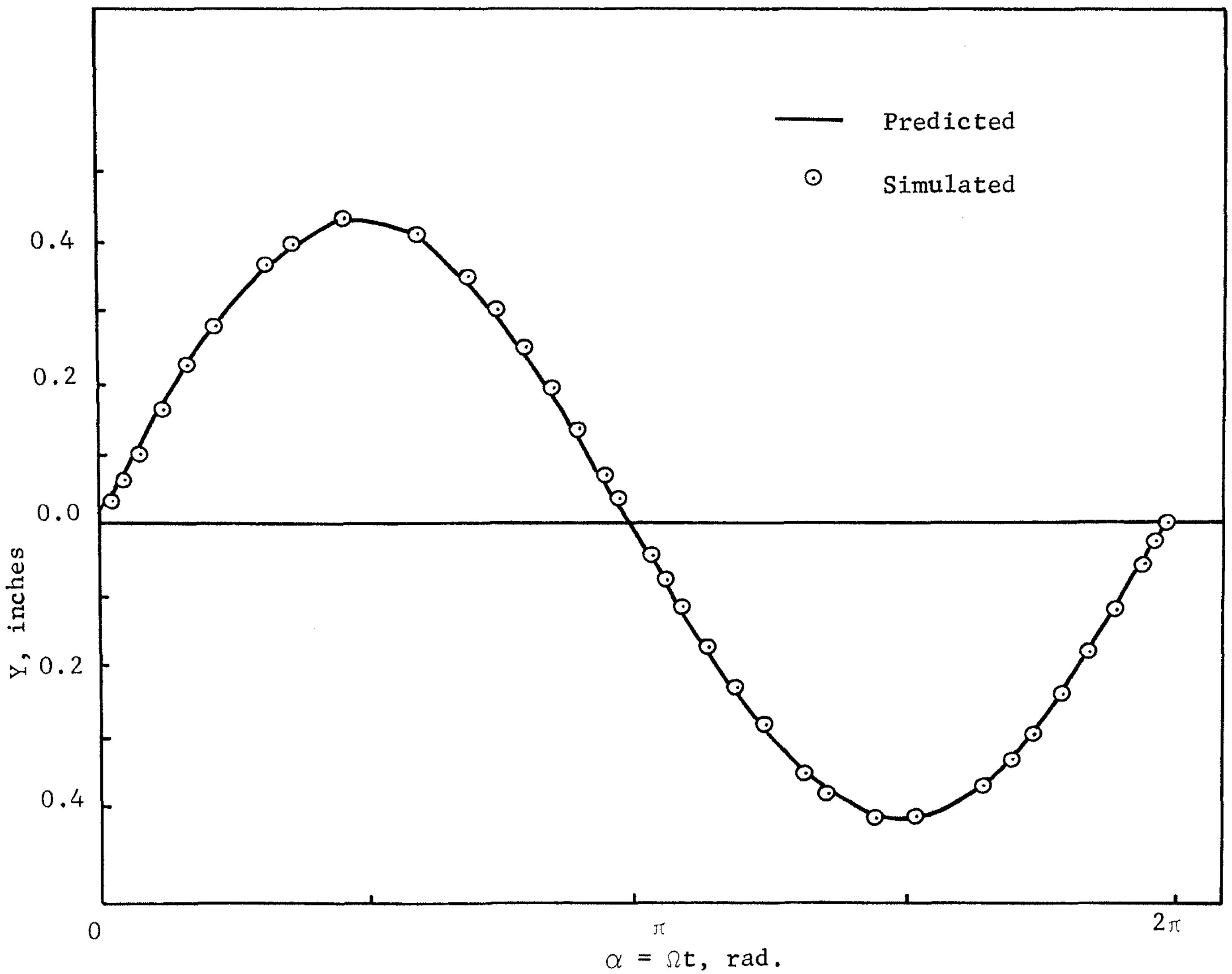
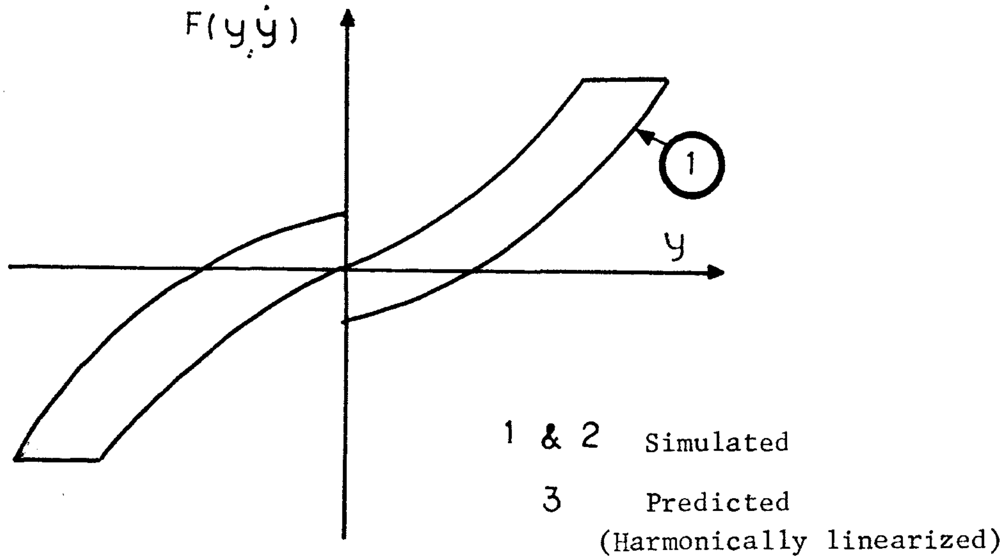
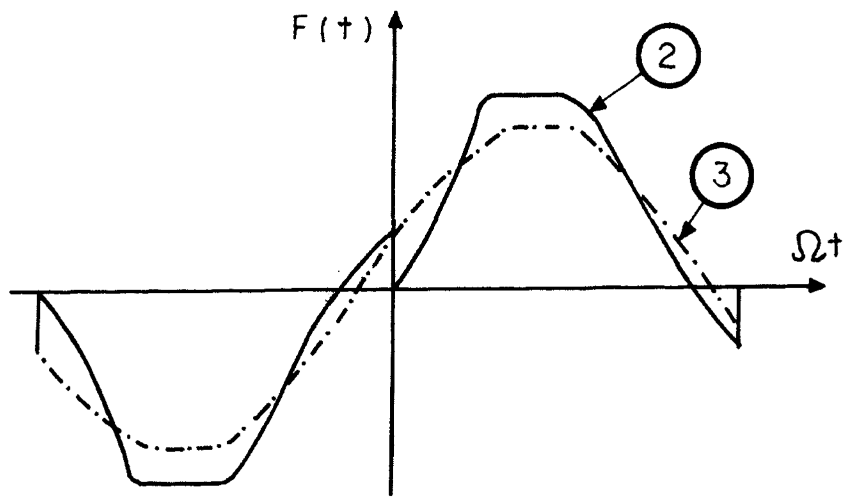


Figure 24. Periodic Solution y(t)



Damping Force in Function of y and ẏ



Damping Force in Function of Time

Figure 25. Damping Force.

$$N_1 = 51.16$$

$$N_2 = 17.7$$

The maximum amplitude Y was predicted equal to 0.425 inch if the system was excited by a force of 49 lbs.

By superposing the functions $N_1Y \sin \Omega t$ and $N_2Y \cos \Omega t$, one may obtain the harmonically linearized damping force, as shown by curve 3 in Fig. 25.

3. Steady-state Solutions

Using the approximate analysis one may predict quasi-sinusoidal solutions of all the variables $x(t)$, $y(t)$ and their derivatives $\dot{x}(t)$, $\dot{y}(t)$ for noiseless periodic operation.

These predictions appear justified for the range of variables examined by the results obtained through digital simulation, as illustrated by Fig. 26, redrawn from computer plot of the case which has just been discussed.

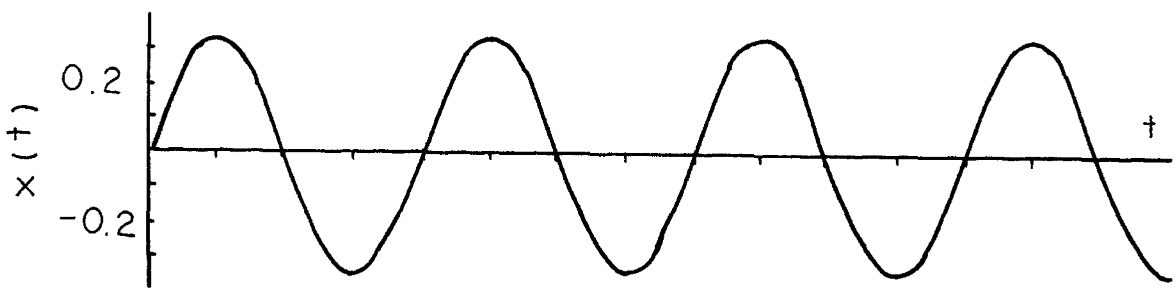
For this case, "steady-state"* solutions have been obtained at 21rst cycle of numerical integration:

$$X = 0.332 \text{ inch}$$

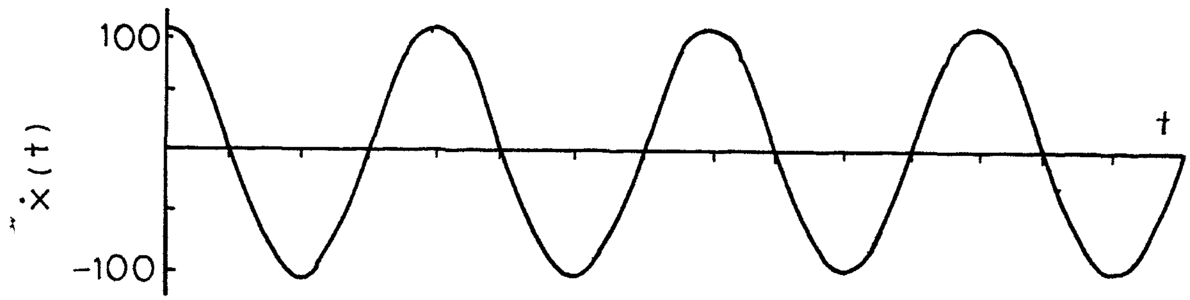
$$\dot{X} = 104 \text{ inches/sec.}$$

---

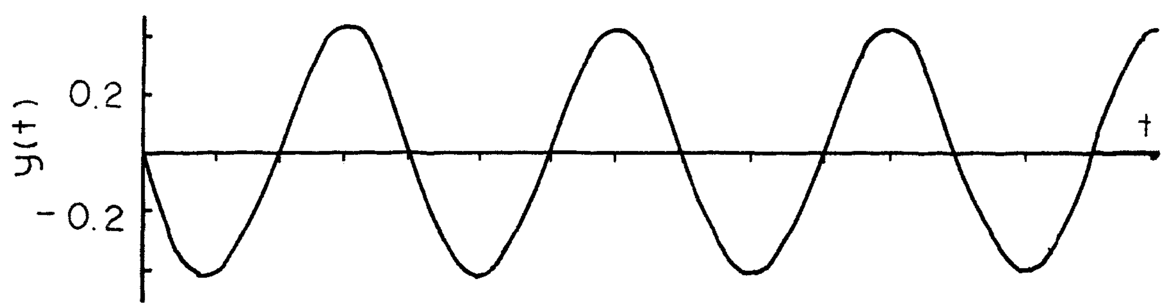
* These peak values were essentially duplicated for each successive cycle until termination of the integartion during the 47th cycle.



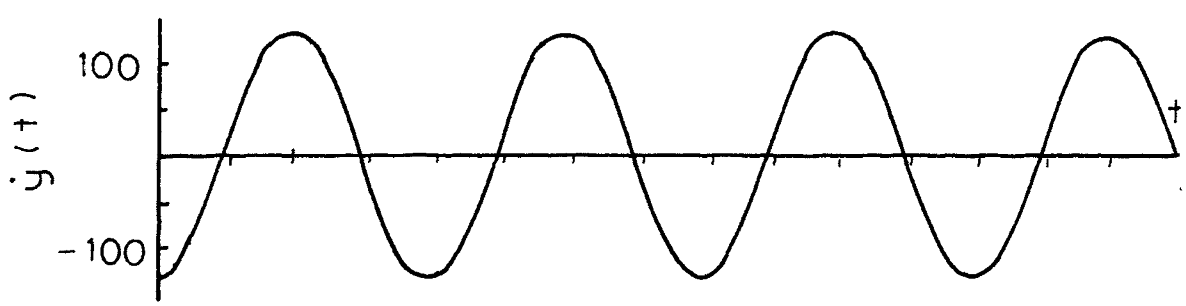
x(t)



ẋ(t)



y(t)



ẏ(t)

Figure 26. Steady-State Solutions

$$Y = 0.427 \text{ inch}$$

$$\dot{Y} = 131 \text{ inches/sec.}$$

compared with predicted values:

$$X = 0.341 \text{ inch}$$

$$\dot{X} = 108 \text{ inches/sec.}$$

$$Y = 0.425 \text{ inch}$$

$$\dot{Y} = 134 \text{ inches/sec}$$

For this case the predicted and the simulated solutions are so close that if they were plotted on the same plot, they would be almost mixed together. The plot of the predicted solutions was therefore not undertaken.

VI. SUMMARY AND CONCLUSIONS

Noiseless periodic operation of an impact type damper was investigated. Results of the analysis were verified by digital simulation of the system. In general there was agreement between the analysis and the simulation.

The following observations can be made on the basis of the present study:

1. Noiseless periodic operation of the damper may be established by selecting appropriate design parameters.

2. Reduction of system response amplitude to less than 1/5 of its value when the system at resonance vibrates without the damper may be obtained.

3. Noiseless periodic operation of the damper may be predicted by the describing function method given in Chapter IV with qualifications and the primary system steady-state amplitude within the range of noiseless periodic operation may be predicted by the describing function method given in Chapter IV.

4. During the course of this investigation, the describing function method predicted noiseless periodic operation wherein noiseless periodic operation was not observed during simulation, rather nonperiodic impacting operation was observed. There is sufficient evidence to believe that the fault is less with the describing function approach and more with the use of the binomial expansion to approximate several integrand functions

in the determination of the expression N_1, the first term of the describing function $N = N_1 + j\,N_2$, used to describe the nonlinear damping force F(y,y).

5. In spite of this problem, the approximate analytical approach described in Chapter IV is valuable in that it gives a comprehensive picture of the global behavior of the nonlinear system. The approach may be used as a good guide for efficient use of computers in the simulation of the motion of the system with damper. Guided by this approach, the digital simulation of the system may lead to the analysis of noiseless periodic operation for a certain predicted operation range of excitation force F_o.

6. Using the approximate analytical approach as a guide and the digital simulation as a tool for verification, one may select appropriate design parameters for a damper to obtain a reduced response amplitude ratio below certain limit, for a certain range of excitation.

7. Once a damper has been designed and attached to a given system, one may widen the operation range of the excitation force F_o by adjusting the preset pressure P_c of the ball valves.

An improved method for the integration involved in the expression N_1 is left as a possible objective for future work, which may include the analysis of periodic operation during which weakened impacts may occur periodically..

BIBLIOGRAPHY

1. Lieber, P. and Jensen, D. P., "An Acceleration Damper Development, Design, and Some Applications," Trans. ASME, Vol. 67 (1945), pp. 523-530.

2. Grubin, C., "On the Theory of the Acceleration Damper," Journal of Applied Mechanics, Vol. 23, Trans. ASME, Vol. 78 (1956), pp. 373-378.

3. Arnold, R. N., "Response of an Impact Vibration Absorber to Forced Vibration," Ninth International Congress of Applied Mechanics (1956).

4. Warburton, G. B., Discussion of "On the Theory of the Acceleration Damper," Journal of Applied Mechanics, Vol. 24, Trans. ASME, Vol. 79 (1957), pp. 322-324.

5. Masri, S. F.,"Analytical and Experimental Studies of Impact Dampers" Ph.D. Thesis, California Institute of Technology, Pasadena, California (1965).

6. McGoldrick, R. T.,"Experiments with an Impact Vibration Damper," David Taylor Model Basin Report No. 816.

7. Lieber, P. and Tripp, F., "Experimental Results on the Acceleration Damper," Rensselaer Polytechnic Institute Aeronautical Laboratory, Report No. TR. AE 5401 (1954).

8. Duckwald, C.S., "Impact Damping for Turbine Buckets," General Engineering Laboratory, General Electric, Report No. R55GL108 (1955).

9. Rocke, R. D., and Masri, S. F., "Application of a Single-Unit Impact Damper to an Antenna Structure," Shock and Vibration Bulletin, Vol 39, 1968.

10. Siljak, D. D., Nonlinear Systems - The Parameter Analysis and Design, John Wiley & Sons, Inc. (1969).

11. Sridhar, Rangasami, "A General Method for Deriving the Describing Functions for a Certain Class of Nonlinearities," IRE-Transactions on Automatic Control, AC5 (June 1960) pp. 135-141.

12. Minorsky, N., Nonlinear Oscillations, Van Nostrand, New York (1962).

Bibliography (continued)

13. Minorsky, N., The Theory of Nonlinear Control System, McGraw-Hill Book Company, (1969).

14. Babuska, I., Numberical Solution of Ordinary and Partial Differential Equations, Interscience Publishers, Prague, (1966).

15. Masri, S. F., "Steady-State Response of a Multidegree System with an Impact Damper," J. Applied Mechanics, Pages 72-APM-39.

VITA

Nguyen Khanh Van was born on April 30, 1938, in Nghe-An, Vietnam. He recieved his primary education in Hanoi, and his secondary education in Saigon, Vietnam. He received his college education from National Technical Center in Saigon, Vietnam. He received a Bachelor of Engineering Degree in Mechanical Engineering in June, 1961.

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APPENDICES

APPENDIX A

COMPUTER PROGRAM LISTING

```
C            * * * * * * * * * * * * * * * * * * * *
C
C
C
C         ANALYSIS OF A SUBSTITUTE FOR THE IMPACT DAMPER
C         TO DAMP
C         NEAR-RESONANT MECHANICAL VIBRATIONS
C
C
C            * * * * * * * * * * * * * * * * * * * *
C
C            * * * * * * * * * * * * * * * * * * * *
C
C         MAIN PROGRAM
C
C            * * * * * * * * * * * * * * * * * * * *
C
C
C                    PARAMETERS
C
C         W1    = WEIGHT OF THE PRIMARY SYSTEM
C         W2    = WEIGHT OF THE MOVING PISTON IN THE
C                   DAMPER
C         XMASS = MASS OF THE PRIMARY SYSTEM
C         YMASS = MASS OF THE MOVING PISTON IN THE DAMPER
C         MU    = YMASS/XMASS = MASS RATIO
C         CREST = COEFFICIENT OF RESTITUTION
C         KK    = MU*(1+CREST)/(1+MU)
C         D     = LENGTH OF THE DAMPER
C         DHALF = HALF THE LENGTH OF THE DAMPER
C         DIAM  = INTERNAL DIAMETER OF THE DAMPER
C         YB    = VALUE OF Y WHEN P2 FIRST REACHES CRITICAL
C                   VALUE PC
C         YC    = VALUE OF Y WHEN EXPANSION BEGINS
C         SO    = VO/AREA = EQUIVALENT TRAVEL OF PISTON
C                   DUE TO DEAD CHAMBER
C         S2    = D/2 - YC
C         C     = CLEARANCE BETWEEN PISTON AND END OF
C                   CONTAINER
C         AREA  = CROSS SECTION OF THE DAMPER
C         VO    = VOLUME OF DEAN CHAMBER
C         PA    = ATMOSPHERIC PRESSURE
C         P1    = PRESSURE ON THE LEFT HAND SIDE OF THE
C                   PISTON
C         P2    = PRESSURE ON THE RIGHT HAND SIDE OF THE
C                   PISTON
C         PC    = CRITICAL PRESSURE
C         GAMMA1= N  =  GAS CONSTANT FOR POLYTROPIC
C                   COMPRESSION
C         GAMMA2= N' =  GAS CONSTANT FOR POLYTROPIC
C                   EXPANSION
C         FO    = AMPLITUDE OF EXCITATION FORCE
```

```
C          A0     = F0/XMASS
C          ZETA   = DAMPING COEFFICIENT
C          OMEGA  = NATURAL FREQUENCY OF THE PRIMARY SYSTEM
C          OMEGAF= FORCING FREQUENCY
C          FREQ   = OMEGA/(2*3.14159)
C          FREQF  = OMEGAF/(2*3.14159)
C          ALFA   = OMEGAF*T
C          DFORCF= F0*SIN(OMEGAF*T)   = DRIVING FORCE
C          AFORCE= A0*SIN(OMEGAF*T)
C                = (F0/XMASS)*SIN(OMEGAF*T)
C          PFORCE= DAMPING FORCE DUE TO THE DAMPER
C                = (P2-P1)*AREA
C          JJJ    = PROBLEM NUMBER
C
C
C                    INCREMENTS  & ERROR TERMS
C
C          H      = INITIAL TIME INCREMENT
C          TF     = UPPER BOUNDARY OF THE INDEPENDENT
C                    VARIABLE T
C          EPS    = ERROR CRITERION FOR X,Y
C          EEF    = RELATIVE ERROR CRITERION FOR THE
C                    CLEARANCE AT IMPACT
C          EEE    = RELATIVE ERROR CRITERION FOR PRESSURE AT
C                    OPENING OF THE VALVE
C          EEG    = ERROR CRITERION FOR YC
C          EEO    = ERROR CRITERION FOR DETERMINATION OF THE
C                    ORIGINE ON Y-AXIS
C          U,V    = ERROR TERMS USED IN HALF STEP TEST
C          UU,VV  = ERROR TERMS USED IN DOUBLE STEP TEST
C
C
C                    VARIABLES  & FUNCTIONS
C
C          T      = TIME
C          G2     = X DOUBLE DOT
C          G4     = Y DOUBLE DOT
C
C
C                    DUMMY VARIABLES & FUNCTIONS
C
C          Z0(1)  = DUMMY INITIAL VALUE OF X
C          Z0(2)  = DUMMY INITIAL VALUE OF XDOT
C          Z0(3)  = DUMMY INITIAL VALUE OF Y
C          Z0(4)  = DUMMY INITIAL VALUE OF YDOT
C          Z(1)   = DUMMY VALUE OF X    DURING INTEGRATION
C          Z(2)   = DUMMY VALUE OF XDOT DURING INTEGRATION
C          Z(3)   = DUMMY VALUE OF Y    DURING INTEGRATION
C          Z(4)   = DUMMY VALUE OF YDOT DURING INTEGRATION
C          TT     = DUMMY VALUE OF T    DURING INTEGRATION
```

```
C          ZC(1) = DUMMY VALUE OF X    AT TIME T+H,COMPUTED
C                    FROM Z(1) AT TIME T
C          ZC(2) = DUMMY VALUE OF XDOT AT TIME T+H,COMPUTED
C                    FROM Z(2) AT TIME T
C          ZC(3) = DUMMY VALUE OF Y    AT TIME T+H,COMPUTED
C                    FROM Z(3) AT TIME T
C          ZC(4) = DUMMY VALUE OF YDOT AT TIME T+H,COMPUTED
C                    FROM Z(4) AT TIME T
C          J1    = DUMMY VALUE OF X    IN SUBROUTINE FORCE
C          J2    = DUMMY VALUE OF XDOT IN SUBROUTINE FORCE
C          J3    = DUMMY VALUE OF Y    USED AS ARGUMENT IN
C                    SUBROUTINE FORCE
C          J4    = DUMMY VALUE OF YDOT USED AS ARGUMENT IN
C                    SUBROUTINE FORCE
C          YY    = ABSOLUTE VALUE OF Y USED IN
C                    SUBROUTINE FORCE
C          YYY   = ABSOLUTE VALUE OF Y USED  IN IMPACT TEST
C          YP    = DUMMY VALUE OF Y AT TIME T USED IN
C                    SUBROUTINE FORCE  = ZO(3)
C          YYDOT = Y*YDOT,USED IN SUBROUTINE FORCE
C          SPEED = Y*YDOT  ,USED TO DETERMINE IMPACT
C          ZC1   = ZC(1) = DUMMY VALUE OF X USED IN
C                    HALF STEP TEST
C          ZP1   = Z (1) = DUMMY VALUE OF X USED IN
C                    HALF STEP TEST
C          ZC3   = ZC(3) = DUMMY VALUE OF Y USED IN
C                    HALF STEP TEST
C          ZP3   = Z (3) = DUMMY VALUE OF Y USED IN
C                    HALF STEP TEST
C          PP    = DUMMY VALUE OF P2
C          K(I,J)= CORRECTOR TERMS IN RUNGE KUTTA-MERSON
C                    FORMULA
C                  I=1,2,3,4,5 = CORRESPONDING TO 5
C                    CORRECTOR TERMS
C                  J= 1,2,3,4  = CORRESPONDING TO X,XDOT,
C                    Y,YDOT
C
C
C                    UNITS
C
C          X,Y,YA,YC,D,...        = INCH
C          P1,P2,PC,PA,...        = P.S.I
C          W1,W2,FO,...           = LB
C          T                      = SECOND
C          FREQUENCY              = C.P.S
C          PFORCE,DFORCE          = LBS
C          AREA                   = SQUARE INCH
C
```

```
C
      DOUBLE PRECISION T,TT,TF
      DOUBLE PRECISION DSIN,DABS
      DOUBLE PRECISION J1,J2,J3,J4
      DOUBLE PRECISION ZP1,ZC1,ZP3,ZC3,ZC4
      DOUBLE PRECISION Z(4),ZC(4),ZO    ,K(5,4)
      DOUBLE PRECISION P1,P2,PC,PA,GAMMA1,GAMMA2
      DOUBLE PRECISION PFORCE,AFORCE,AO,FO,DFORCE
      DOUBLE PRECISION XMASS,YMASS,MU,KK,CREST,W1,W2
      DOUBLE PRECISION OMEGA,OMEGAF,FREQ,FREQF,ALFA,ZETA
      DOUBLE PRECISION YY,YYY,YYDOT,SPEED,G2,G4,YX,YYCDOT
      DOUBLE PRECISION D,DHALF,SO,S2,VO,AREA,C,DIAM
      DOUBLE PRECISION H,EPS,EEE,EEF,U,V,UU,VV,EEG,EEO
      DOUBLE PRECISION PHI
      INTEGER FLAG,VALVE
      COMMON PFORCE,GAMMA1,GAMMA2,H,EEE,EEG,EEO
      COMMON P2,PC,PA,DHALF,AREA,SO,ZO(4)
      COMMON FLAG,VALVE
C


C          * * * * * * * * * * * * * * * * * * * *
C
C     INITIALIZATION
C     INPUT & OUTPUT DATA & PARAMETERS
C
C          * * * * * * * * * * * * * * * * * * * *
C
C
C        INPUT & OUTPUT PROBLEM NUMBER
C
  111 READ (1,1) JJJ
      WRITE(3,2) JJJ
      IF(JJJ.EQ.0) GO TO 1000
C
C        INITIAL VALUES OF X,XDOT,Y,YDOT
C
      ZO(1)      = 0.0D00
      ZO(2)      = 0.0D00
      ZO(3)      = 0.0D00
      ZO(4)      = 0.0D00
      FLAG       = 0
      VALVE      = -1
      IMPACT     = 0
C
C        BOUNDARY VALUES OF TIME - TIME INCREMENT
C
      T          = 0.0D00
      TF         = 0.1D+03
      H          = 0.1D-02
```

```
C
C        ERROR CRITERIA
C
      EPS        = 0.1D-04
      EFF        = 0.1D-04
      EEE        = 0.1D-04
      EEG        = 0.1D-04
      EEU        = 0.1D-04
C
C
C
C        INPUT PARAMETERS
C
      READ(1,1001) PC,D,DIAM
      READ(1,1001) GAMMA1,GAMMA2,W1,W2
      READ(1,1001) FREQ,FREQF,ZETA,FO
      READ (1,7777) PHI
      PA         = 0.1469D+02
      P1         = PA
      XMASS      = W1/0.386D+03
      YMASS      = W2/0.386D+03
      MU         = YMASS/XMASS
      CREST      = 0.8D00
      KK         = MU*(0.1D+01+CREST)/(0.1D+01+MU)
      DHALF      = D/0.2D+01
      AREA       = 0.314159D+01*DIAM*DIAM/0.4D+01
      VO         = (DHALF*AREA)/0.1D+04
      SO         = VO/AREA
      C          = DHALF*(0.1D+01+EEF)
      AO         = FO/XMASS

      OMEGA      = 0.2D+01*0.314159D+01*FREQ
      OMEGAF     = 0.2D+01*0.314159D+01*FREQF
C
C        OUTPUT PARAMETERS
C
      WRITE (3,2000)
      WRITE(3,2001) W1,GAMMA1,TF
      WRITE(3,2002) W2,GAMMA2,H
      WRITE(3,2003) XMASS,PC,EPS
      WRITE(3,2004) YMASS,ZETA,EEE
      WRITE(3,2005) MU,FC,EFF
      WRITE(3,2006) CREST,AO
      WRITE(3,2007) KK,OMEGA
      WRITE(3,2008) AREA,OMEGAF
      WRITE(3,2009) D,FREQ
      WRITE(3,2010) DIAM,FREQF
      WRITE(3,2012) SO,VO
      WRITE (3,8888) PHI
      WRITE (3,2100)
```

```
C
C         FORMAT STATEMENTS
C
    1 FORMAT (I5)
    2 FORMAT (1H1,10X,'PROBLEM',I5//)
 1001 FORMAT (4F20.10)
 2000 FORMAT ( 30X, 'DATA & PARAMETERS '//)
 2001 FORMAT (10X,'W1     =',F15.10,'  LBS',13X,'GAMMA1=',
     1 F15.10,18X,'TF     =',F15.10,'  SECOND'/)
 2002 FORMAT (10X,'W2     =',F15.10,'  LBS',13X,'GAMMA2=',
     1 F15.10,18X,'H      =',F15.10,'  SECOND'/)
 2003 FORMAT (10X,'XMASS =',F15.10,'       ',12X,'PC     =',
     1 F15.10,'  P.S.I.',10X,'EPS    =',F15.10/)
 2004 FORMAT (10X,'YMASS =',F15.10,'       ',12X,'ZETA  =',
     1 F15.10,18X,'EEE    =',F15.10/)
 2005 FORMAT (10X,'MU     =',F15.10,18X,'FO     =',F15.10,
     1 '  LBS',13X,'EEF    =',F15.10/)
 2006 FORMAT (10X,'CREST =',F15.10,18X,'AO     =',F15.10,
     1'  IN/SEC/SEC'/)
 2007 FORMAT (10X,'KK     =',F15.10,18X,'OMEGA =',F15.10,
     1 '  RAD/SEC'/)
 2008 FORMAT (10X,'AREA  =',F15.10,'SQ.INCH',10X,
     1 'OMEGAF=',F15.10,'  RAD/SEC'/)
 2009 FORMAT (10X,'D      =',F15.10,'  INCH',12X,'FREQ  =',
     1 F15.10,'  C.P.S.'/)
 2010 FORMAT (10X,'DIAM  =',F15.10,'  INCH',12X,'FREQF =',
     1 F15.10,'  C.P.S.'/)
 2012 FORMAT (10X,'SO     =',F15.10,'  INCH',12X,'VO     =',
     1 F15.10,'  CUBIC INCH'//)
 7777 FORMAT(F20.10)
 8888 FORMAT(15X,'PHI=',F20.10//)
 2100 FORMAT (1H1,14X,'T',14X,'X',14X,'XDOT',11X,'Y',14X,
     1 'YDOT',11X,'PFORCE',09X,'DFORCE',09X,'P2'//)
C
C
C          * * * * * * * * * * * * * * * * * * * *
C
C      COMPUTATION OF X(T+H),XDOT(T+H),Y(T+H),YDOT(T+H)
C          WHEN X(T),XDOT(T),Y(T),YDOT(T) ARE KNOWN
C
```

```
  100 DO 200   I = 1,5
      DO 160 J=1,4
      GO TO(21,22,23,24,25),I
   21 Z(J) = ZO(J)
      TT = T
      GO TO 150
   22 Z(J)=ZO(J)+K(1,J)/0.3D+01
      TT=T+H/0.3D+01
      GO TO 150
   23 Z(J)=ZO(J)+K(1,J)/0.6D+01+K(2,J)/0.6D+01
      TT=T+H/0.3D+01
      GO TO 150
   24 Z(J)=ZO(J)+K(1,J)/0.8D+01+(0.3D+01/0.8D+01)*K(3,J)
      TT=T+H/0.2D+01
      GO TO 150
   25 Z(J)=ZO(J)+K(1,J)/0.2D+01-(0.3D+01/0.2D+01*K(3,J))+
     10.2D+01*K(4,J)
      TT= T+H
  150 GO TO (31,32,33,34),J
   31 J1 =Z(J)
      GO TO 160
   32 J2=Z(J)
      GO TO 160
   33 J3=Z(J)
      GO TO 160
   34 J4=Z(J)
  160 CONTINUE
      YX = DABS(J3)
      IF(YX.LE.C) GO TO 50
      YX = C
   50 YY = YX
      YYDOT=J3*J4
      ALFA = OMEGAF*TT
      AFORCE = AO*DSIN(ALFA-PHI)
C
C         IF JJJ EQUAL TO 1,SET PFORCE EQUAL TO 0.0
C
      IF (JJJ.EQ.1) GO TO 35
      CALL FORCE(J3,J4,YY,YYDOT,P1)
      GO TO 36
   35 PFORCE = 0.0D00
   36 G2=PFORCE/XMASS+AFORCE-0.2D+01*ZETA*OMEGA*J2-
     1 OMEGA*OMEGA*J1
      G4=-PFORCE/YMASS-G2
C
      K(I,1) =H*J2
      K(I,2) =H*G2
      K(I,3) =H*J4
      K(I,4) =H*G4
  200 CONTINUE
C
      DO 201 J=1,4
  201 ZC(J)=ZC(J) +K(1,J)/0.6D+01+(0.2D+01/0.3D+01)*
     1K(4,J)+K(5,J)/0.6D+01
```

```
C
C         TEST FOR HALF STEP
C



C
C         TEST FOR IMPACT
C
C
      ZC1 =ZC(1)
      ZC3 =ZC(3)
      ZC4 =ZC(4)
      ZP1 =Z(1)
      ZP3 = Z(3)
      YYY=DABS(ZC3)
      YYCDOT = ZC3*ZC4
      U=0.2D00*DABS(ZP1-ZC1)-EPS*DABS(ZC1)
      V=0.2D00*DABS(ZP3-ZC3)-EPS*DABS(ZC3)
      IF((U.LE.0.0D00).AND.(V.LE.0.0D00).AND.(FLAG.EQ.0)
     1.AND.((YYY.LE.DHALF).OR.((YYY.GT.DHALF).AND.
     2(YYY.LE.C)))) GO TO 120
   55 H=H/0.2D+01
      FLAG =0
      GO TO 100
  120 T=T+H
      FLAG=0
      IF(YYY.LE.DHALF) GO TO 210
      SPEED=ZC(3)*ZC(4)
      IF(SPEED.LE.0.0D00) GO TO 210
      WRITE(3,66)
   66 FORMAT(/10X,'IMPACT'/)
      IMPACT    = IMPACT + 1
      ZC(2) =ZC(2) +KK*ZC(4)
      ZC(4) =-CREST*ZC(4)
C
C         NEW VALUES OF X,XDOT,Y,YDOT
C
  210 DO 220 J =1,4
  220 ZO(J) =ZC(J)
C
C
C         OUTPUT
C
C
C
      DFORCE = AFORCE*XMASS
      CALL FORCE (ZC3,ZC4,YYY,YYCDOT,P1)
   78 WRITE(3,40) T,(ZC(J),J=1,4),PFORCE,DFORCE,P2
   40 FORMAT(10X,8(D12.5,3X))
      IF(IMPACT.GT.10) GO TO 1000
C
C
```

```
C
C
C         TEST FOR DOUBLE STEP
C
C
C
      UU=0.128D+02*DABS(ZP1-ZC1)-EPS*DABS(ZC1)
      VV=0.128D+02*DABS(ZP3-ZC3)-EPS*DABS(ZC3)
      IF((UU.GT.0.0D00).OR .(VV.GT.0.0D00)) GO TO 600
  141 H=0.2D+01*H
C
C         TEST FOR UPPER BOUNDARY OF T
C
  600 IF(T-TF) 100,100,999
  999 GO TO 111
 1000 STOP
      END
C
C          * * * * * * * * * * * * * * * * * * *
C
      SUBROUTINE FORCE(J3,J4,YY,YYDOT,P1)
C
C
C          * * * * * * * * * * * * * * * * * * *
C
      DOUBLE PRECISION YP,ZO
      DOUBLE PRECISION P1,P2,PC,PA,PP
      DOUBLE PRECISION PFORCE,GAMMA1,GAMMA2
      DOUBLE PRECISION H,EEE,TEST,SIGN,DABS,EEG,EEO
      DOUBLE PRECISION YC,YY,YYDOT,DHALF,SO,S2,J3,J4,
     1AREA,AYDOT
      INTEGER FLAG,VALVE
      COMMON PFORCE,GAMMA1,GAMMA2,H,EEE,EEG,EEO
      COMMON P2,PC,PA,DHALF,AREA,SO,ZO(4)
      COMMON FLAG,VALVE
C
C
      AYDOT = DABS(J4)
      YP =ZO(3)
      IF(YY.GT.0.0D00) GO TO 1
    2 PFORCE = 0.0D00
      VALVE = -1
      P2 = PA
      GO TO 999
C
C         VALVE = -1   THE VALVE IS CLOSED
C         VALVE = 0    THE VALVE IS OPEN
C         VALVE = 1    THE VALVE IS CLOSED AGAIN
C
    1 IF(VALVE)10,11,12
C
   10 PP = PA*((DHALF+SO)/((DHALF-YY) +SO))**GAMMA1
C
```

```
      IF(PP-PC) 21,21,22
   21 P2=PP
      GO TO 500
C
   22 TEST =DABS(PP-PC)
      IF(TEST-EEE*PC) 31,31,32
   32 FLAG = 1
      GO TO 33
   31 VALVE = 0
   33 P2 =PC
      GO TO 500
C
C
   11 IF(YYDOT) 81,81,82
   82 P2 =PC
      GO TO 500
C
   81 IF(YY.GE.DHALF) GO TO 85
      IF(AYDOT.LE.EEG) GO TO 85
      FLAG = 1
      GO TO 9C
   85 VALVE = 1
   90 S2 = DHALF - YY
      P2 =PC*((S2+SO)/((DHALF-YY)+SO))**GAMMA2
      GO TO 500
C
   12 PP =PC*((S2+SO)/((DHALF-YY)+SO))**GAMMA2
C
      IF(YYDOT.LE.C.D00) GO TC 92
      IF(PP.GT.PC) GO TO 91
      IF(YY.LE.EEO) GO TO 94
      IF(J3*YP) 93,93,92
   94 VALVE = -1
      GO TO 1
   93 FLAG = 1
   92 P2=PP
      GO TO 500
   91 TEST =DABS(PP-PC)
      IF(TEST -EEE*PC) 101,101,102
  102 FLAG = 1
      GO TO 103
  101 VALVE =0
  103 P2 =PC
C
C         GENERALIZE THE COMPUTATIONS TO BOTH HALVES
C                    OF THE DAMPER
C
  500 SIGN =J3/DABS(J3)
  600 PFORCE=SIGN*(P2-P1)*AREA
  999 RETURN
      END
/DATA
```

APPENDIX B

DETAILS OF THE COMPUTER PROGRAM VERIFICATION

1. Impact Test

Since the present damper will be reduced to an impact damper if the damping force $F(y,\dot{y})$ is equal to zero, the program written for the present study and listed in Appendix A must yield the same results as if it were written for impact damper, provided the damping force $F(y,\dot{y})$ is equal to zero.

A typical digital computer output for a given impact damper was shown in "Analytical and Experimental Studies of Impact Damper" by Masri [5]. The results obtained by the program written for the present study were compared with those shown in Masri's studies. The parametric values used in this comparison are:

Input Data:

ω = 1.0 rad./sec	$r = \frac{\Omega}{\omega} = 1.25$
Ω = 1.25 rad./sec	$\zeta = 0.1$
W_1 = 10. lbs.	
W_2 = 4. lbs.	$\mu = \frac{m}{M} = 0.4$
F_o = 0.0259067 lb.	$\frac{F_\Omega}{K} = 1.$
$K = \omega^2 M$ = 0.0259067 lb./in.	
$e = 0.2$	$\frac{D}{F_o/k} = 3.$
D = 3. inches	$F(y,y) = 0.0$
$EPS = EEF = 10^{-11}$	$EEE = 10^{-5}$

IMPACT TEST

Results obtained from the program written for the verification when $F(y,\dot{y}) = 0.0$

Impact # (i)	t_i	x_i	y_i	$\dot{x}_{i_+}$	$\dot{y}_{i_+}$	$\frac{X}{X_A}$ $t_{i-1} < t < t_i$
1	4.89	-1.5000	1.5000	-0.8485	-0.2582	-0.9236
2	6.52	-0.3112	-1.5000	1.0201	0.6473	-1.0514
3	8.93	0.7065	1.5000	-0.3578	-0.6163	0.7852
4	12.28	0.4483	-1.5000	0.5934	0.4771	-0.5420
5	14.34	-0.3485	1.5000	-0.5455	-0.4918	-0.4437
6	16.69	0.2041	-1.5000	0.5959	0.4971	-0.5037
7	19.18	-0.0749	1.5000	-0.6010	-0.5156	0.5522
8	21.74	0.0669	-1.5000	0.5834	0.5174	-0.5431
9	24.29	-0.1240	1.5000	-0.5682	-0.5079	0.5115
10	26.81	0.1675	-1.5000	0.5699	0.5010	-0.4962
11	29.30	-0.1623	1.5000	-0.5791	-0.5022	0.5044
12	31.80	0.1352	-1.5000	0.5827	0.5064	-0.5169
13	34.32	-0.1235	1.5000	-0.5806	-0.5082	0.5197
14	36.84	0.1304	-1.5000	0.5773	0.5071	-0.5150
15	39.36	-0.1405	1.5000	-0.5764	-0.5054	0.5105
16	41.87	0.1429	-1.5000	0.5777	0.5051	-0.5105
17	44.38	-0.1389	1.5000	-0.5789	-0.5057	0.5128
18	46.89	0.1354	-1.5000	0.5789	0.5063	-0.5141
19	49.41	-0.1353	1.5000	-0.5783	-0.5063	0.5137
20	51.92	0.1372	-1.5000	0.5780	0.5060	-0.5128
21	54.43	-0.1383	1.5000	-0.5781	-0.5058	0.5125
22	56.95	0.1379	-1.5000	0.5783	0.5059	-0.5128
23	59.46	-0.1372	1.5000	-0.5784	-0.5060	0.5131
24	61.97	0.1369	-1.5000	0.5783	0.5060	-0.5131
25	64.49	-0.1372	1.5000	-0.5782	-0.5060	0.5030
26	67.00	0.1374	-1.5000	0.5782	0.5060	-0.5029
27	69.51	-0.1375	1.5000	-0.5783	-0.5060	0.5029
28	72.03	0.1374	-1.5000	0.5783	0.5060	-0.5129
29	74.54	-0.1372	1.5000	-0.5783	-0.5060	0.5130
30	77.05	0.1372	-1.5000	0.5783	0.5060	-0.5130
31	79.57	-0.1373	1.5000	-0.5783	-0.5060	0.5130

$$X_A = \frac{F_o/K}{\sqrt{(1 - r)^2 + (2\zeta r)^2}} = 1.624$$

Differences between the results obtained by the present digital simulation and those from Masri's studies [5] are as follows:

$\Delta t_i = 0.01\%$ (at the 31rst impact)

$\Delta x_i = 0.29\%$ (at steady-state operation)

$\Delta \dot{x}_i = 0.03\%$ (at steady-state operation)

$\Delta y = 0.\%$ (at steady-state operation)

$\Delta \dot{y} = 0.03\%$ (at steady-state operation)

$\Delta(\frac{X}{X_A}) = 0.03\%$ (at steady-state operation)

At the 31rst impact of the digital computer output in Masri's study [5], the following values of t, x, y, $\dot{x}$, $\dot{y}$, have been observed:

$t_i = 79.56$

$x_i = -0.1369$

$y_i = 1.5$

$\dot{x}_{i_+} = -0.5785$

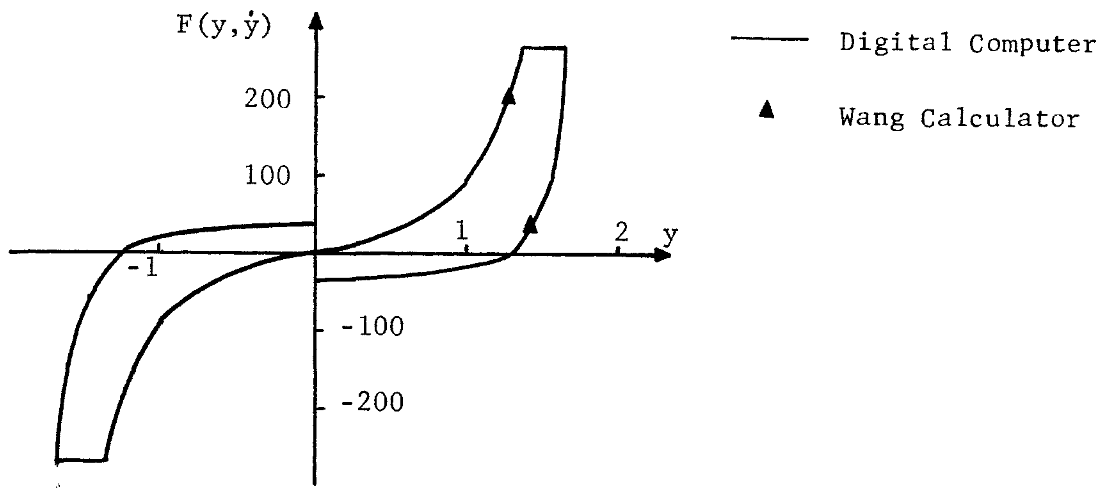
$\dot{y}_{i_+} = -0.5062$

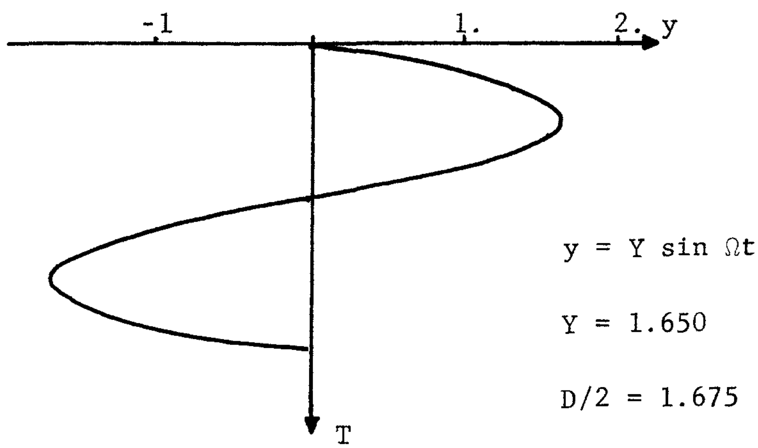
$\frac{X}{X_A} = -0.5132$

2. Damping force test

Since the damping force $F(y,\dot{y})$ is a function of y and $\dot{y}$, if the inputs y and $\dot{y}$ are known, the subroutine written for the damping force $F(y,\dot{y})$ and listed in Appendix A must yield the same results as the results calculated directly by using eq. (2.12).

Comparison of the results obtained using the subroutine FORCE with different inputs, and by using a Wang calculator, is given in Figures 27, 28, 29, and 30.

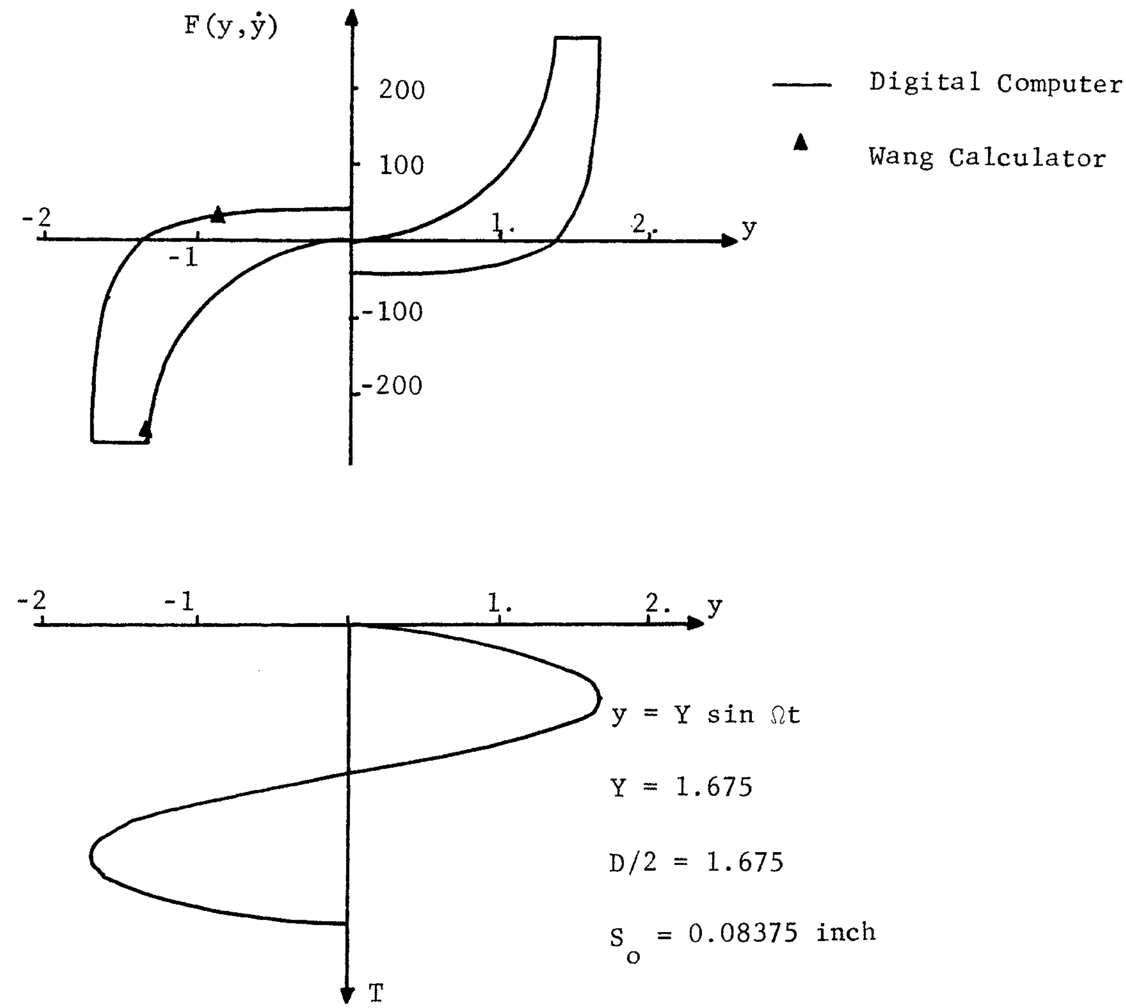




P_c = 100 psi   A = 3.14159 sq. in Ω = 126 rad/sec S_o = 0.08375

	y	P_i	$(P_2-P_1)A$
Digital Computer	1.273822	78.416	200.20
Wang Calculator	1.273822	78.41636	200.20
Digital Computer	1.4437907	25.098	32.697
Wang Calculator	1.4437907	25.09773	32.697

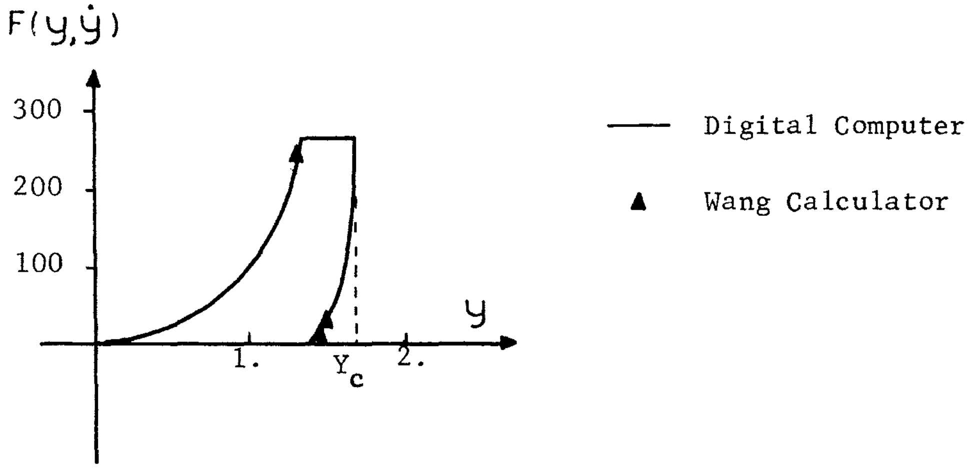
Figure 27. Verification of Subroutine FORCE

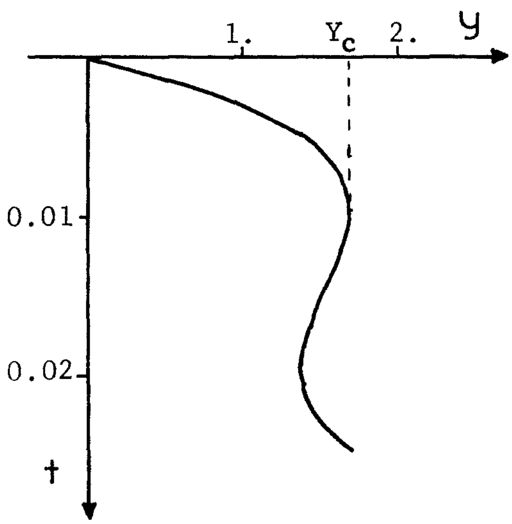


P_c = 100 psi A = 3.14159 sq. in. D/2 = 1.6750 inches

	y	P_i	$(P_i-P_j)A$
Digital Computer	-1.3556528	99.714	-267.11
Wang Calculator	"	99.7136	-267.11
Digital Computer	-0.89370176	4.8054	31.053
Wang Calculator	"	4.80538	31.053

Figure 28. Verification of Subroutine FORCE





$Y_c = 1.675$

$y = B_1 t + B_2 t^2 + B_3 t^3$

$B_1 = 402$

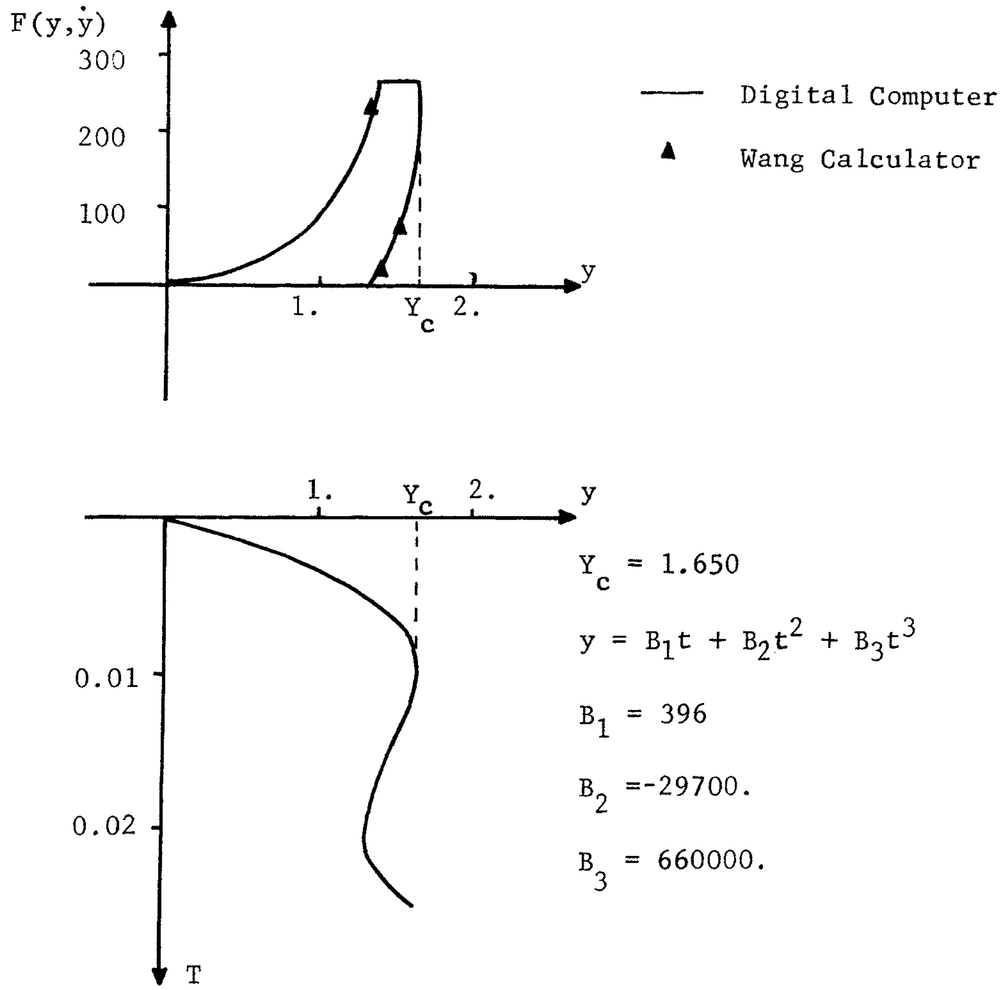
$B_2 = -30150$

$B_3 = 670000$

$P_c = 100$ psi　　$A = 3.14159$ sq. in.　　$D/2 = 1.675$ inches
$S_o = 0.08375$ inch

	y	P_i	$(P_i - P_j)A$
Digital Computer	1.340	94.896	251.97
Wang Calculator	"	94.89555	251.97
Digital Computer	1.5075	23.974	29.167
Wang Calculator	"	23.974103	29.167
Digital Computer	1.44854	18.228	78.390
Wang Calculator	"	18.227526	78.390

Figure 29. Verification of Subroutine FORCE



$Y_c = 1.650$

$y = B_1 t + B_2 t^2 + B_3 t^3$

$B_1 = 396$

$B_2 = -29700.$

$B_3 = 660000.$

$P_c = 100.$ psi A = 3.14159 sq. in. D/2 = 1.675 inches

$S_o = 0.08375$ inch

	y	P_i	$(P_i - P_j)A$
Digital computor	1.320	89.311	234.43
Wang Calculator	"	89.31098	234.43
Digital Computer	1.53384	38.882	76.001
Wang Calculator	"	38.881825	76.001
Digital Computer	1.42692	23.451	27.525
Wang Calculator	"	23.45134	27.525

Figure 30. Verification of Subroutine FORCE