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Bending of isosceles right triangular plates

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BENDING OF ISOSCELES RIGHT TRIANGULAR PLATES

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ABSTRACT

In this thesis the series expansion method is used to obtain the moments and deflections of an isosceles right triangular plate bent by arbitrary transverse loads. The method is based on using an antisymmetric load on a square plate. Two boundary conditions are solved and the numerical results obtained for these cases are tabulated.

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TABLE OF SYMBOLS

a	length of the isosceles edges of a right triangular plate
D	Plate stiffness
E	Young's Modulus
E_m	Coefficient of bending moment
h	Thickness of a plate
M_n	Bending moment per unit length of a section of a plate perpendicular to n direction
M_t	Bending moment per unit length of a section of a plate perpendicular to t direction
M_x, M_y	Bending moment per unit length of a section of a plate perpendicular to x and y axes, respectively
M_{xy}	Twisting moment per unit length of a section of a plate perpendicular to x axis
q	Intensity of lateral load
u	Poisson's ratio
w	Deflection of the plate

I. INTRODUCTION

Many of the present day problems of plates and shells with which the engineer is confronted lead to a two dimensional linear partial differential equation which must satisfy certain boundary conditions. Exact solutions are available for only a few simple mathematical shapes, e.g., circles, squares, ellipses, etc. For those shapes of particular importance such as triangles and rectangles for which exact solutions cannot be obtained, approximate solutions must be employed. One of the approximate methods which is most useful in solving plate problems is Fourier series expansion. Timoshenko used the Fourier series expansion method to get the numerical value of almost all the plate problems throughout his book on the theory of plates and shells.* (1)

A flat plate is a basic structural element of modern engineering structures. It may be thought of as a two-dimensional equivalent of the beam. The flat plate, in general, resists loads applied either transversely or axially, and it resists these by means of direct stresses, shear stresses, bending stresses, and torsional stresses. The complete derivation of equations for flat plates can be found in S. Timoshenko's book (1). The equations are found to be linear partial differential equations up to the fourth order. Timoshenko used Fourier series in solving most of the problems.

In the following pages Fourier series expansion will be used to solve for deflections and bending moments of an isosceles right triangular plate with two kinds of boundary conditions under transverse load.

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*Numbers in parentheses refer to the bibliography

II. REVIEW OF LITERATURE

Triangular plates are used frequently in engineering works, such as the building of tanks, as bottom slabs of bunkers and silos, in large buildings, in delta and swept wings of aircraft and in the bottoms of ships. There are no general exact solutions for triangular plates of any shape which are subjected to lateral loads.

The general method of solving triangular plates consists of using approximate methods which depend on the particular case and the boundary conditions. S. Timoshenko solved the right triangular plate with simply supported edges by using the Navier equation to calculate the deflection and bending moment at any point on the plate (1). S. Woinowsky-Krieger (2), A. Nadai, (3) and B.G. Galerkin (4) solved the equilateral plate and right isosceles triangular plate which are simply supported on all sides and subject to arbitrary transverse loads. H.J. Fletcher used two methods for the solutions of problems of an isosceles right triangular plate bent by arbitrary transverse loads. One method made use of a concentrated diagonal load on a square plate. The other used an antisymmetric load on a square plate. A numerical solution was given for the case in which the hypotenuse is clamped and the legs are simply supported. A table was given indicating how to solve fifteen of the possible eighteen problems in which an edge is free, clamped or supported (5). B. D. Aggarwala had a paper concerning bending of an isotropic triangular plate subjected to concentrated loads. He considered an equilateral triangular plate with one edge clamped and with the other two edges simply supported. The plate is subjected to a concentrated load intensity P acting at the center of the triangular plate (6). Other information concerning triangular plates can be found in H. D. Convey's point matching method, in which he analyzed uniformly loaded triangular plates with either clamped or simply supported edges using a special adaptation of the point-matching technique. The functions satisfying the differential equations must also satisfy exactly the boundary conditions on one edge. Numerical results are tabulated for those three shapes (7). Also, D. E. Ordway and C. Riparbelli presented an application of the method of equivalence to the deflection of a triangular plate. In the first part of this paper the general equations and procedure are worked out for determining the deflection of a thin delta plate under any given normal loading. In the second part the theory is developed and compared with experiments for a delta plate specimen with 45° sweep under uniform load (8).

In the following pages of this thesis, two kinds of boundary conditions for isosceles right triangular plates will be solved by trigonometric series. Numerical results obtained from this method are tabulated.

III. SMALL DEFLECTIONS OF LATERALLY LOADED PLATES

In solving plate problems, it is found that the bending properties of a plate depend greatly on its thickness as compared to its other dimensions. In this thesis, it is assumed that plates are thin and undergo small deflections. The following additional assumptions are made in deriving the differential equations for the laterally loaded thin plate.

1. The material is homogeneous, isotropic, and elastic. Moreover loading is of such magnitude as to restrict the plate to the elastic range.

2. The direction of the load on a plate is normal to the surface of the plate.

3. At the boundary it is assumed that the edges of the plate are free to move in the plane of the plate.

The complete derivation of the differential equations can be found in Timoshenko's book (1). Positive shears and moments acting upon any differential element of the plate are as shown in Figure 1. Moments are all indicated by right hand screw notations. Positive deflection is in the same direction as the loading q.

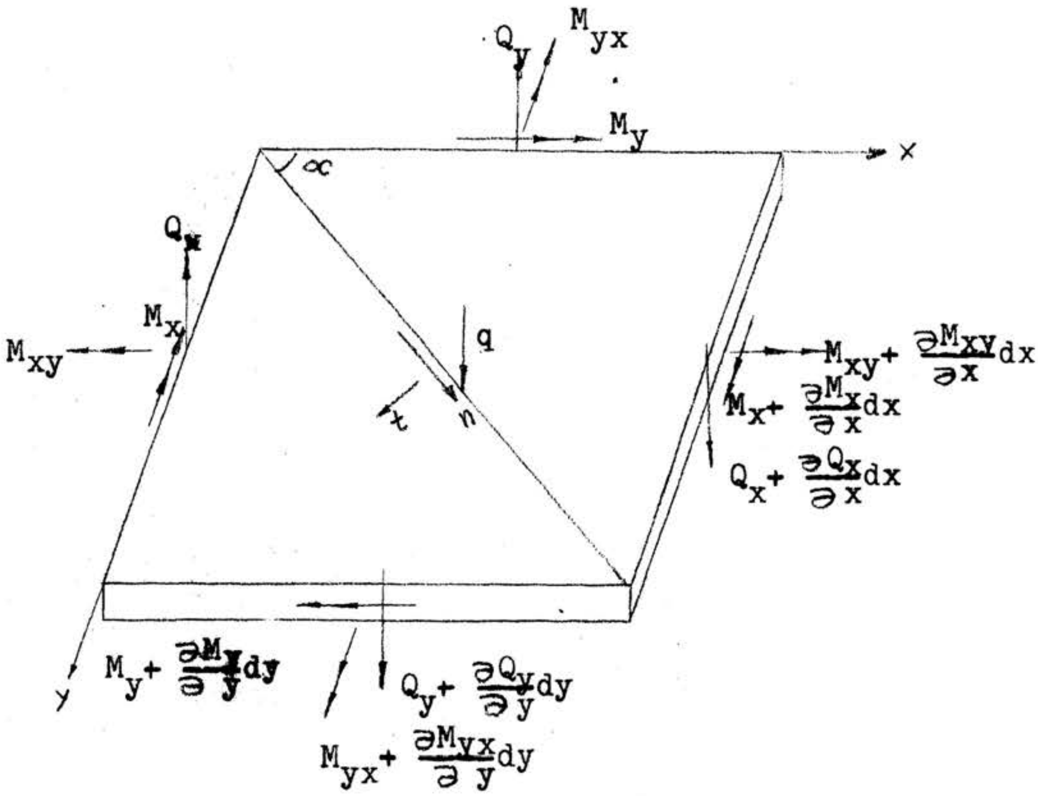

Figure 1. Element of plate with the applied load, shear and moment

Applying Hooke's law and the equations of equilibrium to the free body of the differential element leads to the following set of equations:

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{yx}}{\partial y} = Q_x \qquad (3\text{-}1)$$

$$\frac{\partial M_y}{\partial \alpha} - \frac{\partial M_{yx}}{\partial y} = Q_x \qquad (3\text{-}2)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -q \qquad (3\text{-}3)$$

$$M_{xy} = -M_{yx} = D(1-u)\,\frac{\partial^2 w}{\partial x \partial y} \qquad (3\text{-}4)$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + u\,\frac{\partial^2 w}{\partial y^2}\right) \qquad (3\text{-}5)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + u\,\frac{\partial^2 w}{\partial x^2}\right) \qquad (3\text{-}6)$$

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (3\text{-}7)$$

$$M_n = M_x \cos^2\alpha + M_y \sin^2\alpha - 2\,M_{xy} \sin\alpha\cos\alpha \qquad (3\text{-}8)$$

$$M_t = M_x \sin^2\alpha + M_y \cos^2\alpha + 2\,M_{xy} \sin\alpha\cos\alpha \qquad (3\text{-}9)$$

where q = lateral load which is a function of x and y

w = vertical deflection of any point in the plate

u = Poisson's ratio

D = plate stiffness = $\frac{Et^3}{12(1-u^2)}$

t = the thickness of the plate

E = Young's Modulus

Q_x, Q_y = vertical shears

IV. SOLUTION OF ISOSCELES RIGHT TRIANGULAR PLATES

In this thesis, isotropic isosceles right triangular thin plates of constant thickness subjected to lateral load with small deflections will be analyzed. The main purpose is to find bending moments and deflections under different boundary conditions. For plates under lateral loading the differential equation for deflection will be (1)

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q(x,y)}{D} \qquad (3\text{-}7)$$

The solution of this equation for an isosceles right triangular plate can be obtained from the solution for a square plate by loading a square plate so that the boundary conditions for the isosceles right triangular plate are satisfied.

The solution of an isosceles right triangular plate can be obtained by choosing boundary condtions of a square plate which are symmetrical with respect to the diagonal and loadings symmetrical or antisymmetrical with respect to the diagonal of a square plate. This procedure is illustrated below.

A square plate OABC, as shown in Figure 2, has boundary conditions which are symmetrical with respect to the diagonal AC. A uniformly distributed load antisymmetrical with respect to the diagonal AC is added:

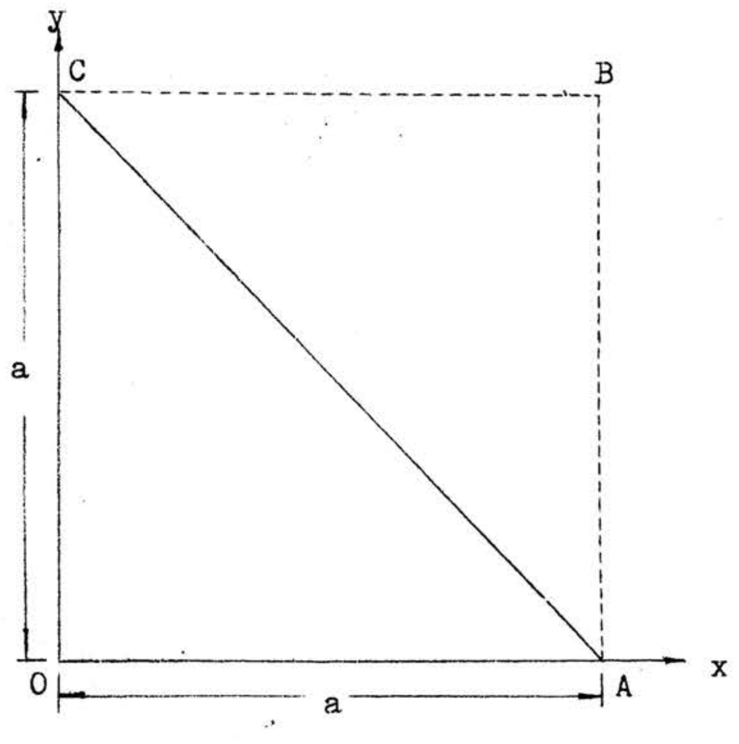

Figure 2. A square plate

$$q(x,y) = -q(a-y,a-x)$$

where $0 \leq x+y \leq a$

These two loads produce a deflection of the square plate such that the diagonal AC becomes a nodal line. Thus the portion OAC of the square plate is in exactly the same condition as a simply supported triangular plate OAC.

If, on the other hand, A square plate OABC, has boundary conditions and loadings which are symmetrical with respect to the diagonal AC, i.e.,

$$q(x,y) = q(a-y,a-x) \qquad \text{where} \quad 0 \leq x+y \leq a$$

then, the deflection of the square plate is symmetrical with respect to the diagonal AC, also, the slope along the perpendicular to the diagonal AC is zero, i.e., $(\frac{\partial W}{\partial n})_{AC}=0$. If one superimposes a concentrated load along the diagonal so that the deflection along the diagonal is also zero, i.e., $W_{AC}=0$, then the diagonal AC of the square plate OABC satisfies the conditions of a clamped edge for both triangles ABC and OAC.

In the following discussion, two sets of boundary conditions are selected as illustrations. Numerical solutions of the values of deflections and moments are presented.

CASE I ONE LEG CLAMPED, HYPOTENUSE AND ONE LEG SIMPLY SUPPORTED

The procedure for solving this case is as follows:

(a) This case may be considered to be one-half of a square plate, as indicated in Figure 3 by dashed lines. If a load p is applied at a point A with coordinates ξ, η, a fictitious load $-p$ is applied at A', which is the image of the point A with respect to the line BC. These two loads produce a deflection of the square plate such that the diagonal BC becomes a nodal line. Thus the portion OBC of the square plate is in exactly the same condition as a simply supported triangular plate OBC.

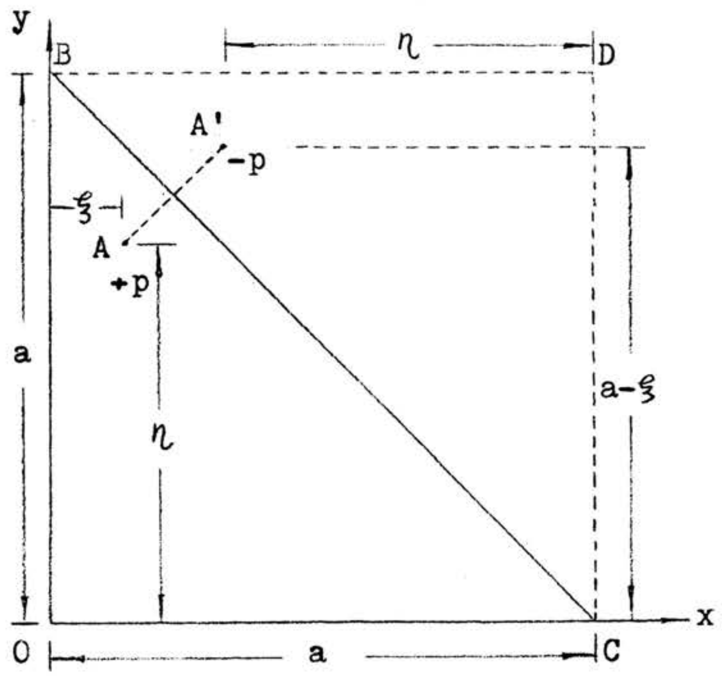

Figure 3. Isosceles right triangular plate

(b) Add an antisymmetrical bending moment with respect to the diagonal BC on the edges DB and BO, i.e.,

$$M_x(y)_{x=o} = -M_y(a-y)_{y=a}$$

such as shown in Figure 4. This antisymmetrical bending moment also produces deflections of the square plate such that the diagonal BC becomes a nodal line.

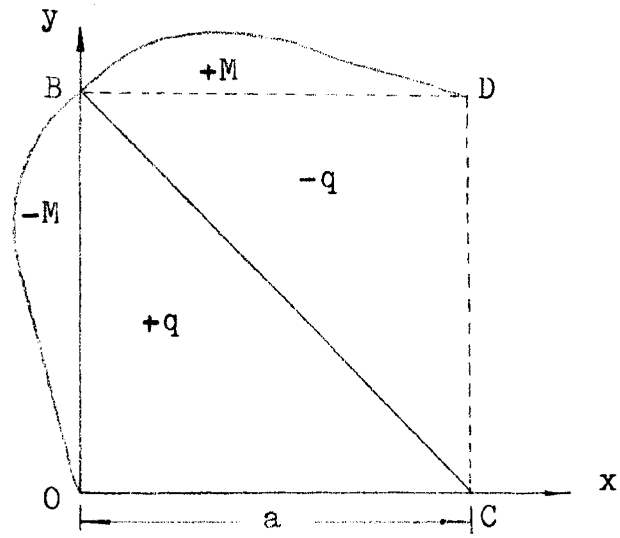

Figure 4. Antisymmetrical bending moments acting along two edges of a square plate

(c) Combine conditions (a) and (b), by using the boundary condition of the clamped leg BO, and using the formulas listed in chapter III, deflections and bending moments at any point of the triangular plate can easily be found. The following section is the numerical solution of this case.

If a load p is applied at a point A with coordinates ξ, η in a simply supported square plate, a load -p is applied at A', where point A' is symmetrical to point A with respect to the diagonal BC as in Figure 3. From Timoshenko's book (1)

$$W_O = \frac{4pa^2}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(m^2+n^2)} \left[\sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{a} - (-1)^{m+n} \sin\frac{m\pi\eta}{a} \sin\frac{n\pi\xi}{a} \right] \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a} \qquad (4\text{-}1)$$

The loading is uniformly distributed over a square plate and is antisymmetrical with respect to the diagonal BC, i.e.,

$$q(\xi,\eta) = -q(a-\eta, a-\xi)$$
$$\text{where} \quad 0 \leq \xi+\eta \leq a \tag{4-2}$$

Substituting this uniform load $q\,d\xi\,d\eta$ for p and integrating equation (4-1) over the area of the triangle BCO,

$$W_1 = \frac{4a^2}{\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_{mn}}{(m^2+n^2)^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} \tag{4-3}$$

where

$$A_{mn} = \int_0^a\int_0^{a-\eta} q(\xi,\eta)\left[\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{a} - (-1)^{m+n}\sin\frac{m\pi\eta}{a}\sin\frac{n\pi\xi}{a}\right]d\xi\,d\eta \tag{4-4}$$

by adding a bending moment along the edge BD

$$M_y(x) = \sum_{m=1}^{\infty} E_m \sin\frac{m\pi x}{a}$$

where E_m is a constant which depends on each particular case. From Timoshenko's book (1) the deflection of the plate will be:

$$W_2 = \frac{a^2}{4\pi^2 D}\sum_{m=1}^{\infty}\frac{E_m}{m^2}\left\{\frac{1}{\cosh\frac{m\pi}{2}}\left[\frac{m\pi}{2}\tanh\frac{m\pi}{2}\cosh m\pi\left(\frac{y}{a}-\frac{1}{2}\right) - m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\sinh m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\right] + \frac{1}{\sinh\frac{m\pi}{2}}\left[\frac{m\pi}{2}\coth\frac{m\pi}{2}\cdot\sinh m\pi\left(\frac{y}{a}-\frac{1}{2}\right) - m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\cosh m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\right]\right\}\sin\frac{m\pi x}{a} \tag{4-5}$$

In the same manner, by adding a bending moment along the edge BO, i.e.,

$$M_x(y) = -M_y(a-y)_{y=a}$$

$$= -\sum_{m=1}^{\infty} E_m \sin m\pi(1-\frac{y}{a}) \qquad (4\text{-}6)$$

the deflection will be:

$$W_3 = \frac{a^2}{4\pi^2 D}\sum_{m=1}^{\infty}\frac{E_m}{m^2}\Big\{\frac{1}{\sinh\frac{m\pi}{2}}\Big[\frac{m\pi}{2}\coth\frac{m\pi}{2}.\sinh m\pi(\frac{x}{a}-\frac{1}{2})$$

$$- m\pi(\frac{x}{a}-\frac{1}{2})\cosh m\pi(\frac{x}{a}-\frac{1}{2})\Big] - \frac{1}{\cosh\frac{m\pi}{2}}.$$

$$\Big[\frac{m\pi}{2}\tanh\frac{m\pi}{2}\cosh m\pi(\frac{x}{a}-\frac{1}{2}) - m\pi(\frac{x}{a}-\frac{1}{2}).\sinh m\pi(\frac{x}{a}-\frac{1}{2})\Big]\Big\}$$

$$.\sin m\pi(1-\frac{y}{a}) \qquad (4\text{-}7)$$

The complete deflection of the triangular plate is obtained by summing up equations (4-3), (4-5) and (4-7), i.e.,

$$W = W_1 + W_2 + W_3 \qquad (4\text{-}8)$$

to satisfy the boundary conditions of the clamped edge BO, the following conditions must be satisfied:

$$(\frac{\partial W}{\partial x})_{x=0} = (\frac{\partial W_1}{\partial x})_{x=0} + (\frac{\partial W_2}{\partial x})_{x=0} + (\frac{\partial W_3}{\partial x})_{x=0} \equiv 0 \qquad (4\text{-}9)$$

where $0 \leq y \leq a$

From equation (4-3), the slope of the deflection along the edge $x=0$ produced by a uniformly distributed load is:

$$(\frac{\partial W_1}{\partial x})_{x=0} = \frac{4a}{\pi^3 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mA_{mn}}{(m^2+n^2)^2}\sin\frac{n\pi y}{a} \qquad (4\text{-}10)$$

From equation (4-5), the moment $M_y(X) = \sum_{m=1}^{\infty} E_m \sin\left(\frac{m\pi x}{a}\right)$ distributed along the edge x=o produces a slope

$$\left(\frac{\partial w_2}{\partial x}\right)_{x=o} = \frac{a}{4\pi D}\sum_{m=1}^{\infty}\frac{E_m}{m}\left\{\frac{1}{\cosh\frac{m\pi}{2}}\left[\frac{m\pi}{2}\tanh\frac{m\pi}{2}\cdot\cosh m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\right.\right.$$

$$\left. - m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\sinh m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\right] + \frac{1}{\sinh\frac{m\pi}{2}}\cdot$$

$$\left[\frac{m\pi}{2}\coth\frac{m\pi}{2}\sinh m\pi\left(\frac{y}{a}-\frac{1}{2}\right) - m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\cdot\right.$$

$$\left.\left.\cosh m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\right]\right\} \qquad (4\text{-}11)$$

by using the following series, equation (4-11) can be much simplified as below.

$$\cosh m\pi\left(\frac{y}{a}-\frac{1}{2}\right) = \frac{4}{\pi}\sum_{n=1,3\ldots}^{\infty}\frac{n}{(m^2+n^2)}\cosh\frac{m\pi}{2}\sin\frac{n\pi y}{a}$$

$$\sinh m\pi\left(\frac{y}{a}-\frac{1}{2}\right) = -\frac{4}{\pi}\sum_{n=2,4\ldots}^{\infty}\frac{n}{(m^2+n^2)}\sinh\frac{m\pi}{2}\sin\frac{n\pi y}{a}$$

$$m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\cosh m\pi\left(\frac{y}{a}-\frac{1}{2}\right) = \frac{4}{\pi}\sum_{n=2,4\ldots}^{\infty}\frac{n}{m^2+n^2}\cdot$$

$$\left(\frac{2m^2}{m^2+n^2}\sinh\frac{m\pi}{2} - \frac{m\pi}{2}\cosh\frac{m\pi}{2}\right)\sin\frac{n\pi y}{a}$$

$$m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\sinh m\pi\left(\frac{y}{a}-\frac{1}{2}\right) = -\frac{4}{\pi}\sum_{n=1,3\ldots}^{\infty}\frac{n}{(m^2+n^2)}\cdot$$

$$\left(\frac{2m^2}{m^2+n^2}\cosh\frac{m\pi}{2} - \frac{m\pi}{2}\sinh\frac{m\pi}{2}\right)\sin\frac{n\pi y}{a} \qquad (4\text{-}12)$$

where $0 < y < a$

$$\left(\frac{\partial w_2}{\partial x}\right)_{x=o} = \frac{2a}{\pi^2 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}mnE_m}{(m^2+n^2)^2}\sin\frac{n\pi y}{a} \qquad (4\text{-}13)$$

From equation (4-7) as before, the slope would be:

$$\left(\frac{\partial W_3}{\partial x}\right)_{x=0} = \frac{a}{8D}\sum_{m=1}^{\infty} E_m\left\{\left[\coth\frac{m\pi}{2} - \frac{2}{m\pi}\right]\coth\frac{m\pi}{2} + \left[\tanh\frac{m\pi}{2} - \frac{2}{m\pi}\right]\tanh\frac{m\pi}{2} - 2\right\}\sin m\pi\left(1-\frac{y}{a}\right) \qquad (4\text{-}14)$$

Now, by substituting equations (4-10), (4-13), (4-14) into (4-9), it becomes:

$$\frac{4}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mA_{mn}}{(m^2+n^2)^2}\sin\frac{n\pi y}{a} + \frac{2}{\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}mnE_m}{(m^2+n^2)^2}\sin\frac{n\pi y}{a}$$

$$+ \frac{\pi}{8}\sum_{n=1}^{\infty}(-1)^{n+1}E_n\left[\left(\coth\frac{n\pi}{2} - \frac{2}{n\pi}\right)\coth\frac{n\pi}{2} + \left(\tanh\frac{n\pi}{2} - \frac{2}{n\pi}\right)\right.$$

$$\left.\tanh\frac{n\pi}{2} - 2\right]\sin\frac{n\pi y}{a} = 0 \qquad (4\text{-}15)$$

if equation (4-15) establishes, the constants of all the terms $\sin\frac{n\pi y}{a}$ should be zero, then equation (4-15) becomes:

$$\frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{mA_{mn}}{(m^2+n^2)^2} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{n+1}mnE_m}{(m^2+n^2)^2} + \frac{(-1)^{n+1}}{8}\left[\left(\coth\frac{n\pi}{2} - \frac{2}{n\pi}\right)\right.$$

$$\left.\coth\frac{n\pi}{2} + \left(\tanh\frac{n\pi}{2} - \frac{2}{n\pi}\right)\tanh\frac{n\pi}{2} - 2\right]E_n = 0 \qquad (4\text{-}16)$$

From equations (4-3) and (4-4), if the loading is uniformly distributed over the triangular plate

$$A_{mn} = \frac{4\,qa^2 n}{m(n^2-m^2)\pi^2} \qquad \text{when } m=1,3,5.....;\ n=2,4,6.....$$

$$A_{mn} = \frac{4\,qa^2 m}{n(m^2-n^2)\pi^2} \qquad \text{when } m=2,4,6.....;\ n=1,3,5.....$$

When the values of A_{mn} are substituted into equation (4-3), it becomes:

$$W_1 = \frac{16qa^4}{\pi^6 D}\Big[\sum_{m=1,3,5\cdots}^{\infty} \sum_{n=2,4,6\cdots}^{\infty} \frac{n \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}}{m(n^2-m^2)(m^2+n^2)^2} + \sum_{m=2,4,6\cdots}^{\infty} \sum_{n=1,3,5\cdots}^{\infty} \frac{m \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}}{n(m^2-n^2)(m^2+n^2)^2} \qquad (4\text{-}18)$$

Now, by substituting equation into (4-16), and assuming

$K = \frac{64qa^2}{\pi^4}$, the following equation is obtained:

$$\frac{\pi}{2}\Big[\Big(\coth\frac{n\pi}{2} - \frac{2}{n\pi}\Big)\coth\frac{n\pi}{2} + \Big(\tanh\frac{n\pi}{2} - \frac{2}{n\pi}\Big)\tanh\frac{n\pi}{2} - 2\Big]E_n + \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{mnE_m}{(m^2+n^2)^2} + \frac{K}{n}\sum_{m=2,4,6\cdots}^{\infty}\frac{m^2}{(m^2-n^2)(m^2+n^2)^2} = 0$$

where $n = 1,3,5.........$ (4-19)

$$\frac{\pi}{2}\Big[\Big(\coth\frac{n\pi}{2} - \frac{2}{n\pi}\Big)\coth\frac{n\pi}{2} + \Big(\tanh\frac{n\pi}{2} - \frac{2}{n\pi}\Big)\tanh\frac{n\pi}{2} - 2\Big]E_n + \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{mnE_m}{(m^2+n^2)^2} - nK\sum_{m=1,3,5\cdots}^{\infty}\frac{1}{(n^2-m^2)(m^2+n^2)^2} = 0$$

where $n = 2,4,6.......$ (4-20)

Substituting the numerical value of the coefficients in equations (4-19) and (4-20), and by considering only the first ten coefficients, $m=1$ through $m=10$, the following system of ten equations with ten unknowns E_1 through E_{10} is obtained.

$$4.1588E_1-0.6400E_2-0.2400E_3-0.1107E_4-0.0592E_5-0.0351E_6$$
$$-0.0244E_7-0.0151E_8-0.0107E_9-0.0078E_{10}=0.182960K$$

$$-0.6400E_1+2.6413E_2-0.2840E_3-0.1600E_4-0.0951E_5-0.0600E_6$$
$$-0.0399E_7-0.0277E_8-0.0199E_9-0.0148E_{10}= -0.075911K$$

$$-0.2400E_1-0.2840E_2+1.8722E_3-0.1536E_4-0.1038E_5-0.0711E_6$$
$$-0.0499E_7-0.0360E_8-0.0267E_9-0.0202E_{10}=0.000012K$$

$$-0.1107E_1-0.1600E_2-0.1536E_3+1.4458E_4-0.0952E_5-0.0710E_6$$
$$-0.0530E_7-0.0400E_8-0.0306E_9-0.0238E_{10}= -0.004819K$$

$$-0.0592E_1-0.0951E_2-0.1038E_3-0.0952E_4+1.1766E_5-0.0645E_6$$
$$-0.0511E_7-0.0404E_8-0.0320E_9-0.0256E_{10}=0$$

$$-0.0351E_1-0.0600E_2-0.0711E_3-0.0710E_4-0.0645E_5+0.9916E_6$$
$$-0.0465E_7-0.0384E_8-0.0316E_9-0.0260E_{10}= -0.000952K$$

$$-0.0244E_1-0.0399E_2-0.0499E_3-0.0530E_4-0.0511E_5-0.0465E_6$$
$$+0.8568E_7-0.0351E_8-0.0298E_9-0.0252E_{10}=0$$

$$-0.0151E_1-0.0277E_2-0.0360E_3-0.0400E_4-0.0404E_5-0.0384E_6$$
$$-0.0351E_7+0.7541E_8-0.0274E_9-0.0238E_{10}= -0.000303K$$

$$-0.0107E_1-0.0199E_2-0.0267E_3-0.0306E_4-0.0320E_5-0.0316E_6$$
$$-0.0298E_7-0.0274E_8+0.6734E_9-0.0220E_{10}=0$$

$$-0.0078E_1-0.0148E_2-0.0202E_3-0.0238E_4-0.0256E_5-0.0260E_6$$
$$-0.0252E_7-0.0238E_8-0.0220E_9-0.6083E_{10}= -0.000123K$$
$$(4\text{-}21)$$

Solving the above ten simultaneous equations, the values of E_m can be obtained and are indicated in Table I.

Table I

Values of the coefficients of moments at clamped edge

E_1	E_2	E_3	E_4	E_5
0.041213K	-0.018617K	0.002315K	-0.001980K	0.000583K
E_6	E_7	E_8	E_9	E_{10}
-0.000565K	0.000219K	-0.000242K	0.000104K	-0.000124K

Substituting the calculated values of the coefficients E_1 through E_{10} into equation (4-6), the bending moments along the clamped edge of the triangular plate will be obtained and are indicated in Table II.

Table II

Bending moments of triangular plate along clamped edge

$\frac{y}{a}$	$M_x(qa^2)$	$\frac{y}{a}$	$M_x(qa^2)$	$\frac{y}{a}$	$M_x(qa^2)$
0	0	0.35	-0.032514	0.70	-0.011292
0.05	-0.010470	0.40	-0.031090	0.75	-0.007956
0.10	-0.018992	0.45	-0.028851	0.80	-0.005157
0.15	-0.025055	0.50	-0.025864	0.85	-0.002910
0.20	-0.029214	0.55	-0.022360	0.90	-0.001245
0.25	-0.031840	0.60	-0.018666	0.95	-0.000315
0.30	-0.032895	0.65	-0.014944	1.00	0

The deflections and bending moments on the perpendicular bisector of the hypotenuse of the isoceles right triangular plate simply supported on all edges under constant load q can be calculated by equation (4-18). These values were also obtained by H.J. Flectcher (5) and are tabulated in Table III.

Now, by substituting all the known values from Table I into equations (4-5) and (4-7), and combining the results with equation (4-18), the deflection on the perpendicular bisector of the hypotenuse of the isosceles right triangular plate with one leg clamped, the hypotenuse and the other leg simply supported will be obtained. Values of moments are also tabulated in Table IV.

Table III

Deflections and moments of right triangular plate simply supported on all edges under constant load q (μ=0.3)

$\frac{x}{a} = \frac{y}{a}$	$W_1(qa^4/D)$	$W_{1xx}(qa^2/D)$	$W_{1xy}(qa^2/D)$	$M_n(qa^2)$	$M_t(qa^2)$
0.00	0	0	0.01914	-0.01340	0.01340
0.05	0.000046	-0.00189	0.01728	-0.00965	0.01455
0.10	0.000167	-0.00514	0.01311	-0.00249	0.01586
0.15	0.000327	-0.00864	0.00848	0.00530	0.01717
0.20	0.000486	-0.01175	0.00383	0.01260	0.01796
0.25	0.000607	-0.01385	-0.00019	0.01813	0.01788
0.30	0.000658	-0.01478	-0.00339	0.02160	0.01684
0.35	0.000619	-0.01404	-0.00522	0.02190	0.01460
0.40	0.000486	-0.01152	-0.00550	0.01882	0.01112
0.45	0.000269	-0.00692	-0.00387	0.01170	0.00629
0.50	0	0	0	0	0

Table IV

Deflections and moments of right triangular plate with one leg clamped, hypotenuse and the other leg simply supported. ($\mu = 0.3$)

$\frac{x}{a} = \frac{y}{a}$	$W(qa^4/D)$	$W_{xx}(qa^2/D)$	$W_{yy}(qa^2/D)$	$W_{xy}(qa^2/D)$	$M_n(qa^2)$	$M_t(qa^2)$
0.00	0	0	0	0	0	0
0.05	0.000010	0.00452	-0.00077	0.00724	-0.00747	0.00249
0.10	0.000061	0.00286	-0.00243	0.00822	-0.00603	0.00557
0.15	0.000151	-0.00103	-0.00465	0.00678	-0.00106	0.00844
0.20	0.000261	-0.00544	-0.00692	0.00392	0.00527	0.01076
0.25	0.000360	-0.00908	-0.00865	0.00076	0.01100	0.01207
0.30	0.000419	-0.01165	-0.00961	-0.00216	0.01535	0.01231
0.35	0.000414	-0.01248	-0.00924	-0.00408	0.01697	0.01126
0.40	0.000337	-0.01140	-0.00735	-0.00465	0.01543	0.00892
0.45	0.000191	-0.00809	-0.00359	-0.00342	0.00998	0.00520
0.50	0	-0.00233	0.00233	0	0	0

CASE II ISOSCELES RIGHT TRIANGULAR PLATE WITH CLAMPED LEGS AND SIMPLY SUPPORTED HYPOTENUSE

Procedures for solving this boundary condition are the same as the procedure for isosceles right triangular plate with one leg clamped, hypotenuse and the other leg simply supported except an extra pair of antisymmetrical bending moments in edges CD and CO with respected to the diagonal CB are necessary as shown in Figure 5.

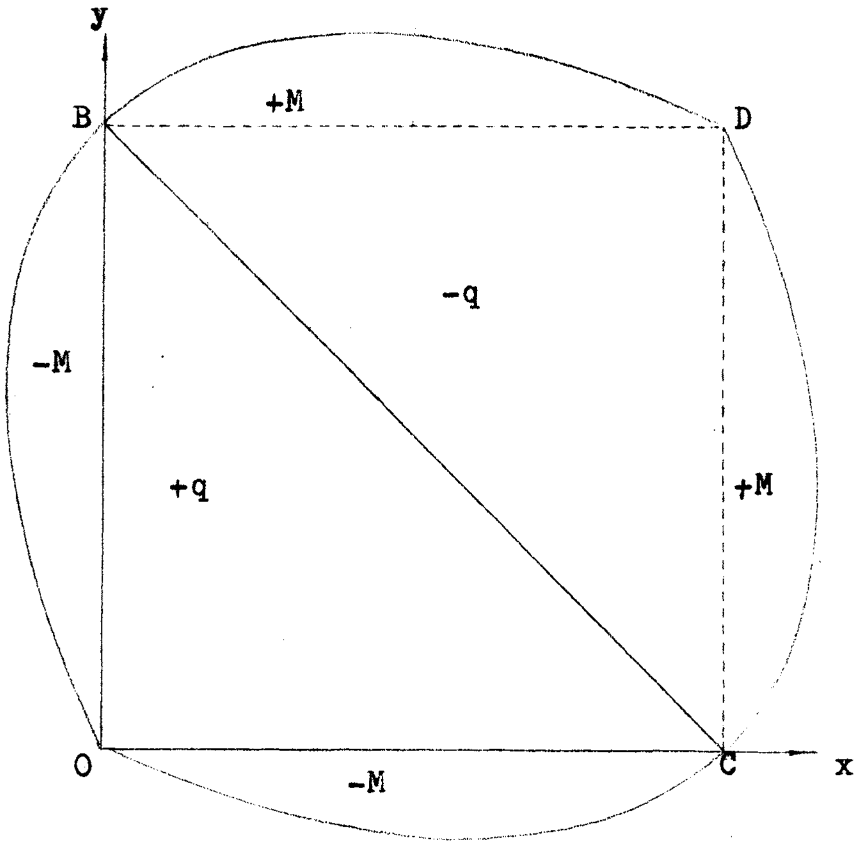

Figure 5. A pair of antisymmetrical bending moments acting along the edges of a square plate.

The moment along edge CD can be represented, as before, by the trigonometric series

$$M_x(y) = \sum_{m=1}^{\infty} E_m \sin\frac{m\pi y}{a}$$

the deflection of the plate caused by this moment will be:

$$W_4 = \frac{a^2}{4\pi^2 D}\sum_{m=1}^{\infty}\frac{E_m}{m^2}\left\{\frac{1}{\cosh\frac{m\pi}{2}}\left[\frac{m\pi}{2}\tanh\frac{m\pi}{2}\cosh m\pi\left(\frac{x}{a}-\frac{1}{2}\right) - m\pi\left(\frac{x}{a}-\frac{1}{2}\right)\cdot\right.\right.$$

$$\left.\sinh m\pi\left(\frac{x}{a}-\frac{1}{2}\right)\right] + \frac{1}{\sinh\frac{m\pi}{2}}\left[\frac{m\pi}{2}\coth\frac{m\pi}{2}\cdot\sinh m\pi\left(\frac{x}{a}-\frac{1}{2}\right) -\right.$$

$$\left.\left. m\pi\left(\frac{x}{a}-\frac{1}{2}\right)\cdot\cosh m\pi\left(\frac{x}{a}-\frac{1}{2}\right)\right]\right\}\sin\frac{m\pi y}{a} \qquad (4\text{-}22)$$

In the same manner, the bending moment along edge OC is

$$M_y(x) = -\sum_{m=1}^{\infty} E_m \sin m\pi\left(1-\frac{x}{a}\right) \qquad (4\text{-}23)$$

The deflection of the plate will be:

$$W_5 = \frac{a^2}{4\pi^2 D}\sum_{m=1}^{\infty}\frac{E_m}{m^2}\left\{\frac{1}{\sinh\frac{m\pi}{2}}\left[\frac{m\pi}{2}\coth\frac{m\pi}{2}\sinh m\pi\left(\frac{y}{a}-\frac{1}{2}\right) - m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\cdot\right.\right.$$

$$\left.\cosh m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\right] - \frac{1}{\cosh\frac{m\pi}{2}}\left[\frac{m\pi}{2}\tanh\frac{m\pi}{2}\cosh m\pi\left(\frac{y}{a}-\frac{1}{2}\right) -\right.$$

$$\left.\left. m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\sinh m\pi\left(\frac{y}{a}-\frac{1}{2}\right)\right]\right\}\sin m\pi\left(1-\frac{x}{a}\right) \qquad (4\text{-}24)$$

Now, the complete deflection of the triangular plate is obtained by summing up expressions (4-3), (4-5), (4-7), (4-22) and (4-24), i.e.,

$$W' = W_1 + W_2 + W_3 + W_4 + W_5 \qquad (4\text{-}25)$$

To satisfy the boundary conditions of the clamped edges BO and CO, the following conditions must be satisfied.

$$\left(\frac{\partial w'}{\partial x}\right)_{x=0} = 0$$

$$\left(\frac{\partial w'}{\partial y}\right)_{y=0} = 0 \qquad (4\text{-}26)$$

Equation (4-26) can be expressed an follows:

$$\frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{mA_{mn}}{(m^2+n^2)^2} + \frac{4}{\pi}\sum_{m=2,4,\cdots}^{\infty}\frac{mnE_m}{(m^2+n^2)^2} + \frac{\pi}{4}\left[\left(\coth\frac{n\pi}{2} - \frac{2}{n\pi}\right)\cdot \coth\frac{n\pi}{2} - 1\right]E_n = 0$$

where $n=1,3,5\ldots\ldots$ (4-27)

$$\frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{mA_{mn}}{(m^2+n^2)^2} - \frac{4}{\pi}\sum_{m=1,3,\cdots}^{\infty}\frac{mnE_m}{(m^2+n^2)^2} - \frac{\pi}{4}\left[\left(\tanh\frac{n\pi}{2} - \frac{2}{\pi n}\right)\cdot \tanh\frac{n\pi}{2} - 1\right]E_n = 0$$

where $n=2,4,6\ldots\ldots\ldots\ldots$ (4-28)

if the loading is uniformly distributed over the triangular plate, by assuming $K = \frac{64qa^2}{\pi^4}$, as before, equations (4-27), (4-28) become

$$\pi\left[\left(\coth\frac{n\pi}{2} - \frac{2}{n\pi}\right)\coth\frac{n\pi}{2} - 1\right]E_n + \frac{16}{\pi}\sum_{m=2,4,\cdots}^{\infty}\frac{mnE_m}{(m^2+n^2)^2} + \frac{K}{n}\sum_{m=2,4,\cdots}^{\infty}\frac{m^2}{(m^2-n^2)(m^2+n^2)^2} = 0$$

where n=1,3,5......... (4-29)

$$\pi\left[\left(\tanh\frac{n\pi}{2} - \frac{2}{n\pi}\right)\tanh\frac{n\pi}{2} - 1\right]E_n + \frac{16}{\pi}\sum_{m=1,3,5\ldots}^{\infty}\frac{mnE_m}{(m^2+n^2)^2} -$$

$$nK\sum_{m=1,3,5\ldots}^{\infty}\frac{1}{(n^2-m^2)(m^2+n^2)^2} = 0$$

where n=2,4,6........... (4-30)

Substituting the numerical values of the coefficients in equations (4-29) and (4-30), and by considering only the first ten coefficients, m=1 through m=10 the following system of ten equations with ten unknows E_1 through E_{10} is obtained:

$$4.9870E_1-1.2800E_2-0.2215E_4-0.0710E_6-0.0303E_8-0.0157E_{10}$$
$$= 0.182960K$$

$$-1.2800E_1+3.2033E_2-0.5680E_3-0.1903E_5-0.0797E_7-0.0399E_9$$
$$= -0.075911K$$

$$-0.5680E_2+2.0915E_3-0.3072E_4-0.1422E_6-0.0721E_8-0.0404E_{10}$$
$$=0.000012K$$

$$-0.2215E_1-0.3072E_3+1.5708E_4-0.1904E_5-0.1060E_7-0.0612E_9$$
$$= -0.004819K$$

$$-0.1903E_2-0.1904E_4-1.2566E_5-0.1290E_6-0.0808E_8-0.0512E_{10}$$
$$=0$$

$$-0.0701E_1-0.1422E_3-0.1290E_5-1.04720E_6-0.0930E_7-0.0631E_9$$
$$= -0.000952K$$

$$-0.0797E_2-0.1060E_4-0.0930E_6-0.8976E_7-0.0702E_8-0.0504E_{10}$$
$$=0$$

$$-0.0303E_1-0.0721E_3-0.0808E_5-0.0702E_7-0.7854E_8-0.0548E_9$$
$$= -0.000303K$$

$$-0.0399E_2-0.0612E_4-0.0631E_6-0.0548E_8-0.6981E_9-0.0440E_{10}$$
$$=0$$

$$-0.0157E_1-0.0404E_3-0.0512E_5-0.0504E_7-0.0440E_9-0.6283E_{10}$$
$$= -0.000123K \tag{4-31}$$

Solving the above ten simultaneous equations, the values of E_m can be obtained and are indicated in Table V.

Table V

Values of the coefficients of moments at clamped edges.

E_1	E_2	E_3	E_4	E_5
0.034001K	-0.010691K	-0.002683K	0.000974K	-0.001354K
E_6	E_7	E_8	E_9	E_{10}
0.000749K	-0.000705K	0.000449K	-0.000405K	0.000286K

Substituting the calculated values of the coefficients E_1 through E_{10} into equations (4-6) and (4-23), the bending moments along the clamped edges of the triangular plate are obtained and are indicated in Table VI.

TABLE VI

Bending moments of triangular plate along clamped edges

$\frac{y}{a}$ on $x=0$ $\frac{x}{a}$ on $y=0$	M_x (qa^2) M_y	$\frac{y}{a}$ on $x=0$ $\frac{x}{a}$ on $y=0$	M_x (qa^2) M_y	$\frac{y}{a}$ on $x=0$ $\frac{x}{a}$ on $y=0$	M_x (qa^2) M_y
0	0	0.35	-0.026626	0.70	-0.010994
0.05	-0.002318	0.40	-0.026706	0.75	-0.007990
0.10	-0.007009	0.45	-0.025416	0.80	-0.005177
0.15	-0.013364	0.50	-0.023410	0.85	-0.002798
0.20	-0.018925	0.55	-0.020933	0.90	-0.001252
0.25	-0.022647	0.60	-0.017820	0.95	-0.000462
0.30	-0.025125	0.65	-0.014319	1.00	0

Now, by substituting all the known values from Table I and Table V into equations (4-5), (4-7), (4-22) and (4-24) and combining the results with equation (4-18) by using equation (4-25), the deflection of the perpendicular bisector of the hypotenuse of the isosceles right triangular plate with clamped legs and simply supported hypotenuse will be obtained. Values of moments are also tabulated in Table VII.

TABLE VII

Deflections and moments of right triangular plate with clamped legs and simply supported hypotenuse (μ=0.3)

$\frac{x}{a}=\frac{y}{a}$	$W'(qa^4/D)$	W'_{xx}, W'_{yy} (qa^2/D)	$W'_{xy}(qa^2/D)$	$M_n(qa^2)$	$M_t(qa^2)$
0.00	0	0	0	0	0
0.05	0.000002	0.00117	0.00313	-0.00373	-0.00066
0.10	0.000025	0.00087	0.00576	-0.00516	0.00290
0.15	0.000077	-0.00109	0.00606	-0.00282	0.00566
0.20	0.000154	-0.00387	0.00429	0.00197	0.00810
0.25	0.000233	-0.00647	0.00186	0.00709	0.00972
0.30	0.000290	-0.00843	-0.00088	0.01159	0.01035
0.35	0.000299	-0.00907	-0.00291	0.01382	0.00975
0.40	0.000252	-0.00812	-0.00379	0.01320	0.00790
0.45	0.000145	-0.00520	-0.00297	0.00883	0.00468
0.50	0	0	0	0	0

By comparing the results of the above three different kinds of boundary conditions, as in Table III, Table IV and Table VII, the deflections are shown in Figure 6, moments are shown in Figure 7 and Figure 8. Also, moments of the clamped edges are compared as shown in Figure 9.

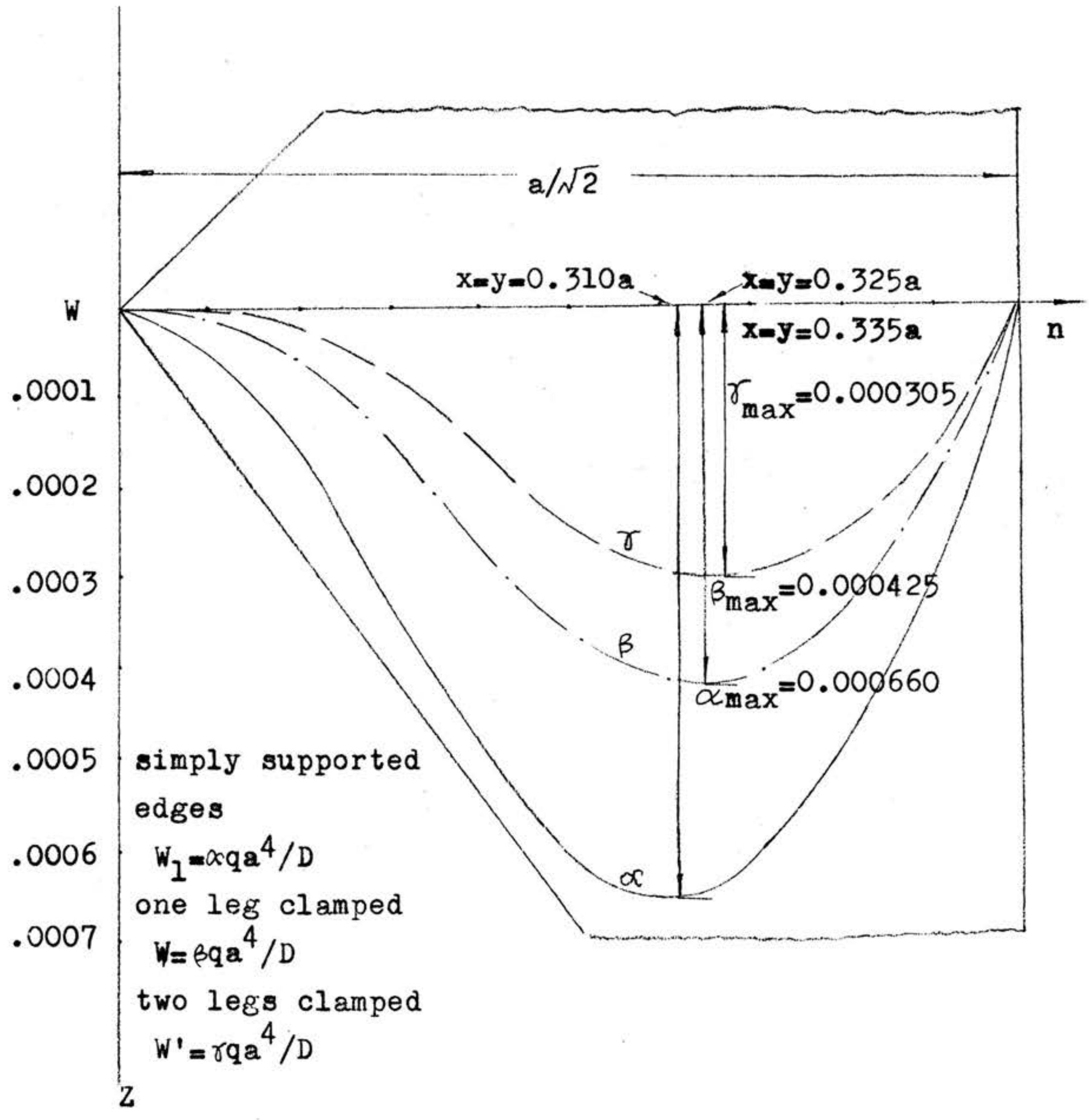

Figure 6. Deflections along the perpendicular bisector of hypotenuse of the triangular plate of various boundary conditions

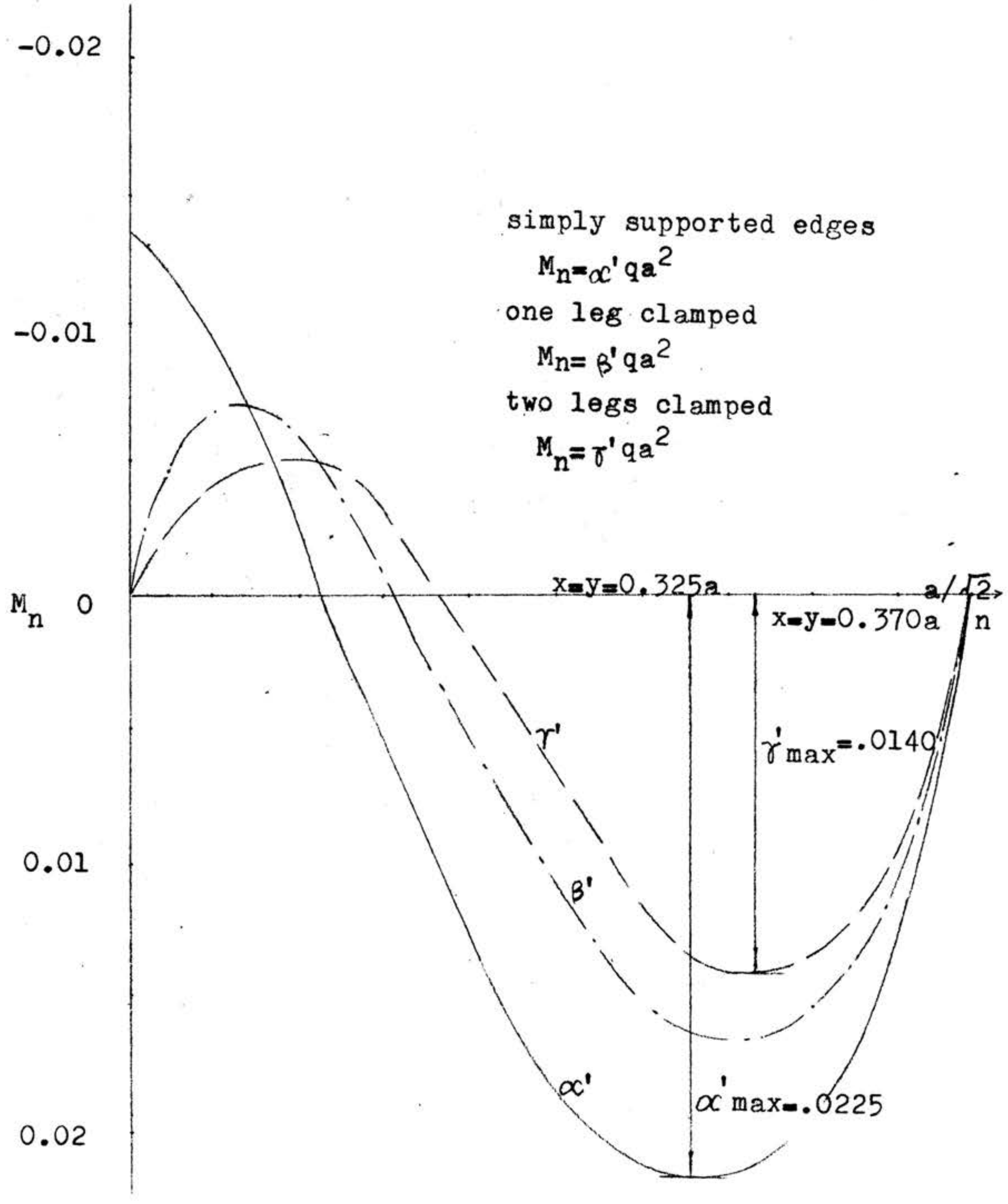

Figure 7. Bending moments along the perpendicular bisector of the hypotenuse of the triangular plate with various boundary conditions

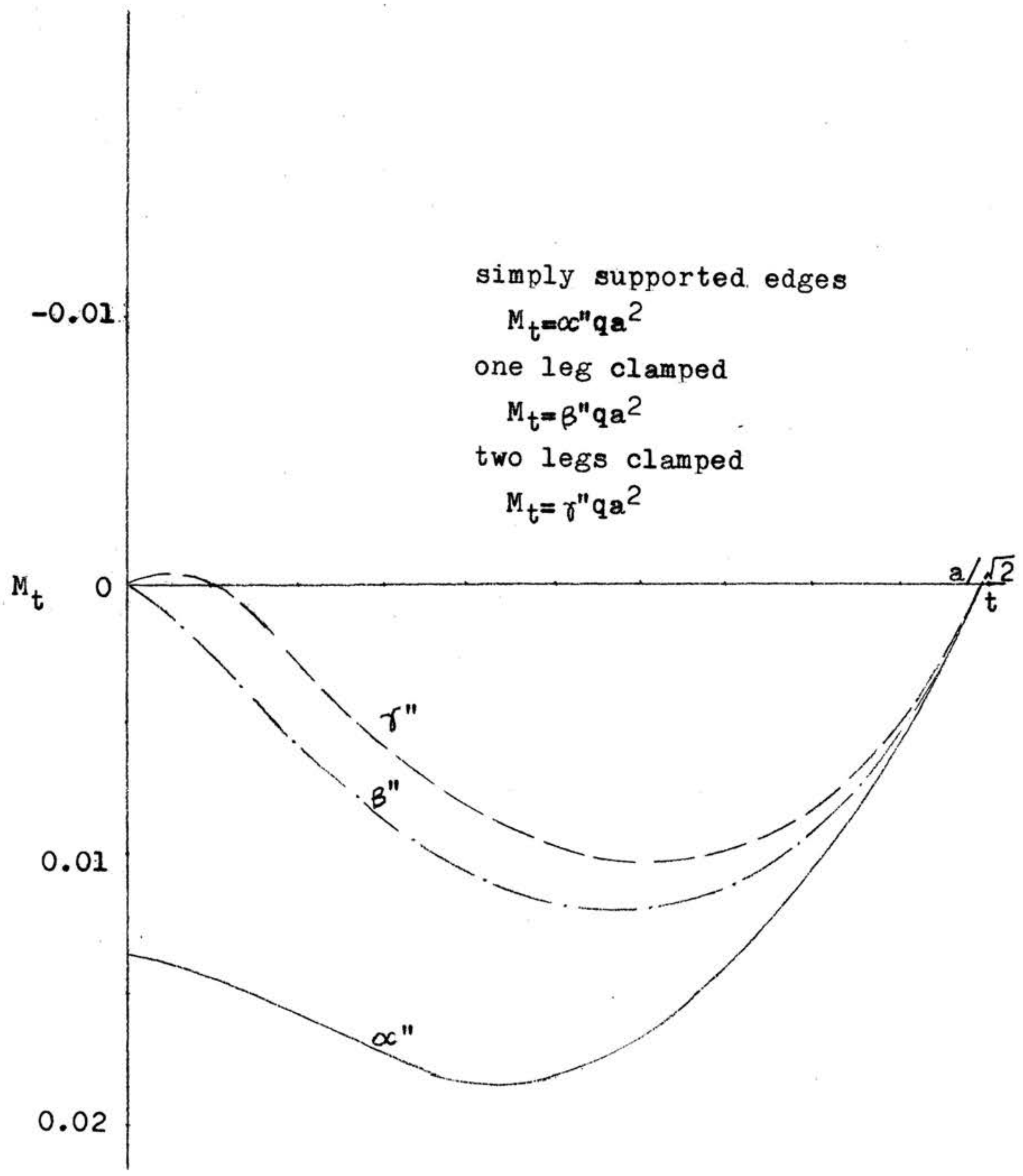

Figure 8. Bending moments along the perpendicular bisector of the hypotenuse of the triangular plate with various boundary conditions

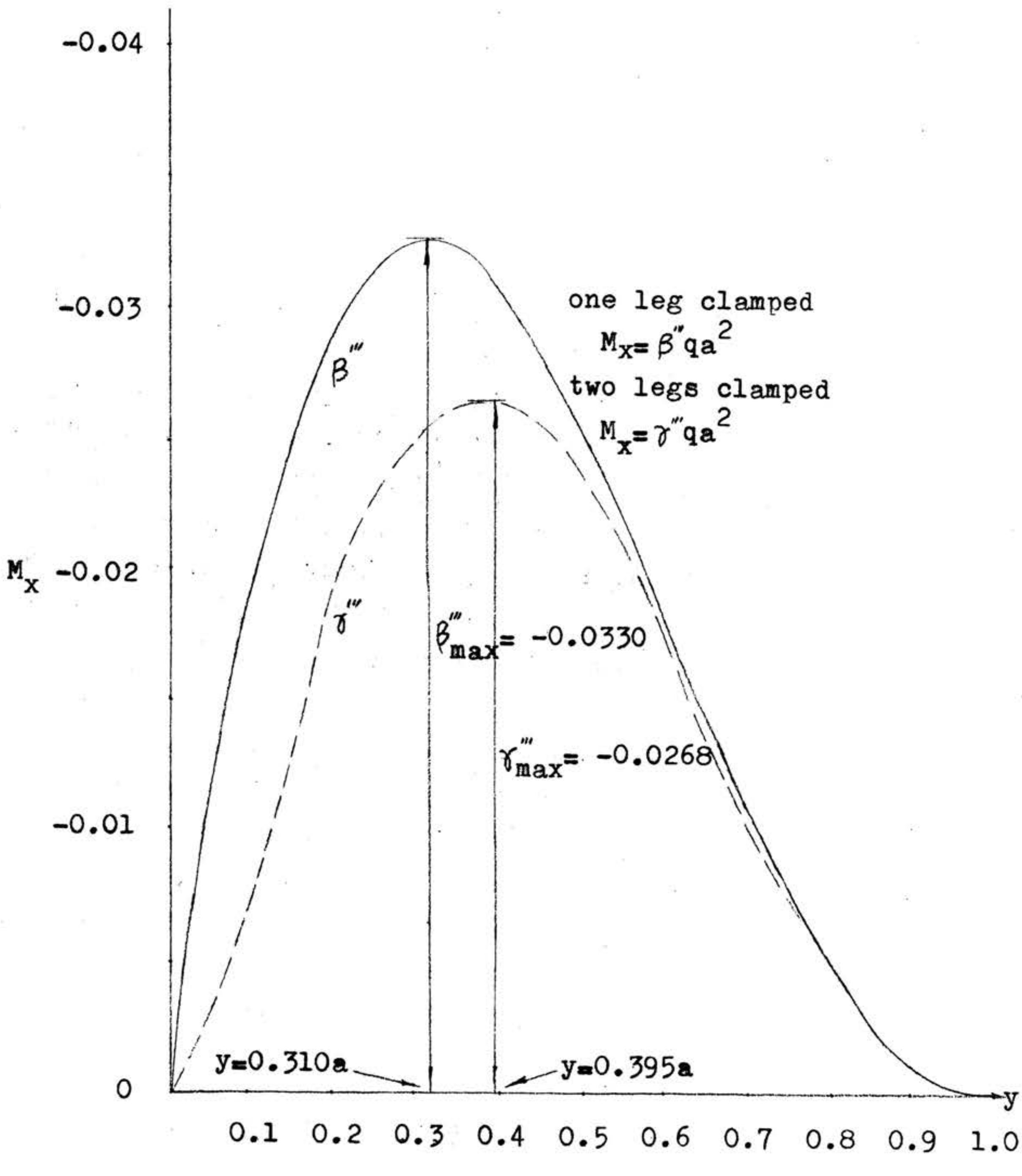

Figure 9. Bending moments along clamped edges

V. DISCUSSION

To find the value of the coefficients E_m, theoretically one has to consider an infinite number of coefficients and thus solve an infinite number of simultaneous equations. In this thesis, it is considered accurate enough to use only the first ten coefficients and solve for the ten unknowns E_1 through E_{10}.

The difference between considering only the first ten coefficients and the infinite coefficients is quite small. As it can be seen, by considering only the first four coefficients in equation (4-21) and the first five coefficients in equation (4-31), the results, which are tabulated in Table VIII and Table IX, are almost the same as those obtained with ten coefficients.

The results from both Case 1 and Case 2 of this thesis can be used to design those isosceles right triangular plates which fit the above boundary conditions. As it can be seen, when the numerical values of the uniformly distributed load, the thickness of the plate and the Young's modulus are known, the corresponding deflections and bending moments at any point of the triangular plates can be found from equations (4-8), (4-25), (4-6), and those formulas listed in Chapter III.

Table VIII

Values of the coefficients of moments at clamped edge by considering only the first four coefficients in expression (4-21)

m	1	2	3	4
E_m	0.041206K	-0.018629K	0.002300K	-0.001995K

Table IX

Values of the coefficient of moments at clamped edges by considering only the first five coefficients in expression (4-31)

m	1	2	3	4	5
E_m	0.033989K	-0.010690K	-0.002749K	0.001010K	-0.001466K

VI. CONCLUSION

From the results obtained in this thesis, the following conclusions were observed:

1. In the case of symmetry the maximum deflection of an isosceles right triangular plate simply supported on all edges under constant load q or simply supported hypotenuse and clamped legs under constant load q occurs in the perpendicular bisector of the hypotenuse of the plate, that is:

In a simply supported triangular plate

$$W_{1max} = 0.000660\ qa^4/D \qquad \text{where } x=y=0.310a$$

In triangular plate with clamped legs and simply supported hypotenuse

$$W'_{max} = 0.000305\ qa^4/D \qquad \text{where } x=y=0.335a$$

2. From H. J. Fletcher's work (5) the maximum deflection of an isosceles right triangular plate clamped along the diagonal and simply supported on the edges is

$$W_{max} = 0.000370\ qa^4/D \qquad \text{where } x=y=0.255a$$

3. Because of unsymmetry the maximum deflection of an isosceles right triangular plate with one leg clamped, hypotenuse and the other leg simply supported does not occur on the perpendicular bisector of the hypotenuse of the plate. The largest deflection in the perpendicular bisector of the hypotenuse is

$W = 0.000425\ qa^4/D$ where $x=y=0.325a$

4. Because of symmetry, $\frac{1}{\gamma_x} = \frac{1}{\gamma_y}$ or $W_{xx}=W_{yy}$ both on the perpendicular bisector of the hypotenuse of an isosceles right triangular plate simply supported on all edges and an isosceles right triangular plate with a simply supported hypotenuse and clamped legs.

$$\tan 2\alpha = \frac{\frac{2}{\gamma_{xy}}}{\frac{1}{\gamma_x} - \frac{1}{\gamma_y}}$$

The principal planes are in the n and t directions in these cases. By comparing Figure 7 and Figure 8, the largest moment of the triangular plate in the perpendicular bisector of the hypotenuse should be:

In a simply supported triangular plate

$(M_n)_{max} = M_{max} = 0.0225\ qa^2$ where $x=y=0.325a$

this is also the maximum moment in the above plate.

In a triangular plate with clamped legs and simply supported hypotenuse

$(M_n)_{max} = 0.0140\ qa^2$ where $x=y=0.370a$

Also, by comparing Figure 7, Figure 8 and Figure 9, the maximum moment in an isosceles right triangular plate with one leg clamped, hypotenuse and the other leg simply supported is on the clamped edge, i.e.,

$$M_{max} = (M_x)_{max} = -0.03155\ qa^2 \qquad \text{where } x=y=0.5a$$

5. By comparing Figure 7 and Figure 8 the bending moments on the perpendicular bisector of the hypotenuse on an isosceles right triangular plate with a simply supported hypotenuse and clamped legs and an isosceles right triangular plate with one leg clamped, hypotenuse and the other leg simply supported are approximately equal, but the difference of the bending moments between both the above two plates and the simply supported plate is large, especially near the right angle of the triangular plate.

6. From Figure 9, the bending moments along the clamped edge between an isosceles right triangular plate simply supported hypotenuse and clamped legs and an isosceles right triangular plate with one leg clamped, hypotenuse and the other leg simply supported are also approximately equal.

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VITA

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