A study of rotational critical speed for a multi-mass system

Richard K. Brockmann

A STUDY OF ROTATIONAL CRITICAL SPEED
FOR A MULTI-MASS SYSTEM

BY

RICHARD K. BROCKMANN

A

THESIS

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Approved by

Charles L. Edwards (advisor) Lyle G. Rhea

[illegible] James A. Jones

ABSTRACT

The object of this thesis is to demonstrate, by use of an IBM-1620 digital computer, a fast efficient method for locating natural frequencies of multi-mass rotational systems. Stress and deflection characteristics are examined at these frequencies also.

The shafts considered in this thesis are simply supported and symmetrically loaded with five concentrated masses. These five masses are placed at increments of 10 inches on a 60 inch shaft. The variable to be examined is shaft diameter, which is varied from 0.2 inch to 1 inch by increments of 0.2 inch. The effect of shaft weight, which is a function of shaft diameter, is examined; and a definite pattern is obtained for critical speeds.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

W	Circular frequency (rad./sec.)
W_n	Natural or critical frequency (rad./sec.)
W_f	Final circular frequency (rad./sec.)
ΔW	Increment of frequency (rad./sec.)
E	Modulus of Elasticity (p.s.i.)
I,Z	Area Moment of Inertia (in.^{4})
DX	Shaft increment between concentrated masses (in.)
D	Diameter of shaft (in.)
NX	Number of Stations
V	Shear Force (lbs.)
θ,TH	Slope (rad.)
M,T	Bending Moment (in.-lb.)
Y	Deflection (in.)
ST	Bending Stress (p.s.i.)
SR	Shear Stress (p.s.i.)
K	Spring Constant (lb./in.)
m	Mass (lb-sec.^{2}/in.)
w	Inertia Force (lbs.)

INTRODUCTION

This thesis utilizes the facilities of the IBM-1620 digital computer to locate natural frequencies of a multi-mass rotational system and examine their effect on deflection and stress characteristics in the shaft. Until the advent of the digital computer, no practical solution to the problem was available for more than two modes of vibration. The method investigated is adaptable to any rotating-circular shaft with either distributed or concentrated masses.

The shafts examined are symmetrically loaded and simply supported. The system, (fig. 2-a) composed of five concentrated masses, has five critical frequencies or five distinct modes of vibration. If the system is allowed to operate at or near one of the natural frequencies, the deflection and stress become infinitely large and failure will result. In the analysis, the system is considered in transverse vibration; thus neglecting gyroscopic effects. Variation between the actual boundary condition for moment and deflection and the mathematical boundary conditions are also neglected.

This subject was chosen by the author because of his interest in the field of vibration and stress analysis and for further knowledge of critical speed. To the author's knowledge no previous work has been attempted in analyzing the stress and deflection distribution curves at various modes or relationships for critical speeds of a system containing concentrated masses.

ACKNOWLEDGMENT

While preparing this thesis a great deal of assistance and encouragement was received from many people. The author would especially like to thank Herbert R. Alcorn for assistance in programing this problem on the digital computer. Thanks also goes to Professor Charles L. Edwards for the suggestion of the subject and his assistance during the solution of the problem. Thanks are also given to my wife, Doris, for the typing of this thesis and the inspiration given during the long hours of its preparation.

REVIEW OF LITERATURE

Stodola was one of the first men to solve the critical speed problem of higher order modes. This method considers inertia loading and influence coefficients but it becomes very lengthy for more than two concentrated masses.

Den Hartog (2)* discusses shaft deflection of a simply supported shaft with one concentrated mass operating at or near the critical frequency.

Church (1) discusses the same system as Den Hartog (2) and also demonstrates a tabular method to calculate critical frequencies of multi-mass systems with concentrated loading.

Macduff and Curreri (3), discuss shaft deflection and bending stress characteristics for a single disk and shaft system.

Prohl (4) was the first to develop a method to obtain mode shapes of shafts which operate at higher order critical speeds.

* Refer to Bibliography for all references.

DISCUSSION

The first section of this discussion is an analysis of a simply supported shaft with one concentrated mass, as illustrated in Church (1), Den Hartog (2). It is placed here in order that the reader may have an understanding of the principles involved.

Figure (1-a) represents the system where the center of gravity and geometric center do not coincide, due to machining of the disk. When the shaft rotates at an angular frequency of W (rad./sec.), an impressed or centrifugal force is set up which causes a deflection equal to y. The force, due only to the concentrated mass, acts at the center of gravity of the concentrated mass and equals $m(y+e)W^2$. It is balanced by the restoring (spring) force of the shaft which acts at the center of the mass and equals Ky for equilibrium:

$$(1) \qquad m(Y+e)W^2 = KY$$

$$mYW^2 - KY = - meW^2$$

divide both sides by mW_n^2

$$(1a) \qquad Y\left(\frac{W}{W_n}\right)^2 - \frac{K}{mW_n^2} = e\left(\frac{W}{W_n}\right)^2$$

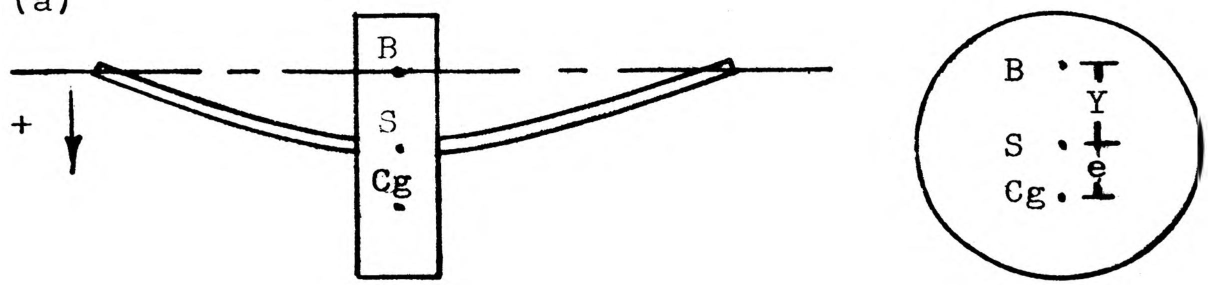
$$\frac{Y}{e} = \frac{(W/W_n)^2}{1 - (W/W_n)^2}$$

This equation is for the case where damping is negligible. From this equation it is interesting to examine the physical action that takes place as the rotational speed of the shaft is increased, Figure (1-b).

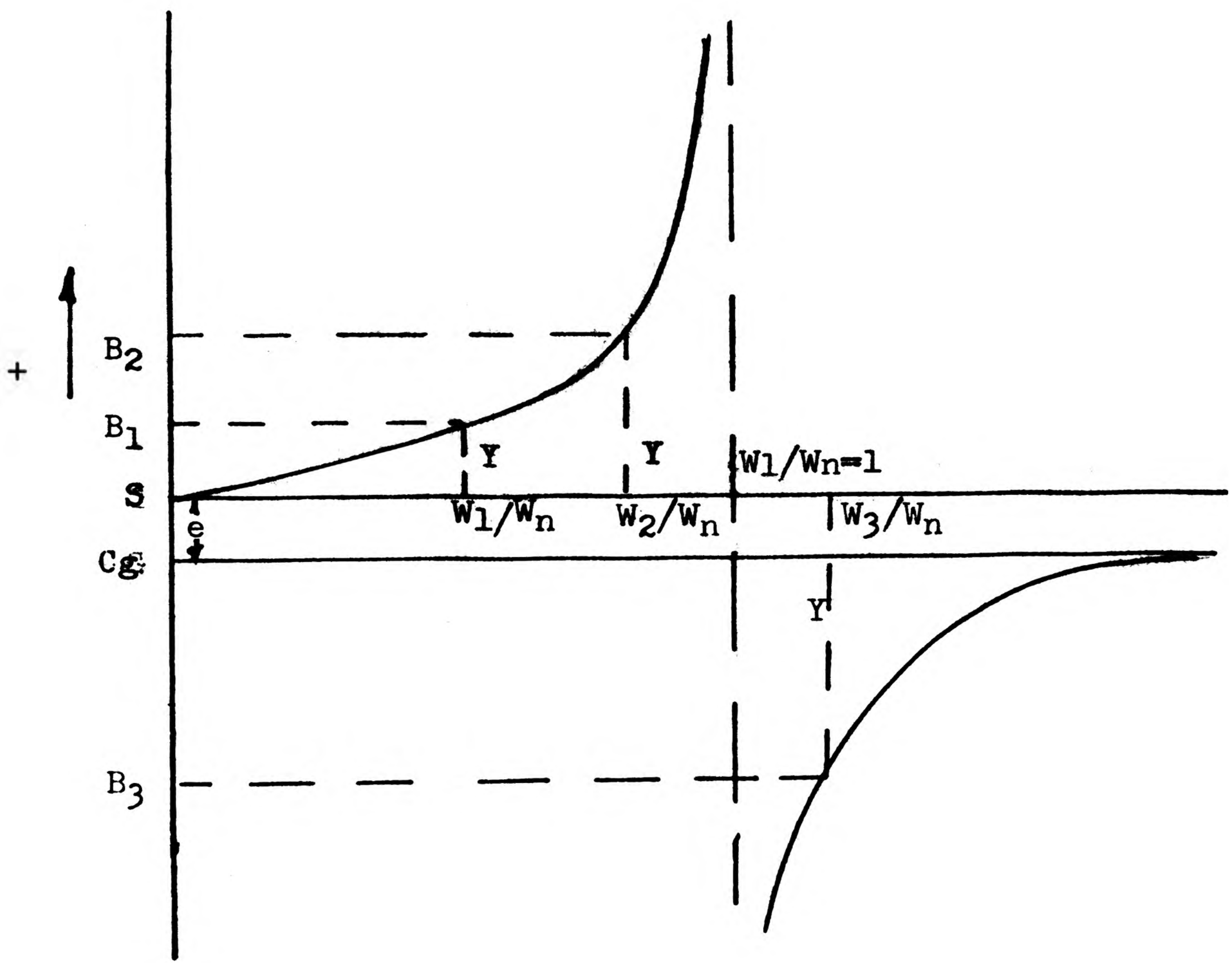
FIGURE 1

CURVES PERTAINING TO THE SINGLE-DISK AND SHAFT SYSTEM

(a)



(b)



When the rotational speed is below the critical, the center of gravity is outside the geometric center of the disk. This can be easily seen, as Y/e will remain positive for all speed ratios less than unity. The path of the geometric center (S) is a circle at a radius Y about the bearing center line (B). The center of gravity also follows a circular path at a radius of Y+e about the bearing center line (B).

This value of Y is required to achieve equilibrium between the spring and inertia forces acting on the system. As the speed ratio increases to unity, the shaft is in a state of "indifferent equilibrium". In other words, the geometric center oscillates about the bearing center line. At speeds greater than the critical, the value of Y becomes negative and below the critical, Y is positive. In the limit Y/e becomes -1 or the center of gravity approaches the center line of the bearings thus eliminating the inertia force. At these high speeds, the shaft is very stable. This sign change represents a change in phase, or angle, between the exciting force vector meW^2 and the radius Y. At low speeds, this angle is 0°, while at high speeds, it approaches 180°.

Bending stress is examined by Macduff and Curreri (3) for this system:

(2) $$ST = \frac{Mc}{I}$$

where: $$M = \frac{Lw}{4}$$

$$c = \frac{D}{2}$$

$$I = \frac{\pi D^4}{64}$$

Eliminating W

$$Y = wL^3/48EI$$

$$w = 48YEI/L^3$$

$$M = 12YEI/L^2$$

$$ST = 6DEY/L^2 \qquad (2a)$$

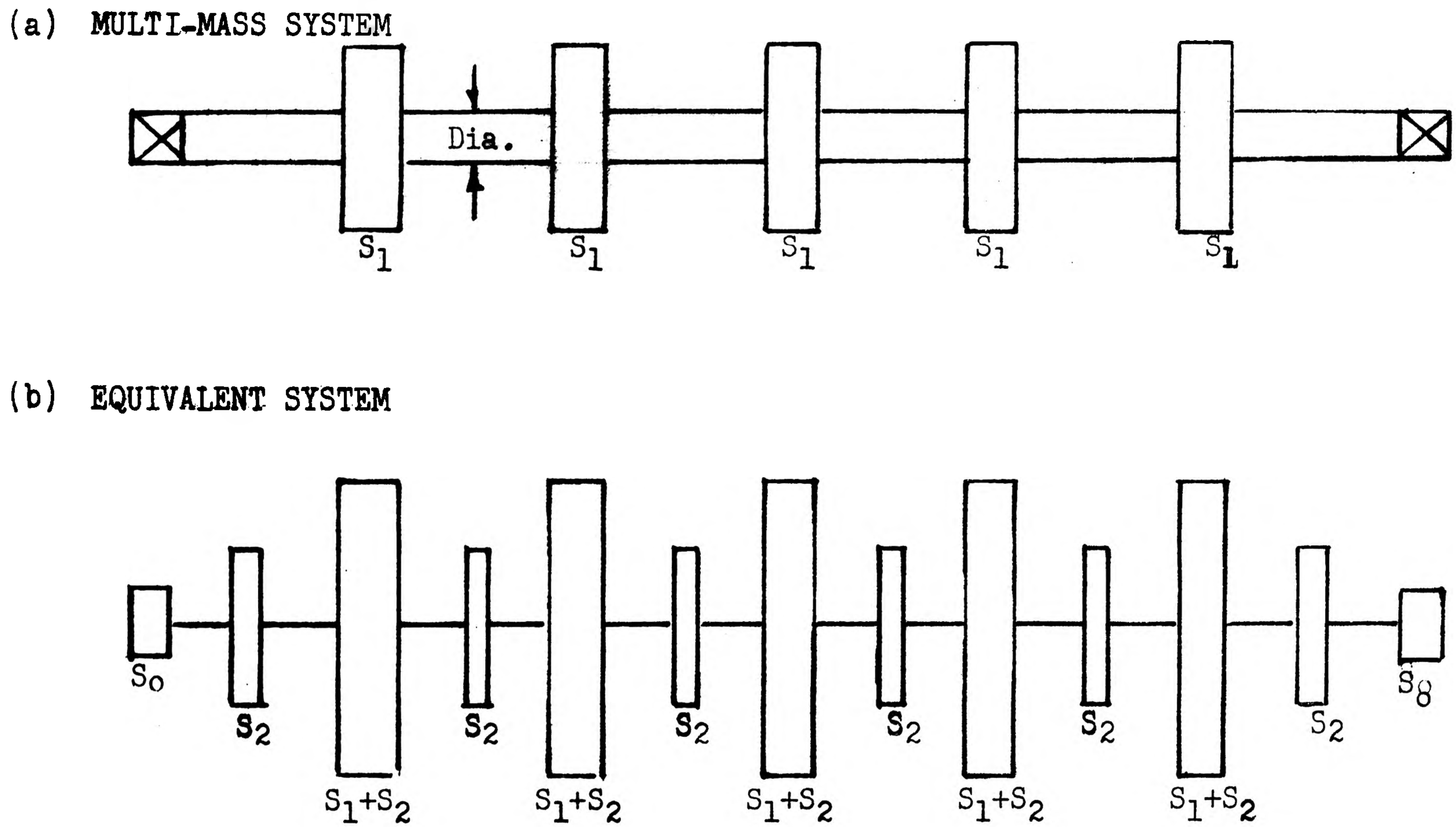
Values for Y/e can be obtained from Equation I. If the eccentricity (e) is approximated, a value for deflection on both sides of the critical frequency can be obtained. With these values, and the above formula, bending stress can be obtained at the critical frequency.

The study of the one mass, simply supported shaft involves all the basic principles necessary to understand the problem. An expansion of these principles is given in the remainder of the discussion.

The system shown in Figure (2-a) represents a problem which might be encountered in the design of an automobile driveshaft or a turbine shaft on a jet engine. This system will be analyzed by M. A. Prohl's (5) method developed in 1945. This method is basically the same as that used for torsional vibration, except that there are four integrations rather than two. It is derived from the basic transverse vibration equation of a beam as discussed by Miller (4).

Due to the length of this problem, it is especially suited for solution by the digital computer. As in the

FIGURE 2



(a) MULTI-MASS SYSTEM
Dia.
S_1
S_1
S_1
S_1
S_1
(b) EQUIVALENT SYSTEM
S_0
S_2
S_1+S_2
S_2
S_1+S_2
S_2
S_1+S_2
S_2
S_1+S_2
S_2
S_1+S_2
S_2
S_8

method for torsional vibration, an estimated natural frequency is used to start the calculation and determine the acceleration and hence the inertia force on the beam. Once the inertia loading is found, shear, bending moment, slope and deflection can be determined at each station in the system. With the boundary conditions known at the first support, it is possible to start at this support and vary the frequency until we obtain satisfactory boundary conditions at the next support. When these boundary conditions are satisfied, the system is in equilibrium.

It is essential to an understanding of this method to recall from strength of materials the following basic equations:

$$
\begin{aligned}
V = \text{Shear} &= \int \text{Load}\, dx \\
M = \text{Moment} &= \int V\, dx \\
\theta = \text{Slope} &= \frac{1}{EI}\int M\, dx \\
Y = \text{Deflection} &= \int \theta\, dx
\end{aligned}
\qquad (3)
$$

To adapt these equations to the problem, it is first necessary to divide the shaft into concentrated masses connected by weightless shaft sections or springs, Figure (2-b). Basically, a numerical integration procedure is used to solve these equations and thus obtain values for shear, moment, slope and deflection along the beam. For a given mode of vibration the masses must be multiplied by the acceleration YW^2 to determine the loading on the beam. The integration or summation of these inertia loads gives the shear diagram for a general case, Figure (3-a). Starting with the initial shear at Station 0 and since the change in shear along the beam is:

(4) $\Delta V = mYW^2$

the shear at Station 1 is:

(4a) $V_1 = V_o + m_o Y_o W^2$

and the shear over Section 2 is:

(4b) $V_2 = V_1 + m_1 Y_1 W^2 = V_o + m_o Y_o W^2 + m_1 Y_1 W^2$

From the values of shear moments at each station and the bending moment diagram can be found Figure (3-b).

Since:

$$M = \int V\,dx$$

(5)
$$M_1 = M_o + V_1 DX$$
$$M_2 = M_1 + V_2 DX = M_o + V_1 DX + V_2 DX$$

To obtain the slope curve, Figure (3-c), note that the moment between Station 0 and 1 is given by:

(5a) $M = M_o + (M_1 - M_o)X/DX$

Therefore:

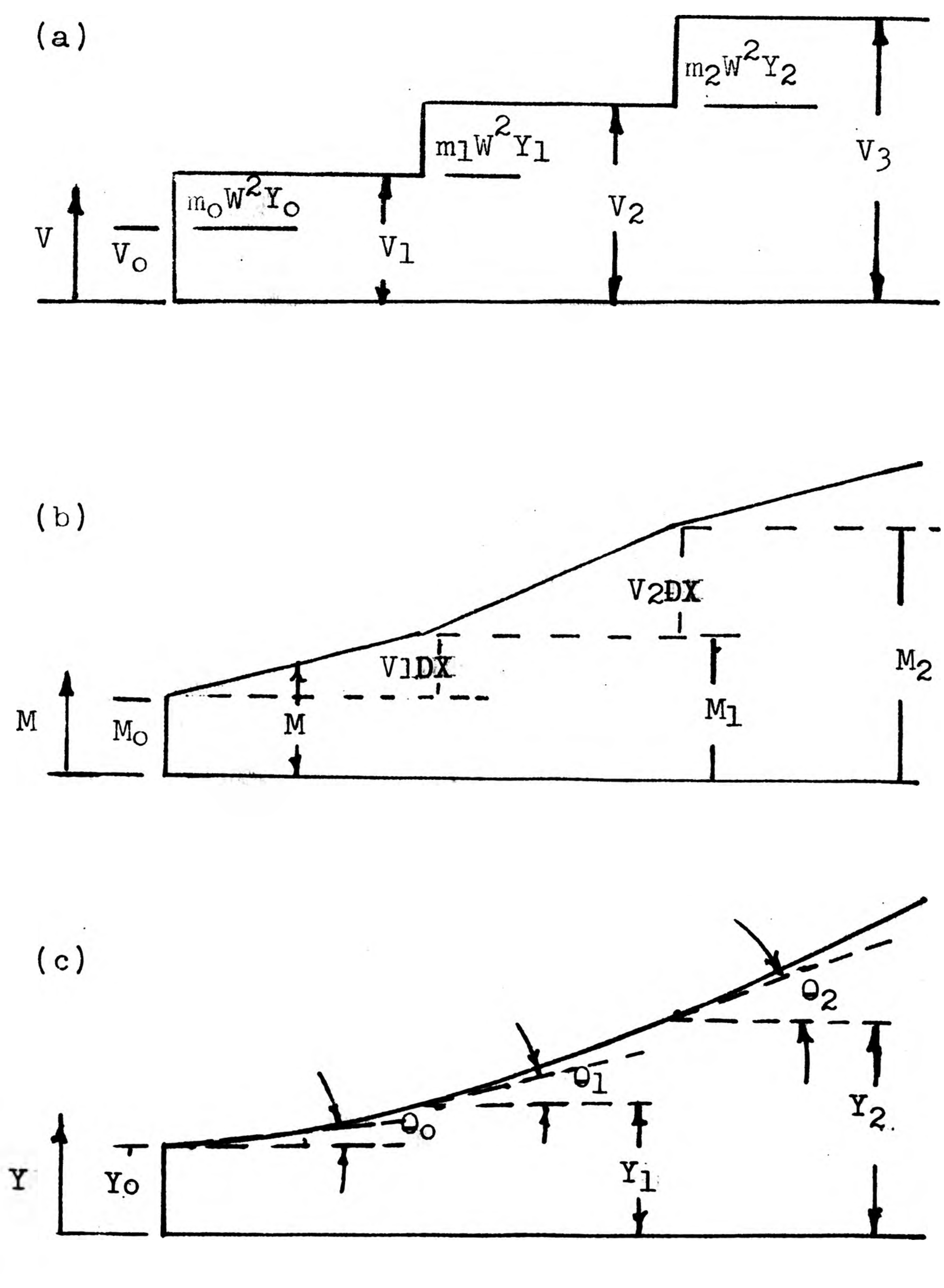
$$\theta = \frac{1}{EI}\int M\,dx$$

(6)
$$\theta = \frac{1}{EI}MX + C$$
$$\theta = \frac{1}{EI}\left(M_o X + \frac{(M_1 - M_o)X^2}{2}\right) + \theta_o$$

but at Station 1:

(6a)
$$X = DX$$
$$\theta_1 = \frac{DX}{EI}\left(\frac{M_o}{2} + \frac{M_1}{2}\right) + \theta_o$$

FIGURE 3
GRAPHICAL REPRESENTATION OF ELEMENTARY BEAM EQUATIONS



The deflection curve, Figure (3-c), can be obtained by integrating the slope equation.

(7)

$$Y = \int \Theta \, dx$$

$$Y = \frac{1}{EI}(\Theta X + C_1)$$

$$Y = \frac{1}{EI}(M_o X^2/2 + (M_1 - M_o)X^3/6DX) + \Theta_o X + Y_o$$

but at Station 1

(7a)

$$X = DX$$

$$Y_1 = DX/EI(M_o/3 + M_1/6)\ DX + \Theta_o DX + Y_o$$

By progressing across the beam general equations can be obtained that are adaptable to the digital computer. These equations are in computer form:

(8)

$$V = V_L + Y_L * S(JX) * W{*}{*}2$$

$$T = T_{L1} + V{*}DX$$

$$\Theta = \frac{DX}{EI}*(\frac{T_{L1}}{2} + \frac{T}{2}) + \Theta_{L1}$$

$$Y = Y_L + \Theta_L DX + \frac{DX}{EI}(\frac{T_{L1}}{3} + \frac{T_L}{6})DX$$

Let:

$$T_{SUM} = T_{SUM} + T_{L1} + T_{L2}$$

$$B = \frac{DX}{EI}$$

$$Y = Y_L + (TH + B(\frac{T}{6} + \frac{T_L}{3} + T_{SUM})){*}DX$$

$$\Theta = B(\frac{T_{L1}}{2} + \frac{T}{2}) + \Theta_{L1}$$

Let: $$T_{SUM\ 1} = T_{SUM\ 1} + T_{L1}$$

$$\Theta = B\left(\frac{T}{2} + T_{SUM\ 1} + \theta_o\right)$$

Normally it is possible to eliminate two of the four basic boundary conditions at the supports, as they are zero. In the case of a simply supported shaft, moment and deflection are zero and the unknown boundary conditions are shear and slope. In the method used, it will be assumed that one of these unknown boundary conditions is zero and the other is unity. With this assumption it is possible to calculate shear force, moment, slope and deflection in terms of the unity boundary condition. The assumption is then reversed and the above steps repeated to obtain shear force, moment, slope and deflection in terms of this boundary condition. Finally the results of both solutions are added to obtain the complete solution.
The equations for the known boundary conditions at the right support are:

(9a) $$Y = A\ V_o + B\ \theta_o$$

(9b) $$M = C\ V_o + D\ \Theta_o$$

By considering either of the above boundary conditions as zero it is possible to obtain V_o in terms of θ_o or vice versa. This value is then substituted into the equation for the second known boundary condition. If both are zero, the boundary conditions are satisfied and the beam is capable of supporting vibration. There are four possible ways that the second boundary condition can be expressed; namely, moment/initial slope, moment/initial shear, deflection/initial slope, or deflection/initial shear. In this thesis the deflection was chosen as zero and the moment was expressed in terms of initial slope. Figures (4-9) represent a plot of moment/initial slope versus frequency for each shaft diameter considered. When the moment/initial slope remainder becomes zero, the boundary conditions are satisfied and a critical frequency is obtained.

Relations can also be obtained for shaft deflection, bending stress and shear stress at the critical speed. Each of these relations will be in terms of either θ_o or V_o. In this thesis the relation with θ_o is used in the mathematical solution for simplicity. The solution for θ_o is not carried out in this thesis but will be shown here for future reference. All deflections will be caused by inertia forces and a deflection equation can be written at each station in terms of these forces. The following equations utilize influence coefficients. An influence coefficient is a deflection caused by a unit load. For example, a_{11} means a deflection at point one due to a unit load at point one. The deflection at station one for this problem is:

$$(10) \qquad Y_1 = a_{11}Y_1m_1W^2 + a_{12}Y_2m_2W^2 + a_{13}Y_3m_3W^2 + a_{14}Y_4m_4W^2 + a_{15}Y_5m_5W^2$$

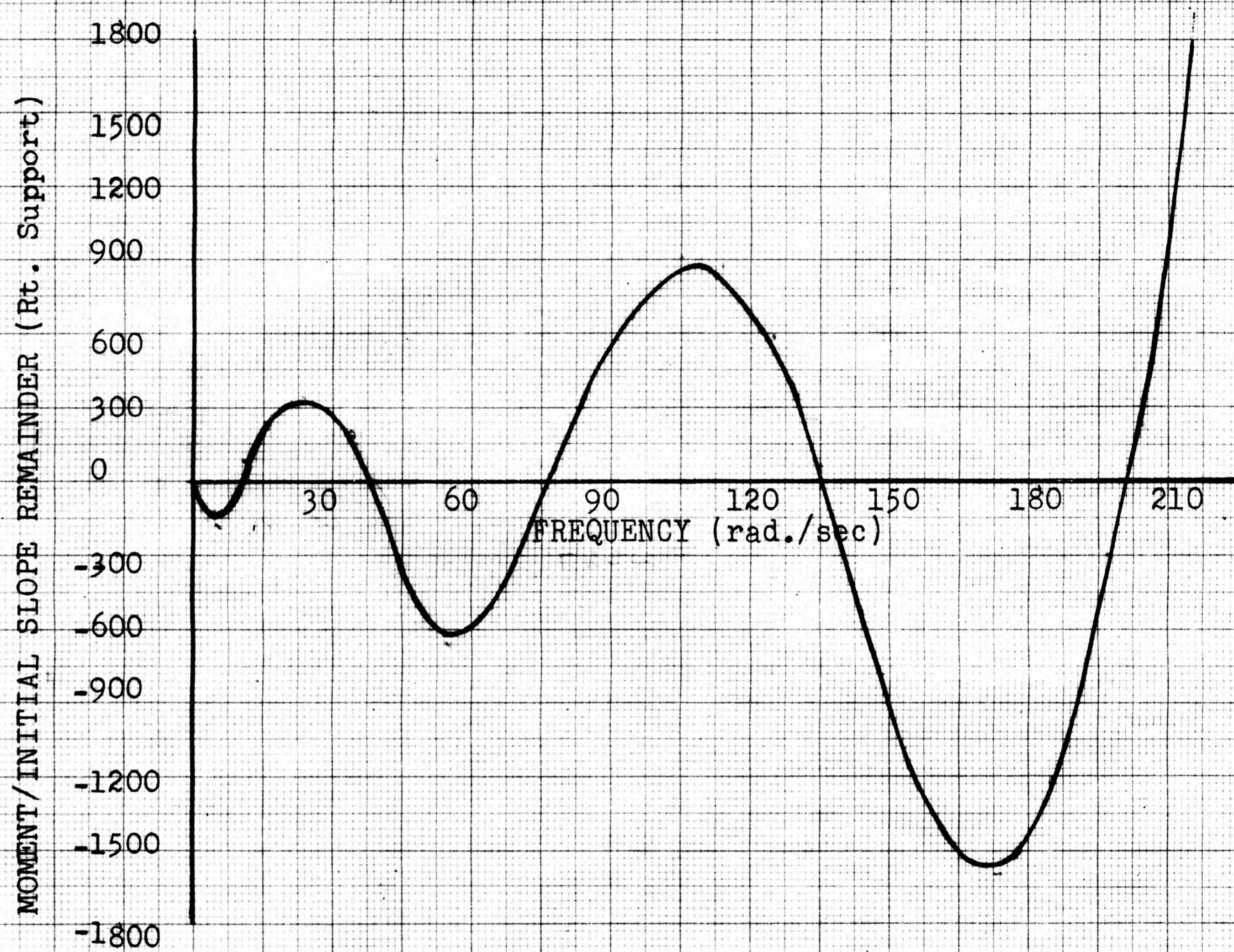


fig. 4

MOMENT/INITIAL SLOPE REMAINDER (Rt. Support) vs. FREQUENCY
FOR A
0.2 INCH DIAMETER SHAFT

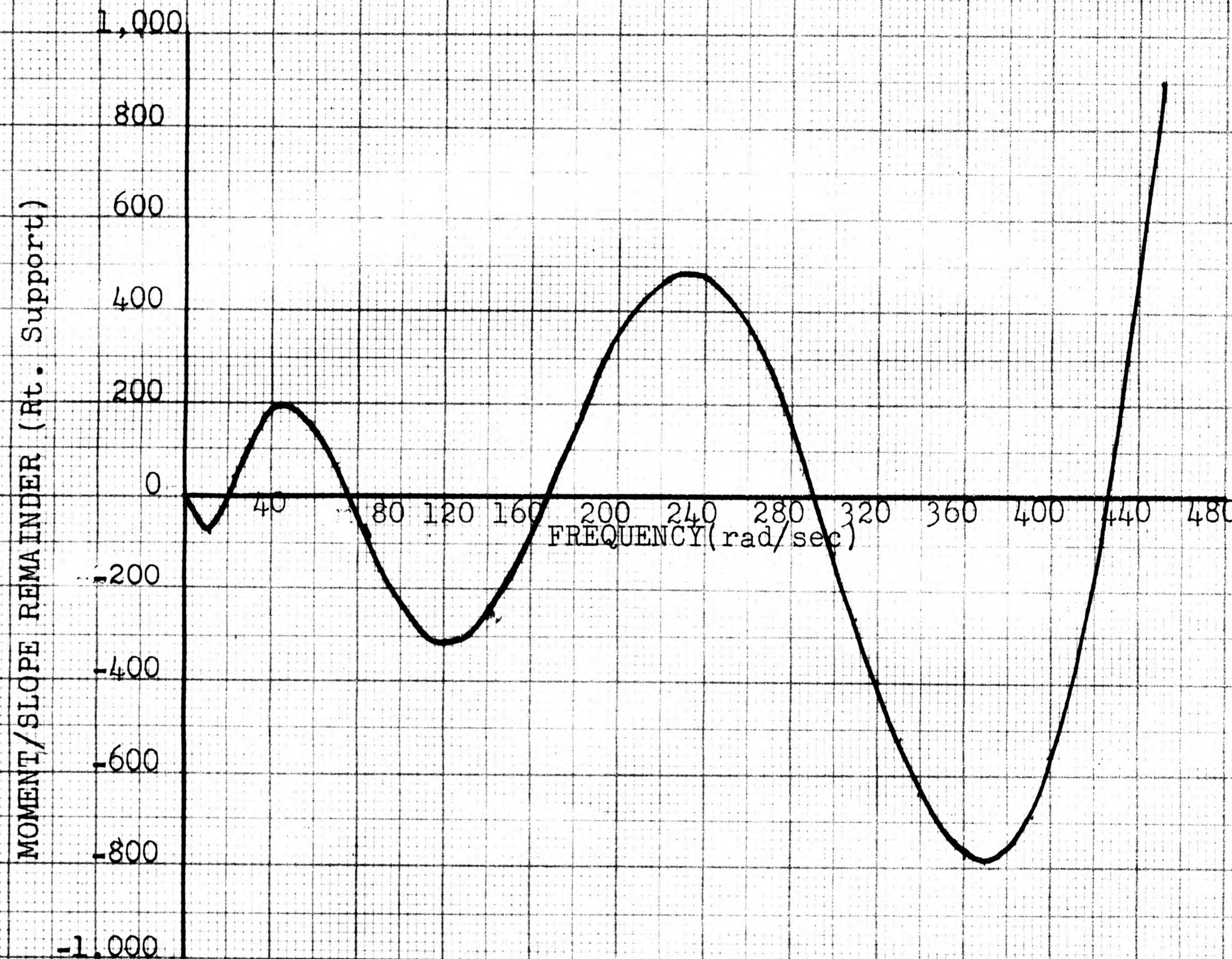


fig. 5

MOMENT/SLOPE REMAINDER (Rt. Support) versus FREQUENCY
FOR A
0.3 INCH DIAMETER SHAFT

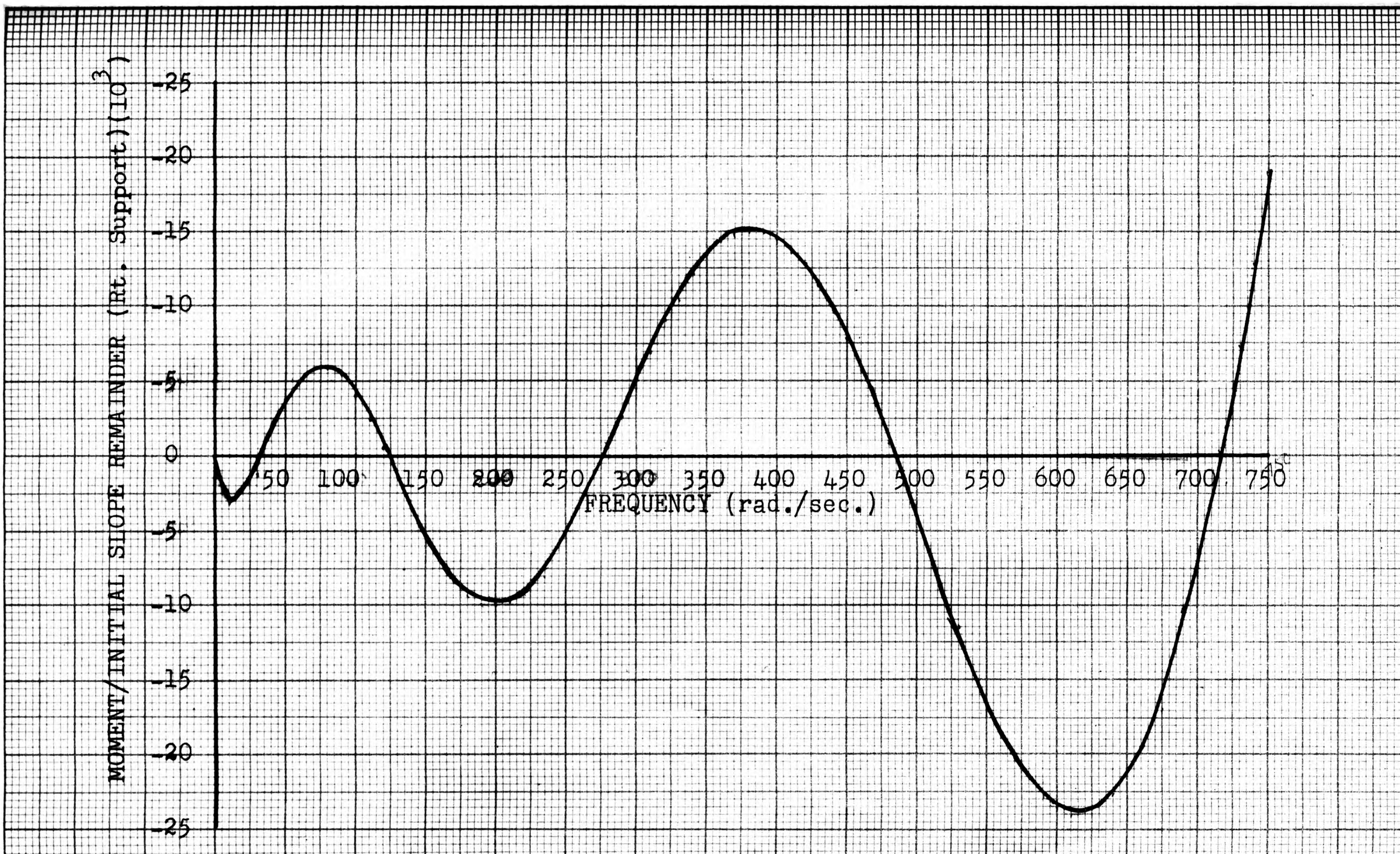


fig. 6

MOMENT/INITIAL SLOPE REMAINDER (Rt. Support) versus FREQUENCY
FOR A
0.4 INCH DIAMETER SHAFT

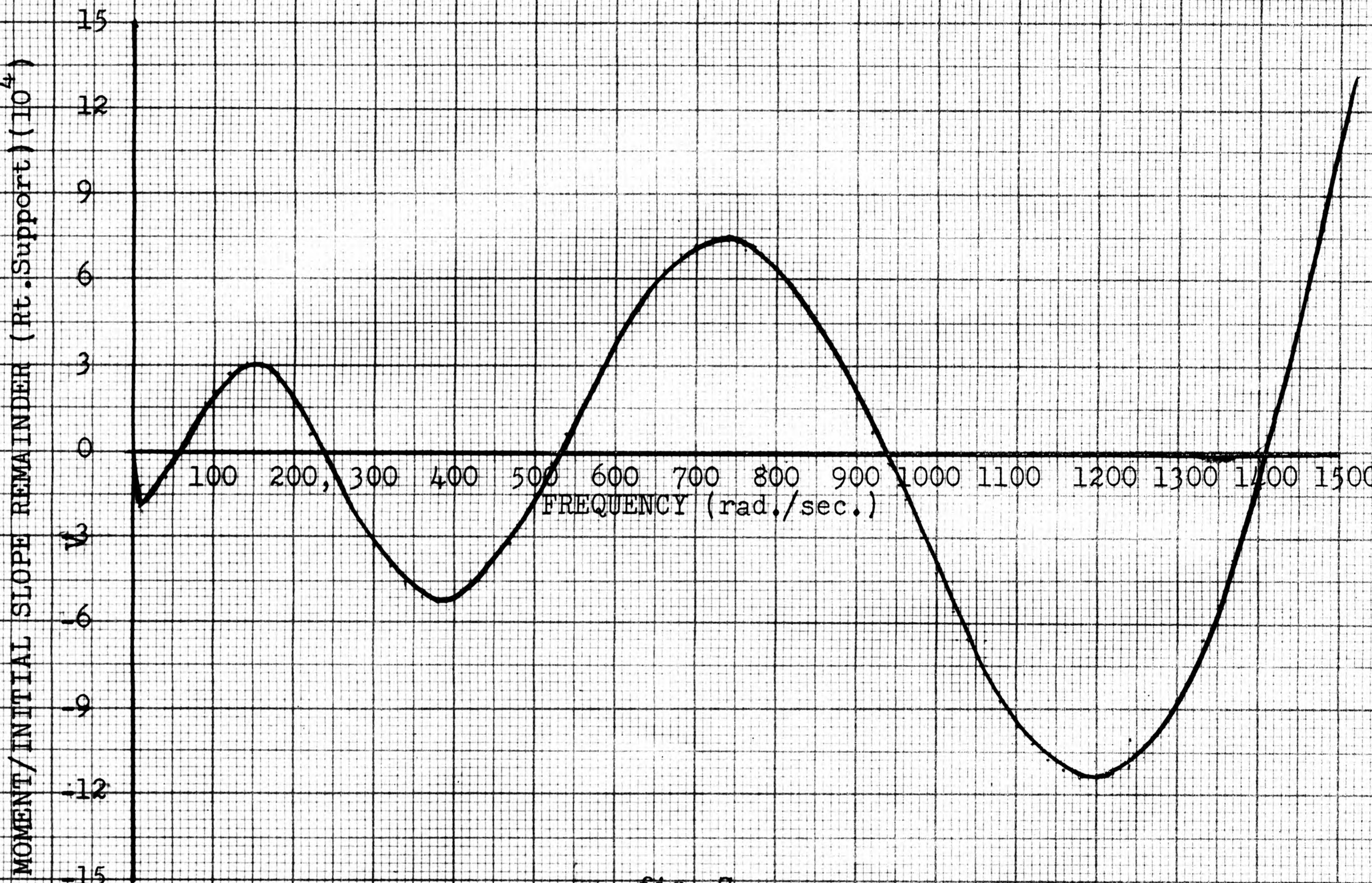


fig. 7

MOMENT/INITIAL SLOPE REMAINDER Rt. Support) versus FREQUENCY
FOR A
0.6 INCH DIAMETER SHAFT

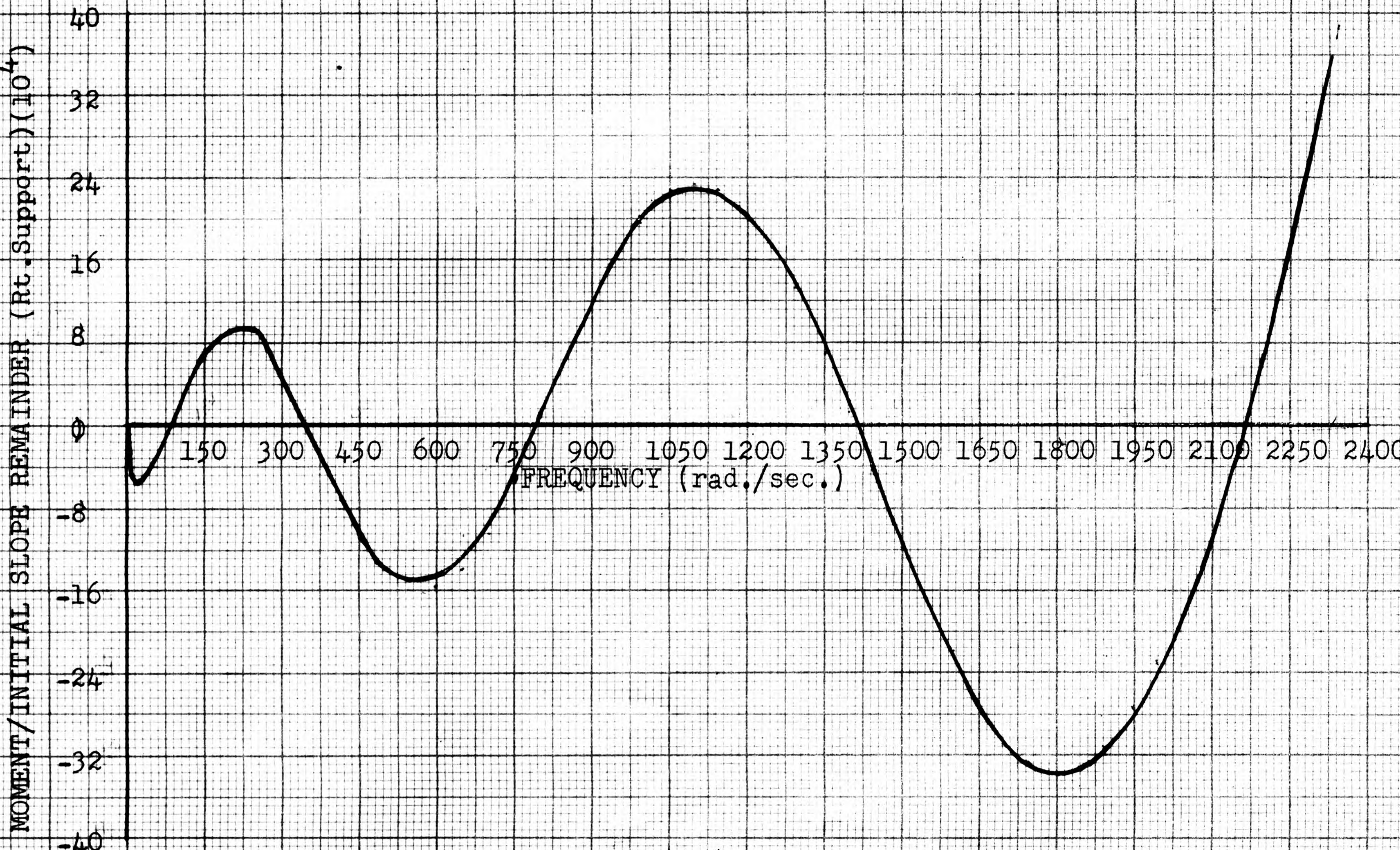


fig. 8

MOMENT/INITIAL SLOPE REMAINDER (Rt. Support) vs. FREQUENCY
FOR A
0.8 INCH DIAMETER SHAFT

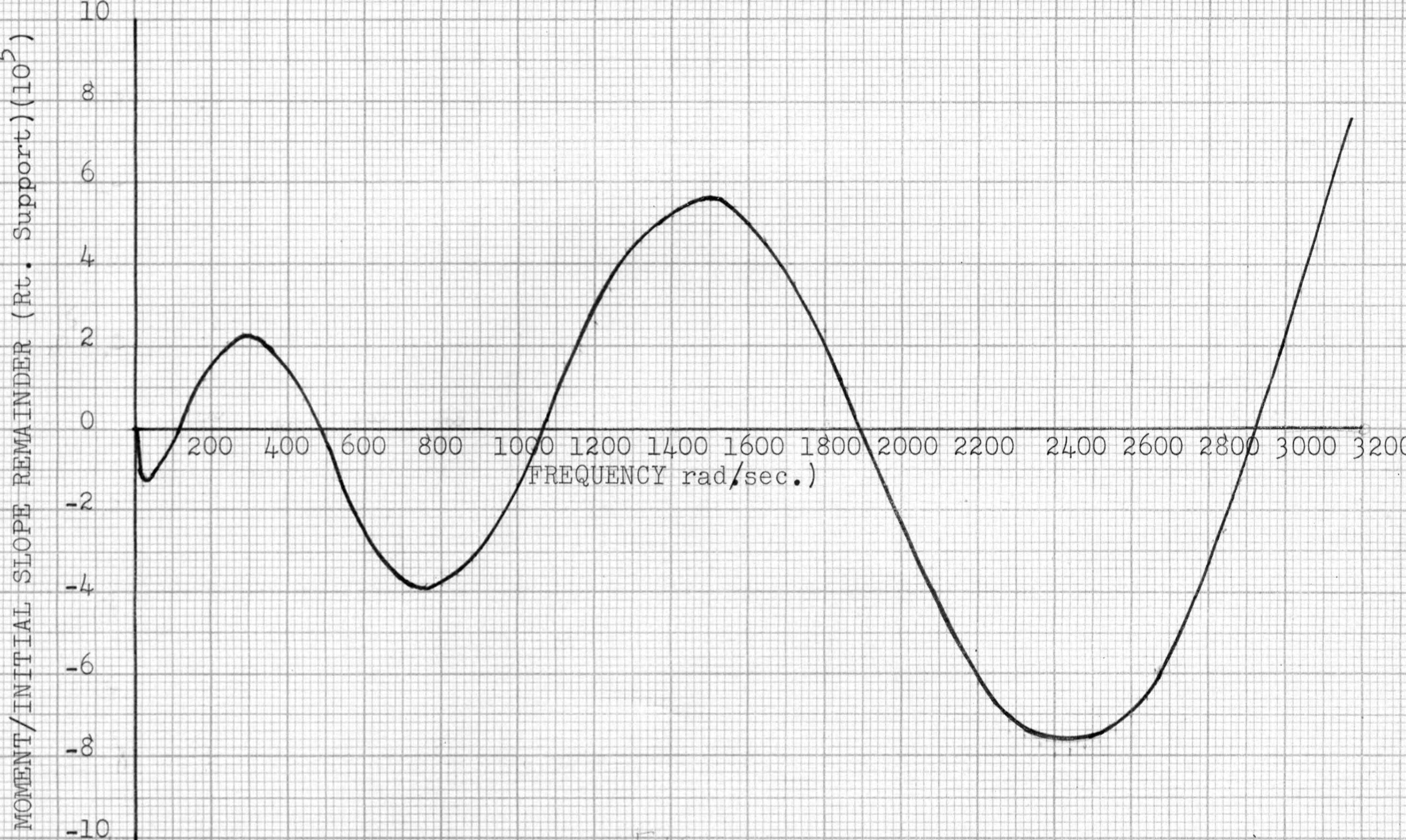


fig. 9

MOMENT/INITIAL SLOPE REMAINDER (Rt. Support) vs. FREQUENCY
FOR A
1.0 INCH DIAMETER SHAFT

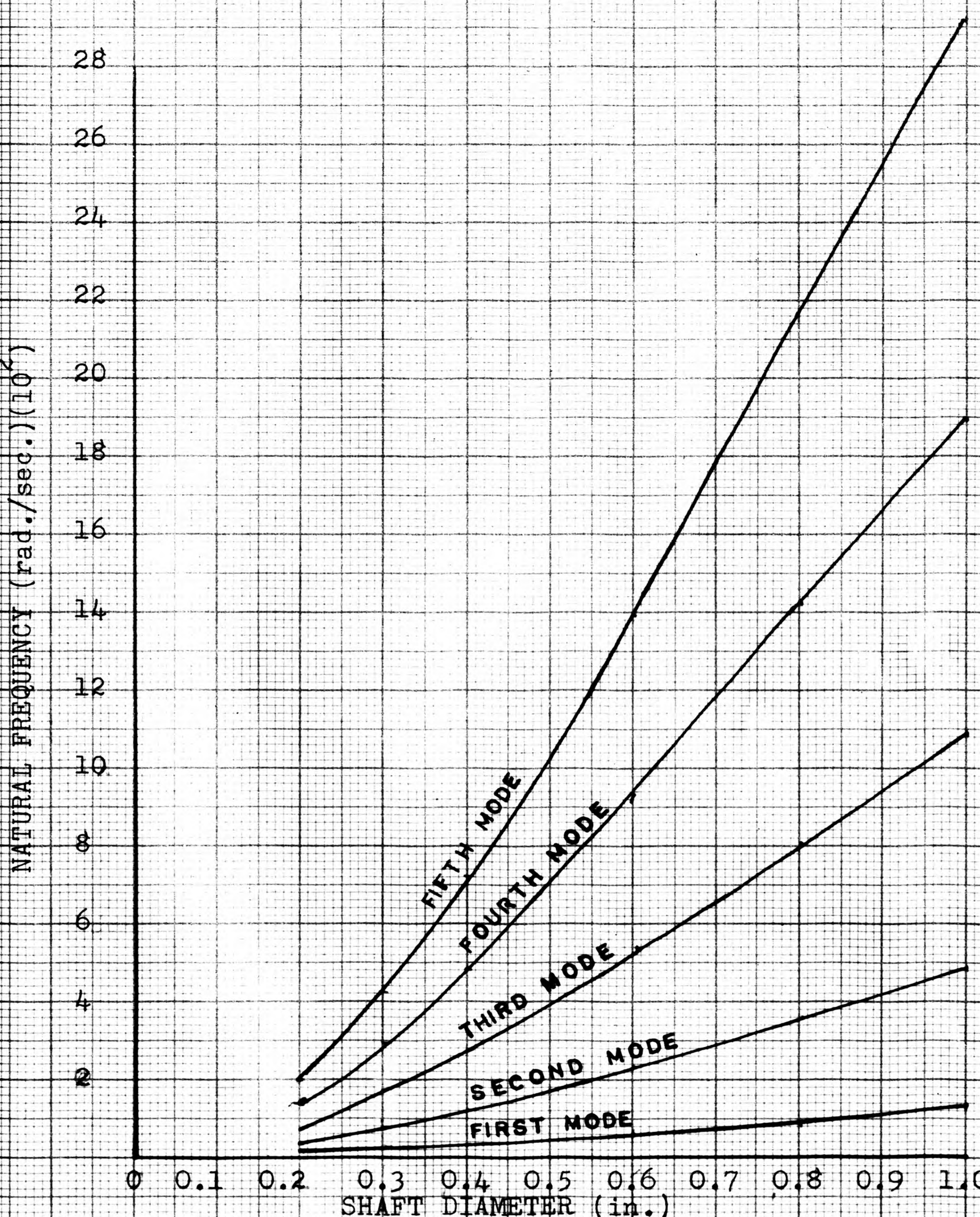


fig. 10

NATURAL FREQUENCY vs. SHAFT DIAMETER
FOR A
ROTATING SHAFT WITH FIVE MASSES

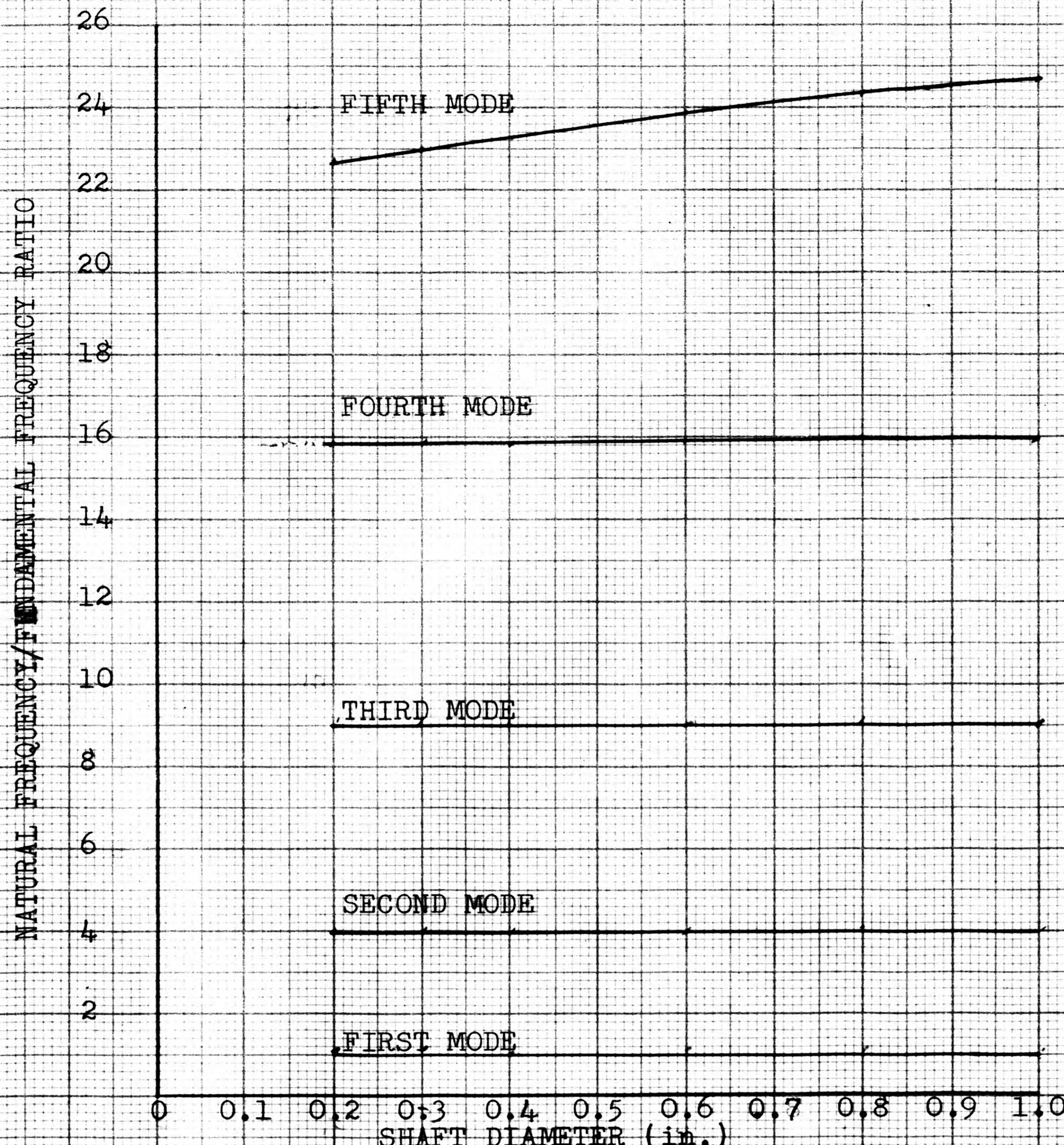


fig. 11

NATURAL FREQUENCY/FUNDEMENTAL FREQUENCY RATIO
vs. SHAFT DIAMETER
FOR A
ROTATING SHAFT WITH FIVE MASSES

A total of five equations can be written with five unknowns. Therefore, a solution for deflection at each station can be obtained. By taking these results for deflection and using the deflection/slope ratio plots in this thesis, a value for θ_o can be obtained for each case considered. With this value of θ_o deflection, shear stress and bending stress can be expressed as absolute values.

The deflection/initial slope versus beam length curves, Figure (12), are found by utilizing the deflection boundary condition (Equation 9a) at the right support. From this, one unknown V_o is found in terms of the second unknown θ_o. This ratio is then substituted at each station to eliminate V_o and obtain deflection in terms of initial slope. These calculations are made at each critical frequency for all diameter shafts.

The same procedure is used to find moment and shear force in terms of initial slope. For each of these a new ratio was set up using the same approach as above. The three ratios (Y_{RAT}, T_{RAT}, V_{RAT}) are equal at the natural frequency. Once moment and slope are obtained at each station, the maximum bending stress and shear stress are found by use of the following formula:

$$ST = \frac{T(NX)(D/2)}{I}$$

(11) $$SR = \frac{V\,Q}{I\,D} \qquad Q = \bar{y}\,dA$$

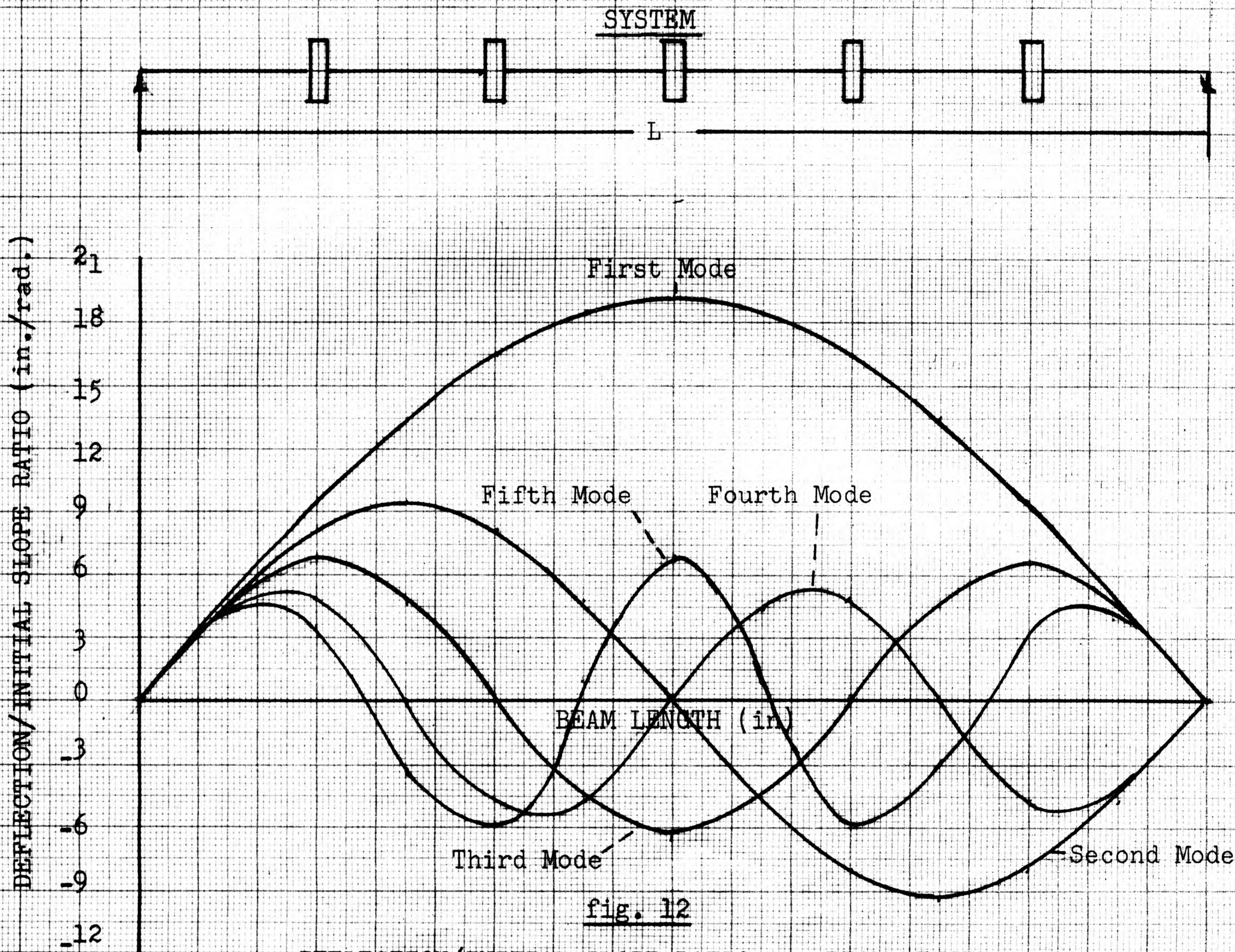


fig. 12

DEFLECTION/INITIAL SLOPE RATIO vs. BEAM LENGTH
FOR A
ROTATING SHAFT WITH FIVE MASSES

The values for bending stress in terms of initial slope are plotted for each mode, Figures(13-17). A curve is also drawn to show the magnitude of both bending, Figure (19), and shear stress, Figure (18), at each mode for each shaft examined.

The first and main problem to be examined was locating critical frequencies of a multi-mass rotational system, without damping. The only variable examined is shaft diameter, which ranges from 0.2 to 1 inch in increments of 0.2 for these relationships. The curves in Figures(4-9) are moment (right support)/initial slope ratio versus frequency. These curves, one for each diameter considered, cross the axis five times and go off to infinity after the last crossing. Each of these crossings represent a critical frequency.

The value of critical speed thus obtained considers the system in transverse vibration, where the system would not be rotating. In the actual case, rotation exists; therefore, if the shaft carries one or more disks the gyroscopic forces must be considered, (6). These forces tend to resist the shaft deflection or result in a moment opposing the inertia moment. This reduces the deflection and tends to raise the critical speed.

A second assumption was made that the bearings are rigid and do not deflect. This is erroneous as every bearing will deflect somewhat, due to loading conditions.

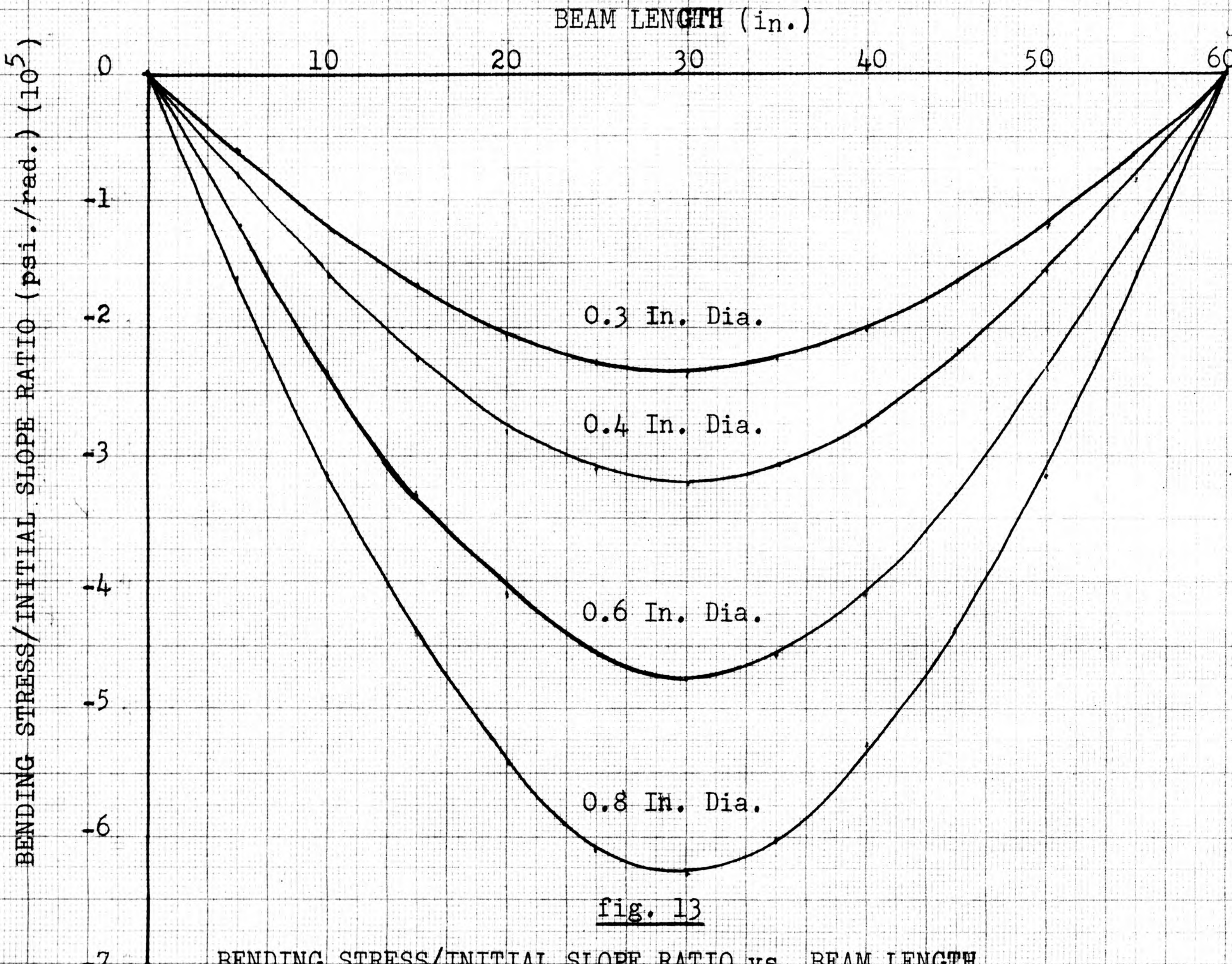


fig. 13

BENDING STRESS/INITIAL SLOPE RATIO vs. BEAM LENGTH
AT FIRST MODE

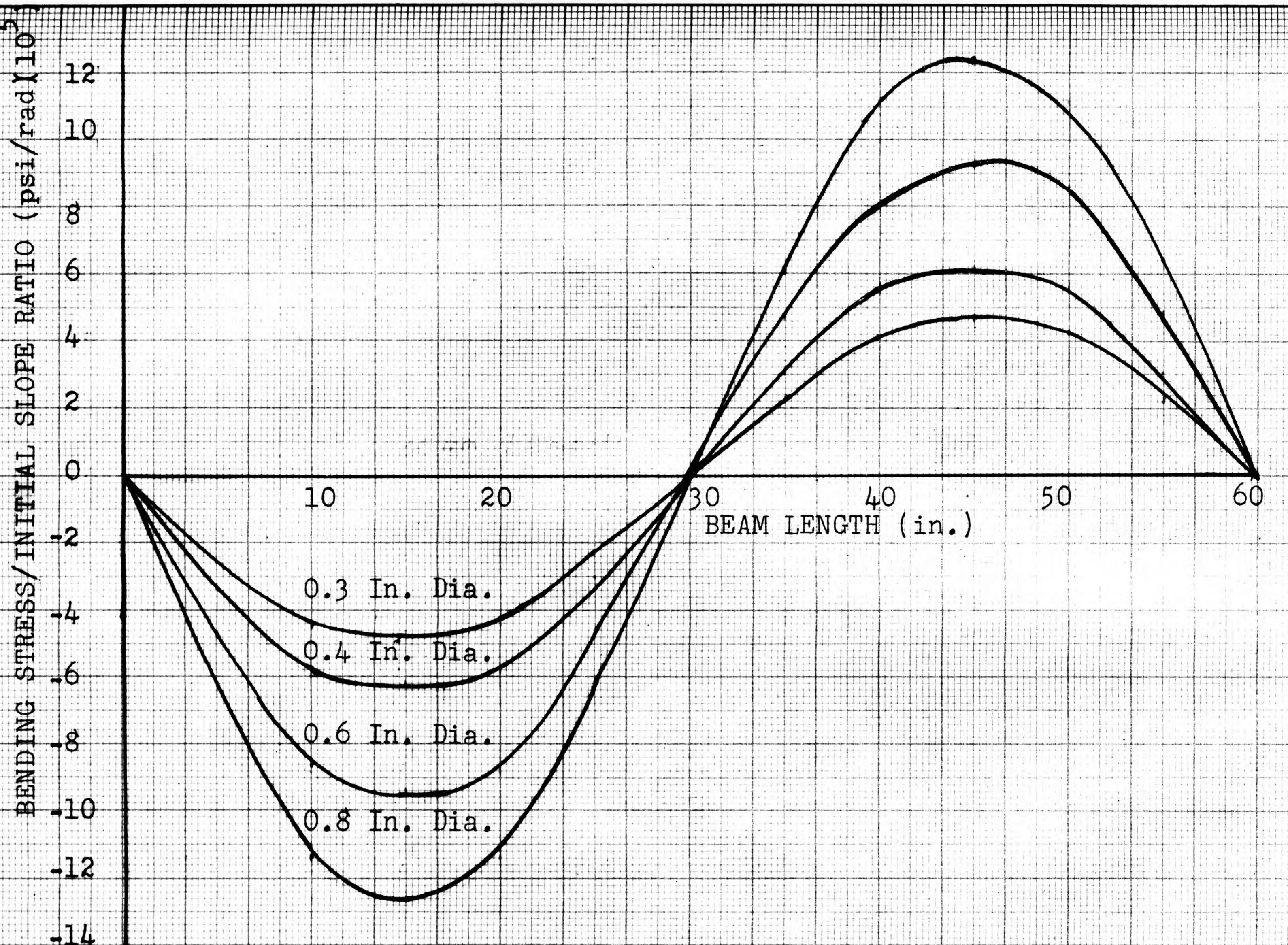


fig. 14

BENDING STRESS/INITIAL SLOPE RATIO vs. BEAM LENGTH
AT SECOND MODE

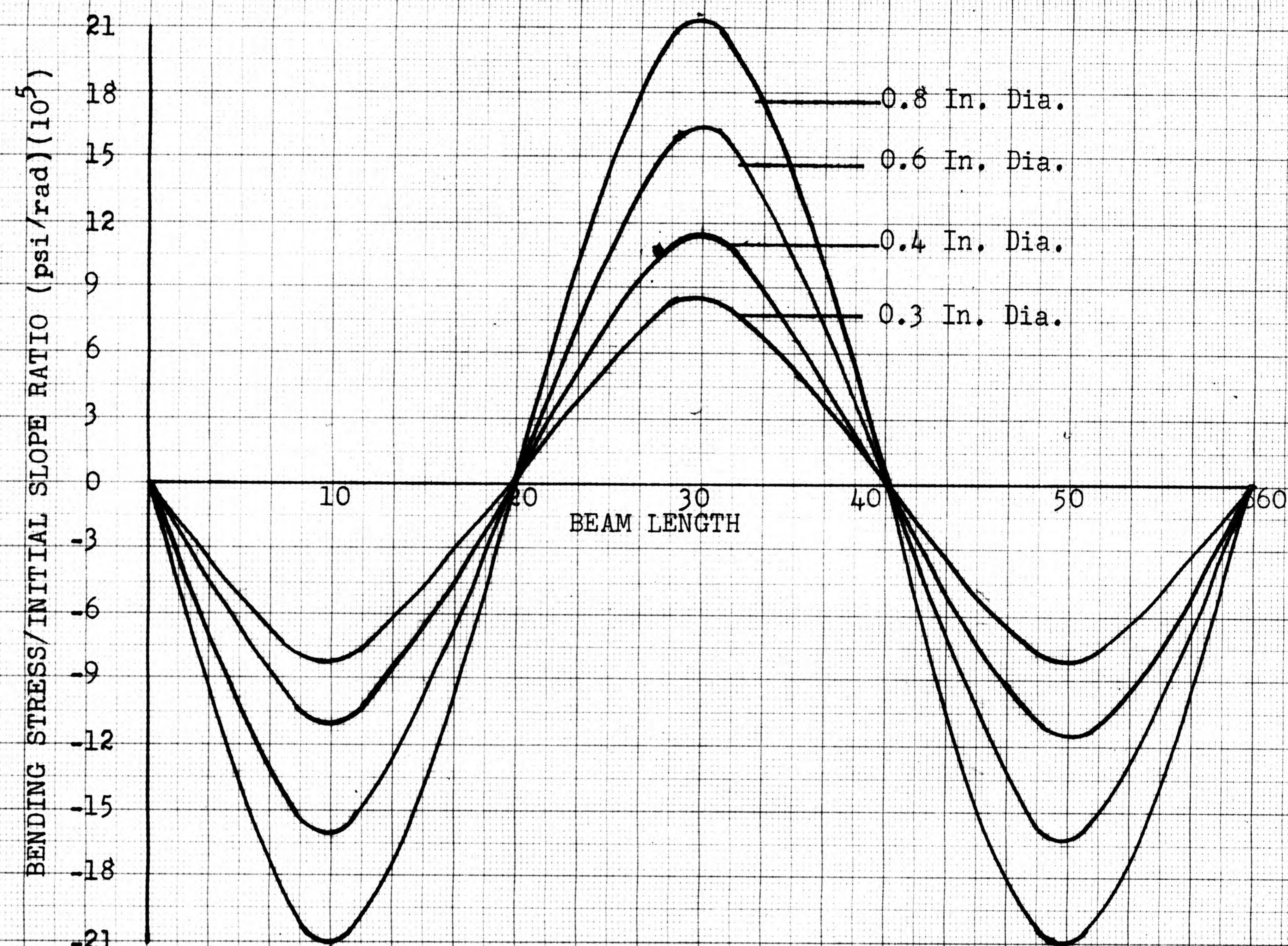


fig. 15

BENDING STRESS/INITIAL SLOPE RATIO vs. BEAM LENGTH
AT THIRD MODE

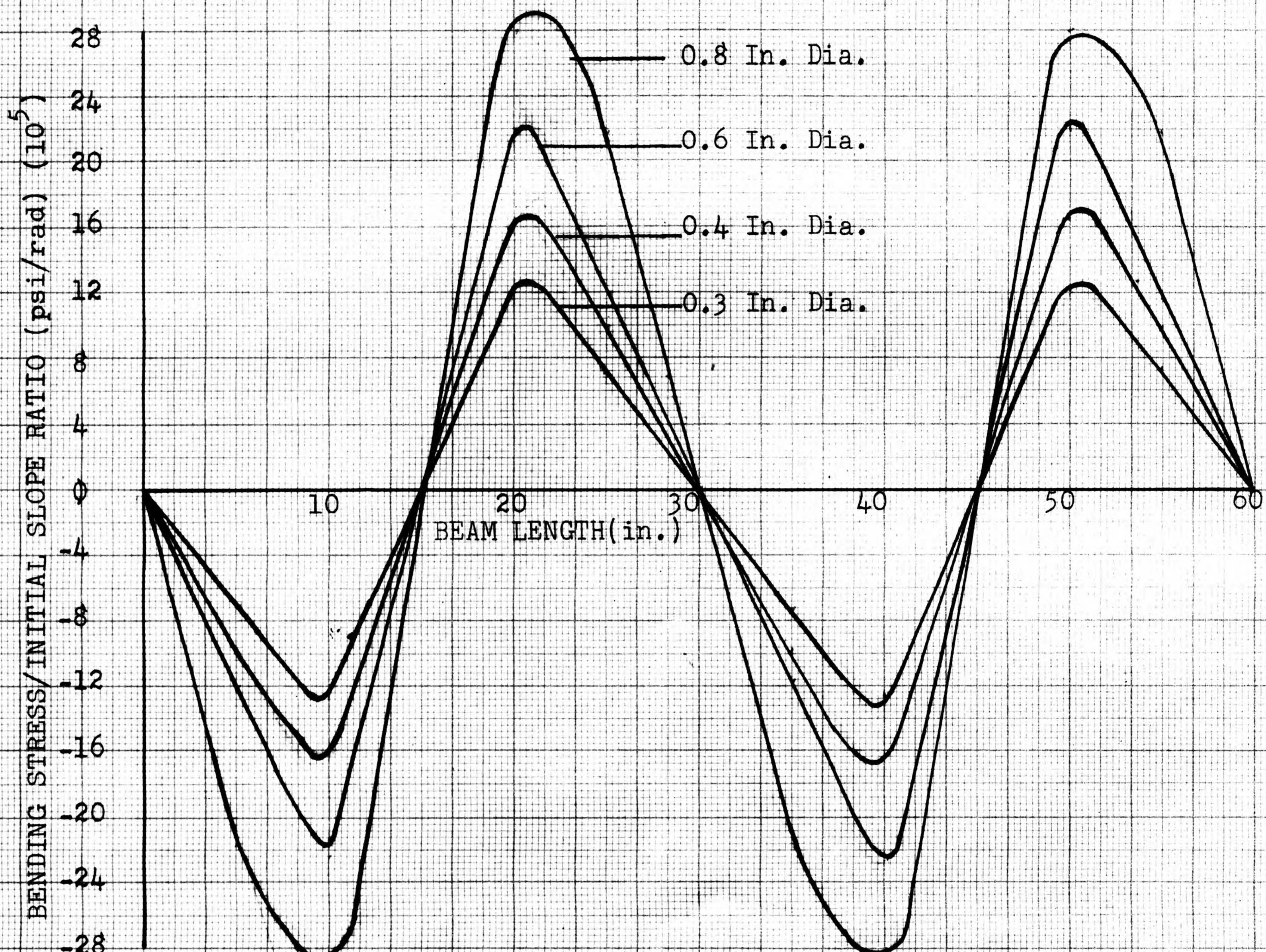


fig. 16

BENDING STRESS/INITIAL SLOPE RATIO vs. BEAM LENGTH AT FOURTH MODE

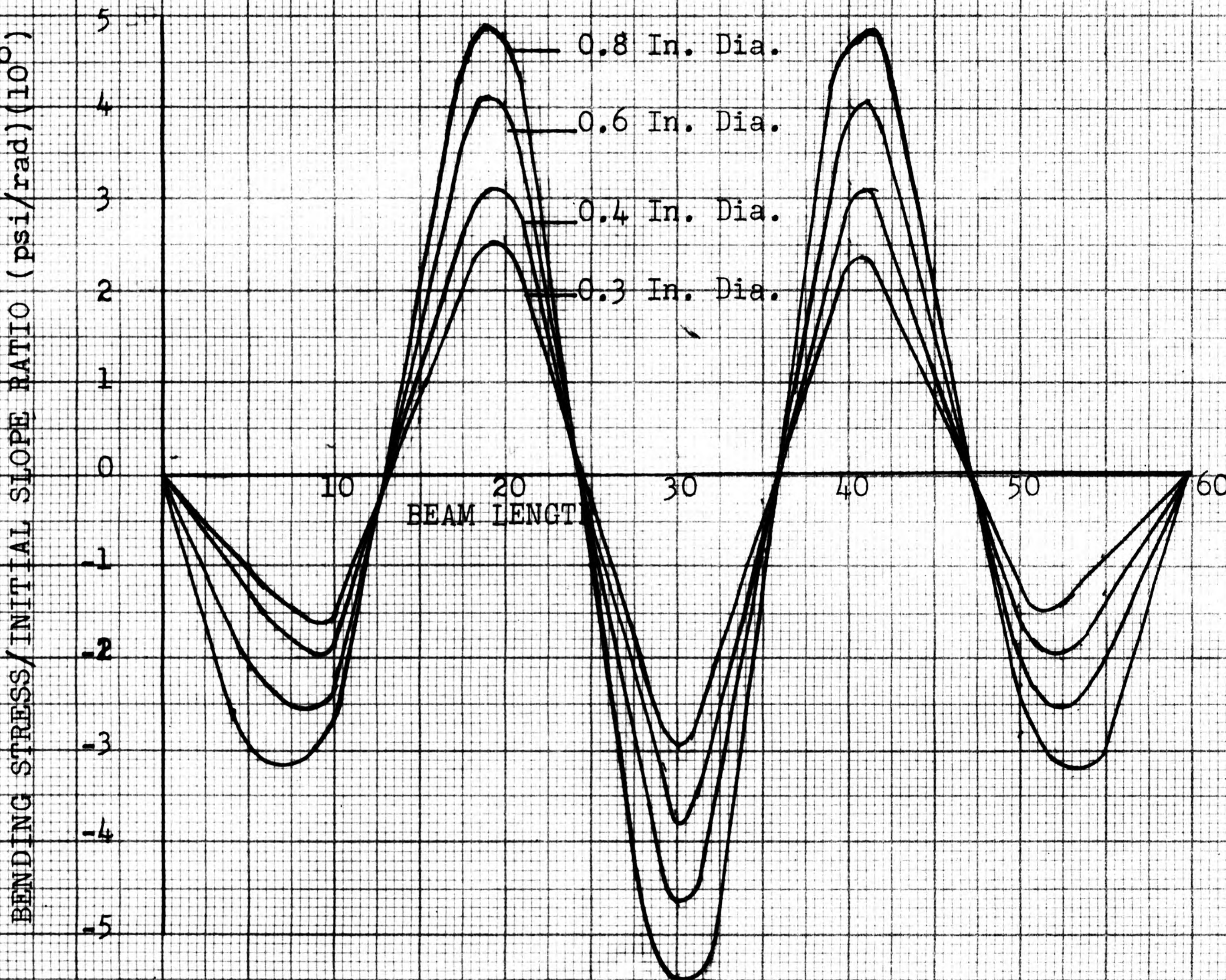


fig. 17

BENDING STRESS/INITIAL SLOPE RATIO vs. BEAM LENGTH
AT FIFTH MODE

This deflection will tend to lower critical speed as the shaft is more flexible.

Another assumption, which is not exactly true, is that of zero moment at the end supports. The moment approaches zero if the initial slope is small. For large slopes the assumption of zero moment is not true as bearings will tend to resist larger deflections. This assumption would tend to lower the critical speed of the system as it becomes a more flexible system when considering the moments (at supports) as zero.

Figure 10, critical speed versus shaft diameter, shows the rate at which the critical speed increases with shaft diameter. When the higher order critical speeds are compared to the fundamental, Figure 11, a definite ratio is obtained. For a diameter of 0.2 inch or a ratio of shaft mass to concentrated mass of 0.13 the following pattern (1:4:9:15.8:22.8) is obtained. At the other extreme the 1 inch diameter or a shaft mass to concentrated mass ratio of 3.3 is (1:4:9:16:24.8) and appears to be asymptotic at a ratio of (1:4:9:16:25), for large diameter shafts.

The shaft deflection/initial slope ratio versus shaft lenght, Figure 12, curves are examined by varying diameters from 0.3 to 0.8 inches. These curves show that the deflection curve has a definite form for each mode regardless of shaft diameter. This means that the deflection/slope ratio is independent of shaft diameter.

By examining the bending stress/initial slope ratio versus beam length curves, Figures (13-17) it is discovered that bending stress is dependent on shaft diameter. Only one value of bending stress is plotted, but both compressive and tensile stress of equal magnitude exist. The values plotted are on the bottom of the shaft relative to the deflection/initial slope ratio curves, Figure 12. It should be kept in mind that the shaft is in a state of indifferent equilibrium and this stress will alternate from positive to negative values, but the sign of stress slope ratio remains constant as slope changes to cancel sign of bending stress. Figure 19 shows a plot of maximum bending stress/initial slope ratio versus mode. It can be seen that this ratio increases greatly between the fourth and fifth modes. This increase is due to direction and magnitude of inertia forces and initial slope value.

From the shear stress/initial slope ratio verses beam length, Figure 18, a definite increase is noted as the mode of vibration increases. This increase is due to the inertia force which is proportional to frequency squared.

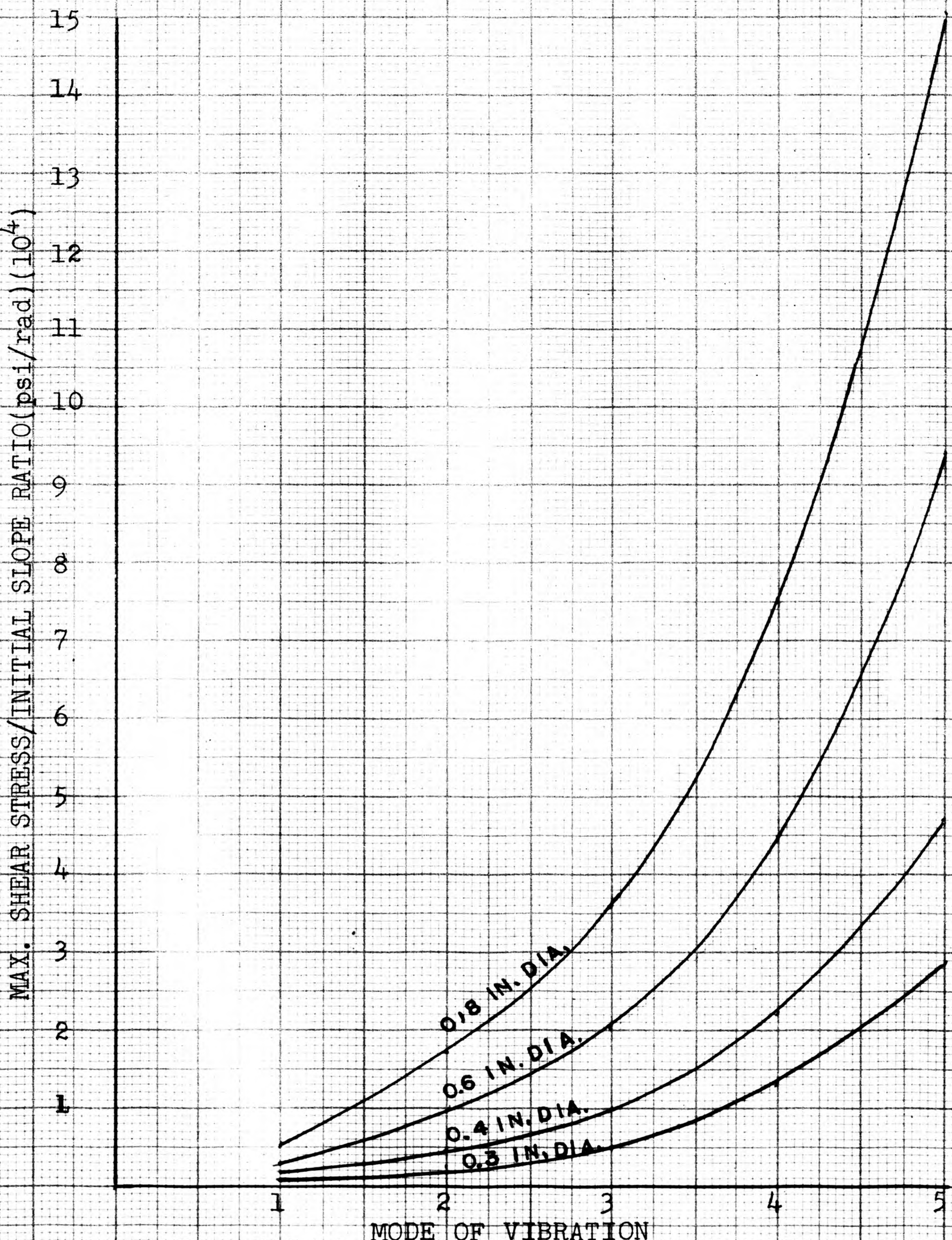


fig. 18

MAX. SHEAR STRESS/INITIAL SLOPE RATIO vs. MODE
FOR A
ROTATING SHAFT WITH FIVE MASSES

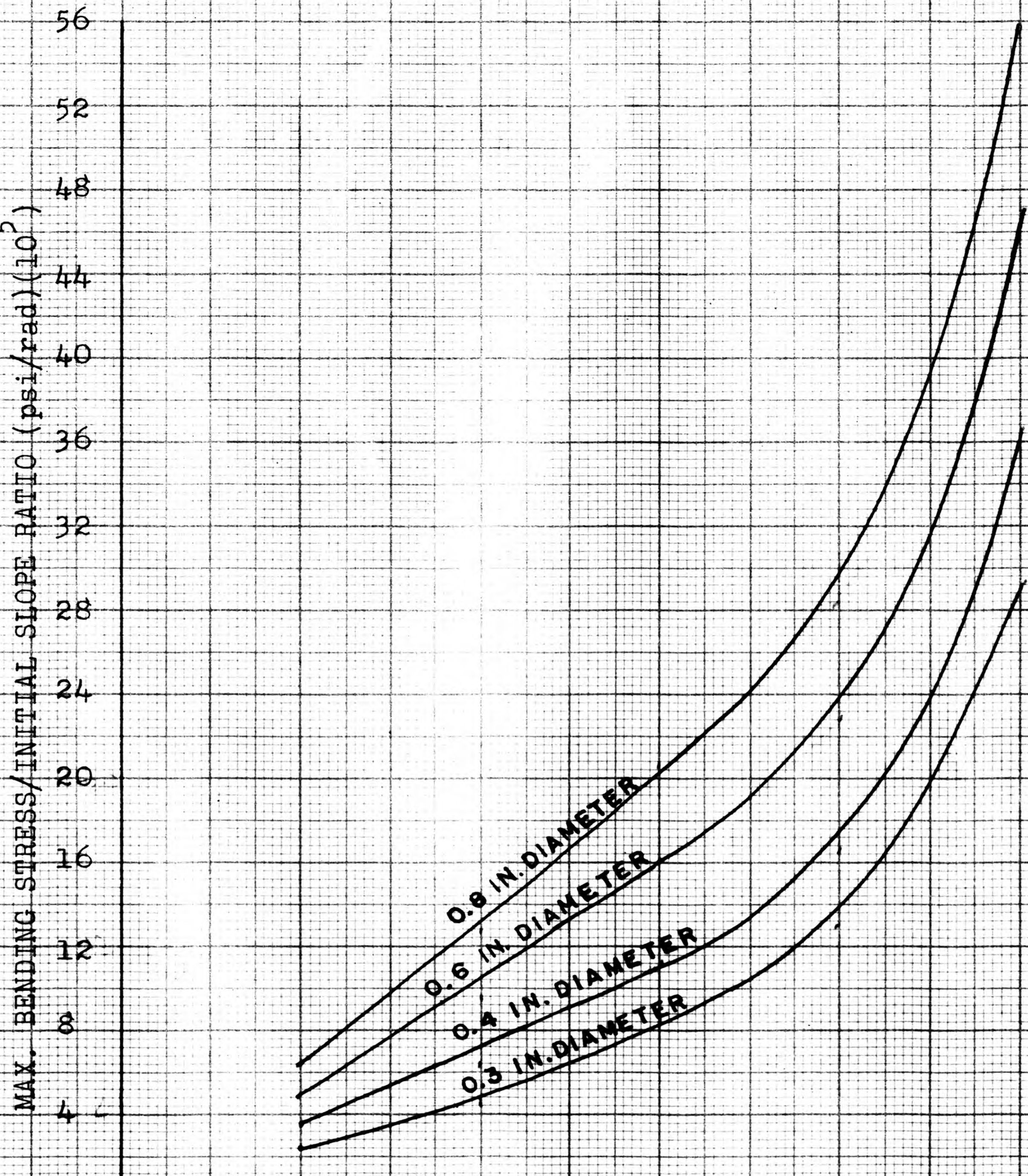


fig. 19

MAX. BENDING STRESS/INITIAL SLOPE RATIO vs. MODE
FOR A
ROTATING SHAFT WITH FIVE MASSES

CONCLUSION

A conclusion can be reached, from the discussion and results shown on the foregoing pages, that this approach to the problem of multiple critical speed can be adapted to a system with numerous supports and an infinite number of concentrated masses.

From an examination of the moment/initial slope remainder versus frequency curves, a definite relationship was discovered between the frequencies at different modes of vibration, Figure (11). By a comparison of the relationship obtained for the 0.2 inch diameter shaft (shaft mass/conc. mass ratio of 0.13) and the 1 inch diameter shaft (shaft mass/conc. mass ratio of 3.3) a maximum deviation of 8.04 percent is obtained at the fifth mode. If the asymptotic condition is compared to the 0.2 inch diameter shaft, a deviation of 8.8 percent exists at the fith mode. These deviations are acceptable when discussing critical speed, since the mathematical solution neglects many variables such as gyroscopic effect, bearing elasticity, and variation in boundary moment which would affect the actual system. Also it is impossible to operate a machine within ten percent of a critical frequency due to the large amplitudes of vibration. With these facts in mind, very good results could be obtained by using the basic energy method (for locating fundamental critical speeds) and the asymptotic relationship (1:4:9:16:25) for either concentrated or distributed loading conditions. A close examination of the above relationship shows that each number stands for the mode of vibration squared and could be extended to systems with more than five masses.

From the deflection/initial slope ratio versus beam length curve both deflection and slope can be expressed as inertia load divided by modulus of elasticity and moment of inertia times a constant. These curves, therefore, are useful for any diameter shaft or material as long as the beam dimensions are not changed.

From the bending stress/initial slope ratio versus beam length, it can be concluded that the only variables are area moment of inertia, and modulus of elasticity. As the diameter increases, the magnitude of this curve increases. By comparing the bending stress and shear stress ratios, the bending stress is shown to be the governing stress for a particular mode; and, therefore, would be the value to design the system by.

In conclusion, the author suggests that the results of this thesis be extended in future thesis to determine stress values and deflection values by the method shown in the discussion.

SUGGESTIONS FOR FUTURE THESIS TOPICS

1. Apply this method to shaft of three supports, two rigid supports or cantilever beam with more than one mass.

2. Electrical analogy of critical speed by using inductors, condenser and resistors to simulate mass, spring constant and friction respectively.

3. Investigate spring scale of simply supported shaft in the multi-mass system, to see if spring scale changes with change in mode.

4. Incorporate friction into the solution of critical speed either by this method, analog computer, or electrical analogy.

5. Continuation of this thesis by the method discussed in the discussion to determine the initial slope. With this value of initial slope and the stress graphs in this thesis find magnitude of bending and shear stress.

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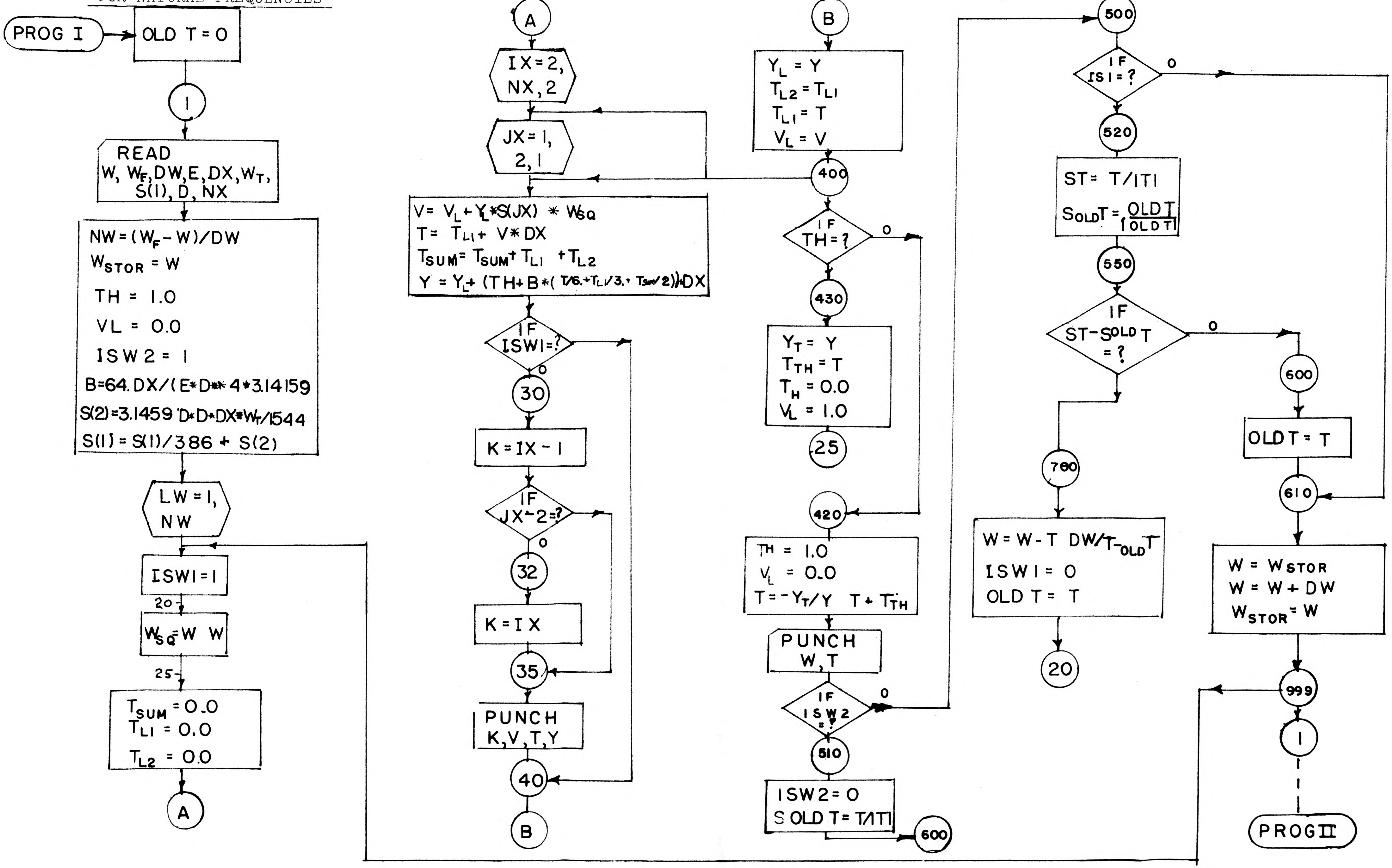
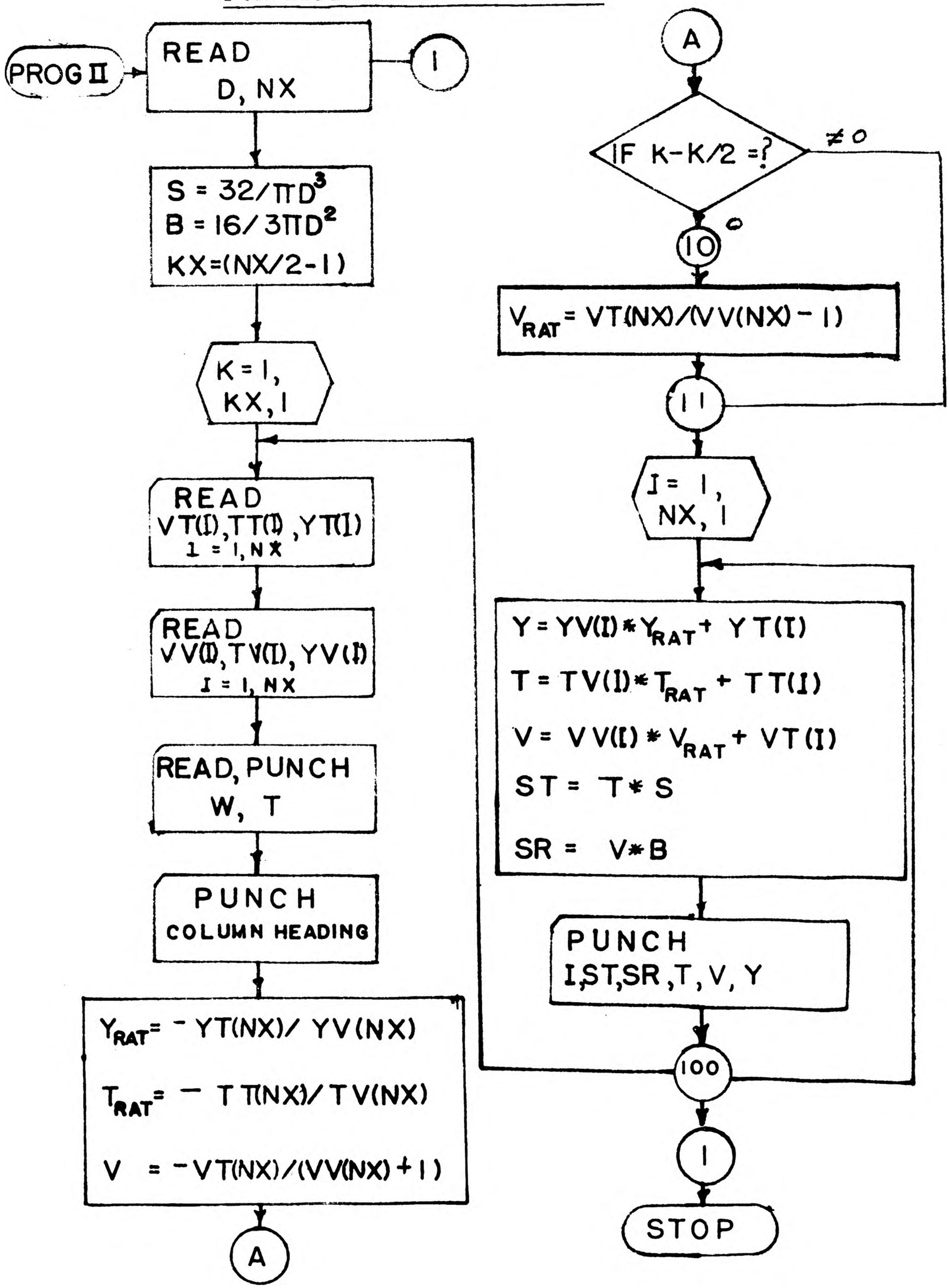
APPENDIX

COMPUTER FLOW CHART
FOR NATURAL FREQUENCIES



COMPUTER FLOW CHART
FOR DEFLECTION AND STRESS



```
C       CRITICAL SPEEDS OF A ROTATIONG MULTI-MASS SYSTEM  (R. J. IN)
C       W IS THE STARTING ROTATIONAL VEL. IN RAD/SEC        (COL  8)
C       WF IS THE FINAL       DITTO                         (COL 16)
C       DW IS THE DESIRED INCREMENT OF ROTATIONAL VEL.      (COL 24)
C       E IS THE MODULUS OF ELASTICITY                      (COL 32)
C       DX IS 1/2 THE DISTANCE BETWEEN THE LOADS            (COL 40)
C       WT IS THE WEIGHT/CUBIC INCH OF THE SHAFT            (COL 48)
C       S(1) IS THE CONCENTRATED LOAD (IN POUNDS)           (COL 56)
C       D IS THE DIAMETER OF THE SHAFT                      (COL 64)
C       NX (INTEGER) IS 2*(NO. OF LOADS +1)                 (COL 67)
C       THE ABOVE INPUT DATA MUST BE PUNCHED IN THE ORDER LISTED
C         WITH FORMAT (8E8.0,I3)
        DIMENSION S(2)
        OLDT = 0.0
      1 READ 1000, W, WF, DW, E, DX, WT, S(1), D, NX
        NW =(WF - W)/DW
        WSTOR = W
        TH = 1.0
        VL = 0.0
        ISW2 = 1
        B = 64.*DX/(E*D**4*3.1415927)
        S(2) = 3.1415927*D*D*DX*WT/1544.
        S(1) = S(1)/386. + S(2)
        DO 999 LW = 1, NW
        ISW1 = 1
     20 WSQ = W*W
     25 TSUM = 0.0
        YL = 0.0
        TL1 = 0.0
        TL2 = 0.0
        DO 400 IX = 2, NX, 2
        DO 400  JX = 1, 2
        V= VL+ YL*S(JX)*WSQ
        T = TL1 + V*DX
        TSUM = TSUM + TL1 + TL2
        Y = YL + (TH + B*(T/6.+TL1/3.+TSUM/2.))*DX
        IF (ISW1)40,30,40
     30 K = IX-1
```

```
      IF (JX -2)35,32,35
   32 K = IX
   35 PUNCH 1001, K, V, T, Y
   40 YL = Y
      TL2 = TL1
      TL1 = T
  400 VL = V
      IF (TH)430,420,430
  430 YT = Y
      TTH = T
      TH = 0.0
      VL = 1.0
      GO TO 25
  420 TH = 1.0
      VL = 0.0
      T = -YT/Y * T + TTH
      PUNCH 1002, W, T
      IF (ISW2)510,500,510
  510 ISW2 = 0
      SOLDT = T/ABSF(T)
      GO TO 600
  500 IF(ISW1) 520,610,520
  520 ST = T/ABSF(T)
      SOLDT = OLDT/ABSF(OLDT)
  550 IF (ST - SOLDT)700,600,700
  600 OLDT=T
  610 W=WSTOR
      W = W + DW
  999 WSTOR = W
      GO TO 1
  700 W = W-T*DW/(T - OLDT)
      ISW1 =0
      OLDT=T
      GO TO 20
 1000 FORMAT (8E8.0,I3)
 1001 FORMAT (3H K=,I3,5H,  V=,E14.7,5H,  T=,E14.7,5H,  Y=,E14.7)
 1002 FORMAT (3HOW=,E14.7,5H,  T=,E14.7)
      END
```

```
 1604
C      CALCULATION OF SHAFT DEFLECTION AND STRESS AT CRITICAL SPEED.
       DIMENSION TT(12),YT(12),TV(12),YV(12),VT(12),VV(12)
     1 READ 1001, D, NX
       S = 32.0/(3.1415926 * D**3)
       B = 16.0/(3.0 * 3.1415926 * D**2)
       KX = NX/2 - 1
       DO 100 K= 1, KX
       READ 1002, (VT(I), TT(I), YT(I),I=1, NX)
       READ 1002, (VV(I), TV(I), YV(I),I=1, NX)
       READ 1003, W, T
       PUNCH 1003, W, T
       PUNCH 1004
       YRAT = -YT(NX)/YV(NX)
       TRAT = -TT(NX)/TV(NX)
       VRAT = -VT(NX)/(VV(NX)+1.0)
       IF(K-(K/2)*2)10,10,11
    10 VRAT=-VT(NX)/(VV(NX)-1.0)
    11 DO 100 I=1, NX
       Y = YV(I)*YRAT + YT(I)
       T = TV(I)*TRAT + TT(I)
       V = VV(I)*VRAT + VT(I)
       ST = T * S
       SR = B * V
   100 PUNCH 1005, I, ST, SR, T, V, Y
       GO TO 1
  1001 FORMAT (56X,E8.0,I3)
  1002 FORMAT (11X,E14.7,5X,E14.7,5X,E14.7)
  1003 FORMAT (3HOW=,E14.7,5H,  T=,E14.7)
  1004 FORMAT (2H K,5X,11HBEND STRESS,3X,12HSHEAR STRESS,6X,6HMOMENT,
      110X,5HSHEAR,7X,10HDEFLECTION)
  1005 FORMAT(1X,I2,5(3X,E12.5))
       END
```

VITA

The author, Richard Kenneth Brockmann, was born in Ozone Park, New York on March 22, 1939. He attended John Adams High School in Queens, New York and graduated in June 1956. In January 1957 he enrolled in the Mechanical Engineering Department at the Missouri School of Mines and Metallurgy. On August 13, 1961 the author was married to Doris June Baier of Huntington, New York. Upon receiving a Bachelor of Science Degree in Mechanical Engineering from the School of Mines and Metallurgy in January 1962 he enrolled at the School as a Graduate Assistant to do graduate work leading to a Master of Science Degree in Mechanical Engineering.