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Triangles, Triangles and, Yes, More Triangles: Explorations in Euclidean Ramsey Theory

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Triangles, Triangles and, Yes, More Triangles:
Explorations in Euclidean Ramsey Theory

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Explorations in Euclidean Ramsey Theory

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Thesis Abstract

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Abstract:

Several important general theorems of Euclidean Ramsey Theory are presented with an emphasis on trying to prove or disprove the 1973 conjecture of Erdös et al. that for all triangles, except for equilateral triangles, it is possible to find a monochromatic coloring of the vertices in any two coloring of the plane. Further investigation included looking at triangles in greater dimensions.

Introduction

In 1928, Frank Plumpton Ramsey published his first and only paper, "On a Problem of Formal Logic" [13], in the field which was later to be named after him. Like Ramsey, other early contributors to Ramsey theory looked at interesting problems, only to continue with research in other fields. Isaac Schur, for example, attempted to prove Fermat's Last Theorem with Ramsey theory, and B. L. van der Waerden solved only one problem before returning to his research in algebraic geometry. It was not until two decades ago that Ramsey theory emerged as a clear sub-discipline of combinatorial analysis. Today, Ramsey theory is used to solve geometrical problems, as well as problems in communication networking and informational retrieval.

The focus of this paper is Euclidean Ramsey theory, which, as the name suggests, deals with n-dimensional Euclidean space, E^n. This space is considered to be r-colored, if each point in the space is randomly colored one of r colors. By connecting a finite number of points, a finite configuration or geometrical shape, K, is created. If all the vertices of K have the same color then the figure is said to be monochromatic. Putting these definitions together, it is possible to define the relation R(K,n,r):

DEFINITION. Let K be a finite configuration. If under any r-coloring of the points of E^n, there exists a monochromatic coloring of the vertices of K′ which is congruent to K then the relation R(K,n,r) holds [10, 116].

DEFINITION. A configuration K is said to be *Ramsey* if, for all r, there exists n′ so that, for $n \geq n'$, R(K,n,r) holds

[10, 117].

In this paper, I will discuss some important theorems of Euclidean Ramsey theory and consider some specific questions for various configurations, with the main emphasis placed on the relation R(K,n,r) where n = r = 2 and K is a triangle.

Background

The simplest question one could ask about Euclidean Ramsey theory is whether or not a line segment of any length is Ramsey. The following theorem shows that the answer is yes when n = r = 2:

Theorem 1: Given any 2-coloring of a plane, it is possible to find two points of the same color at a distance d apart.

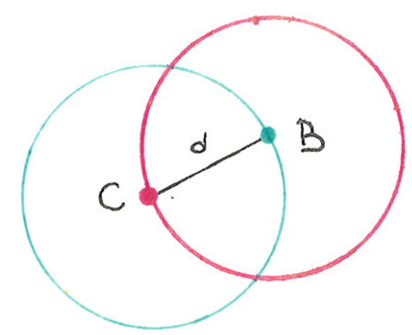


Figure 1

Proof: Assume, to the contrary, that no such monochromatic line segment exists. Let point C be red; this means that there must be a green circle of radius d around C so that no red point exists at distance d from C. However, each green point on the circle must also have a circle of radius d about it, on which no green point exists. Choose an arbitrary point B on the circle about C and draw a circle of radius d about it. This circle will consist of all red points. However, the circles with centers C and B have to intersect in two points, and both points can only be colored once. Thus there exist two points a distance d apart which are monochromatic in any 2-coloring of R^2.□

From here, there are two directions in which questions may be asked. The first, is what happens to a line segment in E^3?

By the same argument of Theorem 1, it is easy to discover that any line segment is Ramsey in E^3. Alternatively, one could ask whether by adding an additional point the resulting triangle is still Ramsey in E^2. This question is more complicated; in fact R(K,2,2) does not hold for all triangles K:

Theorem 2: The relation R(K,2,2) is false when K is an equilateral triangle.

Proof: An equilateral triangle of side α has a minimum height of $\sqrt{3}\alpha/2$. If the plane is striped with two colors in widths just short of $\sqrt{3}\alpha/2$, i.e., each stripe covers the half open interval $[x_0, x_0 + \sqrt{3}\alpha/2)$ then no monochromatic triangle can occur.

However, we will see that the relation R(K,n,r) does hold for 30-60-90 triangles (see Thm. 7). Graham et al. [10, 117], conjectured that R(K,2,2) is true for all triangles except equilateral.

Before continuing our discussion of triangles, let us step back and look at some important theorems of Euclidean Ramsey theory. Again a few definitions are needed to begin.

DEFINITION. A *brick* is the set of vertices of a rectangular parallelepiped.

DEFINITION. A configuration $K = \{v_0, v_1, \ldots, v_n\}$ of points in E^m is *spherical* if it is embedded in the surface of a sphere, that is, if there is a center x and a radius r so that $|v_i - x| = r$ for all $v_i \in K$ [5, 348].

Two very important theorems proved by Erdös et al. are:

Theorem 3: If K is a subset of a brick, it is Ramsey [5, 358].

Theorem 4: If K is not spherical, it is not Ramsey [5, 349].

Theorem 4 is the strongest known restriction on Ramsey configurations and all known Ramsey configurations are subsets of bricks. It is not known, however whether being a subset of a brick is a necessary condition for a Ramsey configuration.

In order to prove that any brick is Ramsey, the compactness principle must be used. This principle is key in all areas of Ramsey theory, not just Euclidean Ramsey theory. Before stating the principle, however, we need some more notation and definitions.

> DEFINITION. Let $H = (V,E)$ be a hypergraph where V equals E^n and E is a family of subsets of V containing a fixed finite number of points. Let $W \subseteq V$. The restriction of H to W, denoted H_W, is the hypergraph $H_W = (W, E_W)$ where $E_W = \{x \in E: x \subseteq W\}$ [10, 13].

For example, let V equal the set of all points in E^2. Let $E = \{\{x, y, z\}: x, y, z$ are the vertices of an equilateral triangle of side $\alpha\}$. If W consists of the points in the first quadrant then E_W is all the equilateral triangles of side α which are in the first quadrant.

> DEFINITION. The *chromatic number* $\chi(H)$ of a hypergraph H is the minimal r such that a map $\chi: V \rightarrow \{1,...r\}$ of H exists where *no* $x \in E$ are monochromatic [10, 11].

For the above example, $\chi(H) = 2$ since Theorem 2 tells us that the relation $R(K,2,2)$ does not hold when K is an equilateral triangle. Now the compactness principle states:

> **Theorem 5:** Let $H = (V,E)$ be a hypergraph where all $x \in E$ are finite. Suppose that, for all $W \subseteq V$, W finite
> $$\chi(H_W) \leq r;$$
> Then $\chi(H) \leq r$ [10, 13].

So, how does the compactness principle apply to bricks? In order to see this, we need a more precise definition of a brick.

DEFINITION. A brick in E^n is any set congruent to a set $B = \{ (x_1, \ldots, x_n) \mid x_i = 0, a_i; a_i \geq 0; 1 \leq i \leq n\}$ [5, 357].

Using this second definition, if we let $K_i = \{0, a_i\}$, a brick can be written as $B = K_1 X \ldots X K_n$. Each set K_i is Ramsey in a finite subset of E^2. So the following theorem applies:

Theorem 6: If K_1 and K_2 are Ramsey, then so is $K_1 X K_2$ [5, 357].

The proof of Theorem 6 requires the use of the compactness principle.

Triangles

Triangles are indeed spherical so from the previous section we know that it is possible that the relation R(K,2,2) holds for all triangles except equilaterals. In trying to prove this conjecture, I looked at "special" triangles, such as right, isosceles and 30-degree triangles. My first successful steps were made with right triangles. I discovered that:

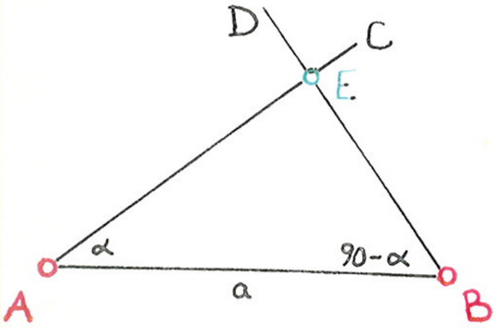
Theorem 7: If $N \equiv 2 \bmod 4$ and $N \geq 6$ then there exists ¼(N - 2) right triangles for which the relation R(K,2,2) holds, namely

$$T_1, T_2, \ldots, T_i, \ldots, T_{¼(N-2)}.$$

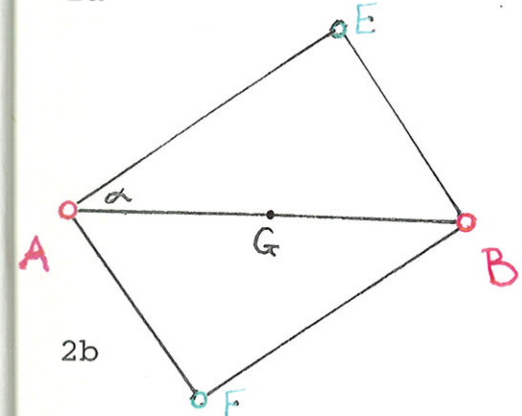
For each i, $\alpha_i = 180i/N$ is the smallest angle of T_i.

To understand the proof, it will be useful to look at the 90-36-54 triangle of Figure 2 alongside the proof. To construct the initial triangle:

Let $|AB| = a$. Create AC at angle $\alpha_i = 180i/N$ to AB, where i=1,2,...,(N-2)/4. Construct BD at angle $90-\alpha_i$ to AB.



2a



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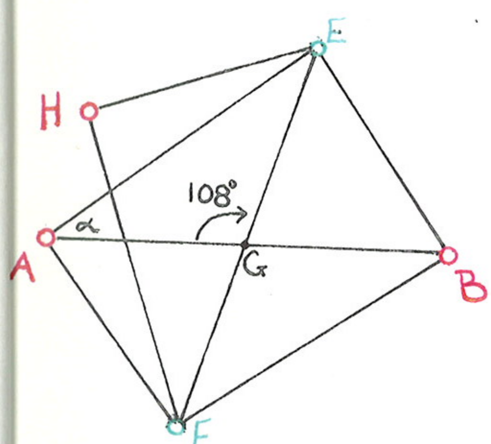

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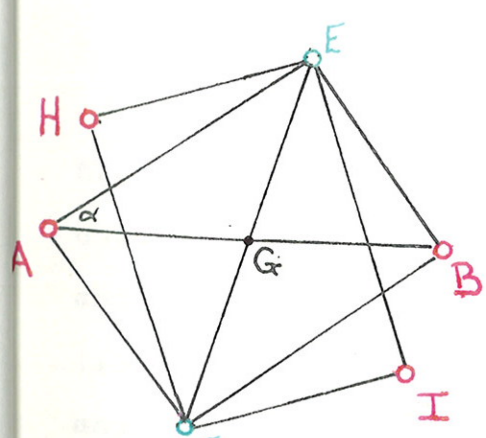
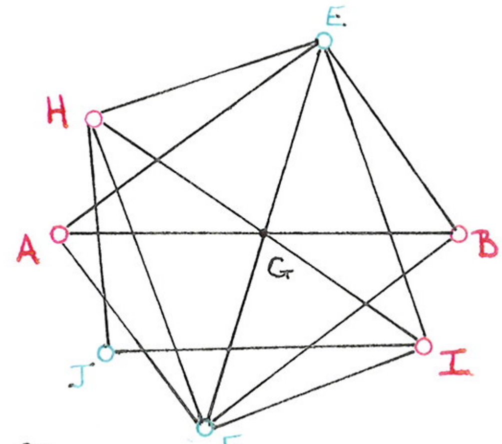
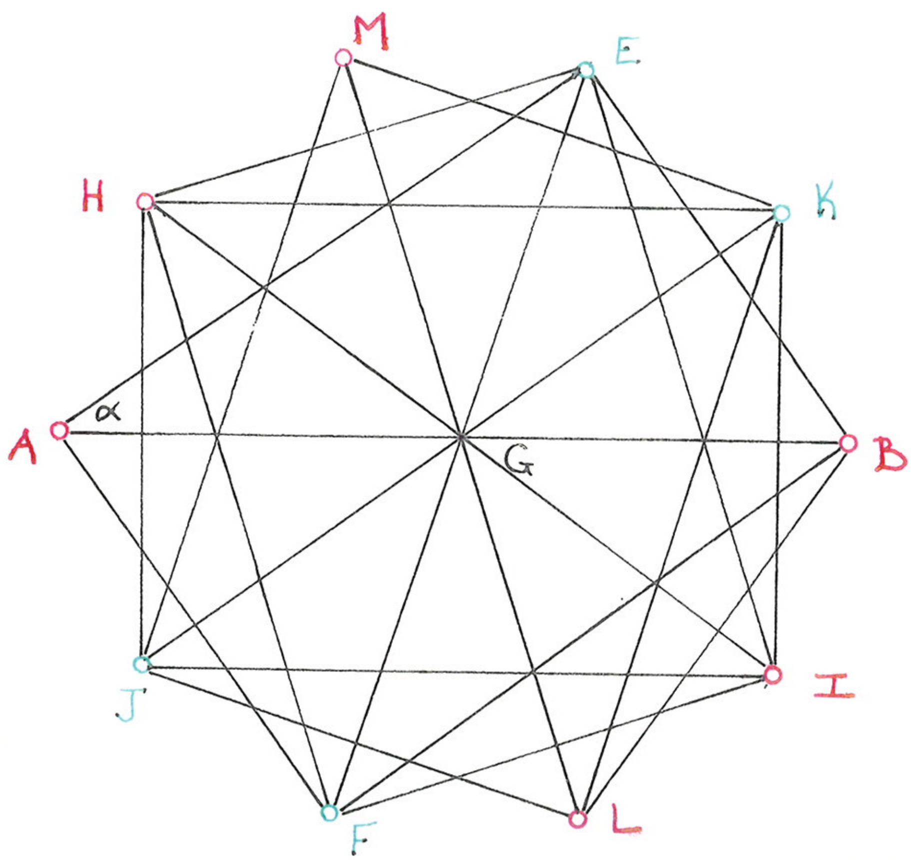


2d



2e



2f

Figure 2

BD will intersect AC at some point E since ∆CAB and ∆DBA are acute. Assume ∆ABE is not monochromatic, so let A and B be red and E be green (Fig. 2a). It must be proven that there exists a triangle congruent to ∆ABE which is monochromatic.

Proof by contradiction: Assume there is not a monochromatic triangle to ∆ABE. Rotate ∆ABE 180° about the midpoint (G) of AB, lining up the hypothenuse. Color the new point, F, green so that ∆ABF is not monochromatic. This creates rectangle ABEF since

$$m\measuredangle AEB = m\measuredangle AFB = 90°,\ m\measuredangle EAF = m\measuredangle BAE + m\measuredangle BAF = \alpha_i + 90 - \alpha_i = 90$$

and similarly for ∆EBF (Fig. 2b). The diagonals of a rectangle are equal so $|EF| = a$. By rotating ∆ABF, $180-2\alpha_i$ about G, the hypothenuse of the new triangle will connect the points E and F (Fig. 2c). *In order not to have a monochromatic triangle, color newly created vertex red. Rotate triangle 180° about G, aligning hypothenuse (Fig. 2d and e). Color new vertex red and rotate triangle $180-2\alpha_i$ about G, forming a new point to be colored green. Repeat process N/2 - 4 times from *, alternating coloring new pairs of vertices red and green so that monochromatic triangles are avoided. Begin coloring with green. The last new vertex will coincide with point A if i is odd and B if i is even, since starting at the 90° angle, E, of the original triangle, the triangle has been rotated through an angle of $(3 + N/2 - 4)(180 + 180 - 2\alpha_i)$. If i is even, E can be obtained by repeating from * once more since the total rotation of E is $[180 + (180 - 2\alpha_i)]N/2$ which is a multiple of 360°. If i is odd, * must be repeated again and then the final triangle must be rotated another 180 degrees in order to return to E, since, the total rotation of E for i odd is thus $[180 + (180 - 2\alpha_i)]N/2 + 180$ which is again a multiple of 360°. All vertices of the polygon will be colored and a monochromatic triangle will exist which is a contradiction.□

After discovering this proof, I found that Erdös et al. [7, 568] used this same method called the "roulette method" to prove this and other theorems involving right triangles. In another article [15, 388], the "ladder method" of Erdös along with a case analysis was used to prove that all right triangles are Ramsey.

For the ladder method [7, 566], let x and y be two points a distance b apart and both colored red (Figure 3). At a perpendicular distance $a'=a/3$ from x and y, create green points x_1 and y_1. Continue this process upwards alternating colors. Let c be the length of the hypothenuse of the triangle with legs a and b. Let c' be the length of the hypothenuse of the triangle with sides a' and b. Thus $a^2 + b^2 = c^2$ and $(a')^2 + b^2 = (c')^2$. If (a,b,c)-triangle with points x,y,z is assumed to be monochromatic then $z = x_3$ or y_3 which indicates triangle (a',b,c') must also be monochromatic. In general, let n be any positive integer with $a' = a/(2n+1)$ and $z = x_{2n+1}$ or y_{2n+1}. Thus if information about one right triangle on the plane is known, it is possible to prove all right triangles have the relation R(K,2,2). However, by Theorem 9 we know there are many right triangles for which the relation R(K,2,2) holds, thus a solid foundation exists on which to build.

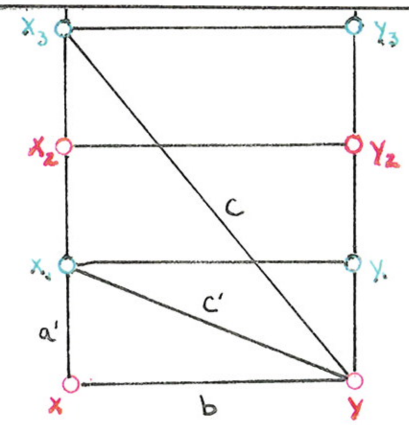
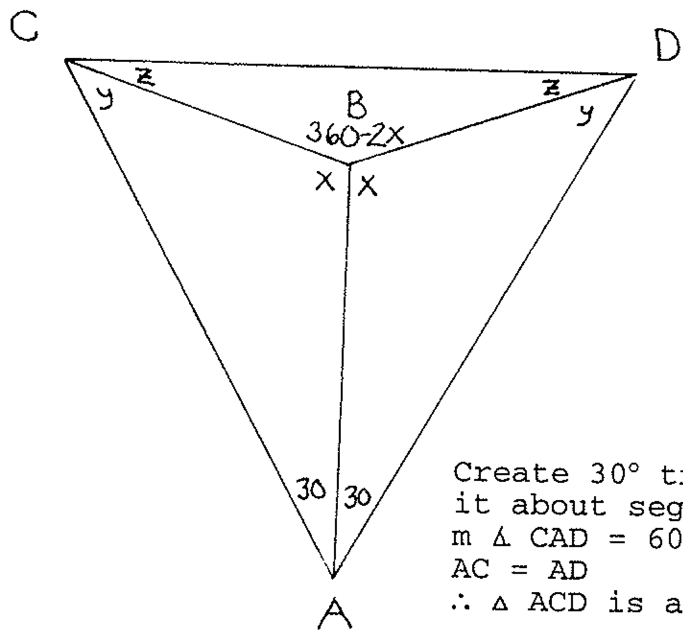


Figure 3

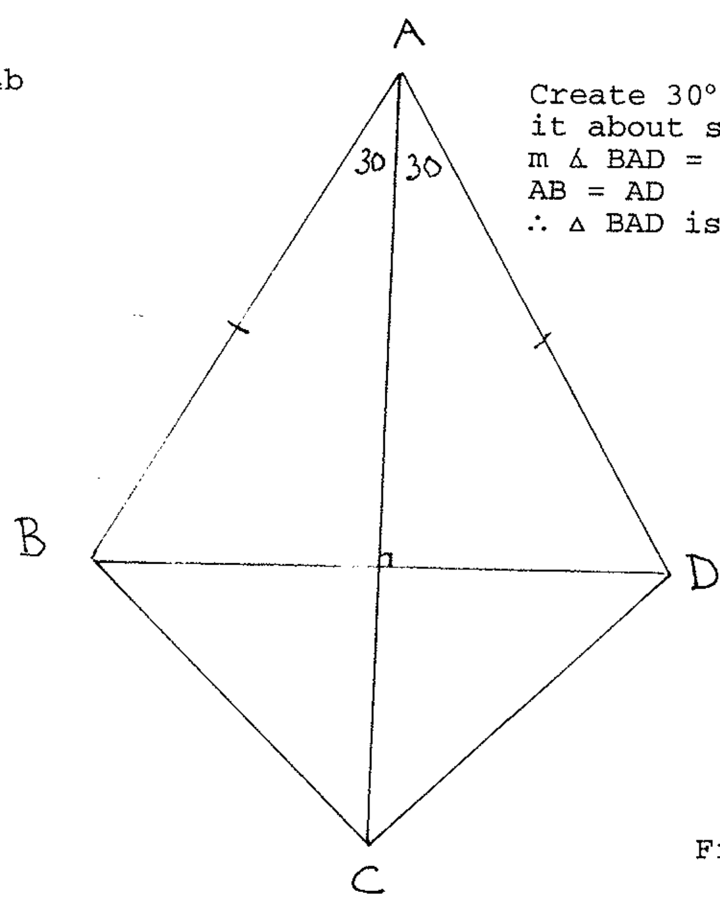
Next, I worked with 30° triangles. These triangles are special because the side opposite the 30 degree angle is one half the length of the hypothenuse. Normally, when trying to force monochromatic triangles, it is possible to build new triangles off of only one edge of every triangle, thus greatly restricting the possibilities. There are two additional ways in which segments of the correct length may be created for all thirty-degree triangles, as illustrated in Figure 4. All three of my

4a



Create 30° triangle ABC and reflect it about segment AB to create △ ABD.
m ∡ CAD = 60
AC = AD
∴ △ ACD is an equilateral triangle

4b



Create 30° triangle ABC and reflect it about segment AC to create △ ADC
m ∡ BAD = 60
AB = AD
∴ △ BAD is an equilateral triangle

Figure 4

proofs depend upon one additional property unique to the particular triangle. These additional properties are shown in Figure 5. By using existing edges and creating new edges in the above fashion, I was able to show:

Theorem 8: The relation R(K,2,2) holds for:
(i) 30-45-105
(ii) 30-40-110
(iii) 30-50-100
triangles.

The proof of this theorem is illustrated in Figures 6-8. At the same time, however, I was able to find just as many triangles for which the relation R(K,2,2) did not appear to hold. As a result of the necessity of using special principles for each triangle, the proofs end up using very different methods as can be seen by comparing Figures 7 and 8. Thus there is no apparent way to generalize the proofs. However, I have recently come across a theorem which states:

Theorem 9: For all 30 degree triangles the relation R(K,2,2) holds.

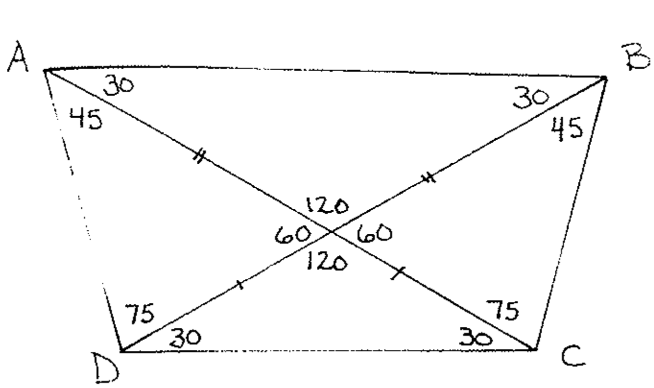
This proof is based on two theorems. The first states:

Theorem 10 [7, 563]: If K is a triple with sides a, a, c and if there exists a coloring for which R(K,2,2) holds true, then R(K*,2,2) is true for any triple with sides a, b, c for which the triangle inequality holds.

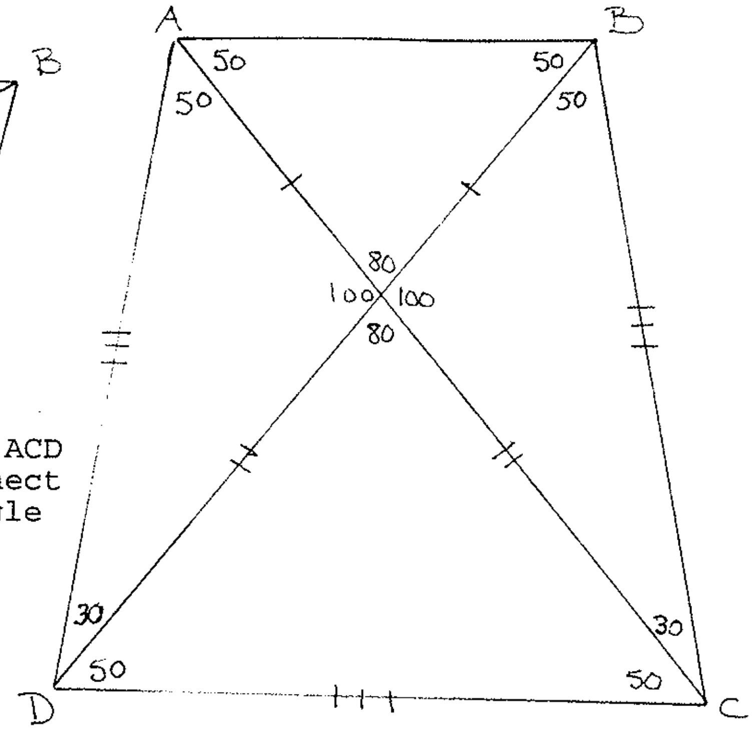
The second theorem needed for the proof of Theorem 9 states:

Theorem 11 [7, 570]: If five points can be found in the plane which have only the distances a, b, c, d and the distance d (not necessarily distinct from a, b, c) occurs only once and a, b, c satisfy the triangle inequality, then R(K,2,2) holds where K is a triangle with sides of length a, b, c.

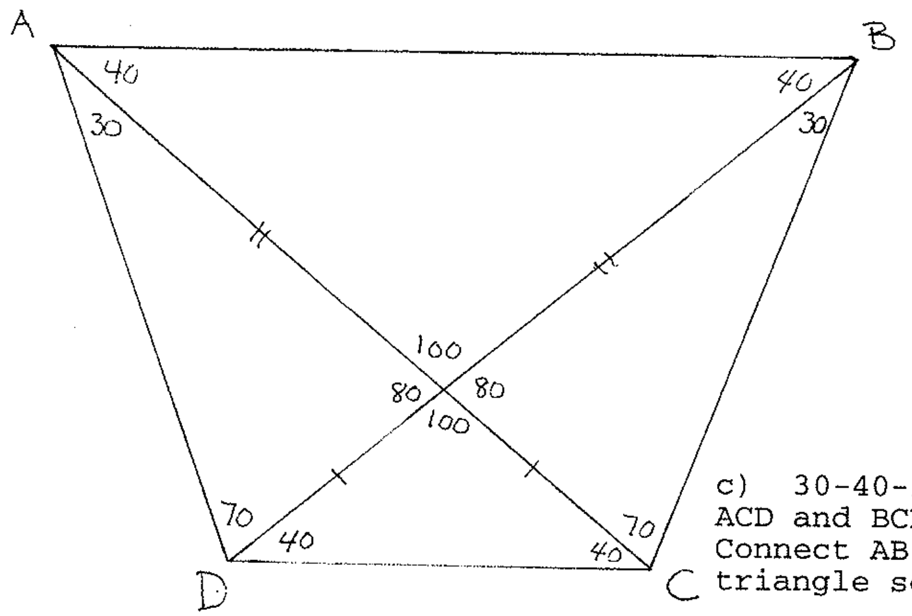
Proof: Given five points and two colors, three points must



a) 30-45-105. Create triangles ACD and BCD off of medium side. Connect AB. △ ABD is an isosceles triangle so AB is congruent to BD.



b) 30-50-100. Create triangles ABD and ABC off of shortest side. Connect DC. △ ADC is an isosceles triangle so DC is congruent to AD.

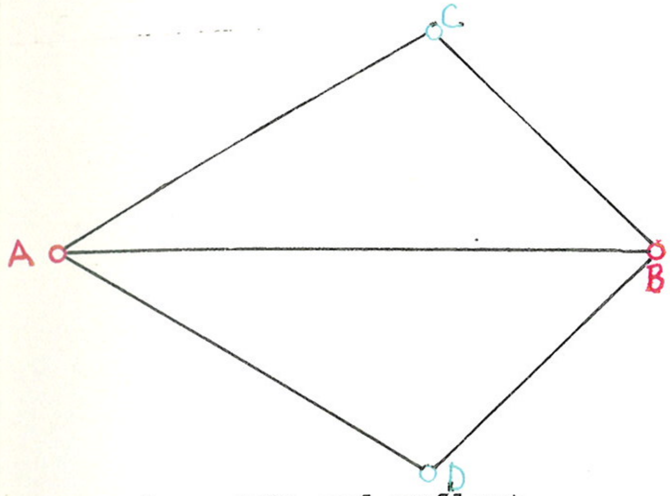


c) 30-40-110. Create triangles ACD and BCD off of shortest side. Connect AB. △ ABC is an isosceles triangle so AB is congruent to AC.

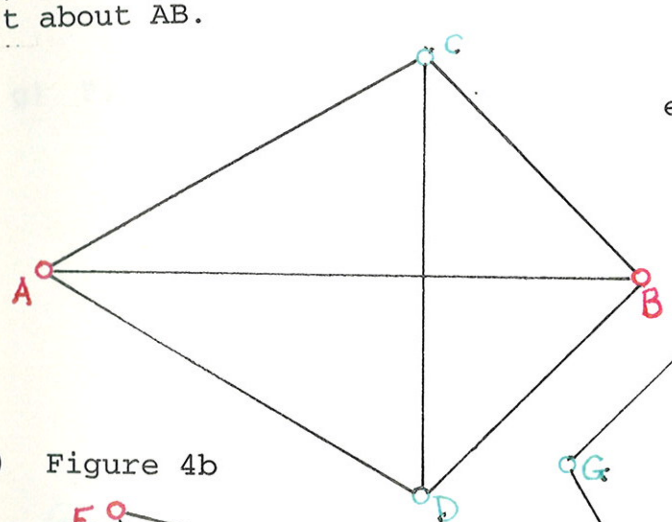
Figure 5

Proof of Theorem 8i

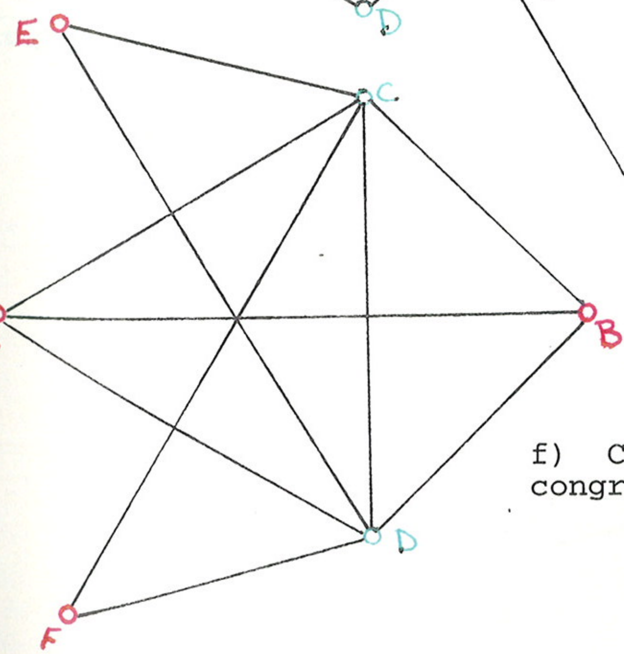
a) Theorem 1



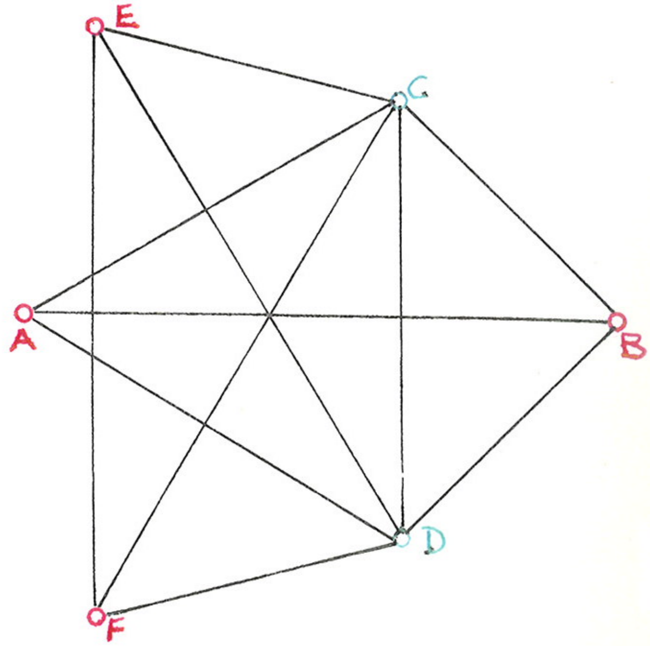
b) Create ∆ ABC and reflect it about AB.



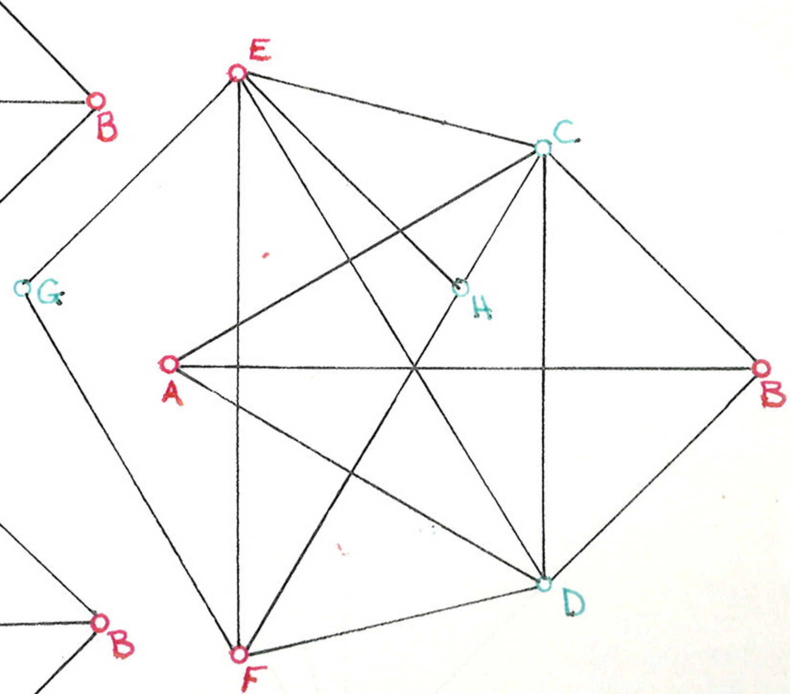
c) Figure 4b



d) Create ∆ CDE and ∆ CDF congruent to ∆ ABC.

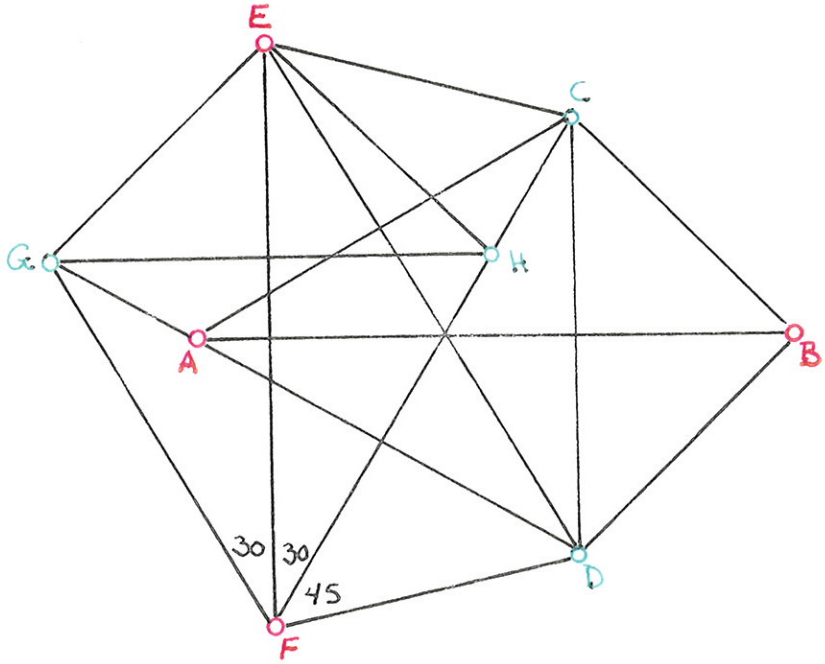


e) Figure 5a

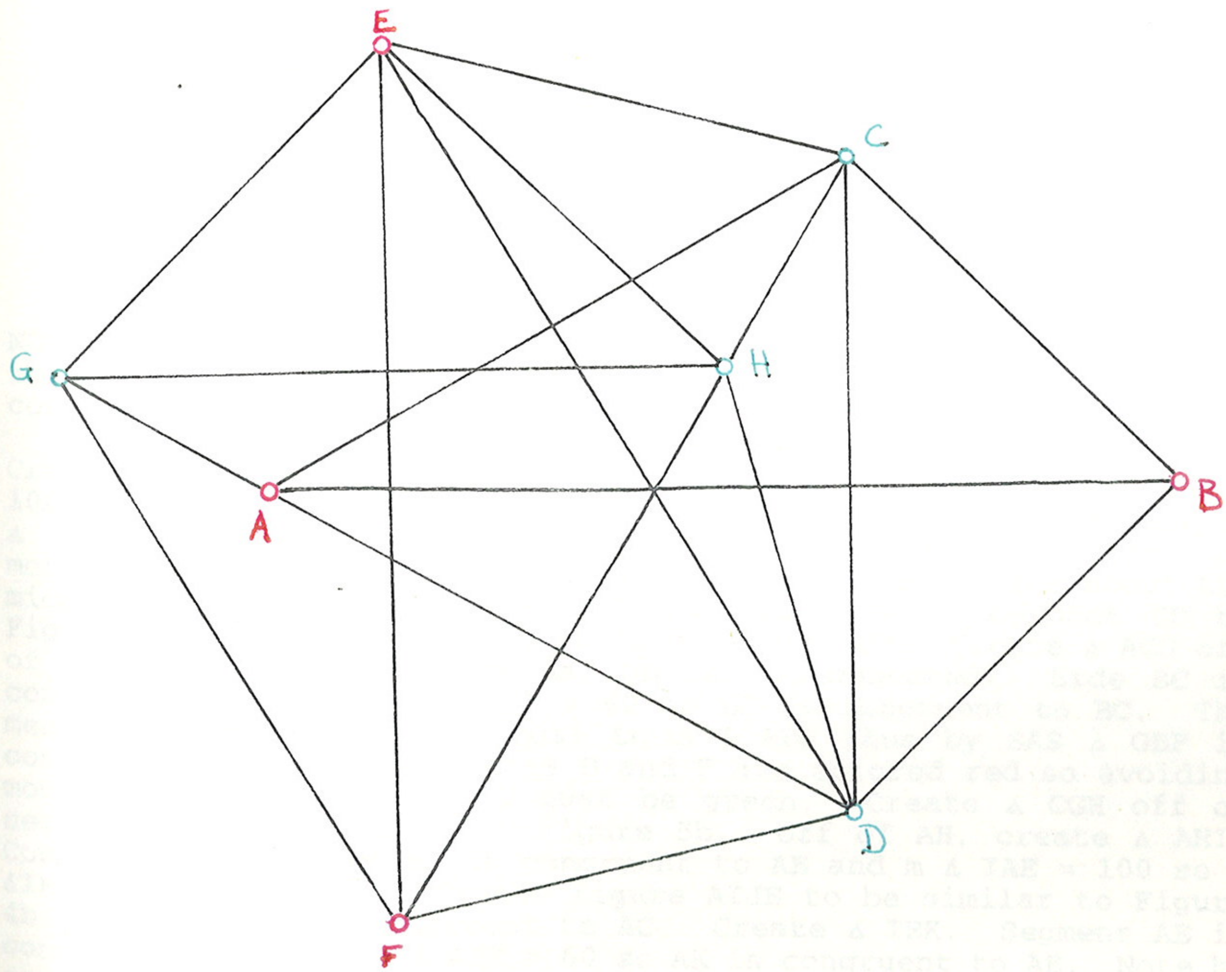


f) Create ∆ EFG and ∆ EFH congruent to ∆ ABC.

Figure 6

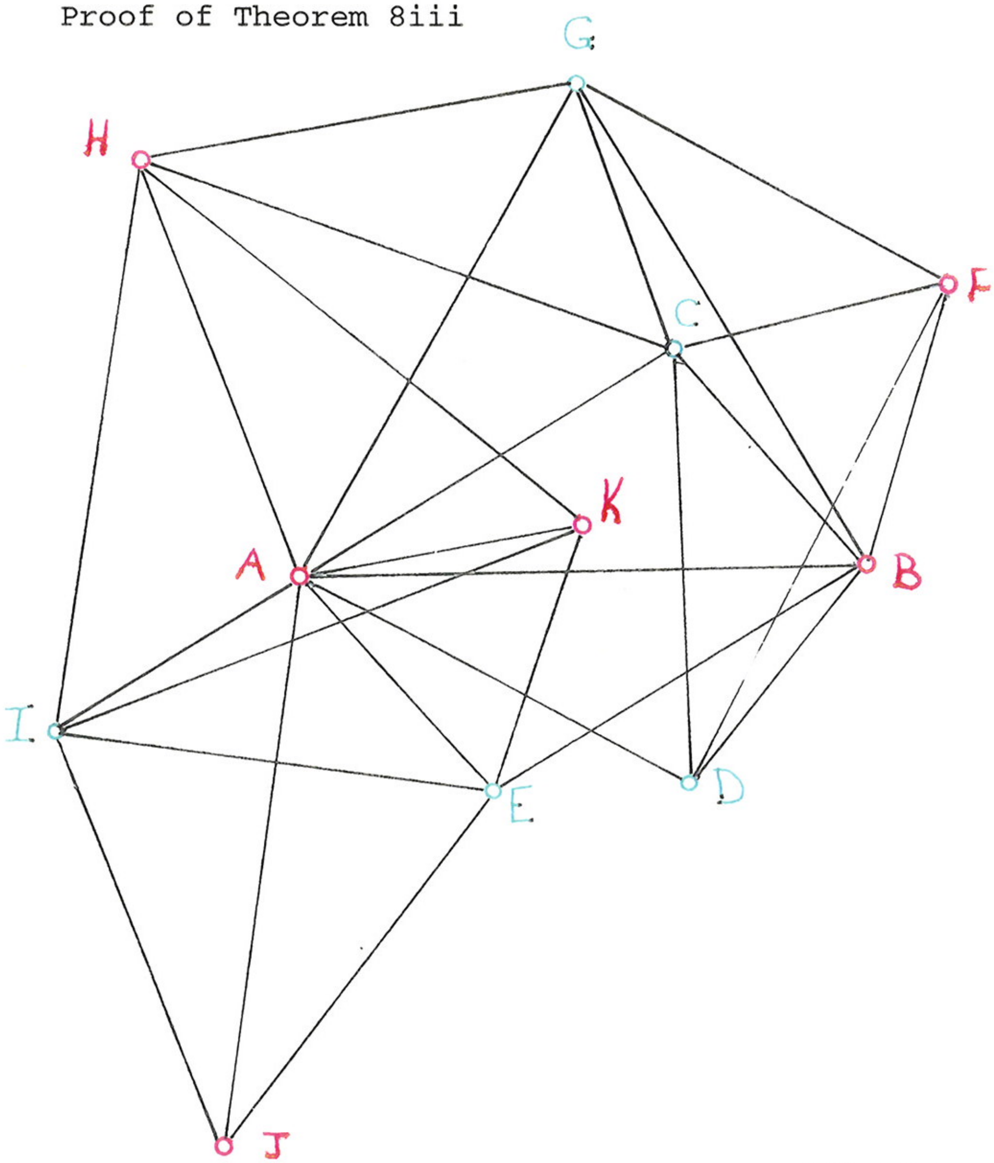


g) Figure 4b and create △ DFG, which is congruent to △ ABC by SAS.



h) Create △ DHG, which is congruent to △ ABC by SAS. Note that △ DHG is monochromatic.

Proof of Theorem 8iii

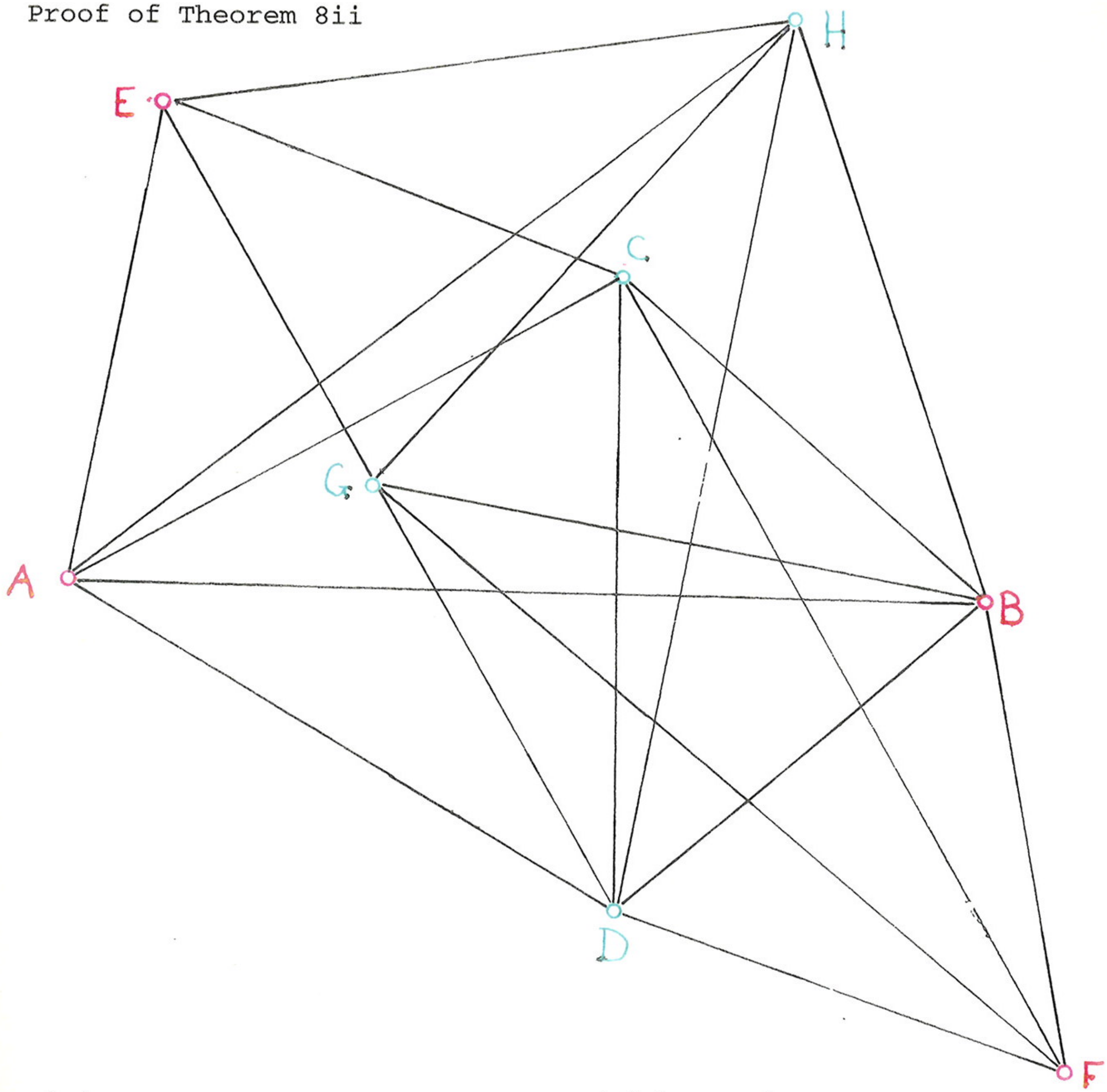


NOTE: Proof by contradiction, so always color vertices in order to avoid monochromatic triangles. Also, all triangles will be constructed to be congruent to original triangle ABC.

Create monochromatic line segment AB by Theorem 1. Create a 30-50-100 triangle ABC and reflect it about AB to get △ ABE congruent to △ ABC. Thus vertices C and D must be green so that no monochromatic triangles exist. Rotate △ ABC 180 degrees about the midpoint of AB. Thus vertex E must be green. Connect CD by Figure 4b. Create △ CDF off of line segment CD. Create △ ACG off of segment AC. By Figure 4a, GB can be conected. Side BC is congruent to CF and m ∠ BCF = 60 so BF is congruent to BC. The measure of angle GBF is equal to m ∠ ABC thus by SAS △ GBF is congruent to △ ABC. Vertices B and F are colored red so avoiding monochromatic triangles, G must be green. Create △ CGH off of segment CG. Connect AH by Figure 5b. Off of AH, create △ AHI. Connect IE. Segment AI is congruent to AE and m ∠ IAE = 100 so m ∠IEA = m ∠ EIA = 40. Create figure AIJE to be similar to Figure 4b. Segment IE is congruent to AC. Create △ IEK. Segment AE is congruent to EK and m ∠ AEK = 60 so AK is congruent to AE. Note by connecting HK, Figure 4a is created and △ AHK is monochromatic.

Figure 7

Proof of Theorem 8ii



NOTE: Proof by contradiction, so always color vertices in order to avoid monochromatic triangles.

Create a monochromatic line segment AB by Theorem 1. Create a 30-40-110 triangle ABC and reflect it about AB. Color vertices C and D green so that no monochromatic triangles exist. Create △ CDE and △ CDF congruent to △ ABC off of segment CD, so E and F must be red. The measure of ∡ADC is 60 degrees and the measure of ∡ CDE is 30 degrees so ∡ ADE = 30°. By SAS, △ ADE is congruent to △ ABC. Create △ AEH congruent to △ ABC. By Figure 5c, DH is congruent to AB. Side DF is congruent to BD and m ∡ BDF = 60° thus BF is congruent to BD. Connect BF. Create △ BFG. The measure of angle GDB is 80 and m ∡ GBD = 50 so m ∡DGB = 50 thus BD is congrent to DG. Angle GDH is 40 so by SAS △ DGH is congruent to △ ABC and is monochromatic.

Figure 8

exist which are monochromatic by the pigeon hole principle. Color the points who are distance d apart opposite colors. Either a monochromatic triangle of sides of length a, b, c exist or a monochromatic isosceles triangle of sides of length a, a, b or a, a, c exist. However, if a monochromatic isosceles triangle exists then by Theorem 10, a monochromatic triangle of sides a, b, c exists.

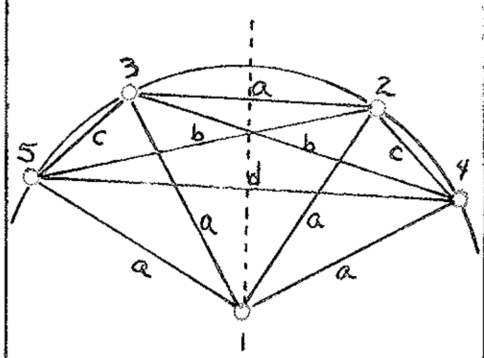


Figure 9

Proof of Theorem 9: We will set up a figure based on Theorem 11, so that d(1,2) = a is defined as the distance between point 1 and point 2. If points 2, 3, 4 are all at distance a from point 1, and d(2,3) = a, then set d(2,4) = b and d(3,4) = c. By reflecting the entire figure about a line of symmetry of the equilateral triangle 1, 2, 3 a fifth point is obtained. So d(5,2) = c, d(5,3) = b and call d(5,4) = d (see Figure 9). Angle 312 is 60 degrees so angles 234 and 243 are 30 degrees since they subtend the same arc as the central angle 312. Choose points 4 and 5 to have opposite colors and continue to color the remaining points as desired. Either a thirty-degree triangle will be monochromatic (in which case the proof is complete) or a triangle with sides a, a, c will be monochromatic. However, if a monochromatic triangle of sides a, a, c exists, then Theorem 10 says that a monochromatic triangle of sides a, b, c exists.□

Throughout my research, I have continued to find articles with new listings of Ramsey triangles. The most extensive theorem, from "Euclidean Ramsey Theorems, III" [7, 572] states:

Theorem 12: R(K,2,2) holds for all triangles K with sides a, b, c which

(i) have a 30° angle,

(ii) have a 150° angle.

Furthermore,

Theorem 13: All triangles (a, b, $(b^2 + 2a^2)^{1/2}$), 2b>a are Ramsey.

Theorem 14: All triangles (a, b, $4b^2 - a^2$), $(3/2)^{1/2}b < a < (5/2)^{1/2}b$ are Ramsey [15, 389].

Conclusion

It is known that for all triangles R(K, 3, 2) holds [12, 345]. However, if three colors are used instead of two in E^3, the question is more difficult. Bóna has proved that 30-60-90 triangles and isosceles-right-triangles are Ramsey in 3-colored E^3. His method of proof is very similar to the methods I used in my proofs, beginning by assuming that no monochromatic triangle exists and proving that one does with the use of case analysis. Even more generally, it is known that given any r-coloring, $r \geq 2$, there exists some dimension for which all triangles are Ramsey [8, 777]. The proof of this is based on Ramsey's original theorem presented in "On a Problem of Formal Logic" [13] and a similar argument used in the proof that all bricks are Ramsey.

I proved that the relation R(K, 2, 2) holds for a limited number of right and thirty-degree triangles. I found other proofs that this relation holds for these and other triangles. Yet, the question whether all triangles except equilaterals have this relation remains open. It appears that there is no one way of answering this question; rather, different proofs must be used for each kind of triangle, thus making the question more difficult to answer. Time will only tell whether a solution is found to this problem.

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