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Counting Cards: Combinatorics, Group Theory, and Probability in War

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COUNTING CARDS: COMBINATORICS, GROUP THEORY, AND PROBABILITY IN WAR

A THESIS

The Honors Program

College of Saint Benedict

In Partial Fulfillment

of the Requirements for the Distinction "All College Honors"

and the Degree Bachelor of Arts

In the Department of Mathematics

by

Angela Chappell

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PROJECT TITLE: Counting Cards: Combinatorics, Group Theory and Probability in War

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INTRODUCTION:

The card game War is simple enough for young children to play. But a young child most likely has no idea what kinds of interesting mathematics are at work determining the events of the game. An observant player may begin to notice patterns in the game and ask questions such as: What are the chances of playing a game of War in which no matches occur? Is it possible for a loop to develop, creating a never-ending game? If so, what determines the length of the loop? What loop lengths are possible? How do cards cycle between players within a loop? What would happen if we changed the number of suits or the total number of cards?

This thesis began when my advisor, Professor Marc Brodie, began to ask these questions while playing War with his children. He shared his ideas with me and I became interested in the topic. Professor Brodie wrote various programs on Mathematica which simulate the game of War allowing for various conventions of play as well as different types of decks. These programs provided a large sample space from which we were able to discern patterns and develop conjectures. One of these programs, as well as a sample printout of a game are included in Appendices 1 and 2.

We will begin by providing the rules and basic definitions used in the card game. A brief review of theorems and ideas from Algebra and Number Theory which are useful in establishing our results is also included. (See Appendix 11.) Finally we will use these definitions and theorems to answer the questions mentioned above.

GAME PLAN:

In this project, I consider variations on the ordinary game of War between two players using a standard deck of 52 cards. In each game, we begin with a deck of kn cards where k is the number of suits in the deck and n is the number of face values in each suit. The cards are divided evenly between players A and B. For this reason we limit ourselves to even values of kn. During a play, player A turns over the top card on his or her pile and player B does the same. If the cards are equal in face value, this is called a match. Otherwise, the player holding the higher of the two cards wins the play and takes both cards, placing them on the bottom of his or her pile according to the convention that player A's card is placed beneath the winner's pile ahead of player B's card during each turn. (See Example 1 below.)

Although in a typical game of War the order of card placement is not prescribed and indeed will vary from play to play, we agree to play a game according to a certain convention in order to create a more orderly process in which patterns clearly present themselves.

In the typical game of War, at any point when the top card in player A's hand matches the top card in player B's hand, then there is some kind of rule to deal with this occurrence. For example, both players will put their next three cards face down, compare the fourth card, and the holder of the higher of the fourth cards would take all of the cards. In this research, we chose to avoid matches in order to create less complex patterns.

EXAMPLE 1:

If cards are dealt 6,1,5 to player A and 2,4,3 to player B at the start of the game, then the next three hands would be the following:

1)		2)		3)	
A	**B**	**A**	**B**	**A**	**B**
1	4	5	3	6	1
5	3	6	1	2	4
6		2	4	5	
2				3	

DEFINITIONS:

match: player A's first card matches player B's first card in face value

win: 1) if player A's first card is higher in face value than player B's first card at a particular play, then player A wins that play

2) if at some point one player holds all of the cards, the game is over and the player holding the cards wins the game

loop: if at any point in the game both players' hands match - in face value - the hands they held at a previous point in the game, this signals a loop

Note: we agree that a loop does not contain any matches

perfect loop: if at any point in the game both players' hands match -- in face value *and* suit -- the hands they held at a previous point in the game, this signals a perfect loop

loop length: number of moves between the first repetition of the hand and the next repetition, e.g. if the hands at the start of play x match the hands at the start of play $x+a$ and the hands at the start of play x don't match any hands before play $x+a$ then the loop has length a plays.

winning / losing card: in a loop, a card that won / lost the previous play in which it was involved

winning / losing position: in a loop, position in the player's pile held by a winning / losing card

EXAMPLE 2:

If a game is played by the convention described using the cards 1♣,2♣,3♣,4♣ and 1♡,2♡,3♡,4♡ and at some point player A holds 4♣, 1♣, and player B holds 2♣, 3♡, 1♡, 4♡, 2♡, 3♣ (in that order), then there is a loop of length 12 plays (player A will then hold 4♣, 1♡ and player B will hold 2♡,3♡,1♣,4♡,2♣,3♣) and a perfect loop of length 24 plays. The 4♣, 3♡, 4♡, 3♣ are winning cards and the 1♣, 2♣, 1♡, 2♡ are losing cards. See Appendix 3 for a listing of the hands during each play of the loop.

TWO-PLAYER WAR RESULTS:

Mathematica printouts similar to the printouts given in Appendices 2 and 4 led us to investigate patterns within loops. We hoped to develop a set of criteria to determine possible loop lengths given a deck of kn cards. As a first step, we simply eliminated odd loop lengths:

THEOREM 1:

Given a deck consisting of kn cards, any loop must consist of an even number of moves.

PROOF:

Let kn be the total number of cards in our deck. Assume (WLOG) that player A holds x of those cards, player B holds kn-x of them and that we are in a loop. After one move, player A holds $x\pm1$ cards and player B holds $(kn-x)\mp1$ cards. The parity of the number of cards held by each player alternates with each play. So, after any odd number of plays, the parity of the number of cards in player A's hand will not match the parity of x. Therefore a loop must consist of an even number of moves. ◆

Eliminating odd loop lengths is a start at determining possible loop lengths for a given deck. However, the Mathematica printouts indicated that the number of possible loop lengths for a deck of kn cards was relatively small. For example, with a deck of cards labeled 1 through 8, the only possible loop lengths are 8 and 24 plays. (See Appendix 4.) By closely investigating the movement of cards from play to play within sample loops generated by Mathematica, we discovered a pattern of alternating wins within each loop. In each sample loop, player A won the play whenever he or she was holding an even number of cards and player B won the alternate plays. (See Appendix 3.) The following theorem and its corollaries show that this pattern must appear in every loop.

THEOREM 2:

If a game is in a loop, then winning must alternate between players with each play.

PROOF:

Let kn be the number of cards in our deck. Assume that at the start of the game the cards are arranged as in Figure 1. When player A wins a play, player A's winning card is placed ahead of player B's card on the bottom of player A's pile. On the other hand, when player B wins a play, player A's losing card is placed in front of player B's winning card on the bottom of player B's pile. After $\frac{kn}{2}$ plays, winning cards are aligned with losing cards. Cards are now aligned as in Figure 2. If we number the positions in each player's hand 1,2,... from top to bottom whenever player A is holding an even number of cards, winning cards hold odd positions in player A's hand and even positions in player B's hand.

FIGURE 1

A	B
a_1	b_1
a_2	b_2
a_3	b_3
$\vdots$	$\vdots$
$a_{\frac{kn}{2}}$	$b_{\frac{kn}{2}}$

FIGURE 2

A	B
w_1	l_{r+1}
l_1	w_{r+1}
w_2	l_{r+2}
l_2	w_{r+2}
$\vdots$	$\vdots$
w_m	l_{r+m}
l_m	w_{r+m}
$\vdots$	
w_r	
l_r	

Note: In Figure 2, each player must hold at least two cards since we're in a loop.

Assume that we are in a loop somewhere after the first v plays (cards aligned as in Figure 2). Suppose, for a contradiction, that WLOG player A wins two consecutive plays. That is, a card in a losing (even) position in player A's hand beats a card in a winning (odd) position in player B's hand. The card from player A's hand is placed in an odd position in player A's hand and the card from player B's hand is placed in an even position in player A's hand. Since we are in a loop, the displaced card (the losing card from player B's hand now holding an even position in player A's hand), or a card of equal face value, must beat a card of lower value in a winning position. This simply results in another displaced card and the process repeats. At some point, because we have finitely many cards, the displaced card will be lower in value than all of the cards holding winning positions and will therefore never be able to return to a winning position. The cards will never again be arranged as in Figure 2, so this contradicts our assumption that we are in a loop. Therefore, in a loop, winning must alternate between players with each play. ◆

COROLLARY 1:

If a game is in a loop, winning cards in player A's hand remain the same as do winning cards in player B's hand. That is, the set of winning cards in player A's hand at any play in a loop is the same as the set of winning cards in player A's hand at any other play in the loop. The same holds for the winning cards in player B's hand.

PROOF:

Assume that we are in a loop. The only way for a winning card to switch hands would be for one player to win two plays in a row. This will not happen because by Theorem 2, winning alternates between players with each play. ◆

EXAMPLE 3:

In Example 2, the winning card in player A's hand at play 3 is 4♣ and the winning cards in player B's hand at play 3 are 3♣, 4♡, and 3♡. The same is true at any other play within the loop. (See Appendix 3.)

COROLLARY 2:

If a game is in a loop, Player A will always win when holding an even number of cards.

PROOF:

Assume that we are playing a game as described in Theorem 2. We know that wins alternate back and forth within a loop. Suppose that we are in a loop and that player A wins while holding t cards where t is odd. Since wins alternate between each play, player A must win the t-th play. The winning card from the first turn has been placed beneath the t-th card in player A's pile, so player A must also win the $(t+1)$-st play. This is a contradiction since player A cannot win two plays in a row. ◆

EXAMPLE 4:

Assume the game is in a loop and cards are arranged as in Figure 3. Suppose player A wins at this point. Player B must win the next turn and player A wins the third turn since wins must alternate with each play. At this point, cards are arranged as in Figure 4. Now, according to the theorem, player B must win the next play. But, according to Corollary 1, player A must win since a_1 is a winning card. This contradiction illustrates how within a loop, player A will win only when holding an even number of cards.

FIGURE 3

a_1	b_1
a_2	b_2
a_3	b_3
	b_4
	b_5

FIGURE 4

a_1	b_4
b_1	b_5
a_3	a_2
b_3	b_2

Mathematica printouts of simulated games seemed to indicate that loop length is a multiple of the total number of pairs of cards. For example, with k=2 and n=6 we found loops of length 6, 12, 24, and 60 plays and no others. (See Appendix 5.) At first it was unclear which multiples of the number of pairs of cards would appear as loop lengths. For example, why was 18 not a loop length in the k=2, n=6 game? Using the fact that winning cards must alternate between the players with each turn, we were able to use permutation groups to explain which loop lengths were possible.

Assume that we are playing a game using a deck of kn cards labeled $1_1, 1_2, \ldots, 1_k, 2_1, 2_2, \ldots, 2_k, \ldots, n_1, n_2, \ldots, n_k$ and that the game is in a loop. Also assume that cards are arranged as in Figure 2. By Theorem 2, we know that winning must alternate between each play. Thus, after two plays, player A will once again hold r pairs of cards, leaving m pairs in player B's hand. That is, after each pair of plays, there are always r pairs of cards in player A's hand and m pairs of cards in player B's hand. This enables us to associate a permutation of the group S_{kn} to the positions held by the cards between pairs of plays. For example, after 2 plays, the cards will appear as in Figure 5. The permutation of the kn positions in each pair of plays is $\phi = (w_1 w_2 \cdots w_r)(w_{r+1} w_{r+2} \cdots w_{r+m})(l_1 l_2 \cdots l_q)$. (See Appendix 6 for details.) The winning cards in A's hand will be acted upon by an r-cycle. Winning cards in B's hand will be acted upon by an m-cycle. Losing cards will be acted upon by a q-cycle, where $q = r + m = \frac{kn}{2}$.

FIGURE 5

A	B
w_2	l_{r+2}
l_2	w_{r+2}
w_3	l_{r+3}
l_3	w_{r+3}
$\vdots$	$\vdots$
w_m	l_{r+m}
l_m	w_{r+m}
w_{m+1}	l_1
l_{m+1}	w_{r+1}
$\vdots$	
w_r	
l_r	
w_1	
l_{r+1}	

Knowing the permutation acting on the positions of the cards between each pair of plays within a loop is useful. It allows us to further restrict the list of possible loop lengths for a deck of *kn* cards. Consecutive pairs of plays are simply repeated applications of ϕ to the *kn* cards. Thus, the order of the cyclic group generated by ϕ will give us the number of pairs of plays needed for the cards to return to their original positions, i.e. the length of a perfect loop.

THEOREM 3:

Given a loop in which there are *kn* total cards, with players A and B holding *r* and *m* pairs of cards respectively after each pair of plays, the length of a perfect loop will be given by 2*LCM (*r, m, q*).

PROOF:

We know that $\phi = (w_1 w_2 \cdots w_r)(w_{r+1} w_{r+2} \cdots w_{r+m})(l_1 l_2 \cdots l_q)$ is the permutation associated with the positions. Using Ruffini's "Order of a permutation" theorem we see that $|\phi|$=LCM (*r, m, q*). Note that we multiply the LCM (*r, m, q*) by a factor of two because the LCM (*r, m, q*) gives us the number of *pairs* of plays needed to return the cards to their original positions. ◆

EXAMPLE 5: In Example 1, $\phi = (w_1)(w_2 w_3 w_4)(l_1 l_2 l_3 l_4)$. The length of a perfect loop is 2*LCM(1,3,4)=2*12=24.

By looking at the permutation $\phi \in S_{kn}$, which acts on the positions of the cards, we can use the theory of cyclic groups to answer some further questions about loops. For example, we can determine how the cards cycle within a loop. If we are given values for *m* and *r* we can determine the number of distinct winning cards which beat each losing card. Even more than that, we can determine exactly which cards line up with each other during a loop.

LEMMA 1:

In a loop, let $\alpha = (w_1 w_2 \cdots w_r)$ be the *r*-cycle acting on the winning positions in player A's hand,

$\beta = (w_{r+1} w_{r+2} \cdots w_{r+m})$ be the *m*-cycle acting on the winning positions in player B's hand, and

$\lambda = (l_1 l_2 \cdots l_q)$ be the *q*-cycle acting on the losing positions.

Also, let *d* =gcd (*r,q*) = gcd (*m, q*) (see proof in Appendix 11) and *x* = LCM (*r, m, q*) = the number of pairs of plays in a perfect loop for the given values of *r* and *m*.

Then 1) there are $\frac{q}{d}$ distinct losing cards which line up with each winning card in player A's hand during each x pairs of plays.

2) there are $\frac{q}{d}$ distinct losing cards which line up with each winning card in player B's hand during each x pairs of plays.

3) there are $\frac{r}{d}$ distinct winning cards in player A's hand which line up with each losing card during each x pairs of plays.

4) there are $\frac{m}{d}$ distinct winning cards in player B's hand which line up with each losing card during each x pairs of plays.

NOTE: This means that there are $\frac{q}{d}$ distinct losing cards which line up with each winning card during each x pairs of plays as well as $\frac{r+m}{d}=\frac{q}{d}$ distinct winning cards which line up with each losing card during each x pairs of plays.

PROOF:

1) We have $\alpha^r(w_i)=w_i$ for all $1\leq i\leq r$. This means that after r plays, w_i will be back to its original position. Also, $\{\theta(l_i):\theta\in<\lambda^r>\}$ will determine which cards line up with the card in position w_i after each r pairs of plays.

Since r is a multiple of d, we have $<\lambda^r>\subseteq<\lambda^d>$. Also, $d=qs+rt$ for some integers s and t so $\lambda^d=(\lambda^q)^s(\lambda^r)^t=(\lambda^r)^t\in<\lambda^r>$. Thus, $<\lambda^r>=<\lambda^d>$. We know $|<\lambda^d>|=\frac{q}{d}$ so $|<\lambda^r>|=\frac{q}{d}$ and for each i, $\{\theta(l_i):\theta\in<\lambda^r>\}$ contains $\frac{q}{d}$ elements. That is, there are $\frac{q}{d}$ distinct losing cards which line up with each winning card in player A's hand during each x pairs of plays.

Similarly:

2) $\beta^m(w_j)=w_j$ for $r+1\leq j\leq r+m$ and $<\lambda^m>=<\lambda^d>$ so there are $\frac{q}{d}$ distinct cards which lose to each card in a winning position in player B's hand during each x pairs of plays.

3) $\lambda^q(l_i)=l_i$ for $1\leq i\leq r$, and $<\alpha^q>=<\alpha^d>$ so there are $\frac{r}{d}$ distinct winning cards in player A's hand which beat each losing card during each x pairs of plays.

4) $\lambda^q(l_j)=l_j$ for $r+1\leq j\leq r+m$ and $<\beta^q>=<\beta^d>$ so there are $\frac{m}{d}$ distinct winning cards in player B's hand which beat each losing card during each x pairs of plays. ◆

NOTE: These conditions describe the movement of cards through x pairs of plays. Since a perfect loop will consist of x pairs of plays these conditions help describe the line-up of the cards within one cycle of a perfect loop. For a loop that is not perfect (fewer than x pairs of plays) these conditions will still hold, but we will need to consider more than one cycle of the loop in order for each card to line up with $\frac{q}{d}$ distinct cards. For example: In Example 2 these conditions hold for the perfect loop of length 24 plays. The conditions also hold for every two iterations of the loop of length 12 plays. (See Appendix 3.)

EXAMPLE 6: Suppose that at some point in a game played with the cards 1♣, 2♣, 3♣, 4♣ and 1♡, 2♡, 3♡, 4♡ player A holds 4♡, 1♡, 3♡, 2♡ and player B holds 2♣, 3♣, 1♣, 4♣. The hands at each play in this loop are given in Appendix 7. In this case α=(4♡ 3♡), β=(3♣ 4♣), λ=(1♡ 2♡ 2♣ 1♣); d=gcd(2,4)=2; x=LCM(2,2,4)=4. There are 2 losing cards which line up with each winning card in player A's hand during each iteration of a perfect loop (8 plays), 2 losing cards which line up with each winning card in player B's hand during each iteration of a perfect loop, 1 winning card in player A's hand which lines up with each losing card during each iteration of a perfect loop, and 1 winning card in player B's hand which lines up with each losing card during each iteration of a perfect loop. See Appendix 7 for more specific details.

LEMMA 2:

If a game played with a deck of kn cards labeled $1_1,1_2,\ldots,1_k,2_1,2_2,\ldots,2_k,\ldots,n_1,n_2,\ldots,n_k$ is in a loop, then, as illustrated in Example 6, (See Appendix 7) the cards that will line up with each other are given by the following chart:

$$\{\psi(w_1):\psi\in\langle\alpha^d\rangle\} \text{ and } \{\rho(w_{r+1}):\rho\in\langle\beta^d\rangle\} \text{ line up with } \{\chi(l_1):\chi\in\langle\lambda^d\rangle\}$$
$$\{\psi(w_2):\psi\in\langle\alpha^d\rangle\} \text{ and } \{\rho(w_{r+2}):\rho\in\langle\beta^d\rangle\} \text{ line up with } \{\chi(l_2):\chi\in\langle\lambda^d\rangle\}$$
$$\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots$$
$$\{\psi(w_d):\psi\in\langle\alpha^d\rangle\} \text{ and } \{\rho(w_{r+d}):\rho\in\langle\beta^d\rangle\} \text{ line up with } \{\chi(l_d):\chi\in\langle\lambda^d\rangle\}$$

One of Professor Brodie's original conjectures was that any game played with a deck of cards labeled $1_1,1_2,2_1,2_2,\ldots,n_1,n_2$ where n is prime cannot have any loops. The conjecture was prompted by Mathematica printouts of games with n=5, 7, and 11, in which no loops were found. Lemmas 1 and 2 gives us the ideas we need to prove that Professor Brodie's conjecture was correct. In addition, we find that loops do exist in these games whenever n is composite.

THEOREM 4a:

If a game is played with a deck of $2n$ cards labeled $1_1,1_2,2_1,2_2,\ldots,n_1,n_2$ where n is an odd prime, then no loops are possible.

PROOF:

Since n is prime, then gcd (r, n) = gcd (m, n) = 1. By Lemma 1, this means that $\frac{n}{1}$ distinct losing cards, that is ALL losing cards, line up with each winning card after each x pairs of plays. So the n cards of lowest face value must be in losing positions. Since n is odd, the two cards with the lowest value greater than $\frac{n}{2}$ will be in a winning position and a losing position. This will eventually result in a match. ◆

THEOREM 4b:

If a game is played with a deck of $2n$ cards labeled $1_1,1_2,2_1,2_2,\ldots,n_1,n_2$ where n is composite, then perfect loops exist.

PROOF:

We give a method of constructing a loop under the stated conditions:

Let $n=pv$ where p is prime and $v>1$. Split the cards so that one player (say, player A) holds $2p$ cards and the other player holds $2p(v-1)$. So, $r=p$, $m=p(v-1)$, $q=n$, and $x=\text{LCM}(r,m,q)=\text{LCM}(p,p(v-1),n)$.

We have $d=\gcd(r,q)=\gcd(p,n)=\gcd(p(v-1),n)=\gcd(m,q)=p$. By Lemma 1 we know that there are $\frac{q}{d}=\frac{n}{p}=v$ distinct losing cards which line up with each winning card during each x pairs of plays within a loop and $\frac{q}{d}=\frac{n}{p}=v$ distinct winning cards which line up with each losing card during each x pairs of plays within a loop. This means that the cards will be partitioned into p sets containing cards which line up against each other in a loop. Also, there will be $\frac{2pv}{p}=2v$ cards in each set. Line the cards up in order as follows: 1, 2, 3, 4, ..., pv, 1, 2, 3, 4, ... , pv. Then choose $2v$ distinct cards for each set by taking $2v$ consecutive cards at a time. If we then place the v cards of highest value in winning positions and the others in losing positions within each set, a loop will develop. ◈

EXAMPLE 7:

Assume that we are playing with a deck of 12 cards labeled $1_1,1_2,2_1,2_2,3_1,3_2,4_1,4_2,5_1,5_2,6_1,6_2$. In this case n=3*2 (WLOG we let p=3 and v=2.) Split the cards so that player A holds $2p$=6 of them and player B holds the remaining 6. There will be gcd(3, 6)=3 mutually exclusive sets of cards lining up with each other with 2*2=4 cards per set. Divide the cards into three sets so that there are 4 distinct cards in each set. For example $\{1_1,2_1,3_1,4_1\}$, $\{5_1,6_1,1_2,2_2\}$, $\{3_2,4_2,5_2,6_2\}$. Place the two highest valued cards from the first set (3_1 and 4_1) in winning positions and the remaining cards in losing positions as shown in Figure 6. Add cards from the other sets in a similar fashion as shown in Figure 7. A loop will result. (See Appendix 8.)

FIGURE 6

A	B
3_1	2_1
1_1	4_1
a_3	b_3
a_4	b_4
a_5	b_5
a_6	b_6

FIGURE 7

A	B
3_1	2_1
1_1	4_1
5_1	2_2
1_2	6_1
5_2	4_2
3_2	6_2

If we have one suit, it is clear that all loops will be perfect loops since any time the hands match in face value the hands at a previous play, they will automatically match in suit the hands held at the same previous play. But, if we have two suits, it is conceivable that hands would match in face value, but not suit, the hands of a previous play. In fact, we have seen this demonstrated in the loop of length 12 plays in Example 2. Up to this point, we have discussed possible lengths of *perfect* loops. Using our knowledge of the movement of cards within a loop combined with our knowledge of cyclic groups, we have found all possible *perfect* loop lengths for a deck of *kn* cards. We know from Example 2 that there are loops that are not perfect. The loop of this type in Example 2 is half the length of the perfect loop. A natural question seems to be whether or not, with a deck of cards labeled $1_1,1_2,2_1,2_2,\ldots,n_1,n_2$, there will be a non-perfect loop of some length other than half of one of the possible perfect loop lengths. In order to answer this question, it is helpful to observe the following: According to Theorem 2, within *any* loop winning must alternate between players with each play. Thus, the permutation ϕ of the positions of the cards in a loop will hold for *any* loop. So, *any* loop must be the result of repeated applications of ϕ to the positions of the cards, implying that a loop that is not perfect must be part of a perfect loop. Keeping these ideas in mind, we show that with a deck of two suits, loops that are not perfect must be half the length of one of the possible lengths of a perfect loop.

THEOREM 5:

If a game is played with cards labeled 1, 1, 2, 2, ..., n, n then the possible loop lengths are LCM (r, m, n) pairs of plays as well as [LCM (r, m, n)]/2 pairs of plays, where $r+m=n$. (Recall: r and m are the number of pairs in player A's and player B's hands, respectively.) The loop of length LCM (r, m, n) pairs will be a perfect loop and the loop of length [LCM (r, m, n)]/2 pairs will not be a perfect loop.

PROOF:

Assume we are in a loop in the game described. Let $\phi=(w_1w_2\cdots w_r)(w_{r+1}w_{r+2}\cdots w_n)(l_1l_2\cdots l_n)$ be the permutation of the $2n$ positions between each pair of plays. Let $x = |\phi| =$ LCM (r,m,n). By Theorem 3 we know that a loop of length x pairs of plays is possible. Also, 2 divides LCM (r, m, n). (Since $r+m = n$, either n is even or one of m or r is even.) That is, 2 divides $|<\phi>|$ so $<\phi>$ must contain an element of order 2, specifically $\phi^{\frac{x}{2}}$. Since $\phi^{\frac{x}{2}}$ has order 2, we know that it is composed of disjoint 2-cycles. Thus after $\frac{x}{2}$ pairs of plays, certain pairs of cards have switched positions. By placing cards of equal face value in the positions designated by the 2-cycles, we can create a loop of length $\frac{x}{2}$ pairs of plays. To see that $2x$ and x are the only possible loop lengths, suppose that ϕ^j with $j<x$, $j\neq\frac{x}{2}$ produces a loop. We know $|\phi^j|>2$ since $<\phi>$ contains only one identity and only one element of order 2. So ϕ^j contains at least one k-cycle where $k>2$. If our k-cycle is $(h_1h_2h_3...h_k)$, for example, this means that the cards have changed positions in the following manner: $h_1\rightarrow h_2\rightarrow h_3\rightarrow ...\rightarrow h_k\rightarrow h_1$. With only two cards of each face value, it is impossible to have cards of equal value in all k positions. Therefore, it is impossible that ϕ^j produces a loop. ◆

EXAMPLE 8:

We return to Example 2. We have r=1 and m=3 so a perfect loop has length LCM(1, 3, 4)=12 pairs of plays and we also have a loop of length 6 pairs of plays. In this case $\phi=(w_1w_2w_3)(w_4)(l_1l_2l_3l_4)$ and $\phi^6=(l_1l_3)(l_2l_4)$. As discussed in Example 6, the 2-cycles act upon the losing cards.

THEOREM 6: If a game is played using a deck of $2n$ cards labeled $1_1,1_2,2_1,2_2,...,n_1,n_2$ then in a loop of length [LCM (r, m, n)]/2 pairs or plays, the 2-cycles will act on positions of either winning cards in player A's hand, winning cards in player B's hand or losing cards, whichever set has a cardinality containing the highest power of 2. Moreover, this set is unique.

PROOF:

Let $\phi = (w_1 w_2 \cdots w_r)(w_{r+1} w_{r+2} \cdots w_n)(l_1 l_2 \cdots l_n)$ be the permutation acting on the positions between each pair of plays in a loop. Let the disjoint *r*-, *m*-, and *n*-cycles of ϕ have lengths $2^a k_1, 2^b k_2$, and $2^c k_3$ respectively where k_1, k_2 and k_3 are odd. Then $|\phi| = 2^{\max\{a,b,c\}} k$ where k =LCM(k_1, k_2, k_3) and $|\phi|/2 = 2^{\max\{a,b,c\}-1} k$. So a cycle with the highest power of 2 dividing its order will become disjoint 2-cycles when raised to the $2^{\max\{a,b,c\}-1} k$ power.

Now suppose that $a=b=\max\{a, b, c\}$. That is, suppose that two cycles contain the maximum power of 2 in their length. We know that $2^a k_1 + 2^b k_2 = 2^c k_3$. Since k_1 and k_2 are odd, $k_1 + k_2$ is even and $2^a k_1 + 2^b k_2 = 2^a (k_1 + k_2) = 2^c k_3$ which means $c > a, b$. This is a contradiction. Similar arguments show that we reach a contradiction if we assume $a=c$ or $b=c$ is equal to $\max\{a, b, c\}$. Thus, we will have exactly one cycle whose length contains the highest power of 2. ◆

EXAMPLE 9:

If we let player A hold 4♣, 1♣ and player B hold 2♣, 3♡, 1♡, 4♡, 2♡, 3♣ (as in play 3 of Example 2) then the winning cards in A's hand will be acted on by a 1-cycle. The winning cards in B's hand will be permuted by a 3-cycle. And the losing cards will be permuted by a 4-cycle. The highest power of 2 is in the length of the cycle of the losing cards. As expected, after 12 moves, the cards of equal face value that have switched position are the 1♣, 1♡, 2♣, and 2♡ all of which held losing positions.

When this project began, the ultimate goal was to determine the likelihood of playing a game of War in which there were no matches. We have yet to reach this goal, but the following theorem answers a simplified version of the question. With the theorems we have developed thus far, it is possible to count the number of deals which result immediately in a loop using a deck of $2n$ cards of one suit.

THEOREM 6: If a game is played with a deck of $2n$ cards labeled 1, 2, ... ,$2n$ then the probability that the deal results immediately in a loop is $\frac{1}{6^{\frac{n}{2}}}$ if n is even and $\frac{n!n!}{(2n)!}$ if n is odd.

PROOF: If n is even then we can count the number of arrangements that result immediately in a loop as cards are dealt n to each player. The loop length will be LCM $(\frac{n}{2}, \frac{n}{2}, n) = n$ pairs of plays. There will be two cards in winning positions beating each card in a losing position, as well as two cards in losing positions losing to each card in a winning position. If we choose cards in sets of four, placing them in the arrangement of Figure 8 with the two highest of the four cards in positions a_{2c-1} and b_{2c} for all c and placing the two other cards in the remaining positions, then a loop develops. This is the only possible way to produce a loop in this case. There are two ways to place the high cards in each set of four cards, as well as two ways to place the low cards.

So we have a total of

$$\binom{2n}{4}2^2\binom{2n-4}{4}2^2\binom{2n-8}{4}2^2\cdots\binom{8}{4}2^2\binom{4}{4}2^2 = \frac{2^n*(2n)!}{4!^{\frac{n}{2}}} = \frac{4^{\frac{n}{2}}*(2n)!}{4!^{\frac{n}{2}}} = \frac{(2n)!}{3!^{\frac{n}{2}}} = \frac{(2n)!}{6^{\frac{n}{2}}}$$

deals which result immediately in a loop.

The total number of deals is $(2n)!$ so the probability that a deal results immediately in a loop is

$$\frac{\frac{(2n)!}{6^{\frac{n}{2}}}}{(2n)!} = \frac{1}{6^{\frac{n}{2}}}.$$

FIGURE 8

A	B
$\vdots$	$\vdots$
a_{2c-1}	b_{2c-1}
a_{2c}	b_{2c}
$\vdots$	$\vdots$

If n is odd, then we need to count loops identified by their arrangement beginning on the second play since we consider loops to begin at a point when both players have an even number of cards. So we consider a loop in which $m = \frac{n+1}{2}$ and $r = \frac{n-1}{2}$. In this case, gcd $(\frac{n+1}{2}, \frac{n-1}{2}, n)$ = gcd $(n+1, n-1, n) = 1$. So all of the n highest valued cards must be in winning positions after the first play. This means that, when the cards are dealt, n-1 of the cards of highest value must hold positions $a_2, a_4, a_6, \ldots, a_{n-1}$ and $b_3, b_5, \ldots, b_{n-2}, b_n$ (as shown in Figure 1). The remaining card of highest value must be in position b_1 since player B must win when players hold an odd number of cards. With this in mind, we see that there are $(n!)^2$ deals that are in a loop beginning with the second turn. ◆

Now that we have an understanding of some of the patterns which occur in the two-player game of War, we will turn to investigate patterns in games involving more than two players.

m-PLAYER WAR RESULTS:

The "typical" game of War is played by two players. It is interesting to investigate what happens when we allow any number of people to play the game. Many of the results which were found in the two-player game generalize nicely to a game involving m players. Before we discuss these results, it is helpful to clarify the notation and rules which will be used.

We call the players player 1, player 2, ... , player m; and play the game according to the convention that at any play, player 1's card is placed ahead of player 2's card which is in turn placed in front of player 3's card, ... , which is in turn placed in front of player m's card at the bottom of the winner's pile. At the start of every game the mk cards are dealt evenly to the m players so that each player holds k cards at the beginning of the game. So, the cards are arranged according to Figure 9 at the start of a game. After the first play, assuming player i wins, the cards are arranged as in Figure 10. After k plays, each player has gained cards at the bottom of his or her pile in sets of m, so at the $(k+1)$st turn each player holds a multiple of m cards.

FIGURE 9:

$$\begin{array}{ccccccccc}
1 & 2 & 3 & \ldots & i & \ldots & m \\
 & & & & & & \\
y_{1_1} & y_{2_1} & y_{3_1} & \cdots & y_{i_1} & \cdots & y_{m_1} \\
y_{1_2} & y_{2_2} & y_{3_2} & \cdots & y_{i_2} & \cdots & y_{m_2} \\
y_{1_3} & y_{2_3} & y_{3_3} & \cdots & y_{i_3} & \cdots & y_{m_3} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
y_{1_k} & y_{2_k} & y_{3_k} & \cdots & y_{i_k} & \cdots & y_{m_k}
\end{array}$$

The first result from the two-player section told us that any loop length must be an even number of moves. That is, the loop length was always a multiple of the number of players. Not surprisingly, when we are playing a game with m players, the loop length is always a multiple of m. A sample game with three players is included in Appendix 9. In this example the loop is length 3*6=18 plays.

THEOREM 7:

In an m-player game played with km cards, loop lengths must be a multiple of m.

PROOF:

Assume that we are in a loop somewhere after the kth play. Assume also that player 1 holds r_1 cards, player 2 holds r_2 cards, etc. with $r_1+r_2+...+r_m=mk$. After one turn, one player, player i keeps all of his r_i cards and gains m-1 cards and the other players each lose 1 card. At each turn, one of the players gains m-1 cards and the other players each lose 1 card. Since we are in a loop, at some point, player 1 must again hold r_1 cards while player 2 holds r_2 cards, etc. Notice that whenever player i wins one turn he must lose m-1 turns in order to return to a total of r_i cards. So, if in a loop player i wins b turns, he must lose $b(m$-1) turns in order to return to a total of r_i cards. Thus, the number of plays will be $b+b(m-1)=bm$. Therefore, the number of plays in a loop must be a multiple of m. ◆

Just as in the two-player game, we now need to look at the pattern of winning in a loop in order to more fully understand the possible loop lengths for a given deck of cards. The following lemma discusses the pattern in the number of cards held by each player after every m plays in a game with a loop. Just as in the two-player game the players held an even number of cards after every pair of plays, here the players will all hold a multiple of m cards after each m plays.

LEMMA 3:

In a game played with m players and mk cards with a loop, players will hold a multiple of m cards at the $(k+1)$-st play and at each m-th play after that. Moreover, no player will hold a multiple of m cards at any other play after the $(k+1)$-st.

PROOF:

Assume that we are playing a game with a deck of km cards and that at the start of the game the cards are dealt k to each player. Also, assume that this game results eventually in a loop. As discussed in the paragraph immediately preceding Figure 9, at the k+1st play of the game, each player holds a multiple of m cards, say player i holds mx_i cards for i=1, 2, ... , m and $x_1+x_2+...+x_m=k$. Furthermore, after m plays, each player will again hold a multiple of m cards. (Since we are in a game with a loop, none of the players ever lose all of their cards.) In m plays, suppose player i (holding x_i cards at the start of the $(k+1)$-st play) wins w plays and loses m-w plays. Each time player i wins a play he gains m-1 cards and keeps all of his own cards. Each time he loses a play, he loses 1 card. Thus, after m plays, player i holds $mx_i+(m-1)w-(m-w)=mx_i+m(w-1)=m(x_i+w-1)$ cards. So, each player holds a multiple of m cards. Also, it is impossible for player i to hold a multiple of m cards at any other play than the $(k+t)$th play where $t\equiv 1 \bmod m$. Suppose that at some point all players are holding a multiple of m cards, player i holding mx_i cards, and that after y plays with $0<y<m$ player i is once again holding a multiple of m cards. Player i gains m-1 cards with each win and loses 1 card with each loss. Suppose that he wins w plays and loses y-w of them. Then player i holds $mx_i+(m-1)w-(y-w)=mx_i+mw-w-y+w=m(x_i+w)-y$ cards after these y plays. Since $0<y<m$, y cannot be a multiple of m, so player i does not hold a multiple of m cards. ◆

Theorem 8 and its corollaries are analogous to Theorem 2 and its corollaries in the two-player game. They describe patterns in the winning that must occur within a loop. The sample loop in a three-player game given in Appendix 9 illustrates the pattern in which player 1 wins, player 2 wins, player 3 wins, player 1 wins, etc. Theorem 8 and its proof show that the analogous pattern must appear in every loop of an m-player game.

THEOREM 8:

Within a loop in an m-player game, wins must alternate between players with each turn in the following way:

Player 1 wins, player 2 wins, player 3 wins, ... , player m wins, player 1 wins, ...

PROOF:

Assume we are playing with a deck of mk cards and that we are playing according to the convention described. At the start of the game, cards are arranged as in Figure 9. Assume that we are in a loop somewhere after k plays and that we are in the $(k+1+cm)$-th play of the game. So, we know that all players are holding a multiple of m cards by Lemma 3. Each winning card is aligned with m-1 losing cards. If we number the positions in each player's hand 1, 2, 3, ... from top to bottom whenever Player 1 is holding a multiple of m cards then winning cards hold positions y_{1_j} where $j \equiv 1 \bmod m$ in player 1's hand, y_{2_j} where $j \equiv 2 \bmod m$ in player 2's hand, and in general, y_{i_j} where $j \equiv i \bmod m$ in player i's hand. Suppose by contradiction that player i wins the t-th play where $t \equiv j \bmod m$ and $i \neq j$. That is, a card in a losing position in player i's hand beats a card in a winning position in player j's hand. Because we are in a loop, this displaced card, (the winning card from player j's hand), or a card of equal face value, must beat a card of lower face value in a winning position in j's hand so as to return to j's hand. This will simply create another displaced card, and the process will repeat. The displaced cards are constantly decreasing in face-value. Since we have finitely many cards, in finitely many plays the most-recently displaced card must be less than or equal in face value to all of the cards in winning position in player j's hand. At this point, the process is forced to end and we reach a contradiction. Thus, player i must win on the t-th play where $t=i \bmod m$. That is, wins will cycle through the players with each play as described in the theorem.◆

Notice that the order of winning matches the order in which the cards are placed underneath the winner's pile at each turn. We place player 1's card followed by player 2's card, ... followed by player m's card beneath the winner's pile at each play. The wins alternate between players with each play by the rule that player 1 wins, followed by player 2, . . . followed by player m, followed by player 1, etc. This pattern is a consequence of the convention which we chose to follow. We could chose any permutation of the numbers 1, 2, . . . m to determine the order for the placement of the cards beneath the winner's pile. The order of winning would match the order of placement beneath the winner's pile. In essence changing the order in which we place the cards under the pile is the same as renaming the players. For example, if we played a four-player game of war by the convention that player 1's card was followed by player 2's card, followed by player 3's card, followed by player 4's card beneath the winning player's pile at each play, then the winning would follow as described in Theorem 8. But, suppose we instead placed player 2's card followed by player 4's card, followed by player 1's card, followed by player 3's card beneath the winner's pile at each play. Theorem 8 still applies, but player 2's card now plays the role of player 1's card, player 4's card plays the role of player 2's card, player 1's card plays the role of player 3's card, and player 3's card plays the role of player 4's card. So, the winning will alternate accordingly: player 2 wins, followed by player 4, followed by player 1, followed by player 3.

The following corollaries to Theorem 8 help us to further understand how our knowledge of loops in a two-player game of War generalizes to loops in an m-player game of War. The corollaries help us develop an idea of the permutations of the positions throughout each m plays of a loop.

COROLLARY 1:

In a loop, winning cards in player i's hand remain in player i's hand throughout the loop.

PROOF:

Assume we are in a loop sometime after the first k plays. Then player i's winning cards hold the x_i positions where $y_i \equiv i \bmod m$. The only way that player i will lose his or her winning card will be for player j ($i \neq j$) to win the t-th play for some $t \equiv i \bmod m$. By Theorem 8, this will not happen, since player j wins only on plays t where $t \equiv j \bmod m$. ◆

COROLLARY 2:

In a loop, player 1 will always win any play when all players hold a multiple of m cards.

PROOF:

Assume that we are in a loop somewhere after the first k plays. Suppose that player 1 wins when holding t cards where t is not a multiple of m. We consider the two cases where $t>m$ or $t<m$. First, if $t>m$, by Theorem 8, player 1 must win again in m plays, so, if $t \equiv i \mod m$, ($i \neq 0 \mod m$) player 1 must win the (t-i+1)-st subsequent play, player 2 must win the (t-i+2)-nd subsequent play, and in general player j must win the (t-i+j)-th subsequent play. So, player i+1 must win the $t-i+i+1=(t+1)$-st subsequent play. Recall, $i \neq 0 \mod m$ so some other player than player 1 must win the (t+1)-st subsequent play. But, by Corollary 1 of Theorem 8, player 1's winning card -- which is now in position t+1-- must win the (t+1)-st subsequent play. This is a contradiction since we are in a position in which two players must win the same play. Similarly, if $k<t<m$, by Corollary 1 of Theorem 8 player 1 must win the (t+1)-st subsequent play since player 1's winning card will hold the (t+1)-st position. But, by Theorem 8 we know that player t+1 must win the (t+1)-st subsequent play. This is a contradiction. Thus, player 1 will win when holding a multiple of m cards.

By Corollary 2 to Theorem 8 we know that all players hold a multiple of m cards at the same time. Thus, player 1 will win when all players hold a multiple of m cards.◆

EXAMPLE:

Turn to the sample 3-player game in Appendix 9 to see an illustration of the corollaries to Theorem 8. Notice that winning cards remain in their respective hands throughout the loop, players hold a multiple of 3 cards on the 5th play and at each 3rd play after that, and that player 1 wins whenever all players hold a multiple of three cards.

In a loop, since the wins must cycle through the players with each turn, if each player i holds mx_i cards at some point in the loop, after m plays each player i will again hold mx_i cards. This knowledge lets us look at the permutation of mk positions which takes place over each m plays. A chart of this permutation for a four-player game is given in Appendix 10. The 4-player game is simple enough to clearly illustrate the cycles which occur between the hands of four players. The cycles generalize to an m-player game. Using that generalization, we get the following result.

THEOREM 9:

If a game played with a deck of mk cards is in a loop in which each player i holds mx_i cards after every m plays, then the loop length will be given by m*LCM$(x_1,x_2,...,x_m,x_1+x_2,x_1+x_3,...,x_1+x_m,x_2+x_3,x_2+x_4,...,x_2+x_m,...,x_{m-1}+x_m)$, that is, m times the least common multiple of the individual x_i and all sums of two distinct x_i.

PROOF:

In order to see that the perfect loop length in the game described will be m*LCM$(x_1,x_2,...,x_m,x_1+x_2,x_1+x_3,...,x_1+x_m,x_2+x_3,x_2+x_4,...,x_2+x_m,...,x_{m-1}+x_m)$ we will examine the lengths of the disjoint cycles composing the permutation of the mk cards. Corollary 1 to Theorem8 tells us that winning cards in player i's hand remain in player i's hand. Player i holds x_i winning cards. After each m plays, winning cards in positions y_{i_j} where $j>m$ will simply move up m positions in player i's hand. The winning card in position y_{i_i} will move to the lowest winning position in player i's hand. We can see that the winning cards in player i's hand form an x_i cycle. So, there are m disjoint cycles of winning cards -- lengths $x_1, x_2, x_3, ..., x_m$ -- which partially compose the permutation of the mk cards after m plays. Now we must consider the cycles formed by the losing cards after m plays. To do this, we choose a losing card in position y_{i_j} where $j \neq i \bmod m$ and examine how many sets of m plays are needed in order for this losing card to return to its original position. We will assume that $j \leq m$ since any cycle formed by losing cards must include a losing card that is involved in one of the m plays that we are considering. In the first set of m plays, the card in position y_{i_j} will lose to the card in player j's hand in position y_{j_j} and be placed in player j's hand in position y_{j_k} where $k \equiv i \bmod m$. (The fact that $k \equiv i \bmod m$ is due to the convention by which we are placing the cards beneath the winning player's pile.) Player j holds x_j sets of m cards, so in x_j-1 sets of m plays, the card in position y_{j_k} will be in position y_{j_i}. In the next set of m plays, the card in position y_{j_i} will lose to player i and move into position y_{i_h} where $h \equiv j \bmod m$. Since player i holds x_i sets of m cards, in x_i-1 sets of m plays, the card in position y_{i_h} will return to position y_{i_j}. The number of sets of m plays needed for our losing card to return to its original position was x_i+x_j. From this we see that each losing card cycles through two players' hands. So, we will have a disjoint cycle of length x_i+x_j for each pair i and j. There can be no other cycles since each card in a position y_{i_j} with $j \leq m$ will be involved in one of these winning or losing cycles. All other cards will eventually move into one of the positions described and will be involved in one of the given cycles. We now have a comprehensive list of the disjoint cycles composing the permutation of the mk cards after m plays. The length of the perfect loop in this game will be given by the least common multiple of the disjoint cycle lengths according to Ruffini's Order of a Permutation Theorem. So, the perfect loop length will be given by m*LCM($x_1,x_2,\ldots,x_m,x_1+x_2,x_1+x_3,\ldots,x_1+x_m,x_2+x_3,x_2+x_4,\ldots,x_2+x_m,\ldots,x_{m-1}+x_m$).

EXAMPLE 10:

A diagram of the permutation of the cards in a sample four-player game after four plays is given in Appendix 10. In this sample game, players hold 3, 2, 2, and 1 sets of four cards. The disjoint cycle lengths are 3, 2, 2, 1, 5, 5, 4, 4, 3, and 3.

Quite obviously, there are many questions waiting for answers. For example, it would be nice to be able to determine the probability of a win versus the probability of a loop in the game played by our convention with cards labeled 1, 2, ... , $2n$. We worked on this problem for some time but failed to develop any conclusions. The difficulty arises in the fact that the only completely random arrangement is that at the start of the game. After the cards are dealt, arrangements depend on the initial arrangement.

Determining the probability of no matches in a game played with a standard deck of cards would be a problem of a different flavor than those discussed in this paper, but is a problem that interests me.

There are other conventions by which it would be possible to play. We spent some time looking into the game played by the convention of placing the losing card under the winner's pile ahead of the winning card in each play as well as the game played by the convention of placing the winning card under the winner's pile ahead of the losing card. We didn't formulate any theorems, but it appears that in these games, loop lengths are always a multiple of 4 and that the wins alternate between players two plays at a time. That is, player A wins twice, followed by player B winning twice.

In the typical game of War, matches are allowed. It might be interesting to develop theory about games allowing matches. I hope to have laid the ground work for future students to further explore the mathematics behind this card game.

APPENDIX 1:

The following is a Mathematica program which plays 3-player War for a deck of one suit. The program takes an input, *n*, for the number of cards in the deck. It splits the cards evenly between the three players and plays the game. The output is either "PlayerAloses", "PlayerBloses", or "PlayerCloses" if one players loses all of his or her cards; or "loop" if the game develops a loop. The program also gives the loop length and the number of moves needed to reach an outcome.

```
ANGE>eve war33.m


r=Table[ i,{  i,  1,  n }] ;
p=Permutations [ r] ;
For[ j=1,j<=Length[ p] ,j++,
a=Drop[ p[ [ j]] ,-2n/3] ;
b=Drop[ Drop[ p[ [ j]] ,-n/3] ,n/3] ;
c=Drop[ p[ [ j]] ,2n/3] ;
dj=p[ [ j]] ;
k=1;
s={ Join[ a,{ q} ,b,{ y} ,c] } ;
While[ Length[ a] *Length[ b] *Length[ c] !=0,
k=k+1;If[ a[ [ 1]] ==Max[ a[ [ 1]] ,b[ [ 1]] ,c[ [ 1]]] ,
u=a[ [ 1]] ;
v=b[ [ 1]] ;
w=c[ [ 1]] ;
x={ u,v,w} ;
a=Join[ Drop[ a,1] ,x] ;
b=Drop[ b,1] ;
c=Drop[ c,1] ;
r=0;
If[ b=={  } ,Print[ dj," ",  PlayerBLoses," ",k-1] ;r=1;Return[ t]] ;
If[ c=={  } ,Print[ dj," ",  PlayerCLoses," ",k-1] ;r=1;Return[ t]] ;
t=Join[ a,{ q} ,b,{ y} ,c] ;
For[ m=1,m<=Length[ s] ,m++,
If[ t==s[ [ m]] ,Print[ dj," ",loop," ",m," ",k] ;r=1;Return[ t]]] ;
If[ r==0,  PrependTo[ s,t]] ,
         If[ b[ [ 1]] >c[ [ 1]] ,
         u=a[ [ 1]] ;
         v=b[ [ 1]] ;
         w=c[ [ 1]] ;
         x={ u,v,w} ;
         a=Drop[ a,1] ;
         b=Join[ Drop[ b,1] ,x] ;
         c=Drop[ c,1] ;
         r=0;
         If[ a=={  } ,Print[ dj," ",PlayerALoses," ",k-1] ;r=1;Return[ t]] ;
         If[ c=={  } ,Print[ dj," ",PlayerCLoses," ",k-1] ;r=1;Return[ t]] ;
         t=Join[ a,{ q} ,b,{ y} ,c] ;
         For[ m=1,m<=Length[ s] ,m++,
         If[ t==s[ [ m]] ,Print[ dj," ",loop," ",m," ",k] ;r=1;Return[ t]]] ;
         If[ r==0,PrependTo[ s,t]] ,
u=a[ [ 1]] ;
v=b[ [ 1]] ;
w=c[ [ 1]] ;
x={ u,v,w} ;
a=Drop[ a,1] ;
b=Drop[ b,1] ;
r=0;
If[ a=={  } ,Print[ dj," ",PlayerALoses," ",k-1] ;r=1;Return[ t]] ;
If[ b=={  } ,Print[ dj," ",PlayerBLoses," ",k-1] ;r=1;Return[ t]] ;
c=Join[ Drop[ c,1] ,x] ;
t=Join[ a,{ q} ,b,{ y} ,c] ;
For[ m=1,m<=Length[ s] ,m++,
If[ t==s[ [ m]] ,Print[ dj," ",loop," ",m," ",k] ;r=1;Return[ t]]] ;
If[ r==0,PrependTo[ s,t]]]]]
         If[ r==0,Print[ dj," ",match," ",k]]]
[ End of file]
```

APPENDIX 2:

The following is a Mathematica printout of the hands at each play of a game using the cards 1, 2, 3, . . . , 8. This particular game developed a loop of length 24, and the first time a hand was repeated was at the 29th play. The cards were dealt 1, 2, 3, 5 to player A and 6, 7, 8, 4 to player B at the start of the game. This information is found in the first line of the printout. The hands are listed in reverse order from the point at which the first hand was repeated. Thus, the first hand of the game is listed last on the list, and the hands are listed in order starting at the end of the list working backwards. Notice that the fifth play from the end matches the first play on the list. In this program, the "X" indicates the division between the two players' hands.

```
In[3]:= <<war25.m
{1, 2, 3, 5, 6, 7, 8, 4}    loop 24 29
{5, 4, X, 1, 6, 2, 7, 3, 8}
{{3, 5, 4, X, 8, 1, 6, 2, 7}, {5, 3, X, 4, 8, 1, 6, 2, 7},

>    {2, 5, 3, X, 7, 4, 8, 1, 6}, {5, 2, X, 3, 7, 4, 8, 1, 6},

>    {1, 5, 2, X, 6, 3, 7, 4, 8}, {5, 1, X, 2, 6, 3, 7, 4, 8},

>    {4, 5, 1, X, 8, 2, 6, 3, 7}, {5, 4, X, 1, 8, 2, 6, 3, 7},

>    {3, 5, 4, X, 7, 1, 8, 2, 6}, {5, 3, X, 4, 7, 1, 8, 2, 6},

>    {2, 5, 3, X, 6, 4, 7, 1, 8}, {5, 2, X, 3, 6, 4, 7, 1, 8},

>    {1, 5, 2, X, 8, 3, 6, 4, 7}, {5, 1, X, 2, 8, 3, 6, 4, 7},

>    {4, 5, 1, X, 7, 2, 8, 3, 6}, {5, 4, X, 1, 7, 2, 8, 3, 6},

>    {3, 5, 4, X, 6, 1, 7, 2, 8}, {5, 3, X, 4, 6, 1, 7, 2, 8},

>    {2, 5, 3, X, 8, 4, 6, 1, 7}, {5, 2, X, 3, 8, 4, 6, 1, 7},

>    {1, 5, 2, X, 7, 3, 8, 4, 6}, {5, 1, X, 2, 7, 3, 8, 4, 6},

>    {4, 5, 1, X, 6, 2, 7, 3, 8}, {5, 4, X, 1, 6, 2, 7, 3, 8},

>    {5, X, 4, 1, 6, 2, 7, 3, 8}, {3, 5, X, 8, 4, 1, 6, 2, 7},

>    {2, 3, 5, X, 7, 8, 4, 1, 6}, {1, 2, 3, 5, X, 6, 7, 8, 4}}
```

APPENDIX 3

Example 2: If a game is played using the cards 1♡,2♡,3♡,4♡ and 1♣,2,♣3♣, 4♣ and at some point (in this Example, at play 3 below) player A holds 4♣,1♣ and player B holds 2♣,3♡,1♡,4♡,2♡,3♣ (in that order) then there is a loop of length 12 plays and a perfect loop of length 24 plays. This is a listing of the hands at each play of the loop.

1) A	B
1♡	4♡
2♡	3♣
4♣	2♣
1♣	3♡

2) A	B
2♡	3♣
4♣	2♣
1♣	3♡
	1♡
	4♡

3) A	B
4♣	2♣
1♣	3♡
	1♡
	4♡
	2♡
	3♣

4) A	B
1♣	3♡
4♣	1♡
2♣	4♡
	2♡
	3♣

5) A	B
4♣	1♡
2♣	4♡
	2♡
	3♣
	1♣
	3♡

6) A	B
2♣	4♡
4♣	2♡
1♡	3♣
	1♣
	3♡

7) A	B
4♣	2♡
1♡	3♣
	1♣
	3♡
	2♣
	4♡

8) A	B
1♡	3♣
4♣	1♣
2♡	3♡
	2♣
	4♡

9) A	B
4♣	1♣
2♡	3♡
	2♣
	4♡
	1♡
	3♣

10) A	B
2♡	3♡
4♣	2♣
1♣	4♡
	1♡
	3♣

11) A	B
4♣	2♣
1♣	4♡
	1♡
	3♣
	2♡
	3♡

12) A	B
1♣	4♡
4♣	1♡
2♣	3♣
	2♡
	3♡

13) A	B
4♣	1♡
2♣	3♣
	2♡
	3♡
	1♣
	4♡

14) A	B
2♣	3♣
4♣	2♡
1♡	3♡
	1♣
	4♡

15) A	B
4♣	2♡
1♡	3♡
	1♣
	4♡
	2♣
	3♣

16) A	B
1♡	3♡
4♣	1♣
2♡	4♡
	2♣
	3♣

17) A	B
4♣	1♣
2♡	4♡
	2♣
	3♣
	1♡
	3♡

18) A	B
2♡	4♡
4♣	2♣
1♣	3♣
	1♡
	3♡

19) A	B
4♣	2♣
1♣	3♣
	1♡
	3♡
	2♡
	4♡

20) A	B
1♣	3♣
4♣	1♡
2♣	3♡
	2♡
	4♡

21) A	B
4♣	1♡
2♣	3♡
	2♡
	4♡
	1♣
	3♣

22) A	B
2♣	3♡
4♣	2♡
1♡	4♡
	1♣
	3♣

23) A	B
4♣	2♡
1♡	4♡
	1♣
	3♣
	2♣
	3♡

24) A	B
1♡	4♡
4♣	1♣
2♡	3♣
	2♣
	3♡

25)A	B
4♣	1♣
2♡	3♣
	2♣
	3♡
	1♡
	4♡

26) A	B
2♡	3♣
4♣	2♣
1♣	3♡
	1♡
	4♡

27) A	B
4♣	2♣
1♣	3♡
	1♡
	4♡
	2♡
	3♣

Notice that hands in play 3 match hands in play 15 in face value, so there is a loop of length 12 plays. Also, hands in play 3 match hands in play 27 in suit and face value, so there is a perfect loop of length 24 plays.

The winning card in player A's hand in each play is 4♣ and the winning cards in player B's hand in each play are 4♡,3♣, and 3♡. The other cards are losing cards in each play.

APPENDIX 4:

The following is a sample list of outputs of games using the deck of 8 cards of one suit. The program used was written to permute the 8 cards, give 4 to the first player and four to the second player, play the game and output the result. If the game ended in a win, the output would indicate which player won as well as the length of the game. If the game resulted in a loop, the output would indicate the length of the loop followed by the number of the play on which the first repeated hand occurs followed by the arrangement of the cards on that play. This sample list does not include all permutations of the 8 cards, but it gives a sense of the loop lengths which are possible with a deck of 8 cards of one suit. There are examples of loops of length 8 plays and 24 plays. In fact, the complete list of all permutations of the 8 cards show that these are the only loop lengths which occur.

```
In[2]:= <<war21.m
{1, 2, 3, 4, 5, 6, 7, 8}   PlayerBwins 4
{1, 2, 3, 4, 5, 6, 8, 7}   PlayerBwins 4
{1, 2, 3, 4, 5, 7, 6, 8}   PlayerBwins 4
{1, 2, 3, 4, 5, 7, 8, 6}   PlayerBwins 4
{1, 2, 3, 4, 5, 8, 6, 7}   PlayerBwins 4
{1, 2, 3, 4, 5, 8, 7, 6}   PlayerBwins 4
{1, 2, 3, 4, 6, 5, 7, 8}   PlayerBwins 4
{1, 2, 3, 4, 6, 5, 8, 7}   PlayerBwins 4
{1, 2, 3, 4, 6, 7, 5, 8}   PlayerBwins 4
{1, 2, 3, 4, 6, 7, 8, 5}   PlayerBwins 4
{1, 2, 3, 4, 6, 8, 5, 7}   PlayerBwins 4
{1, 2, 3, 4, 6, 8, 7, 5}   PlayerBwins 4
{1, 2, 3, 4, 7, 5, 6, 8}   PlayerBwins 4
{1, 2, 3, 4, 7, 5, 8, 6}   PlayerBwins 4
{1, 2, 3, 4, 7, 6, 5, 8}   PlayerBwins 4
{1, 2, 3, 4, 7, 6, 8, 5}   PlayerBwins 4
{1, 2, 3, 4, 7, 8, 5, 6}   PlayerBwins 4
{1, 2, 3, 4, 7, 8, 6, 5}   PlayerBwins 4
{1, 2, 3, 4, 8, 5, 6, 7}   PlayerBwins 4
{1, 2, 3, 4, 8, 5, 7, 6}   PlayerBwins 4
{1, 2, 3, 4, 8, 6, 5, 7}   PlayerBwins 4
{1, 2, 3, 4, 8, 6, 7, 5}   PlayerBwins 4
{1, 2, 3, 4, 8, 7, 5, 6}   PlayerBwins 4
{1, 2, 3, 4, 8, 7, 6, 5}   PlayerBwins 4
{1, 2, 3, 5, 4, 6, 7, 8}   PlayerBwins 4
{1, 2, 3, 5, 4, 6, 8, 7}   PlayerBwins 4
{1, 2, 3, 5, 4, 7, 6, 8}   PlayerBwins 4
{1, 2, 3, 5, 4, 7, 8, 6}   PlayerBwins 4
{1, 2, 3, 5, 4, 8, 6, 7}   PlayerBwins 4
{1, 2, 3, 5, 4, 8, 7, 6}   PlayerBwins 4
{1, 2, 3, 5, 6, 4, 7, 8}   PlayerBwins 4
{1, 2, 3, 5, 6, 4, 8, 7}   PlayerBwins 4
{1, 2, 3, 5, 6, 7, 4, 8}   PlayerBwins 4
{1, 2, 3, 5, 6, 7, 8, 4}    loop 24 29  {5, 4, X, 1, 6, 2, 7, 3, 8}
{1, 2, 3, 5, 6, 8, 4, 7}   PlayerBwins 4
{1, 2, 3, 5, 6, 8, 7, 4}    loop 24 29  {5, 4, X, 1, 6, 2, 8, 3, 7}
{1, 2, 3, 5, 7, 4, 6, 8}   PlayerBwins 4
{1, 2, 3, 5, 7, 4, 8, 6}   PlayerBwins 4
{1, 2, 3, 5, 7, 6, 4, 8}   PlayerBwins 4
{1, 2, 3, 5, 7, 6, 8, 4}    loop 24 29  {5, 4, X, 1, 7, 2, 6, 3, 8}
{1, 2, 3, 5, 7, 8, 4, 6}   PlayerBwins 4
{1, 2, 3, 5, 7, 8, 6, 4}    loop 24 29  {5, 4, X, 1, 7, 2, 8, 3, 6}
{1, 2, 3, 5, 8, 4, 6, 7}   PlayerBwins 4
{1, 2, 3, 5, 8, 4, 7, 6}   PlayerBwins 4
{1, 2, 3, 5, 8, 6, 4, 7}   PlayerBwins 4
{1, 2, 3, 5, 8, 6, 7, 4}    loop 24 29  {5, 4, X, 1, 8, 2, 6, 3, 7}
{1, 2, 3, 5, 8, 7, 4, 6}   PlayerBwins 4
{1, 2, 3, 5, 8, 7, 6, 4}    loop 24 29  {5, 4, X, 1, 8, 2, 7, 3, 6}
{1, 2, 3, 6, 4, 5, 7, 8}   PlayerBwins 4
{1, 2, 3, 6, 4, 5, 8, 7}   PlayerBwins 4
{1, 2, 3, 6, 4, 7, 5, 8}   PlayerBwins 4
{1, 2, 3, 6, 4, 7, 8, 5}    loop 8 15  {6, 1, 5, 4, X, 2, 7, 3, 8}
{1, 2, 3, 6, 4, 8, 5, 7}   PlayerBwins 4
{1, 2, 3, 6, 4, 8, 7, 5}    loop 8 15  {6, 1, 5, 4, X, 2, 8, 3, 7}
{1, 2, 3, 6, 5, 4, 7, 8}   PlayerBwins 4
{1, 2, 3, 6, 5, 4, 8, 7}   PlayerBwins 4
{1, 2, 3, 6, 5, 7, 4, 8}   PlayerBwins 4
{1, 2, 3, 6, 5, 7, 8, 4}    loop 24 29  {6, 4, X, 1, 5, 2, 7, 3, 8}
{1, 2, 3, 6, 5, 8, 4, 7}   PlayerBwins 4
```

APPENDIX 5:

The following Mathematica printout lists results of games played with 12 cards labeled 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6. The program which generated this data was a model of a game in which player A held cards 1, 2, 3, 4, 5, 6 in that order and player B held a derangement of the 6 cards. The program was written so that the game would end when a match occurred. We gave player B a derangement of the cards 1, 2, 3, 4, 5, 6 in order to avoid matches within the first 6 plays. For each game, the printout lists the cards dealt to player B at the start of the game and indicates whether the game ended in a match or developed a loop. If the game ended in a match, we are given the number of plays before the match. And if the game develops a loop, the printout includes the length of the loop followed by the number of the first play of the loop. Also included is the arrangement of the players' hands at the point of the match or the start of the loop. Notice that in this sample there are loops of length 6, 12, 24, and 60. Each sample loop length is a multiple of 6 -- the number of pairs of cards.

```
In[3]:= <<war7.m
{2, 1, 4, 3, 6, 5}  {2, 1, 4, 3, 6, 5, X, 1, 2, 3, 4, 5, 6}  loop 6 13
{2, 1, 4, 5, 6, 3}  {6, 1, 5, 2, X, 3, 4, 2, 4, 1, 5, 3, 6}  loop 24 41
{2, 1, 4, 6, 3, 5}  {2, 1, 5, 3, 6, 5, X, 1, 2, 3, 4, 4, 6}  loop 12 19
{2, 1, 5, 3, 6, 4}  {2, 1, 4, 3, 6, 4, X, 1, 2, 3, 5, 5, 6}  loop 12 19
{2, 1, 5, 6, 3, 4}  {2, 1, 5, 3, 6, 4, X, 1, 2, 3, 5, 4, 6}  loop 6 13
{2, 1, 5, 6, 4, 3}  {2, 1, 5, 4, 6, 3, X, 1, 2, 3, 5, 4, 6}  loop 12 19
{2, 1, 6, 3, 4, 5}  {1, 4, 3, 5, 1, 4, 2, 6, 3, X, 6, 2, 5}  loop 24 40
{2, 1, 6, 5, 3, 4}  {2, 1, 5, 3, 6, 4, X, 1, 2, 3, 6, 4, 5}  loop 6 13
{2, 1, 6, 5, 4, 3}  {2, 1, 5, 4, 6, 3, X, 1, 2, 3, 6, 4, 5}  loop 12 19
{2, 3, 1, 5, 6, 4}  {2, 6, 1, 5, 2, 4, 3, X, 6, 1, 5, 3, 4}  loop 12 42
{2, 3, 1, 6, 4, 5}  {5, 2, 4, 3, 6, 4, 3, 1, X, 5, 6, 1, 2}  match 15
{2, 3, 4, 1, 6, 5}  {2, 4, 1, 6, 2, 5, 3, X, 6, 1, 4, 3, 5}  loop 12 36
{2, 3, 4, 5, 6, 1}  {3, 5, 4, 6, 1, X, 3, 2, 4, 1, 5, 2, 6}  match 28
{2, 3, 4, 6, 1, 5}  {2, 5, 1, 6, 2, 5, 3, X, 6, 1, 4, 3, 4}  loop 12 36
{2, 3, 5, 1, 6, 4}  {2, 4, 1, 6, 2, 4, 3, X, 6, 1, 5, 3, 5}  loop 12 36
{2, 3, 5, 6, 1, 4}  {2, 5, 1, 6, 2, 4, 3, X, 6, 1, 5, 3, 4}  loop 12 36
{2, 3, 5, 6, 4, 1}  {4, 2, 6, 2, 5, 3, X, 4, 6, 1, 3, 1, 5}  match 13
{2, 3, 6, 1, 4, 5}  {5, 2, 4, 3, 6, 3, 4, 1, X, 5, 6, 1, 2}  match 15
{2, 3, 6, 5, 1, 4}  {2, 5, 1, 6, 2, 4, 3, X, 5, 1, 6, 3, 4}  loop 12 36
{2, 3, 6, 5, 4, 1}  {4, 2, 6, 2, 5, 3, X, 4, 5, 1, 3, 1, 6}  match 13
{2, 4, 1, 3, 6, 5}  {3, 1, 4, 3, 6, 5, X, 1, 2, 2, 4, 5, 6}  loop 12 19
{2, 4, 1, 5, 6, 3}  {6, 1, 5, 2, X, 3, 4, 3, 4, 1, 5, 2, 6}  loop 24 41
{2, 4, 1, 6, 3, 5}  {3, 1, 5, 3, 6, 5, X, 1, 2, 2, 4, 4, 6}  loop 12 19
{2, 4, 5, 1, 6, 3}  {2, 6, 1, 5, 2, 4, 3, X, 6, 1, 5, 3, 4}  loop 12 42
{2, 4, 5, 3, 6, 1}  {2, 6, 2, 4, 3, X, 6, 1, 4, 1, 5, 3, 5}  loop 24 38
{2, 4, 5, 6, 1, 3}  {4, 5, 1, 3, 2, 6, 3, X, 4, 1, 5, 2, 6}  match 18
{2, 4, 5, 6, 3, 1}  {2, 6, 2, 5, 3, X, 6, 1, 4, 1, 5, 3, 4}  loop 24 38
{2, 4, 6, 1, 3, 5}  {4, 1, 5, 3, 6, 5, X, 1, 2, 2, 4, 3, 6}  loop 12 19
{2, 4, 6, 3, 1, 5}  {2, 5, 2, 6, 3, 4, 1, X, 6, 1, 4, 3, 5}  loop 12 28
{2, 4, 6, 5, 1, 3}  {4, 5, 1, 3, 2, 6, 3, X, 4, 1, 6, 2, 5}  match 18
{2, 4, 6, 5, 3, 1}  {2, 6, 2, 5, 3, X, 5, 1, 4, 1, 6, 3, 4}  loop 24 38
{2, 5, 1, 3, 6, 4}  {3, 1, 4, 3, 6, 4, X, 1, 2, 2, 5, 5, 6}  loop 12 19
{2, 5, 1, 6, 3, 4}  {3, 1, 5, 3, 6, 4, X, 1, 2, 2, 5, 4, 6}  loop 12 19
{2, 5, 1, 6, 4, 3}  {3, 1, 5, 4, 6, 3, X, 1, 2, 2, 5, 4, 6}  loop 12 19
{2, 5, 4, 1, 6, 3}  {5, 4, 1, 3, 2, 6, 3, X, 5, 1, 4, 2, 6}  match 18
{2, 5, 4, 3, 6, 1}  {2, 6, 2, 4, 3, X, 6, 1, 5, 1, 4, 3, 5}  loop 24 38
{2, 5, 4, 6, 1, 3}  {6, 4, 5, 1, 3, 2, X, 3, 5, 1, 4, 2, 6}  loop 12 29
{2, 5, 4, 6, 3, 1}  {2, 6, 2, 5, 3, X, 6, 1, 5, 1, 4, 3, 4}  loop 24 38
{2, 5, 6, 1, 3, 4}  {4, 1, 5, 3, 6, 4, X, 1, 2, 2, 5, 3, 6}  loop 12 19
{2, 5, 6, 1, 4, 3}  {4, 1, 5, 4, 6, 3, X, 1, 2, 2, 5, 3, 6}  loop 12 19
{2, 5, 6, 3, 1, 4}  {2, 5, 2, 6, 3, 4, 1, X, 6, 1, 5, 3, 4}  loop 12 28
{2, 5, 6, 3, 4, 1}  {4, 1, 3, 2, 5, 2, 6, 3, X, 4, 5, 1, 6}  match 13
{2, 6, 1, 3, 4, 5}  {1, 4, 2, 5, 1, 4, 2, 6, 3, X, 6, 3, 5}  loop 24 40
{2, 6, 1, 5, 3, 4}  {3, 1, 5, 3, 6, 4, X, 1, 2, 2, 6, 4, 5}  loop 12 19
{2, 6, 1, 5, 4, 3}  {3, 1, 5, 4, 6, 3, X, 1, 2, 2, 6, 4, 5}  loop 12 19
{2, 6, 4, 1, 3, 5}  {4, 1, 5, 3, 6, 1, 3, 2, 5, 2, X, 4, 6}  match 21
{2, 6, 4, 3, 1, 5}  {4, 1, 3, 1, 5, 2, 6, 2, 5, 3, X, 4, 6}  match 23
{2, 6, 4, 5, 1, 3}  {6, 4, 5, 1, 3, 2, X, 3, 6, 1, 4, 2, 5}  loop 12 29
{2, 6, 4, 5, 3, 1}  {2, 6, 2, 5, 3, X, 5, 1, 6, 1, 4, 3, 4}  loop 24 38
{2, 6, 5, 1, 3, 4}  {4, 1, 5, 3, 6, 4, X, 1, 2, 2, 6, 3, 5}  loop 12 19
{2, 6, 5, 1, 4, 3}  {4, 1, 5, 4, 6, 3, X, 1, 2, 2, 6, 3, 5}  loop 12 19
{2, 6, 5, 3, 1, 4}  {2, 5, 2, 6, 3, 4, 1, X, 5, 1, 6, 3, 4}  loop 12 28
{2, 6, 5, 3, 4, 1}  {4, 1, 3, 2, 5, 2, 6, 3, X, 4, 6, 1, 5}  match 13
{3, 1, 2, 5, 6, 4}  {2, 1, 6, 3, 5, 4, X, 2, 5, 4, 6, 1, 3}  match 17
{3, 1, 2, 6, 4, 5}  {2, 1, 5, 1, 4, 3, 6, 3, 5, 4, X, 2, 6}  match 15
{3, 1, 4, 2, 6, 5}  {2, 1, 4, 2, 6, 5, X, 1, 3, 3, 4, 5, 6}  loop 12 19
{3, 1, 4, 5, 6, 2}  {6, 1, 5, 3, X, 2, 4, 2, 4, 1, 5, 3, 6}  loop 24 41
{3, 1, 4, 6, 2, 5}  {2, 1, 5, 2, 6, 5, X, 1, 3, 3, 4, 4, 6}  loop 12 19
{3, 1, 5, 2, 6, 4}  {2, 1, 4, 2, 6, 4, X, 1, 3, 3, 5, 5, 6}  loop 12 19
```

{3, 1, 5, 6, 2, 4}	{2, 1, 5, 2, 6, 4, X, 1, 3, 3, 5, 4, 6}	loop 12 19
{3, 1, 5, 6, 4, 2}	{2, 1, 5, 4, 6, 2, X, 1, 3, 3, 5, 4, 6}	loop 12 19
{3, 1, 6, 2, 4, 5}	{1, 4, 3, 5, 1, 4, 3, 6, 2, X, 6, 2, 5}	loop 24 40
{3, 1, 6, 5, 2, 4}	{2, 1, 5, 2, 6, 4, X, 1, 3, 3, 6, 4, 5}	loop 12 19
{3, 1, 6, 5, 4, 2}	{2, 1, 5, 4, 6, 2, X, 1, 3, 3, 6, 4, 5}	loop 12 19
{3, 4, 1, 2, 6, 5}	{3, 1, 4, 2, 6, 5, X, 1, 3, 2, 4, 5, 6}	loop 6 13
{3, 4, 1, 5, 6, 2}	{6, 1, 5, 3, X, 2, 4, 3, 4, 1, 5, 2, 6}	loop 24 41
{3, 4, 1, 6, 2, 5}	{3, 1, 5, 2, 6, 5, X, 1, 3, 2, 4, 4, 6}	loop 12 19
{3, 4, 2, 1, 6, 5}	{3, 2, 4, 1, 6, 5, X, 1, 3, 2, 4, 5, 6}	loop 12 19
{3, 4, 2, 5, 6, 1}	{6, 2, 5, 3, X, 1, 4, 3, 4, 1, 5, 2, 6}	loop 24 41
{3, 4, 2, 6, 1, 5}	{3, 2, 5, 1, 6, 5, X, 1, 3, 2, 4, 4, 6}	loop 12 19
{3, 4, 5, 1, 6, 2}	{3, 6, 5, 4, 1, X, 3, 2, 4, 1, 5, 2, 6}	match 16
{3, 4, 5, 2, 6, 1}	{3, 6, 5, 4, 2, X, 3, 1, 4, 1, 5, 2, 6}	match 16
{3, 4, 5, 6, 1, 2}	{3, 6, 4, 5, 1, X, 3, 2, 4, 1, 5, 2, 6}	match 16
{3, 4, 5, 6, 2, 1}	{3, 6, 4, 5, 2, X, 3, 1, 4, 1, 5, 2, 6}	match 16
{3, 4, 6, 1, 2, 5}	{4, 1, 5, 2, 6, 5, X, 1, 3, 2, 4, 3, 6}	loop 12 19
{3, 4, 6, 2, 1, 5}	{4, 2, 5, 1, 6, 5, X, 1, 3, 2, 4, 3, 6}	loop 12 19
{3, 4, 6, 5, 1, 2}	{3, 6, 4, 5, 1, X, 3, 2, 4, 1, 6, 2, 5}	match 16
{3, 4, 6, 5, 2, 1}	{3, 6, 4, 5, 2, X, 3, 1, 4, 1, 6, 2, 5}	match 16
{3, 5, 1, 2, 6, 4}	{3, 1, 4, 2, 6, 4, X, 1, 3, 2, 5, 5, 6}	loop 12 19
{3, 5, 1, 6, 2, 4}	{3, 1, 5, 2, 6, 4, X, 1, 3, 2, 5, 4, 6}	loop 6 13
{3, 5, 1, 6, 4, 2}	{3, 1, 5, 4, 6, 2, X, 1, 3, 2, 5, 4, 6}	loop 12 19
{3, 5, 2, 1, 6, 4}	{3, 2, 4, 1, 6, 4, X, 1, 3, 2, 5, 5, 6}	loop 12 19
{3, 5, 2, 6, 1, 4}	{3, 2, 5, 1, 6, 4, X, 1, 3, 2, 5, 4, 6}	loop 12 19
{3, 5, 2, 6, 4, 1}	{3, 2, 5, 4, 6, 1, X, 1, 3, 2, 5, 4, 6}	loop 12 19
{3, 5, 4, 1, 6, 2}	{3, 6, 5, 4, 1, X, 3, 2, 5, 1, 4, 2, 6}	match 16
{3, 5, 4, 2, 6, 1}	{3, 6, 5, 4, 2, X, 3, 1, 5, 1, 4, 2, 6}	match 16
{3, 5, 4, 6, 1, 2}	{3, 6, 4, 5, 1, X, 3, 2, 5, 1, 4, 2, 6}	match 16
{3, 5, 4, 6, 2, 1}	{3, 6, 4, 5, 2, X, 3, 1, 5, 1, 4, 2, 6}	match 16
{3, 5, 6, 1, 2, 4}	{4, 1, 5, 2, 6, 4, X, 1, 3, 2, 5, 3, 6}	loop 12 19
{3, 5, 6, 1, 4, 2}	{4, 1, 5, 4, 6, 2, X, 1, 3, 2, 5, 3, 6}	loop 12 19
{3, 5, 6, 2, 1, 4}	{4, 2, 5, 1, 6, 4, X, 1, 3, 2, 5, 3, 6}	loop 12 19
{3, 5, 6, 2, 4, 1}	{4, 2, 5, 4, 6, 1, X, 1, 3, 2, 5, 3, 6}	loop 12 19
{3, 6, 1, 2, 4, 5}	{1, 4, 2, 5, 1, 4, 3, 6, 2, X, 6, 3, 5}	loop 24 40
{3, 6, 1, 5, 2, 4}	{3, 1, 5, 2, 6, 4, X, 1, 3, 2, 6, 4, 5}	loop 6 13
{3, 6, 1, 5, 4, 2}	{3, 1, 5, 4, 6, 2, X, 1, 3, 2, 6, 4, 5}	loop 12 19
{3, 6, 2, 1, 4, 5}	{1, 4, 2, 5, 2, 4, 3, 6, 1, X, 6, 3, 5}	loop 24 40
{3, 6, 2, 5, 1, 4}	{3, 2, 5, 1, 6, 4, X, 1, 3, 2, 6, 4, 5}	loop 12 19
{3, 6, 2, 5, 4, 1}	{3, 2, 5, 4, 6, 1, X, 1, 3, 2, 6, 4, 5}	loop 12 19
{3, 6, 4, 1, 2, 5}	{3, 5, 4, 4, 1, 5, 2, 6, 1, X, 3, 2, 6}	match 18
{3, 6, 4, 2, 1, 5}	{3, 5, 4, 4, 2, 5, 1, 6, 1, X, 3, 2, 6}	match 18
{3, 6, 4, 5, 1, 2}	{3, 6, 4, 5, 1, X, 3, 2, 6, 1, 4, 2, 5}	match 16
{3, 6, 4, 5, 2, 1}	{3, 6, 4, 5, 2, X, 3, 1, 6, 1, 4, 2, 5}	match 16
{3, 6, 5, 1, 2, 4}	{4, 1, 5, 2, 6, 4, X, 1, 3, 2, 6, 3, 5}	loop 12 19
{3, 6, 5, 1, 4, 2}	{4, 1, 5, 4, 6, 2, X, 1, 3, 2, 6, 3, 5}	loop 12 19
{3, 6, 5, 2, 1, 4}	{4, 2, 5, 1, 6, 4, X, 1, 3, 2, 6, 3, 5}	loop 12 19
{3, 6, 5, 2, 4, 1}	{4, 2, 5, 4, 6, 1, X, 1, 3, 2, 6, 3, 5}	loop 12 19
{4, 1, 2, 3, 6, 5}	{5, 2, 1, 4, 1, 6, 3, X, 5, 2, 6, 3, 4}	match 14
{4, 1, 2, 5, 6, 3}	{2, 1, 6, 3, 5, 4, X, 2, 5, 3, 6, 1, 4}	match 17
{4, 1, 2, 6, 3, 5}	{2, 1, 5, 1, 6, 3, 5, 4, X, 2, 6, 3, 4}	match 15
{4, 1, 5, 2, 6, 3}	{2, 1, 4, 2, 6, 3, X, 1, 4, 3, 5, 5, 6}	loop 12 19
{4, 1, 5, 3, 6, 2}	{2, 1, 4, 3, 6, 2, X, 1, 4, 3, 5, 5, 6}	loop 12 19
{4, 1, 5, 6, 2, 3}	{2, 1, 5, 2, 6, 3, X, 1, 4, 3, 5, 4, 6}	loop 12 19
{4, 1, 5, 6, 3, 2}	{2, 1, 5, 3, 6, 2, X, 1, 4, 3, 5, 4, 6}	loop 12 19
{4, 1, 6, 2, 3, 5}	{6, 3, 5, 1, 4, 3, X, 2, 4, 2, 5, 1, 6}	loop 12 41
{4, 1, 6, 3, 2, 5}	{2, 1, 4, 3, 5, 1, 6, 3, X, 2, 4, 5, 6}	match 15
{4, 1, 6, 5, 2, 3}	{2, 1, 5, 2, 6, 3, X, 1, 4, 3, 6, 4, 5}	loop 12 19
{4, 1, 6, 5, 3, 2}	{2, 1, 5, 3, 6, 2, X, 1, 4, 3, 6, 4, 5}	loop 12 19
{4, 3, 1, 2, 6, 5}	{3, 1, 4, 2, 6, 5, X, 1, 4, 2, 3, 5, 6}	loop 6 13
{4, 3, 1, 5, 6, 2}	{4, 6, 2, 5, 3, X, 4, 1, 5, 2, 6, 1, 3}	match 20

{5, 3, 4, 6, 2, 1}	{5, 2, 6, 1, 4, 3, X, 1, 4, 2, 6, 3, 5}	loop 12 31
{5, 3, 6, 1, 2, 4}	{4, 1, 5, 2, 6, 4, X, 1, 5, 2, 3, 3, 6}	loop 12 19
{5, 3, 6, 1, 4, 2}	{6, 2, 4, 1, 5, 2, 4, 3, X, 3, 6, 1, 5}	loop 24 35
{5, 3, 6, 2, 1, 4}	{4, 2, 5, 1, 6, 4, X, 1, 5, 2, 3, 3, 6}	loop 12 19
{5, 3, 6, 2, 4, 1}	{6, 1, 4, 1, 5, 2, 4, 3, X, 3, 6, 2, 5}	loop 24 35
{5, 4, 1, 2, 6, 3}	{3, 1, 4, 2, 6, 3, X, 1, 5, 2, 4, 5, 6}	loop 12 19
{5, 4, 1, 3, 6, 2}	{3, 1, 4, 3, 6, 2, X, 1, 5, 2, 4, 5, 6}	loop 12 19
{5, 4, 1, 6, 2, 3}	{3, 1, 5, 2, 6, 3, X, 1, 5, 2, 4, 4, 6}	loop 12 19
{5, 4, 1, 6, 3, 2}	{3, 1, 5, 3, 6, 2, X, 1, 5, 2, 4, 4, 6}	loop 12 19
{5, 4, 2, 1, 6, 3}	{3, 2, 4, 1, 6, 3, X, 1, 5, 2, 4, 5, 6}	loop 12 19
{5, 4, 2, 3, 6, 1}	{3, 2, 4, 3, 6, 1, X, 1, 5, 2, 4, 5, 6}	loop 12 19
{5, 4, 2, 6, 1, 3}	{3, 2, 5, 1, 6, 3, X, 1, 5, 2, 4, 4, 6}	loop 12 19
{5, 4, 2, 6, 3, 1}	{3, 2, 5, 3, 6, 1, X, 1, 5, 2, 4, 4, 6}	loop 12 19
{5, 4, 6, 1, 2, 3}	{4, 1, 5, 2, 6, 3, X, 1, 5, 2, 4, 3, 6}	loop 6 13
{5, 4, 6, 1, 3, 2}	{4, 1, 5, 3, 6, 2, X, 1, 5, 2, 4, 3, 6}	loop 12 19
{5, 4, 6, 2, 1, 3}	{4, 2, 5, 1, 6, 3, X, 1, 5, 2, 4, 3, 6}	loop 12 19
{5, 4, 6, 2, 3, 1}	{4, 2, 5, 3, 6, 1, X, 1, 5, 2, 4, 3, 6}	loop 12 19
{5, 4, 6, 3, 1, 2}	{4, 3, 5, 1, 6, 2, X, 1, 5, 2, 4, 3, 6}	loop 12 19
{5, 4, 6, 3, 2, 1}	{4, 3, 5, 2, 6, 1, X, 1, 5, 2, 4, 3, 6}	loop 12 19
{5, 6, 1, 2, 3, 4}	{3, 1, 4, 2, 5, 1, 6, 2, X, 3, 5, 4, 6}	match 15
{5, 6, 1, 2, 4, 3}	{1, 4, 2, 5, 1, 6, 2, X, 5, 3, 6, 3, 4}	loop 12 28
{5, 6, 1, 3, 2, 4}	{4, 2, 5, 1, 6, 3, 3, 2, X, 4, 6, 1, 5}	match 17
{5, 6, 1, 3, 4, 2}	{1, 4, 2, 5, 1, 6, 3, X, 5, 2, 6, 3, 4}	loop 12 28
{5, 6, 2, 1, 3, 4}	{3, 1, 4, 2, 5, 2, 6, 1, X, 3, 5, 4, 6}	match 15
{5, 6, 2, 1, 4, 3}	{1, 4, 2, 5, 2, 6, 1, X, 5, 3, 6, 3, 4}	loop 12 28
{5, 6, 2, 3, 1, 4}	{4, 2, 5, 2, 6, 3, 3, 1, X, 4, 6, 1, 5}	match 17
{5, 6, 2, 3, 4, 1}	{1, 4, 2, 5, 2, 6, 3, X, 5, 1, 6, 3, 4}	loop 12 28
{5, 6, 4, 1, 2, 3}	{4, 1, 5, 2, 6, 3, X, 1, 5, 2, 6, 3, 4}	loop 6 13
{5, 6, 4, 1, 3, 2}	{4, 1, 5, 3, 6, 2, X, 1, 5, 2, 6, 3, 4}	loop 12 19
{5, 6, 4, 2, 1, 3}	{4, 2, 5, 1, 6, 3, X, 1, 5, 2, 6, 3, 4}	loop 12 19
{5, 6, 4, 2, 3, 1}	{4, 2, 5, 3, 6, 1, X, 1, 5, 2, 6, 3, 4}	loop 12 19
{5, 6, 4, 3, 1, 2}	{4, 3, 5, 1, 6, 2, X, 1, 5, 2, 6, 3, 4}	loop 12 19
{5, 6, 4, 3, 2, 1}	{4, 3, 5, 2, 6, 1, X, 1, 5, 2, 6, 3, 4}	loop 12 19
{6, 1, 2, 3, 4, 5}	{3, 2, 4, 1, 5, 1, 6, 2, X, 3, 5, 4, 6}	match 27
{6, 1, 2, 5, 3, 4}	{4, 2, 1, 5, 1, 6, 3, X, 4, 2, 5, 3, 6}	match 14
{6, 1, 2, 5, 4, 3}	{2, 1, 5, 1, 6, 3, X, 2, 5, 4, 6, 3, 4}	match 15
{6, 1, 4, 2, 3, 5}	{3, 5, 1, 6, 2, 5, 4, 4, 2, X, 3, 1, 6}	match 18
{6, 1, 4, 3, 2, 5}	{2, 1, 4, 3, 5, 1, 6, 3, 5, 4, X, 2, 6}	match 15
{6, 1, 4, 5, 2, 3}	{2, 1, 5, 2, 6, 3, X, 1, 6, 3, 4, 4, 5}	loop 12 19
{6, 1, 4, 5, 3, 2}	{2, 1, 5, 3, 6, 2, X, 1, 6, 3, 4, 4, 5}	loop 12 19
{6, 1, 5, 2, 3, 4}	{4, 3, 5, 1, 6, 2, X, 4, 5, 2, 3, 1, 6}	match 17
{6, 1, 5, 2, 4, 3}	{1, 4, 3, 5, 1, 6, 2, X, 6, 3, 5, 2, 4}	loop 12 28
{6, 1, 5, 3, 2, 4}	{2, 1, 4, 3, 5, 1, 6, 3, X, 2, 6, 4, 5}	match 15
{6, 1, 5, 3, 4, 2}	{1, 4, 3, 5, 1, 6, 3, X, 6, 2, 5, 2, 4}	loop 12 28
{6, 3, 1, 2, 4, 5}	{1, 4, 2, 5, 1, 6, 2, 5, 3, X, 6, 3, 4}	loop 24 40
{6, 3, 1, 5, 2, 4}	{3, 1, 5, 2, 6, 4, X, 1, 6, 2, 3, 4, 5}	loop 6 13
{6, 3, 1, 5, 4, 2}	{3, 1, 5, 2, 4, 1, 6, 2, X, 3, 6, 4, 5}	match 21
{6, 3, 2, 1, 4, 5}	{1, 4, 2, 5, 2, 6, 1, 5, 3, X, 6, 3, 4}	loop 24 40
{6, 3, 2, 5, 1, 4}	{3, 2, 5, 1, 6, 4, X, 1, 6, 2, 3, 4, 5}	loop 12 19
{6, 3, 2, 5, 4, 1}	{3, 2, 5, 1, 4, 1, 6, 2, X, 3, 6, 4, 5}	match 21
{6, 3, 4, 1, 2, 5}	{4, 1, 5, 2, 6, 1, 5, 2, 4, 3, X, 3, 6}	loop 60 81
{6, 3, 4, 2, 1, 5}	{4, 2, 5, 1, 6, 1, 5, 2, 4, 3, X, 3, 6}	loop 60 81
{6, 3, 4, 5, 1, 2}	{5, 1, 6, 2, 4, 3, X, 1, 4, 2, 5, 3, 6}	loop 6 25
{6, 3, 4, 5, 2, 1}	{5, 2, 6, 1, 4, 3, X, 1, 4, 2, 5, 3, 6}	loop 12 31
{6, 3, 5, 1, 2, 4}	{4, 1, 5, 2, 6, 4, X, 1, 6, 2, 3, 3, 5}	loop 12 19
{6, 3, 5, 1, 4, 2}	{6, 2, 4, 1, 5, 2, 4, 3, X, 3, 5, 1, 6}	loop 24 35
{6, 3, 5, 2, 1, 4}	{4, 2, 5, 1, 6, 4, X, 1, 6, 2, 3, 3, 5}	loop 12 19
{6, 3, 5, 2, 4, 1}	{6, 1, 4, 1, 5, 2, 4, 3, X, 3, 5, 2, 6}	loop 24 35
{6, 4, 1, 2, 3, 5}	{3, 1, 4, 2, 5, 1, 6, 2, 5, 4, X, 3, 6}	match 15
{6, 4, 1, 3, 2, 5}	{2, 4, 1, 5, 2, 6, 1, 5, 3, X, 6, 3, 4}	loop 24 50

APPENDIX 6:

The following figures illustrate the permutation of the positions of the cards throughout each pair of plays. The first figure shows the arrangement of the cards at the start of the two plays. The second figure shows how the cards are arranged after player A wins the first play and player B wins the second play. The colored arrows on the first figure point out the movement of the cards after two plays.

FIGURE 1

A	B
w_1	l_{r+1}
l_1	w_{r+1}
w_2	l_{r+2}
l_2	w_{r+2}
$\vdots$	$\vdots$
w_{m-1}	l_{r+m-1}
l_{m-1}	w_{r+m-1}
w_m	l_{r+m}
l_m	w_{r+m}
$\vdots$	
w_{r-1}	
l_{r-1}	
w_r	
l_r	

FIGURE 2

A	B
w_2	l_{r+2}
l_2	w_{r+2}
w_3	l_{r+3}
l_3	w_{r+3}
$\vdots$	$\vdots$
w_m	l_{r+m}
l_m	w_{r+m}
w_{m+1}	l_1
l_{m+1}	w_{r+1}
$\vdots$	
w_r	
l_r	
w_1	
l_{r+1}	

From this illustration, we see that the winning cards in player A's hand form an r-cycle, the winning cards in player B's hand form an m-cycle and the losing cards form an $r+m=q$-cycle. The permutation is $\phi = (w_1 w_2 \cdots w_r)(w_{r+1} w_{r+2} \cdots w_{r+m})(l_1 l_2 \cdots l_q)$.

APPENDIX 7

The following is a list of the hands during each play of the game described in Example 6.

1) A	B
4♡	2♣
1♡	3♣
3♡	1♣
2♡	4♣

2) A	B
1♡	3♣
3♡	1♣
2♡	4♣
4♡	
2♣	

3) A	B
3♡	1♣
2♡	4♣
4♡	1♡
2♣	3♣

4) A	B
2♡	4♣
4♡	1♡
2♣	3♣
3♡	
1♣	

5) A	B
4♡	1♡
2♣	3♣
3♡	2♡
1♣	4♣

6) A	B
2♣	3♣
3♡	2♡
1♣	4♣
4♡	
1♡	

7) A	B
3♡	2♡
1♣	4♣
4♡	2♣
1♡	3♣

8) A	B
1♣	4♣
4♡	2♣
1♡	3♣
3♡	
2♡	

9) A	B
4♡	2♣
1♡	3♣
3♡	1♣
2♡	4♣

The following list indicates which cards from which hands line up with each other throughout the eight plays of the loop. We use (W) to indicate a winning card and (L) to indicate a losing card.

A's cards	**B's cards**
4♡ (W)	2♣(L), 1♡(L)
3♡ (W)	1♣(L), 2♡(L)
1♡(L), 2♣(L)	3♣(W)
2♡(L), 1♣(L)	4♣(W)

The details of the chart given with Corollary 1 of Theorem 4 as they relate to this example are explained below.

If we look at the permutations and consider the first arrangement of the cards to be the arrangement listed in play 1, above, then the values of α, β, and λ are the following:

$\alpha=(4\heartsuit 3\heartsuit)$, $\beta=(3\clubsuit 4\clubsuit)$, $\lambda=(1\heartsuit 2\heartsuit 2\clubsuit 1\clubsuit)$

In this case, d=gcd(2,4)=2. So, we need to examine the cyclic groups $<\alpha^2>$, $<\beta^2>$, and $<\lambda^2>$. Since $|\alpha|=|\beta|=2$, the cyclic groups generated by α^2 and β^2 will contain only one element, specifically, the identity permutation. But, $<\lambda^2>=\{(1\heartsuit 2\clubsuit)(2\heartsuit 1\clubsuit), (1\heartsuit)(2\clubsuit)(2\heartsuit)(1\clubsuit)\}$.

With this in mind, we see that $\{\psi(w_1):\psi\in<\alpha^2>\}$,$\{\psi(w_2):\psi\in<\alpha^2>\}$, $\{\rho(w_3):\rho\in<\beta^2>\}$, and $\{\rho(w_4):\rho\in<\beta^2>\}$ all contain only one element -- w_1, w_2, w_3, and w_4, which correspond to $4\heartsuit$, $3\heartsuit$, $3\clubsuit$, and $4\clubsuit$ in this example. Slightly more interestingly, $\{\chi(l_1):\chi\in<\lambda^2>\}$ and $\{\chi(l_1):\chi\in<\lambda^2>\}$ will each contain two elements. In this example, l_1 is $1\heartsuit$, and the cyclic group generated by λ^2 will either fix $1\heartsuit$ or permute it to the position held by $2\clubsuit$. Similarly, l_2 is $2\heartsuit$ and the cyclic group generated by λ^2 will fix $2\heartsuit$ or send it to the position held by $1\clubsuit$. Thus, the cards that will line up with each other are exactly those which are listed above.

APPENDIX 8:

The following is a list of the plays of the game from Example 7 in which the cards are dealt $3_1, 1_1, 5_1, 1_2, 5_2, 3_2$ (in that order) to player A and $2_1, 4_1, 2_2, 6_1, 4_2, 6_2$ (in that order) to player B. Clearly, the hands in play 13 (although not listed) will match the hands in play 1.

1)		2)		3)		4)		5)		6)	
A	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**
3_1	2_1	1_1	4_1	5_1	2_2	1_2	6_1	5_2	4_2	3_2	6_2
1_1	4_1	5_1	2_2	1_2	6_1	5_2	4_2	3_2	6_2	3_1	1_1
5_1	2_2	1_2	6_1	5_2	4_2	3_2	6_2	3_1	1_1	2_1	4_1
1_2	6_1	5_2	4_2	3_2	6_2	3_1	1_1	2_1	4_1	5_1	1_2
5_2	4_2	3_2	6_2	3_1	1_1	2_1	4_1	5_1	1_2	2_2	6_1
3_2	6_2	3_1		2_1	4_1	5_1		2_2	6_1	5_2	
		2_1				2_2				4_2	

7)		8)		9)		10)		11)		12)	
A	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**	**A**	**B**
3_1	1_1	2_1	4_1	5_1	1_2	2_2	6_1	5_2	3_2	4_2	6_2
2_1	4_1	5_1	1_2	2_2	6_1	5_2	3_2	4_2	6_2	3_1	2_1
5_1	1_2	2_2	6_1	5_2	3_2	4_2	6_2	3_1	2_1	1_1	4_1
2_2	6_1	5_2	3_2	4_2	6_2	3_1	2_1	1_1	4_1	5_1	2_2
5_2	3_2	4_2	6_2	3_1	2_1	1_1	4_1	5_1	2_2	1_2	6_1
4_2	6_2	3_1		1_1	4_1	5_1		1_2	6_1	5_2	
		1_1				1_2				3_2	

APPENDIX 9:

The following is a list of the hands through a loop of a game of three player War. The list begins with the first play of the loop. The deal was originally 4, 12, 1, 8 to player 1; 6, 3, 10, 5 to player 2; 9, 7, 2, 11 to player 3. Player C won that play and the rest of the hands are listed. (Thus, play 2 is numbered "1", play 3 is numbered "2", and so on.) Notice that the hands in "19" match those in "1", so there is a loop of length 18 plays. Also notice that player 1 wins when all players hold a multiple of three cards. The wins alternate between players with each play in the following manner: player 1 wins, player 2 wins, player 3 wins, player 1 wins, etc. The winning cards (12 in player 1's hand, 10 in player 2's hand, 11 and 9 in player 3's hand) never change hands. The loop length is given by 3*LCM(1,1,2,1+1,1+2,1+2)=3*LCM(1,2,3)=18.

1)

A	B	C
12	3	7
1	10	2
8	5	11
		4
		6
		9

2)

A	B	C
1	10	2
8	5	11
12		4
3		6
7		9

3)

A	B	C
8	5	11
12	1	4
3	10	6
7	2	9

4)

A	B	C
12	1	4
3	10	6
7	2	9
		8
		5
		11

5)

A	B	C
3	10	6
7	2	9
12		8
1		5
4		11

6)

A	B	C
7	2	9
12	3	8
1	10	5
4	6	11

7)

A	B	C
12	3	8
1	10	5
4	6	11
		7
		2
		9

8)

A	B	C
1	10	5
4	6	11
12		7
3		2
8		9

9)

A	B	C
4	6	11
12	1	7
3	10	2
8	5	9

10)

A	B	C
12	1	7
3	10	2
8	5	9
		4
		6
		11

11)

A	B	C
3	10	2
8	5	9
12		4
1		6
7		11

12)

A	B	C
8	5	9
12	3	4
1	10	6
7	2	11

13)

A	B	C
12	3	4
1	10	6
7	2	11
		8
		5
		9

14)

A	B	C
1	10	6
7	2	11
12		8
3		5
4		9

15)

A	B	C
7	2	11
12	1	8
3	10	5
4	6	9

16)

A	B	C
12	1	8
3	10	5
4	6	9
		7
		2
		11

17)

A	B	C
3	10	5
4	6	9
12		7
1		2
8		11

18)

A	B	C
4	6	9
12	3	7
1	10	2
8	5	11

19)

A	B	C
12	3	7
1	10	2
8	5	11
		4
		6
		9

APPENDIX 11

USEFUL THEOREMS FROM ABSTRACT ALGEBRA AND NUMBER THEORY:

* GCD is a linear combination:

For any nonzero integers a and b there exist integers s and t such that gcd $(a, b) = as + bt$. Moreover, gcd (a, b) is the smallest positive integer of the form $as + bt$ (Gallian 5).

* Order of a Permutation (Ruffini, 1799)

The order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles (Gallian 91).

* Fundamental Theorem of Cyclic Groups:

Every subgroup of a cyclic group is cyclic. Moreover, if $|\langle a\rangle| = n$, then the order of any subgroup of $\langle a\rangle$ is a divisor of n; and, for each divisor k of n, the group $\langle a\rangle$ has exactly one subgroup of order k -- namely, $\langle a^{k}\rangle$ (Gallian72).

* Number of Elements of Each Order in a Cyclic Group:

If d is a divisor of n, the number of elements of order d in a cyclic group of order n is $\phi(d)$, where $\phi(d)$ is the number of positive integers less than d which are relatively prime to d (Gallian 74).

* If $a+b=c$ then $\gcd(a,c)=\gcd(b,c)$.

Proof: Assume $a+b=c$. Let $d=\gcd(a,c)$ and $d'=\gcd(b,c)$. We know $d|a,c$ so $d|(c-a)=b$. So, $d|b,c$ and therefore $d|d'$. An analogous argument will show that $d'|d$. Thus, $d=d'$.

REFERENCES

Gallian, Joseph A. Contemporary Abstract Algebra. Lexington, Massachusetts: D.C. Heath and Company, 1994.