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# CONSTRUCTION OF BOUNDS ON THE EFFECTIVE SHEAR MODULUS OF ISOTROPIC MULTICOMPONENT MATERIALS

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**Abstract.** In our previous paper, we constructed bounds on the effective bulk modulus of isotropic multicomponent composites using minimum energy principles and modified Hashin-Shtrikman polarization trial fields. In this paper, following the variational approach, we construct more sophisticated bounds on the effective shear modulus. Applications to particular models are presented.

*Keywords:* Isotropic multicomponent material, effective shear elastic modulus, three-point correlation parameters.

## 1. INTRODUCTION

Macroscopic (effective) elastic moduli $k^{eff}$ and $\mu^{eff}$ of isotropic multicomponent materials are important mechanical properties of the materials. It is difficult to find exactly these moduli because of complicated micro-geometries of composites. The most well-known estimates are the volume-weighted arithmetic or harmonic average formulae of Voigt and Reuss (Hill first order) bounds and Hashin-Shtrikman (second order) bounds [1–5]. Pham [3] extended Hashin-Shtrikmans inequalities to incorporate a number of coefficients depending on the fluctuation fields to improve the bounds.

In [1] we had constructed new bounds for effective bulk elastic modulus of isotropic multicomponent materials which involve three-point correlation parameters. Continuing the research in this direction we will use more general multi-free parameter trial fields to construct new tight bounds on effective shear elastic properties of isotropic multicomponent materials. Applications of the bounds are performed for some representative material models.

## 2. CONSTRUCTION OF NEW BOUNDS

The $\alpha$-component of the multicomponent composite has elastic moduli $k_\alpha, \mu_\alpha, \alpha = 1, ..., N$. The local elastic tensor $\mathbf{C}(\mathbf{x})$ is expressible as

$$\mathbf{C}(\mathbf{x}) = \sum_{\alpha=1}^{N} \mathbf{T}(k_\alpha, \mu_\alpha) I_\alpha(\mathbf{x}) \ , \quad \mathbf{x} \in V, \tag{1}$$

where $\mathcal{I}_\alpha$ is the indicator function

$$\mathcal{I}^\alpha(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in V_\alpha \\ 0, & \mathbf{x} \notin V_\alpha \end{cases} \tag{2}$$

$\mathbf{T}$ is the isotropic fourth rank tensor with components

$$T_{ijkl}(k,\mu) = k\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}), \tag{3}$$

$\delta_{ij}$ is Krönecker symbol. The effective elastic moduli $\mathbf{C}^{eff} = \mathbf{T}(k^{eff}, \mu^{eff})$ of the composite can be defined via the minimum energy expression [1]

$$\boldsymbol{\varepsilon}^0 : \mathbf{C}^{eff} : \boldsymbol{\varepsilon}^0 = \inf_{\langle \boldsymbol{\varepsilon} \rangle = \boldsymbol{\varepsilon}^0} \int_V \boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} d\mathbf{x}, \tag{4}$$

while the strain field is expressible via the displacement field $\mathbf{u}(\mathbf{x})$

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]. \tag{5}$$

To find the best possible upper bound on $\mu^{eff}$ from the minimum energy principle (4), we choose the following admissible compatible strain trial field

$$\varepsilon_{ij} = \tilde{\varepsilon}^0_{ij} + \sum_{\alpha=1}^{N} [a_\alpha \frac{1}{2}(\varphi^\alpha_{,ik}\tilde{\varepsilon}^0_{kj} + \varphi^\alpha_{,jk}\tilde{\varepsilon}^0_{ki}) + b_\alpha \psi^\alpha_{,ijkl}\tilde{\varepsilon}^0_{kl}], \quad i,j = 1,...,3; \tag{6}$$

where $\varepsilon^0_{ij} = \tilde{\varepsilon}^0_{ij}$ $(\tilde{\varepsilon}^0_{ii} = 0)$ is a constant deviatoric strain; $\varphi^\alpha$ and $\psi^\alpha$ are the harmonic and biharmonic potentials, Latin indices after comma designate differentiation with respective Cartesian coordinates;

$$\begin{aligned} \varphi^\alpha(\mathbf{x}) &= \int_{V_\alpha} \Gamma_\varphi(\mathbf{x}-\mathbf{y}) d\mathbf{y} \quad ; \quad \nabla^2 \varphi^\alpha(\mathbf{x}) = \delta_{\alpha\beta}\ , \quad \mathbf{x} \in V_\beta; \\ \Gamma_\varphi(r) &= -\frac{1}{4\pi r} \qquad\qquad , \quad \nabla^2 \Gamma_\varphi = \delta(r); \end{aligned} \tag{7}$$

$r = \mid \mathbf{x} - \mathbf{y} \mid$ ; $\delta(r)$ is the Delta Dirac function;

$$\begin{aligned} \psi^\alpha(\mathbf{x}) &= \int_{V_\alpha} \Gamma_\psi(\mathbf{x}-\mathbf{y}) d\mathbf{y} \quad ; \quad \nabla^4 \psi^\alpha(\mathbf{x}) = \delta_{\alpha\beta}, \quad \mathbf{x} \in V_\beta; \\ \Gamma_\psi(r) &= -\frac{1}{8\pi} r \qquad\qquad , \quad \nabla^4 \Gamma_\psi = \delta(r). \end{aligned} \tag{8}$$

In [3] we have introduced the three-point correlation parameters

$$\begin{aligned} A^{\beta\gamma}_\alpha &= \int_{V_\alpha} \varphi^{\beta\alpha}_{ij} \varphi^{\gamma\alpha}_{ij} d\mathbf{x} \quad , \qquad \varphi^{\beta\alpha}_{ij} = \varphi^\beta_{,ij} - \frac{1}{v_\alpha} \int_{V_\alpha} \varphi^\beta_{,ij} d\mathbf{x}, \\ B^{\beta\gamma}_\alpha &= \int_{V_\alpha} \psi^{\beta\alpha}_{ijkl} \psi^{\gamma\alpha}_{ijkl} d\mathbf{x} \quad , \qquad \psi^{\beta\alpha}_{ijkl} = \psi^\beta_{,ijkl} - \frac{1}{v_\alpha} \int_{V_\alpha} \psi^\beta_{,ijkl} d\mathbf{x}. \end{aligned} \tag{9}$$

$2N$ free scalars $a_\alpha, b_\alpha$ in (6) are subjected to restrictions

$$\sum_{\alpha=1}^{N} v_\alpha a_\alpha = 0, \tag{10}$$

$$\sum_{\alpha=1}^{N} v_\alpha b_\alpha = 0, \tag{11}$$

for the trial field (6) to satisfy the restriction $\langle \boldsymbol{\varepsilon} \rangle = \boldsymbol{\varepsilon}^0$ of Eq. (4). Substituting the trial field (6) into the energy functional of Eq. (4) and taking into account (9) and the respective expressions in [3–5], one gets

$$\begin{aligned} W_\varepsilon = \int_V \boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} d\mathbf{x} &= \sum_{\alpha=1}^{N} \int_{v_\alpha} \left[ k_\alpha \varepsilon_{ii} \varepsilon_{jj} + \mu_\alpha \left( 2\varepsilon_{ij}\varepsilon_{ij} - \frac{2}{3}\varepsilon_{ii}\varepsilon_{kk} \right) \right] d\mathbf{x} \\ &= \Bigg\{ \mu_V + \sum_{\alpha=1}^{N} v_\alpha \mu_\alpha \left[ \frac{2}{3} a_\alpha + \frac{4}{15} b_\alpha + \frac{1}{9}\Big( a_\alpha + \frac{2b_\alpha}{5} \Big)^2 \right] \\ &\quad + \sum_{\alpha,\beta,\gamma=1}^{N} \Bigg[ A_\alpha^{\beta\gamma} \left( \frac{1}{10}(k_\alpha - \frac{2}{3}\mu_\alpha)(a_\beta + b_\beta)(a_\gamma + b_\gamma) + \frac{11}{60}\mu_\alpha a_\beta a_\gamma \right. \\ &\quad \left. + \frac{4}{15}\mu_\alpha a_\beta b_\gamma - \frac{1}{15}\mu_\alpha b_\beta b_\gamma \right) + \frac{1}{5}\mu_\alpha b_\beta b_\gamma B_\alpha^{\beta\gamma} \Bigg] \Bigg\} 2\tilde{\varepsilon}_{ij}^0 \tilde{\varepsilon}_{ij}^0, \end{aligned} \tag{12}$$

where $\mu_V = \sum_{\alpha=1}^{N} v_\alpha \mu_\alpha$ is Voigt arithmetic average.

We minimize the expression (12) over variable $a_\alpha, b_\alpha$ restricted by Eqs. (10), (11) with the help of Lagrange multipliers $\lambda$ and $\kappa$ and get the equations

$$\begin{aligned} \frac{1}{3} v_\alpha \mu_\alpha + \frac{v_\alpha}{9}\Big( a_\alpha + \frac{2b_\alpha}{5} \Big)\mu_\alpha + \sum_{\beta,\gamma=1}^{N} A_\gamma^{\alpha\beta} \Bigg[ \frac{k_\gamma - \frac{2}{3}\mu_\gamma}{10}(a_\beta + b_\beta) \\ + \frac{11}{60}\mu_\gamma a_\beta + \frac{2}{15}\mu_\gamma b_\beta \Bigg] - \lambda v_\alpha = 0, \quad \alpha = 1, ..., N; \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{2v_\alpha \mu_\alpha}{15} + \frac{2v_\alpha \mu_\alpha}{45}\Big( a_\alpha + \frac{2b_\alpha}{5} \Big) + \sum_{\beta,\gamma=1}^{N} \Big\{ A_\gamma^{\alpha\beta} \Big[ \frac{k_\gamma - \frac{2}{3}\mu_\gamma}{10}(a_\beta + b_\beta) \\ + \frac{2\mu_\gamma a_\beta}{15} - \frac{\mu_\gamma b_\beta}{15} \Big] + B_\gamma^{\alpha\beta} \frac{\mu_\gamma b_\beta}{5} \Big\} - \kappa v_\alpha = 0, \quad \alpha = 1, ..., N. \end{aligned} \tag{14}$$

Summing Eqs. (13) multiplied by $\mu_\alpha^{-1}$ on $\alpha$ from 1 to $N$ and taking into account Eq. (10), one gets

$$\frac{1}{3} + \sum_{\alpha=1}^{N} \frac{2b_\alpha v_\alpha}{45} + \sum_{\alpha,\beta,\gamma=1}^{N} A_\gamma^{\alpha\beta} \mu_\alpha^{-1} \left[ a_\beta \left( \frac{k_\gamma}{10} + \frac{7\mu_\gamma}{60} \right) + b_\beta \left( \frac{k_\gamma}{10} + \frac{\mu_\gamma}{15} \right) \right] - \lambda \mu_R^{-1} = 0, \tag{15}$$

where $\mu_R$ is Reuss harmonic average

$$\mu_R = \left(\sum_{\alpha=1}^{N} v_\alpha \mu_\alpha^{-1}\right)^{-1}. \tag{16}$$

Also summing Eqs. (14) multiplied by $\mu_\alpha^{-1}$ on $\alpha$ from 1 to $N$ and taking into account Eq. (11), one obtains

$$\begin{aligned}\frac{2}{15} + \sum_{\alpha=1}^{N} \frac{2a_\alpha v_\alpha}{45} + \sum_{\alpha,\beta,\gamma=1}^{N} \Bigg\{ A_\gamma^{\alpha\beta} \mu_\alpha^{-1} \Bigg[ a_\beta \left( \frac{k_\gamma}{10} + \frac{\mu_\gamma 2}{15} \right) \\ + b_\beta \left( \frac{k_\gamma}{10} - \frac{2\mu_\gamma}{15} \right) \Bigg] + B_\gamma^{\alpha\beta} \mu_\alpha^{-1} \frac{\mu_\gamma b_\beta}{5} \Bigg\} - \kappa \mu_R^{-1} = 0.\end{aligned} \tag{17}$$

Now substituting $\lambda$ and $\kappa$ from Eqs. (15) and (17) into Eqs. (13) and (14), finally leads to equations containing only the unknown $a_\alpha$ and $b_\alpha$

$$\mathbf{v}_\mu + \boldsymbol{\mathcal{A}}_\mu \cdot \mathbf{a} = \mathbf{0}. \tag{18}$$

In (18) we have introduced vectors $\mathbf{v}_\mu, \mathbf{a}$ and matrix $\boldsymbol{\mathcal{A}}_\mu$ in $2N$-space

$$\mathbf{a} = \{a_1, \ldots, a_N, b_1, \ldots, b_N\}^T, \tag{19}$$

$$\mathbf{v}_\mu = \left\{ \frac{v_1}{3}(\mu_1 - \mu_R), \ldots, \frac{v_N}{3}(\mu_N - \mu_R), \frac{2v_1(\mu_1 - \mu_R)}{15}, \ldots, \frac{2v_N(\mu_N - \mu_R)}{15} \right\}^T, \tag{20}$$

$$\boldsymbol{\mathcal{A}}_\mu = \left\{ \mathcal{A}_{\alpha\beta}^\mu \right\}, \quad \alpha, \beta = 1, \ldots, 2N; \tag{21}$$

where (in the following $\alpha, \beta = 1, ..., N; \widehat{\alpha} = N + \alpha; \widehat{\beta} = N + \beta$)

$$\begin{aligned}\mathcal{A}_{\alpha\beta}^\mu &= \frac{v_\alpha}{9} \mu_\alpha \delta_{\alpha\beta} + \sum_{\gamma=1}^{N} \left( A_\gamma^{\alpha\beta} - v_\alpha \mu_R \sum_{\delta=1}^{N} \mu_\delta^{-1} A_\gamma^{\delta\beta} \right) \left[ \frac{k_\gamma}{10} + \frac{7\mu_\gamma}{60} \right], \\ \mathcal{A}_{\widehat{\alpha}\widehat{\beta}}^\mu &= \frac{4v_\alpha}{225} \mu_\alpha \delta_{\alpha\beta} + \sum_{\gamma=1}^{N} \Bigg[ \left( A_\gamma^{\alpha\beta} - v_\alpha \mu_R \sum_{\delta=1}^{N} \mu_\delta^{-1} A_\gamma^{\delta\beta} \right) \left( \frac{k_\gamma}{10} - \frac{2\mu_\gamma}{15} \right) \\ &\quad + \left( B_\gamma^{\alpha\beta} - v_\alpha \mu_R \sum_{\delta=1}^{N} \mu_\delta^{-1} B_\gamma^{\delta\beta} \right) \frac{\mu_\gamma}{5} \Bigg], \\ \mathcal{A}_{\alpha\widehat{\beta}} &= \mathcal{A}_{\widehat{\alpha}\beta} = \frac{2v_\alpha}{45} \left( \mu_\alpha \delta_{\alpha\beta} - \mu_R v_\beta \right) + \sum_{\gamma=1}^{N} \left( A_\gamma^{\alpha\beta} - v_\alpha \mu_R \sum_{\delta=1}^{N} \mu_\delta^{-1} A_\gamma^{\delta\beta} \right) \left[ \frac{k_\gamma}{10} + \frac{\mu_\gamma}{15} \right].\end{aligned} \tag{22}$$

From Eq. (18), we find the necessary solutions for $a_\alpha, b_\alpha$

$$\mathbf{a} = -\boldsymbol{\mathcal{A}}_\mu^{-1} \cdot \mathbf{v}_\mu \,. \tag{23}$$

From Eq. (12), with Eqs. (13), (14) and (23), one finds

$$\begin{aligned} W_\varepsilon = \int_V \boldsymbol{\varepsilon} : \mathbf{C} : \boldsymbol{\varepsilon} d\mathbf{x} &= \left[\mu_V + \frac{1}{3}\sum_{\alpha=1}^{N} v_\alpha \mu_\alpha \left(a_\alpha + \frac{2b_\alpha}{5}\right)\right] 2\tilde{\varepsilon}^0_{ij}\tilde{\varepsilon}^0_{ij} \\ &= \left(\mu_V + \mathbf{v'}_\mu \cdot \mathbf{a}\right) 2\tilde{\varepsilon}^0_{ij}\tilde{\varepsilon}^0_{ij} = \left(\mu_V - \mathbf{v'}_\mu \cdot \boldsymbol{\mathcal{A}}^{-1}_\mu \cdot \mathbf{v}_\mu\right) 2\tilde{\varepsilon}^0_{ij}\tilde{\varepsilon}^0_{ij}\,, \end{aligned} \tag{24}$$

where

$$\mathbf{v'}_\mu = \left\{\frac{v_1\mu_1}{3}, \ldots, \frac{v_N\mu_N}{3}, \frac{2v_1\mu_1}{15}, \ldots, \frac{2v_N\mu_N}{15}\right\}^T. \tag{25}$$

From Eqs. (2), (24), finally we obtain the upper bound on the effective shear modulus

$$\mu^{eff} \le M^U_{AB}(\{k_\alpha, \mu_\alpha, v_\alpha\}, \{A^{\beta\gamma}_\alpha, B^{\beta\gamma}_\alpha\}) = \mu_V - \mathbf{v}'_\mu \cdot \boldsymbol{\mathcal{A}}^{-1}_\mu \cdot \mathbf{v}_\mu\,. \tag{26}$$

To construct the lower bound on the effective shear modulus we use the minimum complementary energy principle

$$\boldsymbol{\sigma}^0 : (\mathbf{C}^{eff})^{-1} : \boldsymbol{\sigma}^0 = \inf_{\langle\boldsymbol{\sigma}\rangle = \boldsymbol{\sigma}^0} \int_V \boldsymbol{\sigma} : \mathbf{C}^{-1} : \boldsymbol{\sigma} d\mathbf{x}\,, \tag{27}$$

where $\boldsymbol{\sigma}^0$ is a constant stress field, and the stress field $\boldsymbol{\sigma}$ should satisfy equilibrium equation

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{0} \quad , \quad \mathbf{x} \in V \tag{28}$$

To find a lower bound on the effective shear modulus $\mu^{eff}$ from the minimum complementary energy principle (27), we take the admissible equilibrated stress trial field

$$\begin{aligned} \sigma_{ij} = \tilde{\sigma}^0_{ij} + \sum_{\alpha=1}^{N} &[a_\alpha(\varphi^\alpha_{,ik}\tilde{\sigma}^0_{kj} + \varphi^\alpha_{,jk}\tilde{\sigma}^0_{ki} - \mathcal{I}_\alpha \tilde{\sigma}^0_{ij}) \\ &- (a_\alpha + b_\alpha)\delta_{ij}\varphi^\alpha_{,kl}\tilde{\sigma}^0_{kl} + b_\alpha \psi^\alpha_{,ijkl}\tilde{\sigma}^0_{kl}], \quad i, j = 1, ..., 3; \end{aligned} \tag{29}$$

where $\sigma^0_{ij} = \tilde{\sigma}^0_{ij}(\tilde{\sigma}^0_{ii} = 0)$ is a constant deviatoric stress, the free scalars $a_\alpha, b_\alpha$ are subjected to the same restrictions (10) and (11). Substituting the trial field (29) into (27) and following procedure similar to that form (12) to (26), one obtains the best possible lower bound on $\mu^{eff}$

$$\mu^{eff} \ge M^L_{AB}\left(\{k_\alpha, \mu_\alpha, v_\alpha\}, \{A^{\beta\gamma}_\alpha, B^{\beta\gamma}_\alpha\}\right) = (\mu_R^{-1} - \bar{\mathbf{v}}'_\mu \cdot \bar{\boldsymbol{\mathcal{A}}}^{-1}_\mu \cdot \bar{\mathbf{v}}_\mu)^{-1}\,, \tag{30}$$

where

$$\bar{\mathbf{v}}_\mu = \left\{-\frac{v_1}{3}(\mu_1^{-1} - \mu_V^{-1}), \ldots, -\frac{v_N}{3}(\mu_N^{-1} - \mu_V^{-1}), \frac{2v_1(\mu_1^{-1} - \mu_V^{-1})}{15}, \ldots, \frac{2v_N(\mu_N^{-1} - \mu_V^{-1})}{15}\right\}^T, \tag{31}$$

$$\bar{\mathbf{v}}'_\mu = \left\{-\frac{v_1\mu_1^{-1}}{3}, \ldots, -\frac{v_N\mu_N^{-1}}{3}, \frac{2v_1\mu_1^{-1}}{15}, \ldots, \frac{2v_N\mu_N^{-1}}{15}\right\}, \tag{32}$$

$$\bar{\boldsymbol{\mathcal{A}}}_\mu = \left\{\bar{\mathcal{A}}^\mu_{\alpha\beta}\right\}, \quad \alpha, \beta = 1, \ldots, 2N; \tag{33}$$

[in (34) $\alpha, \beta = 1, \ldots, N; \widehat{\alpha} = N + \alpha; \widehat{\beta} = N + \beta$]

$$
\begin{aligned}
\bar{\mathcal{A}}^{\mu}_{\alpha\beta} &= \frac{v_\alpha}{9}\mu_\alpha^{-1}\delta_{\alpha\beta} + \sum_{\gamma=1}^{N}\left(A_\gamma^{\alpha\beta} - \frac{v_\alpha}{\mu_V}\sum_{\delta=1}^{N}\mu_\delta A_\gamma^{\delta\beta}\right)\left[\frac{2k_\gamma^{-1}}{45} + \frac{7}{15}\mu_\gamma^{-1}\right],\\
\bar{\mathcal{A}}^{\mu}_{\widehat{\alpha}\widehat{\beta}} &= \frac{4v_\alpha}{225}\mu_\alpha^{-1}\delta_{\alpha\beta} + \sum_{\gamma=1}^{N}\left[\left(A_\gamma^{\alpha\beta} - \frac{v_\alpha}{\mu_V}\sum_{\delta=1}^{N}\mu_\delta A_\gamma^{\delta\beta}\right)\left(\frac{8k_\gamma^{-1}}{45} - \frac{2\mu_\gamma^{-1}}{15}\right)\right.\\
&\qquad\left. + \left(B_\gamma^{\alpha\beta} - \frac{v_\alpha}{\mu_V}\sum_{\delta=1}^{N}\mu_\delta B_\gamma^{\delta\beta}\right)\frac{\mu_\gamma^{-1}}{5}\right],\\
\bar{\mathcal{A}}^{\mu}_{\alpha\widehat{\beta}} &= \bar{\mathcal{A}}^{\mu}_{\widehat{\alpha}\beta} = -\frac{2v_\alpha}{45}\left(\mu_\alpha^{-1}\delta_{\alpha\beta} - \mu_V^{-1}v_\beta\right) + \sum_{\gamma=1}^{N}\left(A_\gamma^{\alpha\beta} - \frac{v_\alpha}{\mu_V}\sum_{\delta=1}^{N}\mu_\delta A_\gamma^{\delta\beta}\right)\left[\frac{4k_\gamma^{-1}}{45} + \frac{2\mu_\gamma^{-1}}{15}\right].
\end{aligned}
\tag{34}
$$

## 3. APPLICATIONS

In the case of symmetric cell material without distinct inclusion and matrix phases [4] (Fig. 1a), the three-point correlation parameters $A_\alpha^{\beta\gamma}$, $B_\alpha^{\beta\gamma}$ have particular forms [4, 5] ($\alpha \neq \beta \neq \gamma \neq \alpha$)

$$
\begin{aligned}
A_\alpha^{\beta\gamma} &= v_\alpha v_\beta v_\gamma (f_1 - f_3), & A_\alpha^{\alpha\alpha} &= v_\alpha(1 - v_\alpha)[(1 - v_\alpha)f_1 + v_\alpha f_3],\\
A_\alpha^{\alpha\beta} &= v_\alpha v_\beta[(v_\alpha - 1)f_1 - v_\alpha f_3], & A_\alpha^{\beta\beta} &= v_\alpha v_\beta[(1 - v_\beta)f_3 + v_\beta f_1],\\
B_\alpha^{\beta\gamma} &= v_\alpha v_\beta v_\gamma (g_1 - g_3), & B_\alpha^{\alpha\alpha} &= v_\alpha(1 - v_\alpha)[(1 - v_\alpha)g_1 + v_\alpha g_3],\\
B_\alpha^{\alpha\beta} &= v_\alpha v_\beta[(v_\alpha - 1)g_1 - v_\alpha g_3], & B_\alpha^{\beta\beta} &= v_\alpha v_\beta[(1 - v_\beta)g_3 + v_\beta g_1],
\end{aligned}
\tag{35}
$$

which depend on just 4 shape parameters $f_1, f_3, g_1, g_3$. One also has

$$
\begin{aligned}
\frac{6}{7}f_1 + \frac{8}{35} &\geq g_1 \geq \frac{6}{7}f_1\\
f_1 + f_3 = \frac{2}{3}, &\quad 0 \leq f_1, f_3 \leq \frac{2}{3},\\
g_1 + g_3 = \frac{4}{5}, &\quad 0 \leq g_1, g_3 \leq \frac{4}{5}.
\end{aligned}
\tag{36}
$$

The three-point correlation bounds (26), (30) are specialized to

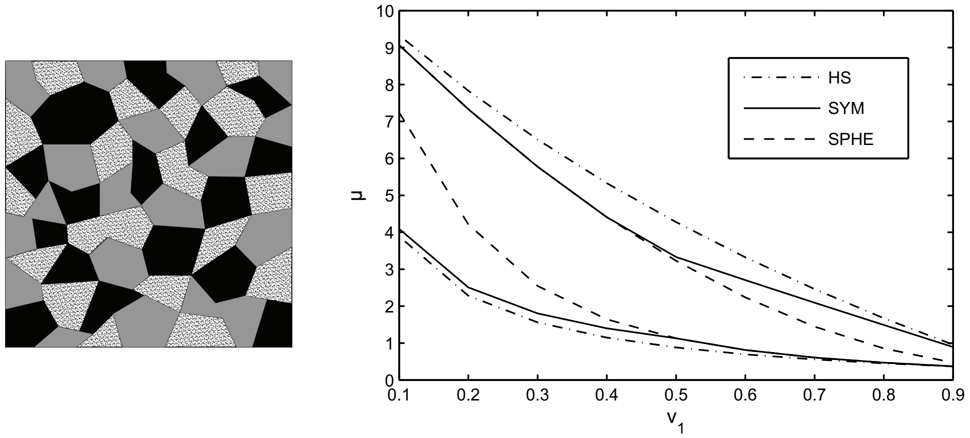
$$
M_{fg}^U \geq \mu^{eff} \geq M_{fg}^L, \tag{37}
$$

where

$$
\begin{aligned}
M_{fg}^U(\{k_\alpha, \mu_\alpha, v_\alpha\}, f_1, g_1) &= M_{AB}^U(\{k_\alpha, \mu_\alpha, v_\alpha\}, \{A_\alpha^{\beta\gamma}, B_\alpha^{\beta\gamma}\} \in (35)),\\
M_{fg}^L(\{k_\alpha, \mu_\alpha, v_\alpha\}, f_1, g_1) &= M_{AB}^L(\{k_\alpha, \mu_\alpha, v_\alpha\}, \{A_\alpha^{\beta\gamma}, B_\alpha^{\beta\gamma}\} \in (35));
\end{aligned}
\tag{38}
$$

and then the shape-unspecified bounds for all symmetric cell materials read

$$
M_{sym}^U \geq \mu^{eff} \geq M_{sym}^L, \tag{39}
$$




(a) (b)

*Fig. 1.* The bounds on the effective shear modulus of three-component symmetric cell materials (SYM), compared to bounds for the specific symmetric spherical cell materials (SPHE) and Hashin-Shtrikman (HS) bounds. (a) A symmetric cell mixture; (b) The bounds

where

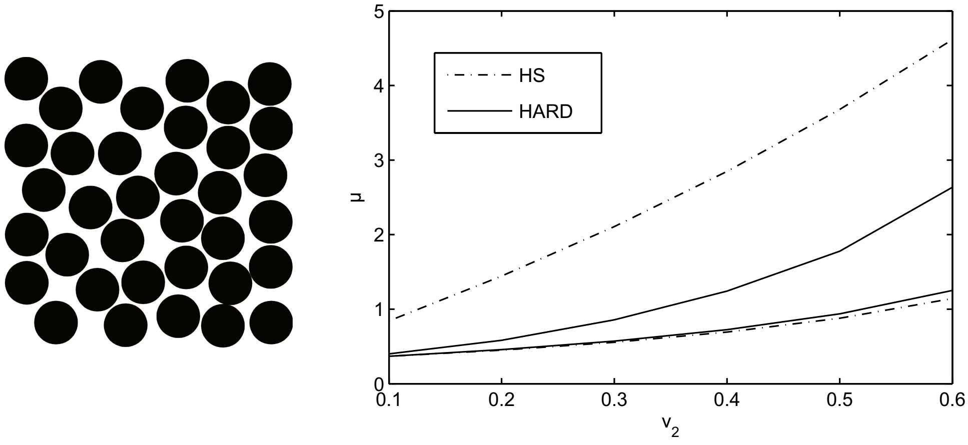
$$\begin{aligned} M^U_{sym}(\{k_\alpha,\mu_\alpha,v_\alpha\}) &= \max_{f_1,g_1\in(36)} M^U_{fg}(\{k_\alpha,\mu_\alpha,v_\alpha\},f_1,g_1),\\ M^L_{sym}(\{k_\alpha,\mu_\alpha,v_\alpha\}) &= \min_{f_1,g_1\in(36)} M^L_{fg}(\{k_\alpha,\mu_\alpha,v_\alpha\},f_1,g_1). \end{aligned} \tag{40}$$

Numerical result for the shape-unspecified bounds on the effective shear modulus of three-phase symmetric cell materials with same data of [1] at the range $v_1 = 0.1 \to 0.9, v_2 = v_3 = \frac{1}{2}(1-v_1)$ with $k_1 = 1, \mu_1 = 0.3, k_2 = 12, \mu_2 = 8, k_3 = 30, \mu_3 = 15$ , are presented in Fig. 1b, which fall inside Hashin-Shtrikman bounds for the larger class of isotropic composites. The bounds $\mu^U_s$, $\mu^L_s$ (with $f_1 = g_1 = 0$) for the specific spherical cell materials are also presented, which lie inside both presented bounds.

The next examples involve two-phase random suspensions of equisized hard spheres (Fig. 2a ) and overlaping spheres (Fig. 3a). The parameters $A^{\beta\gamma}_\alpha$, $B^{\beta\gamma}_\alpha$ are expressed through just two parameters $\zeta_1$ (or $\zeta_2$) and $\eta_1$ (or $\eta_2$) introduced earlier by Milton and Torquato [6–9]

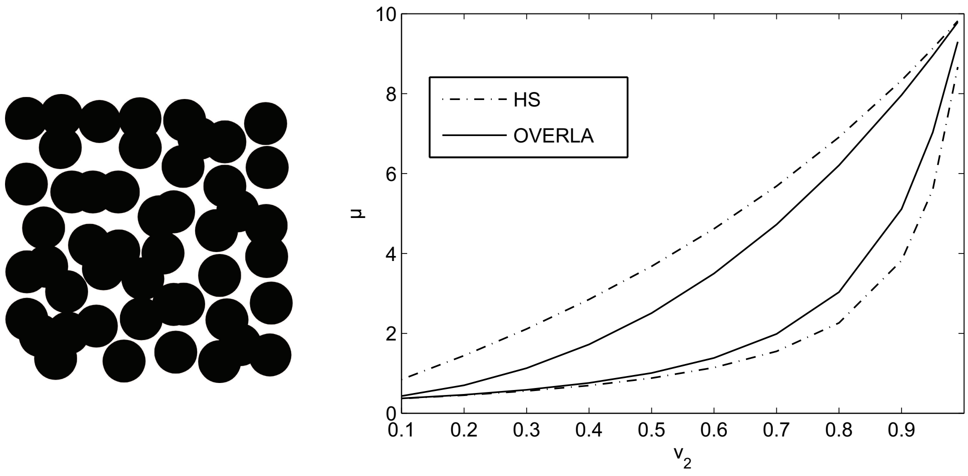
$$\begin{aligned} A^{11}_\alpha = A^{22}_\alpha = -A^{12}_\alpha &= \frac{2}{3}v_1 v_2 \zeta_\alpha\,, \quad \alpha = 1,2;\\ B^{11}_\alpha = B^{22}_\alpha = -B^{12}_\alpha &= \frac{3}{10}v_1 v_2 \eta_\alpha + \frac{1}{2}v_1 v_2 \zeta_\alpha\ . \end{aligned} \tag{41}$$

The bounds (26) and (30) for the models at ranges of $v_2$, with $k_1 = 1, \mu_1 = 0.3, k_2 = 20, \mu_2 = 10$, together with Hashin-Shtrikman bounds are projected in Figs. 2b, 3b.




(a) (b)

*Fig. 2.* Hashin-Strikman bounds (HS) and the bounds (HARD) on the elastic shear modulus of the random suspension of equisized hard spheres. (a) A random suspension of equisized hard spheres; (b) The bounds




(a) (b)

*Fig. 3.* Hashin-Strikman bounds (HS) and the bounds (OVERLA) on the elastic shear modulus of the random suspension of equisized overlapping spheres. (a) A random suspension of equisized overlapping spheres; (b) The bounds

## 4. CONCLUSION

In this paper the authors have constructed three-point correlation bounds on the effective shear elastic modulus of statistically isotropic $N$-component materials from minimum energy principles, using multi-free-parameter trial fields. The bounds are specified to the practical class of symmetric cell materials and random suspensions of equisized spheres, with numerical illustrations.

The trial polarization fields (6), (29) used in this paper depend on $2N-2$ free parameters [i.e., $2N$ parameter $a_\alpha, b_\alpha$ restricted by 2 constraints (10), (11)], hence are more general than the Hashin-Shtrikman ones used [3–5], which contain just 2 free parameters. Therefore the new bounds are more restricting in the cases $N \geq 3$. We remind the particular example of three-phase double-coated-sphere composite [1], where the parameters $A_\alpha^{\beta\gamma}$ have been determined analytically, our new bounds converge to the exact effective bulk modulus, while the old bounds in [3, 5] do not. Note also that the trial fields (6), (29) for the shear modulus containing $2N-2$ free parameters are also more sophisticated than the respective trial fields for the bulk modulus in [1] containing just $N-1$ free parameters.

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