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CONVECTION IN BINARY MIXTURE WITH FREE SURFACE

NGO HUY CAN *and* TRAN THU HA
Institute of Mechanics - Hanoi and Institute of Applied Mechanics - Ho Chi Minh City

Convective motion in a binary mixture without free surface have been the subject of the works [1, 2].

In this paper the convetion in binary mixture with free surface is studied. The existence theorem is proved.

1. BASIC EQUATIONS

For mathematical description of small convective motion in a binary mixture with free surface the following equations and conditions are assumed (see [1, 2, 3]):

$$\frac{\partial v}{\partial t} = \nu\Delta v - \frac{1}{\rho}\nabla p + g\beta_1\gamma T + g\beta_2\gamma C + f_1 \tag{1.1}$$

$$\frac{\partial T}{\partial t} = (\chi + \alpha^2 DN)\Delta T + \alpha DN\Delta C + b_1(v\gamma) \tag{1.2}$$

$$\frac{\partial C}{\partial t} = D\Delta C + \alpha D\Delta T + b_2(v\gamma) \tag{1.3}$$

$$\operatorname{div} v = 0 \tag{1.4}$$

$$v = 0, \quad T = 0, \quad C = 0 \qquad \text{on } S \tag{1.5}$$

$$\frac{\partial v_i}{\partial x_3} + \frac{\partial v_3}{\partial x_i} = 0 \quad (i = 1, 2), \quad \frac{\partial}{\partial t}\Big(p - 2\nu\rho\frac{\partial v_3}{\partial x_3}\Big) = \rho g v_3,$$
$$\frac{\partial T}{\partial t} = b_1 v_3, \quad \frac{\partial C}{\partial t} = b_2 v_3 \quad \text{on } \Gamma \tag{1.6}$$

$$v\Big|_{t=0} = v(0), \quad T\Big|_{t=0} = T(0), \quad C\Big|_{t=0} = C(0), \quad p\Big|_{t=0} = p(0) \tag{1.7}$$

Where the following notations are used: $v = (v_1, v_2, v_3)$ denotes the velocity, p - the pressure, T, C - the temperature and the concentration in the mixture, ρ - the equilibrium state density of the mixture, g - the acceleration of gravity, β_1, β_2 - the heat and concentration coefficient, χ - the coefficient of heat conductivity, α, N - the thermodiffusion and thermodynamics parameters, γ - the unit vector of vertical upward axis Ox_3 in the cartesian coordinate system $Ox_1x_2x_3$, b_1, b_2 - the gradients of temperature and concentration in the equilibrium state of the binary mixture.

2. EXISTENCE THEOREM

The following Hilbert spaces are used throughout

$$L_2(\Omega) = H_2(\Omega) \times H_2(\Omega) \times H_2(\Omega)$$

with the scalar product and norm

$$\left(v, u\right)_{L_2(\Omega)} = \sum_{i=1}^{3} \int_{\Omega} u_i v_i \, d\Omega,$$

$$\| v \|_{L_2(\Omega)} = \left\{ \left(v, v\right)_{L_2(\Omega)} \right\}^{1/2}$$

$$L_2(\Omega) = J(\Omega) + G(\Omega)$$

where

$$J(\Omega) = \left\{ u \in L_2(\Omega), \ \operatorname{div} u = 0, \ u_n = 0 \text{ on } S \right\},$$

$$G(\Omega) = \left\{ v \in L_2(\Omega), \ v = \nabla p, \ p = 0 \text{ on } \Gamma \right\};$$

$$H_{2,00}(\Omega) = \left\{ q \in H_2(\Omega), \ q = 0 \text{ on } S \cup \Gamma \right\},$$

$$H_2^1(\Omega) = \left\{ q \in H_2(\Omega), \ \nabla q \in H_2(\Omega) \right\},$$

$$W_2^1(\Omega) = H_2^1(\Omega) \times H_2^1(\Omega) \times H_2^1(\Omega)$$

The scalar product in $W_2^1(\Omega)$ is defined as follows

$$\left(v, w\right)_{W_2^1(\Omega)} = \sum_{i=1}^{3} \int_{\Omega} \nabla v_{x_i} \nabla w_{x_i} \, d\Omega + \int_{S} vw \, dS$$

$$H_{2,0}^1(\Omega) = \left\{ q \in H_2(\Omega), \ \nabla q \in H_2(\Omega), \ q = 0 \text{ on } S \right\}$$

$$H_{2,00}^1(\Omega) = \left\{ q \in H_2(\Omega), \ \nabla q \in H_2(\Omega), \ q = 0 \text{ on } S \cup \Gamma \right\}$$

$$W_{2,0}^1 = H_{2,0}^1(\Omega) \times H_{2,0}^1(\Omega) \times H_{2,0}^1(\Omega)$$

$$\tilde{W}_{2,0}^1(\Omega) = \left\{ v \in W_{2,0}^1(\Omega), \ \operatorname{div} v = 0, \quad v = 0 \text{ on } S \right\}$$

$$H_0 = H_2(\Gamma) \ominus \{1\}, \quad H_+ = H_0 \cap H_2^{1/2}(\Gamma), \quad H_- = H_0 \cap H_2^{-1/2}(\Gamma)$$

We consider the following auxiliary problems

Problem 1. Let there be given a vector function $g \in J(\Omega)$, we seek $v^{(1)}$ and $p^{(1)}$ so that the following equations and conditions are satisfied:

$$-\nu \Delta v^{(1)} + \frac{1}{\rho} \nabla p^{(1)} = g, \quad \operatorname{div} v^{(1)} = 0 \quad \text{in } \Omega,$$

$$\frac{\partial v_i^{(1)}}{\partial x_3} + \frac{\partial v_3^{(1)}}{\partial x_i} = 0 \quad (i = 1, 2), \quad -p^{(1)} + 2\nu\rho \frac{\partial v_3^{(1)}}{\partial x_3} = 0 \quad \text{on } \Gamma,$$

$$v^{(1)} = 0 \quad \text{on } S$$

Problem 2. Let there be given a function $\psi_1 \in H_-$ we seek a vector function $v^{(2)}$ and a function $p^{(2)}$ so that the following equations and conditions are satisfied

$$-\nu \Delta v^{(2)} + \frac{1}{\rho} \nabla p^{(2)} = 0, \quad \operatorname{div} v^{(2)} = 0 \quad \text{in } \Omega$$

$$\frac{\partial v_i^{(2)}}{\partial x_3} + \frac{\partial v_3^{(2)}}{\partial x_i} = 0 \ (i = 1, 2), \quad -p^{(2)} + 2\nu\rho\frac{\partial v_3^{(2)}}{\partial x_3} = \psi_1 \quad \text{on } \Gamma$$

$$v^{(2)} = 0 \quad \text{on } S.$$

Problem 3. Let there be given a function $h \in H_2(\Omega)$, we seek a function $K^{(1)}$ so that the following equation and condition S are satisfied

$$-\Delta K^{(1)} = h \quad \text{in } \Omega$$
$$K^{(1)} = 0 \quad \text{on } S \cup \Gamma$$

Problem 4. Let there be given a function $\psi_2 \in H_-$ we seek a function $K^{(2)}$ so that the following equation and conditions are satisfied

$$-\Delta K^{(2)} = 0 \quad \text{in } \Omega, \quad K^{(2)} = 0 \quad \text{on } S, \quad K^{(2)} = \psi_2 \quad \text{on } \Gamma$$

The problems 1 - 4 are investigated in the works [4, 5, 6]. Using the lemmas 1 - 4 in [5] we can prove that the system of equations and conditions (1.1) - (1.6) is equivalent to the following system of equations

$$\frac{dv^{(1)}}{dt} = -\nu A_1 v^{(1)} + \nu^{-1} g Q_1 \Gamma(v^{(1)} + v^{(2)}) + V_1(T^{(1)} + T^{(2)}) + V_2(C^{(1)} + C^{(2)}) + \Pi f_1, \tag{2.1}$$

$$\frac{dv^{(2)}}{dt} = -\nu^{-1} g Q_1 \Gamma(v^{(1)} + v^{(2)}) \tag{2.2}$$

$$\frac{dT^{(1)}}{dt} = -(\chi + \alpha^2 DN) A_2 T^{(1)} - \alpha DN A_2 C^{(1)} - V_3(v^{(1)} + v^{(2)}) - b_1 Q_2 \Gamma(v^{(1)} + v^{(2)}) \tag{2.3}$$

$$\frac{dT^{(2)}}{dt} = b_1 Q_2 \Gamma(v^{(1)} + v^{(2)}) \tag{2.4}$$

$$\frac{dC^{(1)}}{dt} = -DA_2 C^{(1)} - \alpha D A_2 T^{(1)} + V_4(v^{(1)} + v^{(2)}) - b_2 Q_2 \Gamma(v^{(1)} + v^{(2)}) \tag{2.5}$$

$$\frac{dC^{(2)}}{dt} = b_2 Q_2 \Gamma(v^{(1)} + v^{(2)}) \tag{2.6}$$

Where A_1, A_2 are self - adjoint, positive definite operators

$$D(A_1) \subset \tilde{W}^1_{2,0}(\Omega), \quad D(A_1^{1/2}) = \tilde{W}^1_{2,0}(\Omega)$$
$$D(A_2) \subset H^1_{2,0}(\Omega), \quad D(A_2^{1/2}) = H^1_{2,0}(\Omega)$$

The operators Q_1, Q_2 are the linear and compact operators

$$Q_1 \ : \ H_- \ \rightarrow \ \tilde{W}^1_{2,0}(\Omega)$$

$$Q_2 \ : \ H_- \ \rightarrow \ H^1_{2,0}(\Omega)$$

$$V_1 T \equiv g\beta_1 \Pi(T\gamma), \quad V_2 C \equiv g\beta_2 \Pi(C\gamma)$$

$$V_3 u \equiv b_1(u\gamma), \quad V_4 u \equiv b_2(u\gamma)$$

Π denotes the projector - operator to $J(\Omega)$. So the problem (1.1) - (1.7) is equivalent to the following problem:

$$\frac{dX}{dt} = -N_1 M_1 A X + B X + f \tag{2.7}$$

$$X\Big|_{t=0} = X(0)$$

where

$$X = \left(v^{(1)}, v^{(2)}, T^{(1)}, T^{(2)}, C^{(1)}, C^{(2)}\right)^{\perp}$$

$$A = \begin{pmatrix} \nu A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad N_1 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & N^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

$$M_1 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \chi + \alpha^2 DN & 0 & \alpha DN & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & \alpha DN & 0 & DN & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

$$B = \begin{pmatrix} \nu\rho^{-1} g Q_1 \Gamma & \nu\rho^{-1} g Q_1 \Gamma & V_1 & V_1 & V_2 & V_2 \\ \nu\rho^{-1} g Q_1 \Gamma & \nu\rho^{-1} g Q_1 \Gamma & 0 & 0 & 0 & 0 \\ V_3 - b_1 Q_2 \Gamma & V_3 - b_1 Q_2 \Gamma & 0 & 0 & 0 & 0 \\ b_1 Q_2 \Gamma & b_1 Q_2 \Gamma & 0 & 0 & 0 & 0 \\ V_4 - b_2 Q_2 \Gamma & V_4 - b_2 Q_2 \Gamma & 0 & 0 & 0 & 0 \\ b_2 Q_2 \Gamma & b_2 Q_2 \Gamma & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$f = \left(\Pi f_1 \ 0\ 0\ 0\ 0\ 0\right)^{\perp}$$

It is clear that the operators N_1, M_1 are positive and limited, the operator A is self-adjoint positive definite and the operator B is limited.

Let us realize in the equation (2.7) the change of variable $X = N_1^{1/2} Y$, we receive

$$\frac{dY}{dt} = -N_1^{1/2} M_1 A N_1^{1/2} Y + N_1^{-1/2} B N_1^{1/2} Y + N_1^{-1/2} f \tag{2.8}$$

Since $AN_1^{1/2} = N_1^{1/2} A$, it follows from (2.8) that

$$\frac{dY}{dt} = -N_1^{1/2} M_1 N_1^{1/2} A Y + N_1^{-1/2} B N_1^{1/2} Y + N_1^{-1/2} f \tag{2.9}$$

It is easy to see that the operator $M_2 = N_1^{1/2} M_1 N_1^{1/2}$ is positive and limited. Realizing in the equation (2.9) the change of variable $Z = M_2^{-1/2} Y$ we get

$$\frac{dZ}{dt} = -M_2^{1/2} A M_2^{1/2} Z + M_2^{-1/2} N_1^{-1/2} B N_1^{1/2} M_2^{1/2} Z + M_2^{-1/2} N_1^{-1/2} f \tag{2.10}$$

$$Z\Big|_{t=0} = Z_0 = M_2^{-1/2} N_1^{-1/2} X_0 \tag{2.11}$$

The operator $M_2^{1/2} A M_2^{1/2}$ is self-adjoint positive definite, the operator $M_2^{-1/2} N_1^{-1/2} B N_1^{1/2} M_2^{1/2}$ is limited, so we get [7].

Theorem. Let $u(0) \in \tilde{W}^1_{2,0}(\Omega)$, $p_\Gamma(0) \in H_-$, $T(0) \in H^1_{2,0}(\Omega)$, $C(0) \in H^1_{2,0}(\Omega)$ then there exists an unique generalized solution of the problem (1.1) - (1.7). *(xem tiếp trang 19)*