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THE INFLUENCE OF SECOND ORDER NARROW-BAND COLORED NOISES ON NON-LINEAR RANDOM VIBRATIONS

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ABSTRACT. Since the effect of some nonlinear terms is lost during the first order averaging procedure, the higher order stochastic averaging method is developed to predict approximately the response of linear and lightly nonlinear systems subject to weakly external excitation of second order coloured noise random processes. Application to Duffing oscillator is considered.

1. Introduction

For many years the well-known averaging method, originally given by Krylov and Bogoliubov and then developed by Mitropolskii (Bogoliubov and Mitropolskii, 1961) has proved to be a very powerful approximate tool for investigating deterministic weakly nonlinear vibration problems. In the field of random vibration the averaging method was extended by Stratonovich (1963) and has a mathematically rigorous proof by Khasminskii (1963). At present, the stochastic averaging method (SAM) is widely used in different problems of stochastic mechanics such as vibration, stability and reliability problems (see e.g. Ariaratnam & Tam,1979; Bolotin, 1984; Ibrahim, 1985; Lin & Cai, 1995; Roberts & Spanos, 1986; Zhu, 1988).

It should be noted that principally only first order SAM has been applied in practice and usually to systems subject to white noise or wideband random processes. It is well-known, however, the effect of some non-linear terms is lost during the first order averaging procedure. In order to over come this insufficiency, different averaging procedures for obtaining approximate solutions have been developed (see e.g. Mitropolskii et al, 1992; Red-Horse & Spanos, 1992; Sri Namachchivaya & Lin, 1988; Zhu & Lin, 1994; Zhu et al, 1997). Recently, a higher order averaging procedure using Fokker-Planck (FP) equation was developed in (Anh, 1993, 1995) and then applied to Van der Pol oscillator under white noise excitation (Anh & Tinh, 1995). In the present paper this procedure is further developed to linear and lightly nonlinear systems subject to weakly external excitation of second order narrow-band coloured noise random processes.

2. Narrow-band coloured noise random processes

We consider a stationary coloured noise random process $c(t)$ which is the result of the passage of a white noise through the linear forming filter L of order $2n$:

$$L(c(t)) = \frac{d^{2n}c(t)}{dt^{2n}} + \sum_{j=0}^{2n-1} \alpha_j \frac{d^j c(t)}{dt^j} = \sigma\dot{\xi}(t) \tag{2.1}$$

where α_j, σ are constants, $\dot{\xi}(t)$ is a the zero-mean white noise process with unit intensity

$$E(\dot{\xi}(t)\dot{\xi}(t+\tau)) = \delta(\tau) \tag{2.2}$$

where $E(\cdot)$ is the operator of expectation. It is supposed that all roots of the characteristic equation for the filter (2.1)

$$\ell(\mu) = \mu^{2n} + \sum_{j=0}^{2n-1} \alpha_j \mu^j = 0 \tag{2.3}$$

have negative real parts. The behaviour of $c(t)$ essentially depends on the roots of the characteristic equation (2.3) and one can get from (2.1) wideband or narrow-band coloured noise processes, respectively.

2.1. Narrow-band coloured noise

Let the filter L can be expressed in the form

$$L(\cdot) \equiv L_0(\cdot) + \varepsilon L_1(\cdot) = \prod_{k=1}^{m} \left[\frac{d^2(\cdot)}{dt^2} + \omega_k^2(\cdot)\right] + \varepsilon L_1(\cdot) \tag{2.4}$$

where L_1 is also a linear filter of order q; $m, q \leq n$; ω_k, $k = 1, 2, \ldots, m$, are distinct positive values and $\omega_k \gg \varepsilon$. In this case it is supposed that one can get from (2.1), (2.4) a narrow- band coloured noise process.

2.2. Second order narrow- band coloured noise

Coloured noise of second order $p(t)$ can be obtained by the filter P of the form

$$P(p(t)) \equiv \ddot{p}(t) + 2\alpha\dot{p}(t) + \omega_1^2 p(t) = 2\sigma_2\sqrt{\alpha}\omega_1\dot{\xi}(t) \tag{2.5}$$

where $\alpha > 0$, ω_1, $\sigma_2 \gg \varepsilon$. The auto-correlation function and the spectral density function of $p(t)$ are, respectively,

$$R_p(\tau) = \sigma_2^2 e^{-\alpha\tau}\Big(\cos\sqrt{\omega_1^2-\alpha^2}\,\tau + \alpha\big(\omega_1^2-\alpha^2\,\big)^{-1}\big|\sin\sqrt{\omega_1^2-\alpha^2}\,\tau\big|\Big)$$
$$S_p(\omega) = \frac{2\sigma_2^2\alpha\omega_1^2}{\pi\left[\left(\omega^2-\omega_1^2\right)^2+4\alpha^2\omega^2\right]} \tag{2.6}$$

The bandwidth of the process $p(t)$ is controlled by the value of the parameter α. For the present analysis the coloured noise $p(t)$ is considered to be a wideband random process if $\alpha \gg \varepsilon$,. If $\alpha \sim \varepsilon$, i.e.

$$\alpha = \varepsilon\alpha_1 \tag{2.7}$$

$p(t)$ is a narrow - band process and the filter (2.5) has the form (2.4), where

$$P_0(\cdot) = \frac{d^2(\cdot)}{dt^2} + \omega_1^2(\cdot), \quad P_1(\cdot) = 2\alpha_1\frac{d(\cdot)}{dt} \tag{2.8}$$

3. Excitation of second order narrow-band coloured noise

Consider a lightly damping system subject to external narrow-band second order coloured noise excitation

$$\ddot{x} + \omega_0^2 x = \varepsilon f_1(x,\dot{x}) + \varepsilon^2 f_2(x,\dot{x}) + p(t) \tag{3.1}$$

where ω_0 is a positive constant, $\varepsilon > 0$ is a small parameter, $p(t)$ is determined from (2.5) and (2.8), f_1 and f_2 are functions of $(x,\dot{x})$. In this case one gets $m = 1$.

Eliminating $p(t)$ from (3.1), using (2.4), one gets

$$\left[\frac{d^2(\cdot)}{dt^2} + \omega_1^2(\cdot)\right](\ddot{x} + \omega_0^2 x) = \varepsilon F_1 + \varepsilon^2 F_2 + \varepsilon^3 + \cdots + \sqrt{\varepsilon}\sigma_1\dot{\xi}(t) \tag{3.2}$$

where it is denoted

$$F_1 = L_0 f_1 - L_1(\ddot{x} + \omega_0^2 x), \quad F_2 = L_0 f_2 + L_1 f_1. \tag{3.3}$$

Suppose

$$\|\omega_0 - \omega_\rho\| \gg \varepsilon, \quad s, \rho = 0, 1, \ s \neq \rho. \tag{3.4}$$

According to the averaging concept, a stationary solution of (3.2) is found in the form

$$\frac{d^i x(t)}{dt^i} = a_0\omega_0^i \cos\left(\varphi_0 + i\frac{\pi}{2}\right) + a_1\omega_1^i \cos\left(\varphi_1 + i\frac{\pi}{2}\right), \quad i = 0,1,2,3 \tag{3.5}$$

where as, a_s, $\varphi_s = 0, 1$ are new variables. By using Ito differentiation formula the equation (3.2) is transformed into the following system of equations [11]:

$$\begin{aligned}\dot{a}_s &= \varepsilon A_{1s}(a,\varphi) + \varepsilon^2 A_{2s}(a,\varphi) + \varepsilon^3 + \cdots - \sqrt{\varepsilon}\,\sigma_1 \frac{\sin\varphi_s}{\omega_s\Omega_s}\dot{\xi}(t) \\ \dot{\varphi}_s &= \omega_s + \varepsilon B_{1s}(a,\varphi) + \varepsilon^2 B_{2s}(a,\varphi) + \varepsilon^3 + \cdots - \sqrt{\varepsilon}\sigma_i \frac{\cos\varphi_s}{a_s\omega_s\Omega_s}\dot{\xi}(t)\end{aligned} \tag{3.6}$$

where it is denoted

$$\begin{aligned}A_{1s}(a,\varphi) &= \Big\{ -\frac{F_1(a,\varphi)}{\omega_s\Omega_s}\sin\varphi_s + \frac{\sigma_1^2\cos^2\varphi_s}{2a_s\omega_s\Omega_s^2}\Big\} \\ B_{1s}(a,\varphi) &= \Big\{ -\frac{F_1(a,\varphi)}{\omega_s\Omega_s a_s}\cos\varphi_s - \frac{\sigma_1^2\sin\varphi_s\cos\varphi_s}{\omega_s^2\Omega_s^2 a_s^2}\Big\} \\ A_{2s}(a,\varphi) &= -\frac{F_2(a,\varphi)}{\omega_s\Omega_s a_s}\cos\varphi_s \\ B_{2s}(a,\varphi) &= -\frac{F_2(a,\varphi)}{\omega_s\Omega_s a_s}\cos\varphi_s \\ \sigma_1 &= 2\sigma_2\sqrt{\alpha_1}\,\omega_1 \\ \Omega_0 &= -\Omega_1 = \omega_1^2 - \omega_0^2, \quad a = (a_0, a_1), \quad \varphi = (\varphi_0, \varphi_1) \\ F_1 &= \ddot{f}_1 + \omega_1^2 f_1 - 2\alpha_1(\dddot{x} + \omega_0^2\dot{x}) \\ F_2 &= \ddot{f}_2 + \omega_1^2 f_2 + 2\alpha_1\dot{f}_1\end{aligned} \tag{3.7}$$

in which

$$x^{(i)}(t) = \sum_{\rho=0}^{1} a_\rho\omega_\rho^i\cos\Big(\varphi_\rho + i\frac{\pi}{2}\Big), \quad i = 0, 1, 2, 3. \tag{3.8}$$

The Fokker-Planck equation for the stationary probability density function $W(a,\varphi)$ corresponding to the system (3.6) takes form

$$\sum_{s=0}^{1}\omega_s\frac{\partial W}{\partial\varphi_s} = -\varepsilon[A_1, B_1]L(W) - \varepsilon^2[A_2, B_2]L(W) + \varepsilon^3\ldots \tag{3.9}$$

where the operators $[A_v, B_v]L(\cdot)$, $v = 1, 2$, are defined as follows

$$\begin{aligned}[A_1, B_1]L(W) &= \sum_{s=0}^{1}\Big[\frac{\partial}{\partial a_s}(A_{1s}W) + \frac{\partial}{\partial\varphi_s}W)\Big] - \\ &-\frac{\sigma_1^2}{2}\sum_{s=0}^{1}\sum_{\rho=0}^{1}\Big\{\frac{\partial^2}{\partial a_s\partial a_\rho}\Big(\frac{\sin\varphi_s\sin\varphi_\rho}{\omega_s\omega_\rho\Omega_s\omega_\rho}W\Big) + \frac{\partial^2}{\partial a_s\partial\varphi_\rho}\Big(\frac{\sin\varphi_s\cos\varphi_\rho}{a_\rho\omega_s\omega_\rho\Omega_s\Omega_\rho}W\Big) + \\ &+\frac{\partial^2}{\partial\varphi_s\partial\varphi_\rho}\Big(\frac{\cos\varphi_s\cos\varphi_\rho}{a_s a_\rho\omega_s\omega_p\Omega_s\Omega_\rho}W\Big)\Big\}\end{aligned} \tag{3.10}$$

$$[A_2, B_2]L(W) = \sum_{s=0}^{1}\Big[\frac{\partial}{\partial a_s}(A_2, W) + \frac{\partial}{\partial\varphi_s}(B_2, W)\Big]$$

where as, a_s, $\varphi_s = 0, 1$ are new variables. By using Ito differentiation formula the equation (3.2) is transformed into the following system of equations [11]:

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where it is denoted

$$\begin{aligned} A_{1s}(a,\varphi) &= \Big\{ - \frac{F_1(a,\varphi)}{\omega_s \Omega_s}\sin\varphi_s + \frac{\sigma_1^2 \cos^2\varphi_s}{2a_s\omega_s\Omega_s^2} \Big\} \\ B_{1s}(a,\varphi) &= \Big\{ - \frac{F_1(a,\varphi)}{\omega_s \Omega_s a_s}\cos\varphi_s - \frac{\sigma_1^2 \sin\varphi_s \cos\varphi_s}{\omega_s^2\Omega_s^2 a_s^2} \Big\} \\ A_{2s}(a,\varphi) &= -\frac{F_2(a,\varphi)}{\omega_s\Omega_s a_s}\cos\varphi_s \\ B_{2s}(a,\varphi) &= -\frac{F_2(a,\varphi)}{\omega_s\Omega_s a_s}\cos\varphi_s \\ \sigma_1 &= 2\sigma_2\sqrt{\alpha_1}\;\omega_1 \\ \Omega_0 &= -\Omega_1 = \omega_1^2 - \omega_0^2, \quad a = (a_0, a_1), \quad \varphi = (\varphi_0, \varphi_1) \\ F_1 &= \ddot{f}_1 + \omega_1^2 f_1 - 2\alpha_1(\dddot{x} + \omega_0^2 \dot{x}) \\ F_2 &= \ddot{f}_2 + \omega_1^2 f_2 + 2\alpha_1 \dot{f}_1 \end{aligned} \tag{3.7}$$

in which

$$x^{(i)}(t) = \sum_{\rho=0}^{1} a_\rho \omega_\rho^i \cos\left(\varphi_\rho + i\frac{\pi}{2}\right), \quad i = 0,1,2,3. \tag{3.8}$$

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$$\sum_{s=0}^{1} \omega_s \frac{\partial W}{\partial \varphi_s} = -\varepsilon[A_1, B_1]L(W) - \varepsilon^2[A_2, B_2]L(W) + \varepsilon^3 \ldots \tag{3.9}$$

where the operators $[A_v, B_v]L(\cdot)$, $v = 1, 2$, are defined as follows

$$\begin{aligned} [A_1, B_1]L(W) &= \sum_{s=0}^{1} \Big[\frac{\partial}{\partial a_s}(A_{1s}W) + \frac{\partial}{\partial \varphi_s} W) \Big] - \\ &- \frac{\sigma_1^2}{2} \sum_{s=0}^{1}\sum_{\rho=0}^{1} \Big\{ \frac{\partial^2}{\partial a_s \partial a_\rho}\Big(\frac{\sin\varphi_s \sin\varphi_\rho}{\omega_s\omega_\rho\Omega_s\omega_\rho} W\Big) + \frac{\partial^2}{\partial a_s \partial \varphi_\rho}\Big(\frac{\sin\varphi_s\cos\varphi_\rho}{a_\rho\omega_s\omega_\rho\Omega_s\Omega_\rho} W\Big) + \\ &+ \frac{\partial^2}{\partial\varphi_s\partial\varphi_\rho}\Big(\frac{\cos\varphi_s\cos\varphi_\rho}{a_s a_\rho \omega_s \omega_p \Omega_s \Omega_\rho} W\Big)\Big\} \end{aligned} \tag{3.10}$$

$$[A_2, B_2]L(W) = \sum_{s=0}^{1} \Big[\frac{\partial}{\partial a_s}(A_2, W) + \frac{\partial}{\partial \varphi_s}(B_2, W) \Big]$$

It can be shown that the operators $[A_v, B_v]L(\cdot)$ are linear ones. We seek then the solution of (3.9) in the form

$$W(a,\varphi) = W_0(a,\varphi) + \varepsilon W_1(a,\varphi) + \varepsilon^2 W_2(a,\varphi) + \dots \tag{3.11}$$

Substituting (3.11) into (3.9) yields

$$\sum_{s=0}^{1} \omega_s \Big[\frac{\partial W_0}{\partial \varphi_s} + \varepsilon \frac{\partial W_1}{\partial \varphi_s} + \varepsilon^2 \frac{\partial W_2}{\partial \varphi_s} + \dots\Big] = -\varepsilon[A_1, B_1]L(W_0) -$$
$$- \varepsilon^2\big\{[A_2, B_2]L(W_0) + [A_1, B_1]L(W_1)\big\} + \dots \tag{3.12}$$

Comparing the coefficients of like powers of ε one obtains

$$\varepsilon^0: \quad \sum_{s=0}^{1} \omega_s \frac{\partial W_0}{\partial \varphi_s} = 0, \tag{3.13}$$

$$\varepsilon^1: \quad \sum_{s=0}^{1} \omega_s \frac{\partial W_1}{\partial \varphi_s} = -[A_1, B_1]L(W_0), \tag{3.14}$$

$$\varepsilon^2: \quad \sum_{s=0}^{1} \omega_s \frac{\partial W_2}{\partial \varphi_s} = -\big\{[A_2, B_2]L(W_0) + [A_1, B_1]L(W_1)\big\}, \tag{3.15}$$

From (3.13) it gives a periodic solution with respect to φ as follows

$$W_0 = W_0(a). \tag{3.16}$$

Substituting (3.16) into (3.14) yields

$$\sum_{s=0}^{1} \omega_s \frac{\partial W_1}{\varphi_s} = -[A_1, B_1]L(W_0(a)). \tag{3.17}$$

The arbitrary integration function $W_0(a)$ must be chosen from the condition for the function $W_1(a,\varphi)$ to be periodic with respect to φ. Thus, one gets from (3.17)

$$\big\langle \big[[A_1, B_1]L(W_0(a))\big]\big\rangle = 0 \tag{3.18}$$

where $\langle\cdot\rangle$ is the averaging operator with respect to phase φ

$$\langle\cdot\rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi}\!\!\int_0^{2\pi} \cdot(\cdot)\, d\varphi_0 d\varphi_1 \tag{3.19}$$

Substituting (3.10) into (3.18) yields

$$\sum_{s=0}^{1}\left\{\frac{\partial}{\partial a}\left(\langle A_{1s}\rangle W_0(a)\right)-\frac{\sigma_1^2}{4\omega_s^2\Omega_s^2}\frac{\partial^2 W_0(a)}{\partial a_s^2}\right\}=0. \tag{3.20}$$

The averaged Fokker - Planck equation (3.20) can be considered as an extension into the case of narrow - band excitation of the well - known first order averaged FP equation obtained for the white noise excitation. Further, it is seen that the averaged FP equation (3.20) is obtained by equating zero the averaged value of the coefficient of power ε in the original FP equation (3.9).

The second term $W_1(a,\varphi)$ in (3.11) is determined from (3.14), using Fourier expansion

$$[A_1,B_1]L(W_0(a))=W_0(a)\sum_{k_0}\sum_{k_1}C_{k_0k_1}(a)e^{i(k_0\varphi_0+k_1\varphi_1)}, \tag{3.21}$$

where

$$C_{k_0k_1}(a)=\frac{1}{W_0(a)(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}[A_1,B_1]L(W_0(a)\cdot\cdot e^{-i(k_0\varphi_0+k_1\varphi_1)}d\varphi_0d\varphi_1. \tag{3.22}$$

Substituting (3.21) into (3.14) yields

$$W_1(a,\varphi)=W_0(a)\Big[W_{10}(a)+\sum_{k_0}\sum_{k_1}\frac{C_{k_0k_1}}{k_0\varphi_0+k_1\varphi_1}e^{i(k_0\varphi_0+k_1\varphi_1)}\Big], \tag{3.23}$$

where

$$k_0\omega_0+k_1\omega_1\neq 0. \tag{3.24}$$

The arbitrary integration function $W_{10}(a)$ must be chosen from the condition for the function $W_2(a,\varphi)$ to be periodic. Analogously, one can find third term $W_2(a,\varphi)$ in (3.11).

4. Application

In order to illustrate the procedure proposed one considers the Duffing system whose equation of motion takes the form:

$$\ddot{x}+\omega^2x=-2\varepsilon h\dot{x}-\varepsilon^2\gamma x^3+p(t) \tag{4.1}$$

Substituting (3.10) into (3.18) yields

$$\sum_{s=0}^{1} \left\{ \frac{\partial}{\partial a}\left(\langle A_{1s}\rangle W_0(a)\right) - \frac{\sigma_1^2}{4\omega_s^2\Omega_s^2}\frac{\partial^2 W_0(a)}{\partial a_s^2} \right\} = 0. \tag{3.20}$$

The averaged Fokker - Planck equation (3.20) can be considered as an extension into the case of narrow - band excitation of the well - known first order averaged FP equation obtained for the white noise excitation. Further, it is seen that the averaged FP equation (3.20) is obtained by equating zero the averaged value of the coefficient of power ε in the original FP equation (3.9).

The second term $W_1(a,\varphi)$ in (3.11) is determined from (3.14), using Fourier expansion

$$[A_1, B_1]L(W_0(a)) = W_0(a)\sum_{k_0}\sum_{k_1} C_{k_0k_1}(a)e^{i(k_0\varphi_0+k_1\varphi_1)}, \tag{3.21}$$

where

$$C_{k_0k_1}(a) = \frac{1}{W_0(a)(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}[A_1,B_1]L(W_0(a)\cdot\cdot e^{-i(k_0\varphi_0+k_1\varphi_1)}d\varphi_0 d\varphi_1. \tag{3.22}$$

Substituting (3.21) into (3.14) yields

$$W_1(a,\varphi) = W_0(a)\Big[W_{10}(a) + \sum_{k_0}\sum_{k_1}\frac{C_{k_0k_1}}{k_0\varphi_0 + k_1\varphi_1}e^{i(k_0\varphi_0+k_1\varphi_1)}\Big], \tag{3.23}$$

where

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The arbitrary integration function $W_{10}(a)$ must be chosen from the condition for the function $W_2(a,\varphi)$ to be periodic. Analogously, one can find third term $W_2(a,\varphi)$ in (3.11).

4. Application

In order to illustrate the procedure proposed one considers the Duffing system whose equation of motion takes the form:

$$\ddot{x} + \omega^2 x = -2\varepsilon h\dot{x} - \varepsilon^2\gamma x^3 + p(t) \tag{4.1}$$

where $h,\ \gamma > 0$, $p(t)$ is a narrow - band random process obtained by the filter (2.5). In this case one gets:

$$f_1(x,\dot{x}) = -2h\dot{x}, \quad f_2(x,\dot{x}) = -\gamma x^3. \tag{4.2}$$

Substituting (4.2) into (3.7) yields:

$$\begin{cases} F_1 = -2(\alpha_1 + h)\dddot{x} - 2(\alpha_1\omega_0^2 + h\omega_1^2)\dot{x}, \\ F_2 = -2h\alpha_1\ddot{x} - \gamma\omega_1^2 x^3 - 3\gamma(x^2\ddot{x} + x\dot{x}^2). \end{cases} \tag{4.3}$$

Using (2.10), (3,4) and (4.3) for equation (2.24) one gets:

$$W_0(a) = Ca_0a_1 \exp\Big\{ - \frac{\left(\omega_1^2 - \omega_0^2\right)^2}{2\sigma_2^2}\Big(a_1^2 + \frac{h\omega_0^2}{\alpha_1\omega_1^2}a_0^2\Big)\Big\} \tag{4.4}$$

($C = \text{const}$). This solution is the same as the result obtained by using the classical SAM. Substituting (4.4) into (3.14) and using (3.7), (3.8) and (4.3) after some calculations one obtains:

$$W_{11}(a,\varphi) =$$
$$= -\frac{h}{\omega_0}\sin^2\varphi_0 - \frac{\alpha_1}{\omega_1}\sin^2\varphi_1 + \Big(\frac{2h\omega_0\Omega_0}{\sigma_2^2}a_0a_1 - \frac{4\alpha_1\omega_1^2\sigma_2^2}{\omega_0\Omega_0^3}\,\frac{1}{a_0a_1}\Big)\sin\varphi_0\cos\varphi_1$$
$$- \Big(\frac{2h\omega_0^2\Omega_0}{\sigma_2^2\omega_1}a_0a_1 - \frac{4\alpha_1\omega_1\sigma_2^2}{\omega_0^3}\,\frac{1}{a_0a_1}\Big)\cos\varphi_0\sin\varphi_1. \tag{4.5}$$

Substituting (4.4) and (4.5) into (3.15) one gets the equation for the arbitrary function $W_{10}(a)$ in the form:

$$\sum_{s=0}^{1}\Big\{\frac{\partial}{\partial a_s}[\langle A_{1s}\rangle W_0(a)W_{10}(a)] - \frac{\sigma_1^2}{4\omega_s^2\Omega_s^2}\,\frac{\partial^2}{\partial a_s^2}[W_0(a)W_{10}(a)]\Big\} =$$
$$= -\sum_{s=0}^{1}\frac{\partial}{\partial a_s}[\langle A_{2s}\rangle W_0(a)] = \frac{3\gamma}{4}\Big[\frac{\partial}{\partial a_0}\Big(\frac{\omega_0}{\Omega_0}a_0^3W_0\Big) + \frac{\partial}{\partial a_1}\Big(\frac{\omega_1}{\Omega_1}a_1^3W_0\Big)\Big]. \tag{4.6}$$

From (4.6) one gets:

$$W_{10}(a) = -\frac{3\gamma}{4\sigma_1^2}(\omega_0^3\Omega_0a_0^4 + \omega_1^3\Omega_1a_1^4). \tag{4.7}$$

Thus, the second order apprpximate solution of the FP equation (3.9) for the Duffing system (4.1) takes the form:

$$W(a,\varphi) = W_0(a)\{1 + \varepsilon[W_{10}(a) + W_{11}(a)]\} \tag{4.8}$$

where $W_0(a)$, $W_{11}(a,\varphi)$ and $W_{10}(a)$ are defined in (4.4), (4.5) and (4.7) respectively. It is seen from (4.4) and (4.8) that the solution $W(a,\varphi)$ in (4.8) is different from the solution obtained by using the classical SAM and the effect of the non - linear term $\varepsilon^2\gamma x^3$ is shown in (4.8).

The corresponding approximate mean square $E[x^2]$ is to be found

$$E[x^2] = \int_0^{2\pi}\int_0^{2\pi}\int_0^{\infty}\int_0^{\infty} x^2 W(a,\varphi)d\varphi_0 d\varphi_1 da_0 da_1. \tag{4.9}$$

Substituting (4.8) and $x = a_0\cos\varphi_0 + a_1\cos\varphi_1$ into (4.9), after calculations one obtains:

$$E(x^2) =$$

$$= \frac{\int_0^{\infty}\int_0^{\infty}(a_0^2+a_1^2)\left[1+\varepsilon W_{10}(a)\right]a_0a_1\exp\left\{-\frac{\left(\omega_1^2-\omega_0^2\right)^2}{2\sigma_2^2}\left(a_1^2+\frac{h\omega_0^2}{\alpha_1\omega_1^2}a_0^2\right)\right\}da_0da_1}{2\int_0^{\infty}\int_0^{\infty}[1+\varepsilon W_{10}(a)]a_0a_1\exp\left\{-\frac{\left(\omega_1^2-\omega_0^2\right)^2}{2\sigma_2^2}\left(a_1^2+\frac{h\omega_0^2}{\alpha_1\omega_1^2}a_0^2\right)\right\}da_0da_1}$$

$$= \frac{\sigma_2^2(h\omega_0^2+\alpha_1\omega_1^2)}{h\omega_0^2\left(\omega_1^2-\omega_0^2\right)^2} - 6\varepsilon\gamma\omega_1\sigma_2^4(h^3\omega_0^3+\alpha_1^3\omega_1^3)+\varepsilon^2\ldots \tag{4.10}$$

In the case $\gamma = 0$ (linear system) one gets:

$$E[x^2] = \frac{\sigma_2^2(h\omega_0^2+\alpha_1\omega_1^2)}{h\omega_0^2\left(\omega_1^2-\omega_0^2\right)^2}\,. \tag{4.11}$$

It is seen from (4.10) and (4.11) that in the case of Duffing system the mean square $E[x^2]$ reduces in comparison with the linear case.

5. Conclusion

For many years the stochastic averaging method has been a very useful tool for investigating non-linear random vibration systems. However, the effect of some non-linear terms cannot be investigated by using the classical first order SAM. In this paper, the higher order stochastic averaging method is developed to predict approximately the response of linear and lightly nonlinear systems subject to weakly external excitation of second order narrow-band coloured noise random processes.

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ẢNH HƯỞNG CỦA KÍCH ĐỘNG ỒN MÀU DẢI HẸP BẬC HAI LÊN DAO ĐỘNG NGẪU NHIÊN PHI TUYẾN

Phương pháp trung bình ngẫu nhiên bậc nhất kinh điển đã được áp dụng rộng rãi đối với các hệ cơ học phi tuyến. Tuy nhiên, hiệu ứng của nhiều số hạng phi tuyến không được thể hiện khi sử dụng phương pháp này. Để khắc phục nhược điểm trên, phương pháp trung bình ngẫu nhiên bậc cao đã được phát triển đối với các hệ cơ học phi tuyến chịu kích động ngẫu nhiên dạng ồn trắng. Trong bài báo này, phương pháp tiếp tục được trình bày đối với các hệ phi tuyến yếu chịu kích động ngẫu nhiên dạng ồn màu dải hẹp bậc hai. Sau đó phương pháp được áp dụng để xác định nghiệm xấp xỉ bậc hai của phương trình Fokker - Planck đối với hệ dao động dạng Duffing.

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