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CHANGE IN MODE SHAPE NODES OF MULTIPLE CRACKED BAR: I. THE THEORETICAL STUDY

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Abstract. In present paper change in position of mode shape nodes induced by multiple cracks in bar is studied with purpose to use for the multiple crack detection from measured mode shape nodes. First, there is derived an explicit expression for natural modes in axial vibration of multiple cracked bar that allows obtaining exact positions of the node in the case of single and double crack. The change in mode shape nodes induced by multiple cracks provides an important indicator for crack localization in bar. Finally, a procedure for multiple crack detection by using mode shape nodes has been proposed and examined in an example of application.

Keywords: Multiple cracked bar, crack detection, mode shape nodes, vibration method, modal analysis.

1. INTRODUCTION

The damage detection problem in structure has received recently a great research interest and the vibration-based approach to solution of the problem has been proved to be the most efficient and promising. Numerous ideas were proposed and a lot of results has been obtained regarding to the problem as can be found in the comprehensive surveys given in [1-4], but it so far faces the unsolved difficulties associated mainly with the erroneousness of measured data and the lack of information on current structure configuration. Until a total solution of the difficulties has been found the best choice is to make use additionally the unemployed attributes of structure that are more sensitive to damage and less contaminated by noise for the damage diagnosis.

The features frequently used as the diagnostic tool for the damage detection problem consist mostly of the dynamical characteristics of structure such as natural frequencies and mode shapes that are often extracted from the original measured data. The natural frequencies are most easily and accurately measured, therefore, they have been early and successfully used for assessing structural integrity [5-12]. Due to the fact that different changes in structure may produce the identical frequency shift and amount of measured frequencies is often limited, the frequency-based solution of the damage detection problem is usually non-unique. In the circumstance, using the anti-resonances in addition to the resonant frequencies is helpful to overcome the non-uniqueness of the crack identification in beam-like structure [13-15]. As a spatial characteristic of structure the mode shape in principle should be useful indication for structural damage localization [16-19]. However, the mode shape is more difficult to measure, not very much sensitive to small damage and strongly contaminated by measurement noise. A number of attempts based on the mode shape curvature or strain energy and the modern signal processing procedures [20-22] have been accomplished to improve the mode shape sensitivity to damage but the problem associated with the incompleteness and erroneousness of measured data remains still unsolved. Recently, the change in mode shape node position that is more easily and accurately measured than the mode shape has been productively used by Gladwell and Morassi [23] for crack localization in structures. By considering a rod with single crack modeled by an axial spring the authors have shown that nodes of undamaged mode shape move toward damage position. This fact provides the simple procedure for localizing single crack in rod: crack would be located between the adjacent nodes which move each to other. This result was extended then for beam by Dilena and Morassi [24], who have shown that the node transition due to cracking in a beam is not monotonic as in the rod. Despite the fact, the authors have achieved in plotting the domains in the plane of crack position and magnitude where a node moves by the same direction. The attainment could be used to find out the beam segment where crack is located with known direction of nodal displacement. Both the papers mentioned above [23, 24], among very little number of similar studies, left unsolved the question to extend the result to the case of multiple cracks.

This paper is devoted to study change in node position induced by multiple cracks in bar with purpose to use the change for multi-crack detection from given mode shape nodes. First, using the axial model of crack [25] a new form of the problem for free longitudinal vibration of a bar with arbitrary number of cracks [26] is derived. Next, the explicit expression of characteristic equation and mode shape has been used to obtain exact position of mode shape node for a bar with single and double crack. The dislocation of node located intermediately between two cracks is thoroughly investigated by numerical simulation for symmetric and asymmetric boundary conditions. Finally, a procedure is proposed for crack localization from given mode shape nodes and illustrated by using the experimental data given in [23].

2. AN EXPLICIT EXPRESSION FOR MODE SHAPE OF MULTIPLE CRACKED BAR

Let's consider an uniform bar with Young's modulus E, density ρ, cross section area A and length $L = 1$ that is assumed to be cracked at the locations $e_1, ..., e_n$, as shown in Fig. 1. Suppose that crack at position e_j is represented by an axial spring of stiffness K_j determined as function of crack depth a_j [25].

Free longitudinal vibration of the bar is described by the equation in nondimensional

$$\Phi''(x) + \lambda^2\Phi(x) = 0,\ x \in (0, 1),\ \lambda = \omega/c_0,\ c_0 = \sqrt{E/\rho} \tag{1}$$

with given boundary conditions at both ends $x = 0$, $x = 1$. The compatibility conditions at the crack positions are

$$\Phi(e_j + 0) = \Phi(e_j - 0) + \gamma_j\Phi'(e_j),\ \gamma_j = EA/LK_j,\ j = 1, ..., n. \tag{2}$$

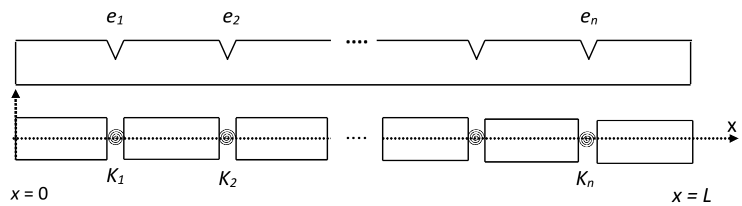

Fig. 1. Model of a multiple-cracked bar

By using notation $\Phi_j(x)$ for solution of Eq. (1) in the segment (e_j, e_{j+1}), $j = 0, ..., n$, $e_0 = 0, e_{n+1} = 1$, the conditions (2) can be rewritten as

$$\Phi'_j(e_j + 0) = \Phi'_{j-1}(e_j - 0) = \Phi'_{j-1}(e_j),\ \Phi_j(e_j + 0) = \Phi_{j-1}(e_j - 0) + \gamma_j \Phi'_{j-1}(e_j), j = 1, ..., n. \quad (3)$$

It's easily to verify that solution $\Phi_j(x)$ of Eq. (2) in the segment (e_j, e_{j+1}) satisfying the condition (3) can be represented in the form

$$\Phi_j(x) = \Phi_{j-1}(x) + \gamma_j \Phi'_{j-1}(e_j) \cos\lambda(x - e_j),\ j = 1, ..., n. \quad (4)$$

In the latter equation $\Phi_j(x)$ is solution of Eq. (1) in the segment (e_{j-1}, e_j) continuously expanded to the segment (e_j, e_{j+1}); $\Phi_0(x)$ is general solution of Eq. (1) in the segment $(0, e_1)$ that has the same form of general solution of Eq. (1) for intact bar. Recurrent relationships (4) allow to represent solution of Eqs. (1)-(2) all over the bar length as

$$\Phi(x) = \Phi_0(x) + \sum_{j=1}^{n} \mu_j K(x - e_j, \lambda), \quad (5)$$

where

$$\mu_j = \gamma_j[\Phi'_0(e_j) + \sum_{k=1}^{n} \mu_k K'(e_j - e_k, \lambda)], j = 1, ..., n; \quad (6)$$

$$K(x,\lambda) = \begin{cases} 0\ , & x \le 0 \\ \cos\lambda x, & x \succ 0 \end{cases} \quad ; \quad K'(x,\lambda) = \begin{cases} 0\ , & x \le 0 \\ -\lambda\sin\lambda x, & x \succ 0 \end{cases} \quad (7)$$

Eqs. (5)-(7) show that at the left end of beam function $\Phi(x)$ coincides with $\Phi_0(x)$ so that the latter can be represented by $\Phi_0(x) = \mathrm{C}L_0(x, \lambda)$, where C is a constant and $L_0(x, \lambda)$ is the solution of Eq. (1) satisfying the left boundary condition. Namely, in the case of general boundary condition $\alpha_0\Phi(0) + \beta_0\Phi'(0) = 0$ the shape function is $L_0(\lambda x) = \alpha_0 \sin\lambda x - \beta_0\lambda\cos\lambda x$. If the bar is fixed at the left end, $x = 0$, $L_0(x, \lambda) = \sin\lambda x$ and $L_0(x, \lambda) = \cos\lambda x$ for the free one. The constant C would be determined from the boundary condition at the right end $x = 1$ that can be expressed in the form $\alpha_1\Phi(1) + \beta_1\Phi'(1) = 0$. Substituting Eq. (5) with function $\Phi_0(x) = \mathrm{C}L_0(x, \lambda)$ into the boundary condition at the right end $x = 1$ leads to

$$\mathrm{C}[\alpha_1 L_0(1, \lambda) + \beta_1 L'_0(1, \lambda)] + \sum_{j=1}^{n} \mu_j[\alpha_1 K(1 - e_j, \lambda) + \beta_1 K'(1 - e_j, \lambda)] = 0$$

from that one gets

$$\mathrm{C} = -\left[1/D_0(\lambda)\right]\sum_{j=1}^{n}\mu_j K_1(e_j,\lambda); \tag{8}$$

$$D_0(\lambda) = \alpha_1 L_0(1,\lambda) + \beta_1 L_0'(1,\lambda); \quad K_1(e_j,\lambda) = \alpha_1 K(1-e_j,\lambda) + \beta_1 K'(1-e_j,\lambda).$$

Using Eq. (8), Eqs. (5) and (6) can be rewritten as

$$\Phi(x) = -[1/D_0(\lambda)]\sum_{j=1}^{n}\mu_j\alpha(x,e_j,\lambda)\,, \tag{9}$$

$$\alpha(x,e,\lambda) = K_1(e,\lambda)L_0(x,\lambda) - D_0(\lambda)K(x-e,\lambda) \tag{10}$$

and

$$D_0(\lambda)\mu_j - \gamma_j\sum_{k=1}^{n}\mu_k a(e_j,e_k,\lambda)] = 0, j = 1,...,n; \tag{11}$$

$$a(e_j,e_k,\lambda) = D_0(\lambda)K'(e_j-e_k,\lambda) - L_0'(e_j,\lambda)K_1(e_k,\lambda);\ j,k = 1,...,n. \tag{12}$$

By introducing the vector and matrix notations $\boldsymbol{\mu} = (\mu_1,...,\mu_n)^T$, $\boldsymbol{\gamma} = (\gamma_1,...,\gamma_n)^T$, $\mathbf{e} = (e_1,...,e_n)^T$, $\mathbf{A} = [a_{jk} = a(e_j,e_k,\lambda);\ j,k = 1,...,n]$, $\boldsymbol{\Gamma} = diag\{\gamma_1,...,\gamma_n\}$ Eq. (11) is written in the matrix form

$$[\boldsymbol{\Gamma}(\boldsymbol{\gamma})\mathbf{A}(\mathbf{e},\lambda) - D_0(\lambda)\mathbf{I}]\boldsymbol{\mu} = 0. \tag{13}$$

For existence of non-trivial vector $\boldsymbol{\mu}$ as a solution of Eq. (13), it must be satisfied the condition

$$D(\lambda,\mathbf{e},\boldsymbol{\gamma}) \equiv \det[\boldsymbol{\Gamma}(\boldsymbol{\gamma})\mathbf{A}(\mathbf{e},\lambda) - D_0(\lambda)\mathbf{I}] = 0. \tag{14}$$

This is a new form of the frequency equation in axial vibration for multiple cracked bar [26]. Obviously, the latter equation is determined by the determinant of order identical to the number of cracks. Solving Eq. (14) with respect to λ results in so-called eigenvalues $\lambda_p, p = 1,2,...$ that relate to the natural frequencies $\omega_p, p = 1,2,...$ by the relationship $(\lambda_p/L)\sqrt{E/\rho} = \omega_p$. Each eigenvalue λ_p associates with a non-trivial solution of Eq. (13) that is so-called eigenvector μ_p of p-th mode. Thus the couple (λ_p,μ_p) enables to express the mode shape in the explicit form

$$\Phi_p(x) = \mathrm{C}_p\sum_{j=1}^{n}\mu_{pj}\alpha(x,e_j,\lambda_p)\,. \tag{15}$$

The constant C_p is introduced in the latter equation because the eigenvector μ_p determined as a solution of Eq. (13) should contain an arbitrary constant that can be determined by choosing a normality condition, for example, $\max\{\Phi_p(x)\} = 1$. This expression of mode shape is utilized in subsequent section for finding node position of multiple cracked bar of the classical boundary conditions.

3. MODE SHAPE NODES OF MULTIPLE CRACKED BAR

3.1. Bar with single crack

For a bar with single crack the characteristic Eq. (14) gets the form

$$D_0(\lambda) + \gamma L_0'(e, \lambda) K_1(e, \lambda) = 0 \tag{16}$$

and general expression for mode shape (15) is simplified to

$$\phi_k(x) = C\{K_1(e, \lambda_k) L_0(x, \lambda_k) - D_0(\lambda) K(x - e, \lambda_k)\} \tag{17}$$

or

$$\phi_k(x) = C\{L_0(x, \lambda_k) + \gamma L_0'(e, \lambda) K(x - e, \lambda_k)\}. \tag{18}$$

From characteristic Eq. (16), it is easily to verify that natural frequency of cracked bar would be unchanged if crack occurred at position e satisfying equation $L_0'(e, \lambda) K_1(e, \lambda) = 0$. In such the case, eigenvalues of both the cracked and uncracked bar, denoted by $\lambda_k^0, k = 1, 2, 3, ...$, are determined from equation $D_0(\lambda) = 0$. Moreover, Eqs. (16) and (18) show that both the natural frequency and mode shape of the bar are unchanged due to crack at the point $\bar{e}$ where

$$L_0'(\bar{e}, \lambda_k^0) = 0. \tag{19}$$

Such the position inside the bar is called Inactive Crack Point (ICP) for vibration. Obviously, the ICPs are zeros (nodes) of axial force and different natural modes, in general, have dissimilar sets of ICP.

On the other hand, zeros of a mode shape referred to as the mode shape nodes (MSN) are typical for the mode shape that can be utilized to make a distinction between mode shapes. For an intact bar the node of k-th mode shape denoted by x_k^0 is solution of equation $L_0(x, \lambda_k^0) = 0$ with respect to x. In the case of single crack the nodes are sought from the equation

$$K_1(e, \lambda_k) L_0(x, \lambda_k) - D_0(\lambda) K(x - e, \lambda_k) = 0, \tag{20}$$

that can be rewritten in the form

$$\begin{cases} L_0(x, \lambda_k) = 0, \ x \in (0, e]\,; \\ K_1(e, \lambda_k) L_0(x, \lambda_k) - D_0(\lambda) \cos \lambda_k (x - e) = 0, \ x \in (e,\, 1)\,. \end{cases}$$

For the case of freely ended bar, function $L_0(x, \lambda) = \cos \lambda x$ allows for obtaining the undamaged eigenvalues and mode shape nodes $\lambda_k^0 = k\pi, k = 1, 2, 3, ...; x_{kn}^0 = (2n - 1)/2k$, $n = 1, ..., k$ and the characteristic Eq. (16) in the form

$$\sin \lambda - \gamma \lambda \sin \lambda e \sin \lambda (1 - e) = 0 \tag{21}$$

Eq. (19) in this case takes the form $\sin \lambda e \sin \lambda (1 - e) = 0$ that gives the ICPs $e_{kn} = n/k\,;\ 1 \leq n \prec k$. The Eq. (20) for finding the MSN now is

$$\begin{cases} \cos \lambda_k x = 0; \quad x \leq e; \\ \sin \lambda_k (1 - e) \cos \lambda_k x - \sin \lambda_k \cos \lambda_k (x - e) = 0; \ x \succ e. \end{cases} \tag{22}$$

It is not difficult to verify that the first equation in the latter system results in solution $x_{kn}^+ = x_{kn}^0 (\lambda_k^0/\lambda_k)$ that is the MSN located on the left of crack position. The second equation gives the nodes $x_{kn}^- = 1 - (1 - x_{kn}^0)(\lambda_k^0/\lambda_k)$ located on the right of crack.

For the bar with fixed ends, function $L_0(x,\lambda) = \sin\lambda x$ gives rise also

$$\lambda_k^0 = k\pi, k = 1,2,3,...,\quad x_{kn}^0 = n/k,\ n = 1,...,k-1, \tag{23}$$

$$\sin\lambda + \gamma\lambda\cos\lambda e\cos\lambda(1-e) = 0. \tag{24}$$

The ICPs are found from equation $\cos\lambda e\cos\lambda(1-e) = 0$ as $e_{kn} = (2n-1)/2k;\ n = 1,...,k$ and the MSNs for k-th mode are roots of the equations

$$\begin{cases} \sin\lambda_k x = 0;\quad x \le e; \\ \cos\lambda_k(1-e)\sin\lambda_k x - \sin\lambda_k\cos\lambda_k(x-e) = 0;\ x \succ e. \end{cases}$$

It can be shown that $x_{kn}^+ = x_{kn}^0(\lambda_k^0/\lambda_k)$, $x_{kn}^- = 1-(1-x_{kn}^0)(\lambda_k^0/\lambda_k)$ are solution of the latter equations with λ_k^0, x_{kn}^0 given in (23).

Since $L_0(x,\lambda) = \sin\lambda x$ in the case of cantilevered bar the eigenvalues and node of undamaged bar are $\lambda_k^0 = (2k-1)\pi/2, k = 1,2,3,...;\ x_{kn}^0 = 2n/(2k-1),\ n = 1,...,k-1$ and the characteristic equation is

$$\cos\lambda + \gamma\lambda\cos\lambda e\cos\lambda(1-e) = 0. \tag{25}$$

The ICPs found from the equation $\cos\lambda e\sin\lambda(1-e) = 0$ are

$$e_{kn} = (2n-1)/(2k-1);\ n = 1,...,k-1.$$

The nodes of cracked mode shape satisfying the equations

$$\begin{cases} \sin\lambda_k x = 0;\quad x \le e; \\ \sin\lambda_k(1-e)\sin\lambda_k x + \cos\lambda_k\cos\lambda_k(x-e) = 0;\ x \succ e \end{cases}$$

have the same form as was obtained in the previous cases.

So, exact mode shape nodes for a bar with single crack in different cases of boundary conditions can be expressed as

$$x_{kn}^+ = x_{kn}^0(\lambda_k^0/\lambda_k),\quad x_{kn}^- = 1-(1-x_{kn}^0)(\lambda_k^0/\lambda_k), \tag{26}$$

satisfying conditions

$$x_{kv}^0 \prec x_{kn}^+ \le e\quad;\quad e \prec x_{kn}^- \prec x_{kn}^0\quad. \tag{27}$$

Obviously, dislocation of mode shape node caused by a crack can be exactly calculated as

$$\delta x_{kn}^- = x_{kn}^0 - x_{kn}^- = (1-x_{kn}^0)\delta\lambda_k/\lambda_k = \delta^-, \delta x_{kn}^+ = x_{kn}^+ - x_{kn}^0 = x_{kn}^0\delta\lambda_k/\lambda_k = \delta^+, \delta\lambda_k = \lambda_k^0 - \lambda_k, \tag{28}$$

where the upper index plus implies that the node moves to the right and the minus sign – to the left. Eqs. (27) demonstrate again the result obtained by Gladwell and Morassi [23] for free-free rod that all nodes of cracked mode shape move toward the crack position. Moreover, Eqs. (28) show that dislocation of node is proportional to the relative shift of frequency and the ratios of dislocations of two nodes are independent on the presence of crack. Actually, the ratios can be calculated as

$$\delta x_{kn}^+/\delta x_{km}^+ = x_{kn}^0/x_{km}^0;\quad \delta x_{kn}^-/\delta x_{km}^- = (1-x_{kn}^0)/(1-x_{km}^0);\quad \delta x_{kn}^-/\delta x_{km}^+ = (1-x_{kn}^0)/x_{km}^0. \tag{29}$$

The Eqs. (26) and (27) allow making also the following discussion. Since the node of cracked mode shape x_k is always located between the generic node x_k^0 and crack position e, see Eqs. (27), $x_k^\pm$ should be approaching x_k^0 together with crack. On the other hand, Eqs. (28) show that the frequency shift must be vanishing when $e \to x_k^0$. This means that x_k^0 should be not only the common node of undamaged and damaged mode shape but also a crack-unaffected point for the mode. As the generic nodes and crack-unaffected points of a mode are definitely different, the node of a mode shape might be broken down when crack occurred at generic node of the mode.

The obtained above results exhibit the fact that measured mode shape nodes provide more useful indicator for crack detection, especially, in combination with the natural frequencies. The crack-unaffected points and generic nodes for five modes in different cases of end conditions are tabulated in Tab. 1.

Table 1. Generic nodes and crack-unaffected points for bar in different boundary conditions

Mode	**Free-Free Ends**		**Fixed-Fixed Ends**		**Fixed-Free Ends**	
	ICP	Generic Node	ICP	Generic Node	ICP	Generic Node
1	non	1/2	1/2	non	non	non
2	1/2	1/4; 3/4	1/4; 3/4	1/2	1/3	2/3
3	1/3; 2/3	1/6; 1/2; 5/6	1/6; 1/2; 5/6	1/3; 2/3	1/5; 3/5	2/5; 4/5
4	1/4; 1/2; 3/4	1/8; 3/8; 5/8; 7/8	1/8; 3/8; 5/8; 7/8	1/4; 1/2; 3/4	1/7; 3/7; 5/7	2/7; 4/7; 6/7
5	0.2;0.4;0.6;0.8	0.1;0.3;0.5;0.7;0.9	0.1;0.3;0.5;0.7;0.9	0.2;0.4;0.6;0.8	1/9;3/9;5/9;7/9	2/9;4/9;6/9;8/9

3.2. Bar with two cracks

For a bar with a double crack the frequency Eq. (14) leads to

$$\begin{aligned} D_0(\lambda) + [\gamma_1 L_0'(e_1,\lambda)K_1(e_1,\lambda) + \gamma_2 L_0'(e_2,\lambda)K_1(e_2,\lambda)] + \\ + \gamma_1\gamma_2 L_0'(e_1,\lambda)K_1(e_2,\lambda)K'(e_2-e_1,\lambda)] = 0. \end{aligned} \tag{30}$$

The expression (15) for mode shape now can be written as

$$\phi(x,\lambda) = C[\mu_1\alpha(x,e_1,\lambda) + \mu_2\alpha(x,e_2,\lambda)], \tag{31}$$

where $\alpha(x,e,\lambda) = K_1(e,\lambda)L_0(x,\lambda) - D_0(\lambda)K(x-e,\lambda)$ and constants μ_1, μ_2 determined from the system of equations, see Eq. (13),

$$\begin{aligned} &[\gamma_1 L_0'(e_1,\lambda)K_1(e_1,\lambda) + D_0(\lambda)]\mu_1 + [\gamma_1 L_0'(e_1,\lambda)K_1(e_2,\lambda)]\mu_2 = 0, \\ &\gamma_2[L_0'(e_2,\lambda)K_1(e_1,\lambda) - D_0(\lambda)K'(e_2-e_1,\lambda)]\mu_1 + \\ &\qquad + [\gamma_2 L_0'(e_2,\lambda)K_1(e_2,\lambda) + D_0(\lambda)]\mu_2 = 0. \end{aligned} \tag{32}$$

Using Eqs. (32) for excluding the constants μ_1, μ_2 from (31) one is able to express mode shape of the bar as

$$\begin{aligned} \phi(x,\lambda) = C[\mu_1\alpha(x,e_1,\lambda) + \mu_2\alpha(x,e_2,\lambda)] = C\{L_0(x,\lambda) + \gamma_1 L_0'(e_1,\lambda)K(x-e_1) + \\ + \gamma_2[L_0'(e_2,\lambda) + \gamma_1 L_0'(e_1,\lambda)K'(e_2-e_1,\lambda)]K(x-e_2)\}. \end{aligned} \tag{33}$$

Eq. (30) shows that frequency of double cracked bar would be unchanged if positions of the cracks satisfy the following system of equations

$$L_0'(e_1,\lambda)K_1(e_1,\lambda) = 0;\ L_0'(e_2,\lambda)K_1(e_2,\lambda) = 0.$$

The latter equations imply that both the cracks should coincide with ICP of the bar and the vibration mode with less than two ICPs cannot be unaffected by double crack.

A mode shape would be unaffected by crack, see Eq. (33), if the crack positions satisfy conditions

$$L_0'(e_1,\lambda)=0;\ L_0'(e_2,\lambda)=0.$$

Mode shape node of the double cracked bar is determined as root of the equation

$$L_0(x,\lambda)+\gamma_1 L_0'(e_1,\lambda)K(x-e_1)+\gamma_2[L_0'(e_2,\lambda)+\gamma_1 L_0'(e_1,\lambda)K'(e_2-e_1,\lambda)]K(x-e_2)=0,$$

that can be rewritten in the form

$$L_0(x,\lambda)=0,\quad 0\prec x\le e_1; \tag{34}$$

$$L_1(x,\lambda)\equiv L_0(x,\lambda)+\gamma_1 L_0'(e_1,\lambda)\cos\lambda(x-e_1)=0,\ e_1\prec x\le e_2; \tag{35}$$

$$\begin{gathered}L_2(x,\lambda)\equiv L_1(x,\lambda)+\gamma_2[L_0'(e_2,\lambda)+\gamma_1 L_0'(e_1,\lambda)K'(e_2-e_1,\lambda)]\cos\lambda(x-e_2)=0,\\ e_2\prec x\prec 1.\end{gathered} \tag{36}$$

In the case of free-free bar, when $L_0(x,\lambda)=\cos\lambda x$, the frequency equation is

$$\begin{aligned}&\sin\lambda-\lambda[\gamma_1\sin\lambda e_1\sin\lambda(1-e_1)+\gamma_2\sin\lambda e_2\sin\lambda(1-e_2)]+\\&\qquad+\lambda^2\gamma_1\gamma_2\sin\lambda e_1\sin\lambda(e_2-e_1)\sin\lambda(1-e_2)=0.\end{aligned} \tag{37}$$

The system of Eqs. (34), (35), (36) is reduced to

$$\cos\lambda_k x=0,\quad 0\prec x\le e_1; \tag{38}$$

$$\cos\lambda_k x-\gamma_1\lambda_k\sin\lambda_k e_1\cos\lambda_k(x-e_1)=0,\ e_1\prec x\le e_2; \tag{39}$$

$$\begin{aligned}&\cos\lambda_k x-\gamma_1\lambda_k\sin\lambda_k e_1\cos\lambda(x-e_1)-\gamma_2\lambda_k[\sin\lambda_k e_2-\\&-\gamma_1\lambda_k\sin\lambda_k e_1\sin\lambda_k(e_2-e_1)]\cos\lambda(x-e_2)=0,\ e_2\prec x\prec 1.\end{aligned} \tag{40}$$

Eq. (38) has the roots $x_{kn}^+=x_{kn}^0(\lambda_k^0/\lambda_k)$ satisfying the condition $0\prec x_{kn}^0\prec x_{kn}^+\le e_1$, consequently, the node dislocation is

$$\delta x_{kn}^+=x_{kn}^+-x_{k0}^0=(\delta\lambda_k/\lambda_k)x_{k0}^0=\delta_k^+. \tag{41}$$

Using Eq. (37), the Eq. (40) can be simplified to $\tan\lambda_k x=-\cot\lambda_k,\ x\in(e_2,1)$, that has the root $x_{kn}^-=1-(1-x_{kn}^0)(\lambda_k^0/\lambda_k)$ satisfying the condition $e_2\prec x_{kn}^-\prec x_{kn}^0\prec 1$. So that node dislocation in this case is calculated as

$$\delta x_{kn}^-=x_{kn}^0-x_{kn}^-=(\delta\lambda_k/\lambda_k)(1-x_{kn}^0)=\delta_k^-. \tag{42}$$

It remains to solve Eq. (39) that can be rewritten as

$$\tan\lambda_k x=\frac{1-\lambda_k\gamma_1\sin\lambda_k e_1\cos\lambda_k e_1}{\lambda_k\gamma_1\sin^2\lambda_k e_1};x\in(e_1,\,e_2]$$

or

$$\tan\lambda_k^0\theta_k=\frac{\lambda_k\gamma_1\sin^2\lambda_k e_1}{1-\lambda_k\gamma_1\sin\lambda_k e_1\cos\lambda_k e_1},\ \ \theta_k\in(\Delta_1,\Delta_2), \tag{43}$$

where $x=\lambda_k^0(x_{kn}^0-\theta_k)/\lambda_k$ and $\Delta_1=x_{kn}^0-\lambda_k e_2/\lambda_k^0$; $\Delta_2=x_{kn}^0-\lambda_k e_1/\lambda_k^0$. Owning solution $\hat\theta_k$ of Eq. (43) the mode shape node determined from Eq. (39) can be calculated as $x_{kn}=\lambda_k^0(x_{kn}^0-\hat\theta_k)/\lambda_k$, consequently, dislocation of the node from the generic one is

$$\delta x_{kn}\equiv x_{kn}-x_{kn}^0=(\delta\lambda_k/\lambda_k)x_{kn}^0-(\lambda_k^0/\lambda_k)\hat\theta_k. \tag{44}$$

where $\delta\lambda_k = \lambda_k^0 - \lambda_k$. Therefore, one obtains

$$\begin{cases} e_1 \prec x_{kn} \equiv x_{kn}^- \prec x_k^0 \prec e_2 \text{ as } \hat{\theta}_k \succ (\delta\lambda_k/\lambda_k^0)x_{kn}^0, \\ e_1 \prec x_k^0 \prec x_{kn} \equiv x_{kn}^+ \prec e_2 \text{ as } \hat{\theta}_k \prec (\delta\lambda_k/\lambda_k^0)x_{kn}^0. \end{cases} \tag{45}$$

If the first crack is small, Eq. (43) gives the approximate solution

$$\hat{\theta}_k = (\lambda_k/\lambda_k^0)\gamma_1 \sin^2 \lambda_k e_1$$

that leads Eqs. (45) to be more apparent

$$\begin{cases} e_1 \prec x_{kn}^- \prec x_k^0 \prec e_2 \text{ as } \gamma_1 \sin^2 \lambda_k e_1 \succ (\delta\lambda_k/\lambda_k)x_{kn}^0, \\ e_1 \prec x_k^0 \prec x_{kn}^+ \prec e_2 \text{ as } \gamma_1 \sin^2 \lambda_k e_1 \prec (\delta\lambda_k/\lambda_k)x_{kn}^0. \end{cases}$$

Obviously, presence of two cracks in both sides of a generic node leads to modification of the node dislocation so that direction of node transition may be changed in comparison with the case of single crack. For instance, generic node x_k^0 located between cracks moves to the left if $\gamma_1 \sin^2 \lambda_k e_1 \succ (\delta\lambda_k/\lambda_k)x_k^0$ or to the right if $\gamma_1 \sin^2 \lambda_k e_1 \prec (\delta\lambda_k/\lambda_k)x_k^0$. Note that the value $(\delta\lambda_k/\lambda_k)x_k^0$ is similar to dislocation of the node caused by single crack (e_2, γ_2). Moreover, in the case of two nodes found between cracks so that $e_1 \prec x_{k1}^0 \prec x_{k2}^0 \prec e_2$ both the nodes would move to the right if $\hat{\theta}_k \prec (\delta\lambda_k/\lambda_k^0)x_{k1}^0$ and to the left if $\hat{\theta}_k \succ (\delta\lambda_k/\lambda_k^0)x_{k2}^0$. Otherwise, the nodes can move by different directions.

If the bar is fixed at both the ends, $L_0(x, \lambda) = \sin \lambda x$, the frequency and node are sought from the equations

$$\begin{aligned} &\sin \lambda + \lambda[\gamma_1 \cos \lambda e_1 \cos \lambda(1 - e_1) + \gamma_2 \cos \lambda e_2 \cos \lambda(1 - e_2)] - \\ &\qquad - \lambda^2 \gamma_1 \gamma_2 \cos \lambda e_1 \sin \lambda(e_2 - e_1) \cos \lambda(1 - e_2) = 0 \end{aligned} \tag{46}$$

and

$$\sin \lambda_k x = 0, \quad 0 \prec x \leq e_1; \tag{47}$$

$$\sin \lambda_k x + \gamma_1 \lambda_k \cos \lambda_k e_1 \cos \lambda_k (x - e_1) = 0, \ e_1 \prec x \leq e_2; \tag{48}$$

$$\begin{aligned} &\sin \lambda_k x + \gamma_1 \lambda_k \cos \lambda_k e_1 \cos \lambda(x - e_1) + \gamma_2 \lambda_k [\cos \lambda_k e_2 - \\ &- \gamma_1 \lambda_k \cos \lambda_k e_1 \sin \lambda_k (e_2 - e_1)] \cos \lambda(x - e_2) = 0, \ e_2 \prec x \prec 1. \end{aligned} \tag{49}$$

Similarly to the previous case, it can be shown that Eqs. (47), (49) yield the solutions

$$x_{kn}^+ = x_{kn}^0(\lambda_k^0/\lambda_k), 0 \prec x_{kn}^0 \prec x_{kn}^+ \leq e_1; x_{kn}^- = 1 - (1 - x_{kn}^0)\lambda_k^0/\lambda_k, e_2 \prec x_k \prec x_k^0 \prec 1$$

and node located between cracks can be found as $x_{kn} = \lambda_k^0(x_{kn}^0 - \theta_k)/\lambda_k$, where θ_k is determined from the equation

$$\tan \lambda_k^0 \theta_k = \frac{\lambda_k \gamma_1 \cos^2 \lambda_k e_1}{1 + \lambda_k \gamma_1 \sin \lambda_k e_1 \cos \lambda_k e_1}, \quad \theta_k \in (\Delta_1, \Delta_2) \tag{50}$$

with $\Delta_1 = x_{kn}^0 - \lambda_k e_2/\lambda_k^0$; $\Delta_2 = x_{kn}^0 - \lambda_k e_1/\lambda_k^0$. Letting $\hat{\theta}_k$ be solution of Eq. (50), one can obtain exactly equations of the form (43), (44) for bar with fixed ends so that the comments followed from the Eqs. (43), (44) remain truthfully also for the case of fixed end bar. A difference between the cases of boundary condition appears only in the right hand side of Eqs. (43) and (50) that gives unlike node dislocations. Namely, while the former equation gives $\hat{\theta}_k = (\lambda_k/\lambda_k^0)\gamma_1 \sin^2 \lambda_k e_1$ the latter yields $\hat{\theta}_k = (\lambda_k/\lambda_k^0)\gamma_1 \cos^2 \lambda_k e_1$

that may lead to dissimilar behavior of node when first crack approaches to the left end of bar.

For the *bar with fixed-free ends*, the characteristic equation is

$$\begin{aligned}\cos\lambda - \lambda[\gamma_1\cos\lambda e_1\sin\lambda(1-e_1)+\gamma_2\cos\lambda e_2\sin\lambda(1-e_2)]+\\ +\lambda^2\gamma_1\gamma_2\cos\lambda e_1\sin\lambda(e_2-e_1)\sin\lambda(1-e_2)=0.\end{aligned} \tag{51}$$

Though the Eq. (51) is different from Eq. (46), the equations for the mode shape node in this case of boundary condition give solutions identical to those of the fixed-fixed bar.

Hence, one is able to make the general conclusion for double cracked bar with arbitrary classical boundary condition as follows: First, the generic node located in the same side of all cracks moves by the same direction toward the cracks and ratios of dislocations of two nodes are dependent only on the generic nodes but not the cracks; Second, the node being found between two cracks can move either to the left or to the right in dependence on position and severity of crack preceding the node and frequency shift induced by all cracks. This is typical for multiple cracked bar with a generic node located between two cracks that is numerically investigated in more detail below.

4. MULTIPLE CRACK IDENTIFICATION BY USING MEASURED MODE SHAPE NODES

Assume that for a given bar natural frequency and nodes of k-th mode have been given so that there are available λ_k^*, $x_1^*,...,x_n^*$; $n \le k$. The first problem is to find out those from the segments $\{(0,x_1^*),(x_1^*,x_2^*),...,(x_n^*,1)\}$ that may contain crack and such the segments are called cracked ones. Suppose, furthermore, that there are given also the generic nodes $(x_1^0,...,x_n^0)$ and eigenvalue λ_k^0 of k-th mode that allow for calculating the eigenvalue shift $\varepsilon_k = (\lambda_k^0-\lambda_k^*)/\lambda_k \succ 0$, node dislocations $\delta x_j = x_j^* - x_j^0, j=1,...,n$ and also values $\delta_j^- = \varepsilon_k(1-x_j^0) \succ 0$, $\delta_j^+ = \varepsilon_k x_j^0 \succ 0$. The sign of the node dislocations (direction of the node movement) and their absolute values compared to the quantities δ_j^-, δ_j^+ (the node dislocations in the case of single crack) are mostly important indicators for crack localization.

Indeed, if two neighboring generic nodes move each to other then crack should be located between them. So that crack must be found in the segment $(x_{r_0}^*, x_{r_1}^*)$ if $\delta x_{r_0} \succ 0$, $\delta x_{r_1} \prec 0$ and only the segment contains crack if the condition $\delta x_{r_1}/\delta x_{r_0} \approx (x_{r_1}^0-1)/x_{r_0}^0$ were also satisfied. Otherwise, it has to check for crack all the remained segments from left to right as follows.

Namely, consider the first segment $(0,x_1^*)$ with given δx_1 and $\delta_1^\pm$. Under the condition $\delta x_1 \prec 0$ a crack should be in this segment and if, moreover, $|\delta x_1| \approx \delta_1^-$ then no crack outside the segment; If $\delta x_1 \succ 0$, then a crack should be on the right of x_1^*, i. e. in the next segment and no crack located in $(0,x_1^*)$ if δx_1 is not less than δ_1^+.

Let the segment (x_{j-1}^*, x_j^*) have been checked for crack and subsequent segment (x_j^*, x_{j+1}^*) is examined as follows: (a) In the case of no crack foregoing x_j^* , i. e. $\delta x_j \succ 0$, the condition $\delta x_{j+1} \prec 0$ confirms presence of a crack in this segment; if both δx_j, δx_{j+1} are positive and $\delta x_{j+1}/\delta x_j \approx x_{j+1}^0/x_j^0$, it implies no crack in the segment. (b) In the case of crack preceding x_j^*, segment (x_j^*, x_{j+1}^*) has no crack when $\delta x_{j+1} - \delta x_j \approx \varepsilon_k(x_{j+1}^0 - x_j^0)$

and no crack would be also on the right of x^*_{j+1} under the condition $\delta x_{j+1}/\delta x_j \approx (1 - x^0_{j+1})/(1 - x^0_j)$.

The last segment $(x^*_n, 1)$ can be examined simply by checking the condition $\delta x_n \succ 0$ justification of which verifies presence of a crack in this interval. Otherwise, i. e. $\delta x_n \prec 0$, no crack exists there if $|\delta x_1| \approx \delta^-_n$.

The crack localization can be conducted for every mode whose frequency and nodes have been given and, as result a collection of the segments that have been detected to be cracked is found for each mode. Different modes give different collections of the cracked segments and cracks if they exist in bar should be located in intersection of the cracked segments from different collections. To specify the crack position and extent in the segments detected to be cracked the subsequent step of the crack identification problem is accomplished as follows. Suppose that the segment (x^*_{k1}, x^*_{k2}) has been predicted to contain a crack of magnitude γ at the position $e \in (x^*_{k1},\, x^*_{k2})$. Letting $\phi^*_k(x)$ and λ^*_k be mode shape and eigenvalue of k-th mode respectively. Since the function $\phi^*_k(x)$ has the nodes x^*_{k1}, x^*_{k2}, the conditions are satisfied

$$\phi'^{\,*}_k(e-0) = \phi'^{\,*}_k(e+0); \phi_k{}^*(e-0) + \gamma\phi'^{\,*}_k(e-0) = \phi_k{}^*(e+0); \phi_k{}^*(x_{k1}{}^*) = \phi_k{}^*(x_{k2}{}^*) = 0. \quad (52)$$

Recalling that function $\phi^*_k(x)$ is a solution of Eq. (2), from the latter conditions one can obtain the equation

$$\sin \lambda^*_k (x^*_{k2} - x^*_{k1}) + \lambda^*_k \gamma \cos \lambda^*_k (e - x^*_{k1}) \cos \lambda^*_k (x^*_{k2} - e) = 0. \quad (53)$$

The obtained Eq. (53) is valid also for the specimen $(0, x^*_1)$ and $(x^*_n, 1)$ if the bar is fixed at the end $x = 0$ and $x = 1$ and it is reduced respectively to

$$\sin \lambda^*_k x^*_1 + \lambda^*_k \gamma \cos \lambda^*_k e \cos \lambda^*_k (x^*_1 - e) = 0 \text{ for } (0, x^*_1), \quad (54)$$

$$\sin \lambda^*_k (1 - x^*_n) + \lambda^*_k \gamma \cos \lambda^*_k (e - x^*_n) \cos \lambda^*_k (1 - e) = 0 \text{ for } (x^*_n, 1). \quad (55)$$

In the case of free ends the equations similar to (54) and (55) are respectively

$$\cos \lambda^*_k x^*_{k1} - \lambda^*_k \gamma \sin \lambda^*_k e \cos \lambda^*_k (x^*_{k1} - e) = 0, \quad (56)$$

$$\cos \lambda^*_k (1 + x^*_{kn} - 2e) + \lambda^*_k \gamma \sin \lambda^*_k (1 - e) \cos \lambda^*_k (e - x^*_{kn}) = 0. \quad (57)$$

The Eqs. (53)-(57) for the k th mode can be represented in general form

$$\varphi_{k1}(e) + \gamma \varphi_{k2}(e) = 0. \quad (58)$$

On the other hand, it is assumed that a crack has been detected to occur in the segment (x^*_{r1}, x^*_{r2}) for r-th mode with the measured nodes x^*_{r1}, x^*_{r2} and the segment (x^*_{r1}, x^*_{r2}) contains the same crack of position and magnitude e, γ as the segment (x^*_{k1}, x^*_{k2}) does. Consequently, an equation similar to (58) for the r-th mode can be obtained as

$$\varphi_{r1}(e) + \gamma \varphi_{r2}(e) = 0, \quad (59)$$

$$e \in (a,\, b), a = \max(x^*_{k1},\, x^*_{r1}), b = \min(x^*_{k2},\, x^*_{r2}).$$

Excluding γ from Eqs. (58), (59) leads to

$$f_{kr}(e) \equiv \varphi_{k1}(e)\varphi_{r2}(e) - \varphi_{r1}(e)\varphi_{k2}(e) = 0. \quad (60)$$

A root e^*_{kr} of Eq. (60), if it exists in the interval (a, b), is the desirable crack position that allows for estimating the crack magnitude

$$\gamma^*_{kr} = -\varphi_{k1}(e^*_{kr})/\varphi_{k2}(e^*_{kr}) = -\varphi_{r1}(e^*_{kr})/\varphi_{r2}(e^*_{kr}). \quad (61)$$

Thus, a single crack position and magnitude in the segments $(x^*_{k1}, x^*_{k2}) \cap (x^*_{r1}, x^*_{r2})$ have been identified. By the same manner one can find out crack position and extent in other pairs of segments that have been predicted to include crack. Hence, the problem of multiple crack identification for a bar by using measured frequencies and mode shape nodes is thus completed.

For illustration the experimental model for a free-free bar studied in [23] is taken into consideration herein. For compatibility of the results obtained in the reference with the theory developed above, the given data for the model have been normalized by unite length so that non-dimensional actual crack position is 0.3752. For the second mode the measured dislocation of two nodes are $\delta x_{21} = 0.0231 \succ 0$, $\delta x_{22} = -0.022264 \prec 0$ so that crack should be in segment (0.2731, 0.7273). Also, since $\delta x_{22}/\delta x_{21} = -0.98 \approx (x^0_{22} - 1)/x^0_{21} = -1.0$ one can be surely that no crack outside the segment. For the third mode, $\delta x_{31} = 0.00447 \succ 0$, $\delta x_{32} = -0.02253 \prec 0$, $\delta x_{33} = -0.00448 \prec 0$ that confirms presence of a crack in the segment (0.1710, 0.4775). On the other hand, because $\delta x_{33}/\delta x_{31} = -1.0 = (x^0_{33} - 1)/x^0_{31}$ one can be definitely that crack would be only in the interval (0.1710, 0.8288). Moreover, comparing the measured dislocation (0.02253) of the node $x^0_{32} = 1/2$ with the theoretical one, $\delta^-_{32} = 0.0197$, calculated for the case of no crack on the right of x^0_{32} leads to the conclusion that only the segment (0.1710, 0.4775) contain a crack. Consequently, it is predicted that crack should be only in the segment (0.2731, 0.4775), the intersection of two segments (0.2731, 0.7273); (0.1710, 0.4775) and measured data needed for specifying crack position in the segment (0.2731, 0.4775) are

$$\lambda^*_2 = 5.73145; x^*_{21} = 0.2731; x^*_{22} = 0.7273, \lambda^*_3 = 9.04365; x^*_{31} = 0.1710; x^*_{32} = 0.4775. \quad (62)$$

Obviously, Eq. (60) with the data given in (62) has unique solution $e^* = 0.37505865$ in the segment [0.2731, 0.4775] and, then, crack magnitude computed by using Eq. (61) is $\gamma^* = 0.247268$ matching very well to 0.259 that has been given in reference [23].

5. CONCLUSIONS

An explicit expression for natural modes of bar with arbitrary number of cracks has been derived and used for analysis of change in mode shape node induced by the multiple cracks. In addition to the well known fact that mode shape nodes move toward the crack, it has been shown herein that dislocation of nodes caused by a single crack in bar is determined by the product of frequency shift and generic position of the nodes. So that ratio of dislocations for two nodes of a particular mode is independent on the presence of crack but dependent only on the node positions. This is valid also for the case of multiple cracks when the nodes under the consideration lie on the same side of all cracks. The node located between two cracks referred to as intermediate node, in general, may move either to the left or to the right dependently upon the position and depth of cracks in both sides of the node. The detailed analysis of change in mode shape nodes induced by multiple cracks has been applied to propose a procedure for multiple crack identification for bar from measured mode shape nodes. An illustrating example has shown that the mode shape nodes are helpful for obtaining unique solution of multiple crack identification.

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CONTENTS