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PATTERN AVOIDANCE

EMMA CHRISTENSEN

ABSTRACT. A set partition avoids a pattern if no subdivision of that partition standardizes to the pattern. There exists a bijection between set partitions and restricted growth functions (RGFs) on which Wachs and White defined four statistics of interest to this work. We first characterize the restricted growth functions of several avoidance classes based on partitions of size four, enumerate these avoidance classes, and consider the distribution of the Wachs and White statistics across these avoidance classes. We also investigate the equidistribution of statistics between avoidance classes based on multiple patterns.

1. INTRODUCTION

Pattern avoidance in set partitions has become an increasingly popular area of research. In 2010 Bruce Sagan enumerated the avoidance classes for patterns of size three and also characterized the restricted growth functions in these classes [Sagan (2010)]. Not only are the sizes of different avoidance classes interesting, statistics on restricted growth functions in avoidance classes also offer a rich area of research. In 1991 Wachs and White defined four statistics of interest to this paper on the words associated with set partitions [Wachs and White (1991)]. In a Research Experience for Undergraduates at Michigan State University in 2014, students investigated the distribution of the Wachs and White statistics over avoidance classes based on patterns of size three [Dahlberg et al. (2015)]. This group also investigated multiple pattern avoidance focusing on two patterns of length three, or one pattern of length three and one of length four [Dahlberg et al. (2016)]. Vit Jelinek, Toufik Mansour, and Mark Shattuck expanded the idea of pattern avoidance in 2013 by investigating avoidance classes determined by two patterns, especially the enumeration of (3,4)-pairs and the enumeration of (4,4)-pairs [Jelinek et al. (2013)].

This paper will begin by considering avoidance classes based on one pattern of size four. We will first characterize the words in each avoidance class. Then using these word characterizations we will enumerate the avoidance classes of interest, develop equations for the Wachs and White statistics of each class, and finally compare the distributions of these statistics on avoidance classes. Following the investigation of the avoidance classes based on patterns of size four, we will consider classes avoiding multiple patterns. This focus on multiple pattern avoiding classes of set partitions will continue to center around the investigation of the equidistribution of statistics. Finally this paper will introduce topics generated by this research which we will investigate in the future.

2. Definitions and Notation

A set partition of $S = \{1, 2, ..., n\}$ is a collection of disjoint subsets (or blocks) $B_1, B_2, ..., B_m$ where $B_1 \cup B_2 \cup ... \cup B_m = S$ and

$$min(B_1) < min(B_2) < ... < min(B_m)$$

A set partition, σ, of S is denoted:

$$\sigma = B_1/B_2/.../B_m$$

The collection of all set partitions of $[n]$ is denoted:

$$\Pi_n = \{\sigma : \sigma \vdash [n]\}$$

Before defining avoidance and containment, one necessary structure to develop will be a subpartition of a set partition σ, denoted σ'.

Definition 2.1. When $S = \{1, ..., n\}$ and $A \subseteq S$, then a subpartition σ' of a partition σ is defined as $\sigma' = A \cap \sigma$.

For example, if $\sigma = 1347/25/68 \vdash [8]$ and $A = \{1, 2, 4, 7\}$, then $\sigma' = \{1, 2, 4, 7\} \cap 1347/25/68 = 147/2$. Our subpartition σ' does not have consecutive elements in this example and σ' does not appear to be an element of Π_n. To deal with a subpartition with nonconsecutive elements we standardize σ'. To do this we assign the smallest element to 1, the next smallest to 2, and so on. So $147/2$ standardizes to $134/2$.

Standardizing subpartitions becomes useful when comparing a set partition to a pattern, where a pattern is another set partition under consideration. If a partition has at least one subpartition that standardizes to a given pattern π, then the partition is said to *contain* π. If no subpartition standardizes to π, then the partition *avoids* π.

Examples are helpful in understanding the basic concepts of pattern avoidance and containment. Let $\pi = 134/2$ and $\sigma = 1347/25/68$. Now consider $A = \{1, 2, 4, 7\}$ such that $\sigma' = A \cap \sigma$ which standardizes to $\pi = 134/2$. We can conclude that σ contains π. Another helpful example is determining if $\sigma = 12/3/469/578$ contains $\pi = 13/2/4$. To decide if σ avoids or contains π we must consider every possible σ' until either we find some σ' that standardizes to π or we show that no such σ' exists. This is a long and time consuming process; however there is a much more elegant approach. Instead of considering every possible σ', if we first find a basic form of partitions that always avoid π, this process will become much more efficient. For this example, $\sigma = 12/3/469/578$ avoids $\pi = 13/2/4$ because for $i, j, k \in \mathbb{N}$ there are no $a, b \in B_i$ where there exist $c \in B_j$ and $d \in B_k$ such that $a < c < b < d$.

Definition 2.2. The collection of the set partitions that avoids a specific pattern, π, is called an avoidance class and is denoted:

$$\Pi_n(\pi) = \{\sigma \in \Pi_n : \sigma \text{ avoids } \pi\}$$

As stated in the introduction, we are not only interested in the types of elements in, and size of the avoidance class, we also want to determine the relationship between avoidance classes and Wachs and White statistics. However, before we can find Wachs and White statistics for partitions in avoidance classes, these set partitions must be converted into restricted growth functions.

Definition 2.3 (Restricted Growth Function)**.** If $w = a_1 a_2 ... a_n$ where $a_1 = 1$ and when $i \geq 2$, then $a_i \leq 1 + \max\{a_1, ..., a_{i-1}\}$, then w is a restricted growth function (RGF).

There is a well known correspondence between set partitions of n and restricted growth functions. The letter assigned to each position is determined as follows: if i is an element in $\sigma = B_1/B_2/.../B_m$ and $i \in B_j$, then $w_i = j$. The following example is a set partition labeled with the block number for each element and the corresponding word labeled with the position of each letter.

$$\sigma = \underset{1}{1} / \underset{2\,2\,2}{235} / \underset{3\,3}{46}$$

$$w(\sigma) = \underset{1\,2\,3\,4\,5\,6}{122323}$$

The collection of RGFs of length n is denoted:

$$R_n = \{w : w \text{ is an RGF of length } n\}$$

The collection of RGFs associated with the avoidance class for a pattern π is denoted:

$$R_n(\pi) = \{w(\sigma) : \sigma \in \Pi_n(\pi)\}$$

.

It will be useful to note that we say that $w = 1^{a_1} 2^{a_2} ... m^{a_m}$ when w is weakly increasing. When w of type $1^{a_1} 2^{a_2} ... m^{a_m}$ this simply requires w to have a_i is for every $1 \leq i \leq m$.

Because our statistics of interest are defined on the RGFs, and not on set partitions, one of the goals of this study is to characterize the words in $R_n(\pi)$ for $|\pi| = 4$ and for multiple patterns. Some work in this area has already been completed and will be useful in characterizing our avoidance classes.

Lemma 2.1 (Word Characterizations based on patterns of size 3)**.** [Sagan (2010)]
1. $R_n(1/2/3) = \{w \in R_n : w \text{ consists of only 1s and 2s}\}$
2. $R_n(1/23) = \{w \in R_n : w \text{ is obtained by inserting a single 1 into a word of the form } 1^l 23...m \text{ for some } l \geq 0 \text{ and } m \geq 1\}$
3. $R_n(13/2) = \{w \in R_n : w \text{ is layered}\}$
4. $R_n(12/3) = \{w \in R_n : w \text{ has initial run } 1...m \text{ and } a_{m+1} = ... = a_n \leq m\}$
5. $R_n(123) = \{w \in R_n : w \text{ has no element repeated more than twice}\}$

Another goal will be to enumerate the avoidance classes. The following theorem is a helpful result:

Lemma 2.2. [Sagan (2010)] *If $w \in R_n$ is weakly increasing with m distinct letters, then there is a total of $\sum_m \binom{n-1}{n-m}$ possibilities for w.*

Proof. If w is weakly increasing, we are not concerned with order when enumerating. Then there are m choices for each letter and $n - m$ letters not assigned. So for each m there are $\binom{m+(n-m)-1}{n-m} = \binom{n-1}{n-m}$ weakly increasing words of length n with m distinct letters. $\qquad\square$

There are several statistics of interest to this paper based on the word given by the restricted growth function. Once we have a characterization for the words in an avoidance class we can determine the distribution of the statistics across avoidance classes. This paper will focus on the four Wachs and White statistics, where the letters $l, r, s,$ and b indicate left, right, smaller, and bigger respectively

[Wachs and White (1991)]. Before defining the Wachs and White statistics across entire words, however, we will begin by defining the contributions to the statistics for individual letters:

Definition 2.4. For a position $i \in [n]$, the associated letter w_i will contribute the following to each statistic:

$$rs_i(w) = |\{w_j < w_i : j > i\}|$$
$$ls_i(w) = |\{w_j < w_i : j < i\}|$$
$$lb_i(w) = |\{w_j > w_i : j < i\}|$$
$$rb_i(w) = |\{w_j > w_i : j > i\}|$$

For example, consider $w = 1123124255$, then $ls_4(w) = 2$ because we are only counting distinct terms to the left and smaller than $w_4 = 3$. Now we will consider the statistics across an entire word.

Definition 2.5 (Wachs and White Statistics). [Wachs and White (1991)] If $w \in R_n$, then

$$rs(w) = \sum_{i=1}^{n} rs_i(w)$$
$$rb(w) = \sum_{i=1}^{n} rb_i(w)$$
$$lb(w) = \sum_{i=1}^{n} lb_i(w)$$
$$rs(w) = \sum_{i=1}^{n} rs_i(w)$$

So the right and smaller statistic counts the number of distinct letters to the right and smaller of each letter in w. To see this more clearly, consider the word, w, labeled with $rs_i(w)$ for all $i \in [n]$:

$$w \quad = \quad 123243125$$
$$rs_i(w): \ 012132000$$

From the definition of the rs statistic we know $rs(w)$ is simply the sum of these individual contributions $rs_i(w)$ for every $i \in n$, so

$$rs(w) = 0 + 1 + 2 + 1 + 3 + 2 + 0 + 0 + 0 = 9$$

Next consider the lb statistic for w, we first label each letter below with $lb_i(w)$ for all $i \in [n]$ and then find $lb(w)$.

$$w \quad = \quad 123243125$$
$$lb_i(w): \ 000101320$$

$$lb(w) = 0 + 0 + 0 + 1 + 0 + 1 + 3 + 2 + 0 = 7$$

The two other Wachs and White statistics for w are:

$$rb(w) = 4 + 3 + 2 + 3 + 1 + 1 + 2 + 1 + 0 = 17$$
$$ls(w) = 0 + 1 + 2 + 1 + 3 + 2 + 0 + 1 + 4 = 14$$

One of the intentions of this research is to find equations for the four Wachs and White statistics on specific avoidance classes. Before focusing on specific classes however, it will be useful in this work to recognize that there is a simple formula for the left and smaller statistic of any word.

Lemma 2.3. *If $w \in R_n$ such that w is of type $1^{a_1}2^{a_2}...m^{a_m}$, then*

$$ls(w) = \sum_{i=1}^{m} ia_i - n$$

Proof. We claim that for each letter, $i \in w$, there will be $i-1$ distinct elements that are smaller and occur before i. Suppose that w is a word of type $1^{a_1}2^{a_2}...m^{a_m}$.

Now consider some $j \in w$ where w does not have all of the integers that occur between 1 and j in some order in the word before a_j. Then for some k where $k < j$ and k first occurs after j, $k+1 \leq j$ then $j \not\leq 1 + max\{1,...,j-1\}$. Therefore no element can appear without all integers before that element also appearing in the word by our definition of RGF. Because w will have a_i occurrences of i, for every i, there will be a_i letters, each contributing $i-1$ to the ls statistic. Then by the definition of ls,

$$ls(w) = \sum_{i=1}^{m} a_i(i-1)$$
$$= \sum_{i=1}^{m} ia_i - n$$

Therefore when $w \in R_n$ and has m distinct letters, $ls(w) = \sum_{i=1}^{m} ia_i - n$. $\qquad \square$

There is a simple parallel equation for the rb statistic of a word, however this formula only applies to weakly increasing words.

Lemma 2.4. *If $w \in R_n$ where w is weakly increasing such that $w = 1^{a_1}2^{a_2}...m^{a_m}$, then*

$$rb(w) = \sum_{i=1}^{m} (m-i)a_i$$

Proof. Let w be a weakly increasing word where $w \in R_n$ such that $w = 1^{a_1}2^{a_2}...m^{a_m}$, and let b be a position in w where $w_b = c$ with a_c occurrences of c in w. Then $rb_b(w) = (m-c)$. There will be a_c of these letters contributing the same value for $rb_i(w)$ to the rb statistic, therefore $rb(w) = \sum_{i=1}^{m}(m-i)a_i$. $\qquad \square$

As we investigate these statistics across avoidance classes, we first identify avoidance classes of the same size. To begin, we use word characterizations to help us enumerate the avoidance classes. The complement of a set partition, defined below, is helpful in enumerating avoidance classes.

Definition 2.6. We define the *complement* of σ to be the set partition σ^c where every element $l \in \sigma$ becomes $(n - l + 1) \in \sigma^c$.

For example, let $\sigma = 1246/37/5$. Then $\sigma^c = 15/2467/3$.

From [Sagan (2010)], we have the following theorem:

Theorem 2.1 (Cardinality of the Avoidance Class of the Complement of π). *If $\pi \in \Pi_k$ and π^c is the complement of π, then $\#\Pi_n(\pi) = \#\Pi_n(\pi^c)$.*

Definition 2.7. If $\pi_1, \pi_2 \in \Pi_n$ such that $\#\Pi_n(\pi_1) = \#\Pi_n(\pi_2)$, then π_1 is *Wilf-equivalent* to π_2. We write $\pi_1 \equiv \pi_2$.

Before considering whether any statistic is equidistributed on two avoidance classes, we first determine if the two classes are Wilf-equivalent. By Theorem 2.2,

for all σ and σ^c, $\sigma \equiv \sigma^c$. This reduces the number of cases to consider when determining the equidistribution of statistics between avoidance classes.

Here is another important lemma to note that will help us in our word characterizations for the avoidance classes.

Lemma 2.5. *If $w(\sigma)$ is weakly increasing, then $w(\sigma^c)$ is also weakly increasing.*

Proof. Let $\sigma \in \Pi_n$ where σ has m blocks and $w(\sigma)$ is weakly increasing. Then $w(\sigma) = 1^{a_1}2^{a_2}...m^{a_m}$ for some $a_1, a_2, ..., a_m$ such that $\sum_{i=1}^{m} a_i = n$. By the definition of RGF, we know that $\sigma = 1...a_1/.../(\sum_{i=1}^{m-1} a_i + 1)...n$, where each block contains only consecutive terms. Then $\sigma^c = (n+1-1)...(n+1-a_1)/.../(n+1-\sum_{i=1}^{m-1} a_i - 1)...1$, which we can simplify to $\sigma^c = 1...a_m/.../(\sum_{i=2}^{m} a_i + 1)...n$ where each block contains consecutive terms. Therefore by the definition of RGF we know that $w(\sigma^c) = 1^{a_m}2^{a_{m-1}}...m^{a_1}$. So $w(\sigma^c)$ is weakly increasing when $w(\sigma)$ is weakly increasing. $\square$

3. Word Characterizations and Cardinalities of Single Pattern Avoidance Classes

There are 15 set partitions of size four. Sagan enumerated and found the word characterization for the avoidance classes based on $1/2/3/4$ and 1234 [Sagan (2010)]. Following this the Research Experience for Undergraduates at Michigan State University found the cardinality and word characterizations of avoidance classes based on $14/2/3$, $13/24$, and $14/23$ [Dahlberg et al. (2016)]. This group also investigated the Wachs and White statistics for these avoidance classes.

In this paper we will enumerate, characterize and investigate the equidistribution of statistics for the avoidance classes based on $12/3/4$, $1/2/34$, $1/234$, $123/4$, $134/2$, $124/3$, $13/2/4$, and $1/24/3$. We chose these avoidance classes because they have not yet been investigated. We did not get the chance to investigate the avoidance classes based on the patterns $12/34$ and $1/23/4$. After finding the word characterizations and the cardinality for each of these eight avoidance classes we find which classes are Wilf-Equivalent. Recognizing these Wilf-Equivalent classes will be important in the study of the distribution of the Wachs and White Statistics. Before explaining our word characterization and cardinality results, we will begin with two avoidance classes with some known results [Sagan (2010)].

Theorem 3.1. [Sagan (2010)] *The word characterization and cardinality of the avoidance class based on the pattern $1/2/3/4$ are as follows:*

  (i) $R_n(1/2/3/4) = \{w \in R_n: w$ consists of only 1's, 2's, and 3's$\}$
  (ii) $\#\Pi_n(1/2/3/4) = 2^{n-1} + S(n,3)$

Proof.   (i) The word characterization is due to Sagan [Sagan (2010)].
  (ii) Let $\sigma \in \Pi(1/2/3/4)$. We know that if σ avoids $1/2/3/4$, σ can have at most three blocks. By the definition of the Stirling numbers of the second kind there are $S(n,1) = 1$ ways for σ to have one block, $S(n,2) = 2^{n-1} - 1$ ways for σ to have two blocks, and $S(n,3)$ ways for σ to have three blocks. Therefore $\#\Pi_n(1/2/3/4) = 2^{n-1} + S(n,3)$.

$\square$

Theorem 3.2. [Sagan (2010)] *The word characterization and cardinality of the avoidance class based on the pattern 1234 are as follows:*

(i) $R_n(1234) = \{w \in R_n \colon w$ has no letter repeated more than three times$\}$

(ii) When $n \leq 12$, $\#\Pi_n(1234) = 1 + \binom{n}{2} + S(n, n-2) + \sum\limits_{i=3}^{n} S(n, n-i) -$
$$\sum\limits_{j=4}^{i+1} \binom{n}{j} S(n-j, n-i-1) - \sum\limits_{k=4}^{n-j-1} \binom{n-j}{k} S(n-j-k, n-i-2)$$

Proof. (i) The word characterization is due to Sagan [Sagan (2010)].

(ii) From the result for $R_n(1234)$ we know that if σ avoids 1234, σ cannot have more than three elements in the same block. If there are n blocks, there can only be one element in each block and there are $S(n, n) = 1$ ways to have n blocks. If there are $n-1$ blocks, there are $n-2$ blocks that have have one element and one block with two elements and there are $\binom{n}{2}$ ways to do this. When there are $n-2$ blocks, there are at most three items in one block and there are $S(n, n-2)$ ways to do this. When there are fewer than $n-2$ blocks, there is the potential for more than three elements in one block, which would allow any σ of this type to contain 1234. The total number of ways to partition the set $\{1, ..., n\}$ into $n-i$ blocks where i can be anything larger than two will be $S(n, n-i)$. Now we must remove the partitions which allow more than three elements in the same block. Whenever a block has four or more elements it will standardize to 1234. To make a block of four or more we must choose four or more elements. When j is the number of elements in the block, there are $\binom{n}{j}$ ways to choose the elements in this block. The other elements are not in this block, so there are $(m-1)$ blocks to choose from where $m = n-i$ and $n-j$ elements left to assign to these $n-i-1$ blocks. Therefore there are $\binom{n}{j} S(n-j, n-i-1)$ ways for the set partition to have at least one block of four or more. However this overcounts the number of possible partitions with four or more elements in one block because it also will count when more than one block has four or more elements, so to account for those set partitions that were removed more than once we must add back the cases with two different blocks each with four or more elements. There are $(n-j)$ elements to choose from and k elements to choose. Then there are $n-j-k$ other elements to place into $n-i-2$ blocks because the first block and the second block that we found both had more than four elements so there are $S(n-j-k, n-i-2)$ ways of assigning the other elements to blocks. This formula will work up to $n = 12$ because there can be at most three blocks with four elements and once the first two blocks of four are found, there will only be one option for the third, after this, as n becomes larger we will have to consider when three or more blocks are larger than four where the last block is not necessarily determined by the other two blocks already found. $\qquad\square$

Next we consider avoidance classes that have not been investigated. The proofs of these theorems will be very similar. Therefore, we will present the proof of Theorem 3.3 in the body of this paper and refer the reader to the appendix for complete proofs of Theorems 3.4 through 3.10.

We first consider avoidance classes based on the pattern $12/3/4$. It is easier to begin by considering an example of words found in $R_n(12/3/4)$. It is simple to see that the set partition $168/2/37/4/5$ avoids $12/3/4$ because any block that contains

two or more elements has at most one other block with a larger element. Therefore $12345131 \in R_n(12/3/4)$. We can describe these words found in $R_n(12/3/4)$ more generally in the following theorem.

Theorem 3.3. *The word characterization and cardinality of the avoidance class based on the pattern $12/3/4$ are given by:*

    i. $R_n(12/3/4) = \{w \in R_n :$

        (A) $m \leq 2$, or

        (B) $m \geq 3$ and $w = uv$ such that u is strictly increasing, and v begins at the first repeated letter. The suffix v contains at most two distinct letters (v can be empty).}

    ii. $\#\Pi_n(12/3/4) = 2^{n-1} + n(n-2) - \frac{(n-1)(n-2)}{2} + \sum\limits_{|v|=2}^{n-2} \sum\limits_{j=1}^{i-1} \binom{n-|v|}{2}\binom{|v|}{j} + $

    $\sum\limits_{|v|=2}^{n-2} \sum\limits_{j=1}^{i-1} (n-i)\binom{i-1}{j}$

Proof.   (i) Word Characterization:

  (Case A) Let $w(\sigma)$ have no letter larger than 2. Then σ can only have one or two blocks. In general if σ has k blocks it must avoid any pattern with more than k blocks. Therefore $\sigma \in \Pi_n(12/3/4)$ and $w(\sigma) \in R_n(12/3/4)$.

  (Case B) Let $w(\sigma)$ be of the form uv such that u is strictly increasing, and v begins at the first repeated letter. When u is strictly increasing, then u has no repeated letter. Then because v can have at most two distinct letters, there are at most two blocks that can have two or more elements in σ, by the definition of an RGF. The repeated elements in w are only in v and there can be at most two distinct letters in v. So when there is a block with two elements (a and b), then σ can have at most one distinct block that contains elements that are larger than a and b. Therefore no subpartition of σ will standardize to $12/3/4$ because there must be at least two larger elements in different blocks after a block of size two or larger. Therefore σ will avoid the pattern.

      Next suppose $w(\sigma)$ is not of form (A) or (B). Then $m \geq 3$ and v must contain more than two distinct letters. There is no way to violate the strictly increasing property of u because v must begin at the first repeated term.

      Then using the definition of the restricted growth function, consider the subpartition constructed by taking the element a where a is the first letter in v, b the other element that occurs in the same block as a, along with c the element associated with the second distinct letter in v, and d the element associated with the third distinct letter in v. This subpartition will have three blocks: there will be a block of the two smaller elements (a and b), followed by a block containing the second largest element, and then a block with the largest element. So the subpartition is $ba/c/d$ where $b < a < c < d$ which therefore standardizes to $12/3/4$. For example if $v(\sigma) = 12123$ then $\sigma = 13/24/5$ and one subpartition created through this process would be $\sigma' = 13/4/5$. It is clear that in this example σ' standardizes to $12/3/4$.

      Therefore when σ avoids $12/3/4$ then $w(\sigma)$ has either form (A) or (B).

  (ii) Next we consider the cardinality of the avoidance class. By the word characterization there are two main cases to consider:

(Case A) Let $m \leq 2$. There are $S(n,2) = 1 + (2^{n-1} - 1) = 2^{n-1}$ words in (Case A) by the definition of the Stirling numbers of the second kind.

(Case B) Let $w = uv$ as defined in the proof of (Case B) of the word characterization. Now we have to consider three possible forms for v.

    (1) Let v have at most one letter. Then the length of v, denoted as $|v|$, can be from 0 to $n-3$ because $m \geq 3$. When $|v| = 0$ there is one form for w; w is strictly increasing. When $|v| > 0$, there are m choices for the letter in v, And $m = n - |v|$ because m is found in u in this case. So there are $1 + \sum_{|v|=1}^{n-3} (n - |v|)$ possible w of this form.

    (2) Let v be of type $p^j q^i$ where $p > q$ and $p, q \in u$. There are $\binom{n-|v|}{2}$ ways to choose p, q. There are $|v|$ positions in v to choose from and j p's in v, so there are $\binom{|v|}{j}$ ways to arrange v where $1 \leq j \leq |v-1|$ because there must be two letters present in v. Therefore there are $\sum_{|v|=2}^{n-3} \sum_{j=1}^{|v|-1} \binom{n-|v|}{2}\binom{|v|}{j}$ possible w of this form.

    (3) Let v be of the form $p^j m^i$ where p also occurs in u. The first letter in v must be p because v begins with the repeated letter. Because there are $n - |v|$ letters in u, there are $n - |v|$ options for this first letter in v. We know already that the other letter in v must be m, and that this m occurs j times. There are $|v| - 1$ options for where these j $m's$ can occur; because v must begin with the first repetition, it must begin with the smaller letter in v, then any other position in v is available. So there are $\binom{|v|-1}{j}$ ways of arranging v in this case. Therefore there are $\sum_{|v|=2}^{n-2} \sum_{j=1}^{|v|-1} (n - |v|)\binom{|v|-1}{j}$ possible w of this form.

Therefore $\#\Pi_n(12/3/4) = 1 + 2^{n-1} + \sum_{|v|=1}^{n-3}(n - |v|) + \sum_{|v|=2}^{n-3}\sum_{j=1}^{|v|-1}\binom{n-|v|}{2}\binom{|v|}{j} + \sum_{|v|=2}^{n-2}\sum_{j=1}^{|v|-1}(n - |v|)\binom{|v|-1}{j}$.

$\square$

Theorem 3.4. *The word characterization and cardinality of the avoidance class based on the pattern $1/2/34$ are given by:*

(i) $R_n(1/2/34) = \{w \in R_n :$
(A) $m \leq 2$
(B) w is a word of the form $u3v$ where u is a prefix of type $1^{a_1}2^{a_2}$, and v contains b_1 $1s$, b_2 $2s$, b_3 $4s$, and so on such that $b_i \leq 1$}

(ii) $\#\Pi_n(12/3/4) = 2^{n-1} + n(n-2) - \frac{(n-1)(n-2)}{2} + \sum_{|v|=2}^{n-2}\sum_{j=1}^{i-1}\binom{n-|v|}{2}\binom{|v|}{j} + \sum_{|v|=2}^{n-2}\sum_{j=1}^{i-1}(n - i)\binom{i-1}{j}$

Theorem 3.5. *The word characterization and cardinality of the avoidance class based on the pattern* $1/234$ *are given by:*

(i) $R_n(1/234) = \{w \in R_n : w$ *is of the form* uv *such that* u *is of the form* $1^{a_1}2$ *and* v *has* b_1 $1s$, b_2 $2s$, *and so on, such that* $b_1, b_3, ..., b_m \leq 2$ *and* $b_2 \leq 1\}$

(ii) $\#\Pi_n(1/234) = 1 + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \prod_{j=0}^{r-1} \binom{n-2j}{2} + \sum_{i=1}^{n-2} \sum_{r=1}^{\lfloor \frac{n-i}{2} \rfloor} \prod_{j=0}^{r-1} r\binom{n-2j-i}{2}$

Theorem 3.6. *The word characterization and cardinality of the avoidance class based on the pattern* $123/4$ *are given by:*

(i) $R_n(123/4) = \{w \in R_n: w$ *is of the form* uv *where* u *has* a_1 $1s$, a_2 $2s$, ..., a_m ms *such that* $a_1, ..., a_m \leq 2$ *and* v *can be any size of one letter repeated*$\}$

(ii) $\#\Pi_n(123/4) = 1 + \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} \prod_{j=0}^{r-1} \binom{n-2j}{2} + \sum_{i=1}^{n-2} \sum_{r=1}^{\lfloor \frac{n-i}{2} \rfloor} \prod_{j=0}^{r-1} r\binom{n-2j-i}{2}$

Before characterizing and enumerating the following two avoidance classes it will be useful to define three special types of letters that will aid in describing the word characterizations.

Definition 3.1 (Block Letter). We define a block letter as a letter $a \in w$ such that the only occurrence of a is in a block of as where this block contains one or more as.

Definition 3.2 (Block-Singleton Letter). We define a block-singleton letter as a letter $a \in w$ where the first occurrence of $a \in w$ is the beginning of a block of as, and after some distinct letter there is exactly one more occurrence of $a \in w$.

Definition 3.3 (Singleton-Block Letter). We define a singleton-block letter as a letter $b \in w$ where the first occurrence of $b \in w$ is followed by a distinct letter, and the only other occurrences of $b \in w$ are found in a block after this.

To clarify these definitions we include the following examples:

a. The block letters of the word $w_1 = 1112234334354$ are 1, 2, and 5.
b. The block-singleton letters of $w_2 = 1233322443124$ are 1, 3 and 4
c. The singleton-block letters of $w_3 = 122311123$ are 1 and 3.

Lemma 3.1 (Sinlgeton-Block/Block-Singleton Avoidance). *If the RGF for* σ *contains a singleton-block or a block-singleton letter and* w *is weakly increasing, then* $w \in R_n(\sigma)$.

Theorem 3.7. *The word characterization of the avoidance class based on the pattern* $134/2$ *is:*

• $R_n(134/2) = \{w \in R_n :$ *every letter in* w *is either a block letter or a block-singleton letter*$\}$

Theorem 3.8. *The word characterization of the avoidance class based on the pattern* $124/3$ *is:*

• $R_n(124/3) = \{w \in R_n :$ *every letter in* w *is either a block letter or a singleton-block letter*$\}$

Theorem 3.9. *The word characterization and cardinality of the avoidance class based on the pattern* $13/2/4$ *are given by:*

(i) $R_n(13/2/4) = \{w \in R_n$:
  (A) $m \leq 2$, or
  (B) $m > 3$ and $w = uv$ where u is weakly increasing and v begins at the first m and contains m and $(m-1)$ in any order, or
  (C) $m > 3$ and $w = uv$ where u is weakly increasing and v begins at the first m and contains a block of ms followed by a block of zs where $z < (m-1)\}$

(ii) $\#\Pi_n(13/2/4) = 2^{n-1} + \sum\limits_{m=3}^{n} \binom{n-1}{n-m} + \sum\limits_{m=3}^{n-1} \sum\limits_{|u|=m-1}^{n-2} \binom{|u|-1}{|u|-m+1} [(2^{(n-|u|-1)} - 1) + (m-2)(n-|u|-1)]$

Theorem 3.10. *The word characterization and cardinality of the avoidance class based on the pattern $1/24/3$ are given by:*

(i) $R_n(1/24/3) = \{w \in R_n$:
  (A) $m \leq 2$, or
  (B) $m > 3$ and w is a weakly increasing word with a block of $1s$ of any size inserted between two distinct terms, or
  (C) $m > 3$ and w is a word beginning with $1s$ and $2s$ in any order followed with a weakly increasing suffix that begins with the first occurrence of $3\}$

(ii) $\#\Pi_n(1/24/3) = 2^{n-1} + \sum\limits_{m=3}^{n} \binom{n-1}{n-m} + \sum\limits_{m=3}^{n-1} \sum\limits_{|u|=m-1}^{n-2} \binom{|u|-1}{|u|-m+1} [(2^{(n-|u|-1)} - 1) + (m-2)(n-|u|-1)]$

This completes our study of the word characterization and cardinality for the avoidance classes: $R_n(1/2/3/4)$, $R_n(1234)$, $R_n(12/3/4)$, $R_n(1/2/34)$, $R_n(1/234)$, $R_n(123/4)$, $R_n(13/2/4)$, $R_n(1/24/3)$, $R_n(134/2)$, and $R_n(124/3)$. Next we will investigate the Wachs and White statistics. We found especially promising results for the distribution of Wachs and White statistics for $R_n(\sigma)$ and $R_n(\sigma^c)$.

4. CHARACTERIZATION OF STATISTICS ACROSS SINGLE PATTERN AVOIDANCE CLASSES

Before considering the distribution of statistics across avoidance classes it will be helpful to obtain formulas for the Wachs and White statistics for individual words based on the avoidance class where each word can be found. Theorems 4.1 through 4.7 describe the statistics for words in certain avoidance classes that showed promising equidistribution results with other avoidance classes. The proofs for these equations for statistics are dependent on the word characterizations and can be found in the appendix. We begin with the results of the Michigan State REU for statistics when $m \leq 2$.

Theorem 4.1. [Dahlberg et al. (2015)] *If $m \leq 2$ and l is the number of $1s$ present in w, then we have the following equations for the Wachs and White statistics:*

$$lb(w) = \begin{cases} 0 & w \text{ weakly increasing} \\ l - i & w = 1^i w' 1^j 2^k \end{cases}$$

$$ls(w) = n - l$$

$$rb(w) = \begin{cases} l & w \ \textit{weakly increasing} \\ l - \delta_{k,0}j & w = 1^i w' 1^j 2^k \end{cases}$$

$$rs(w) = \begin{cases} 0 & w \ \textit{weakly increasing} \\ n - l - k & w = 1^i w' 1^j 2^k \end{cases}$$

Now we will use previous theorems to give more useful characterizations of the words to find equations for the Wachs and White statistics. For the ls and lb statistics for $w \in R_n(1/2/3/4)$ it is helpful to restate the word characterization. By Theorem 3.1 we know that for all $w \in R_n(1/2/3/4)$, $m \leq 3$, so we can restate this as,

$R_n(1/2/3/4) = \{w \in R_n :$

(A) $m \leq 3$ and $w = 1^{h_1} 2^{h_2} 3^{h_3}$ (meaning w is weakly increasing), or

(B) $1 < m \leq 3$ and $w = 1^v w_a 1^x w_b 1^y w_c 1^z$ where w_a begins and ends with a 2 and has a_1 1s and a_2 2s, w_b begins with the first 3 and ends with the last 2, and w_c begins with a 3 and ends with the last 3, or

(C) $1 < m \leq 3$ and $w = 1^w w_a 1^x w_b 1^y w_c 1^z$ where w_1 begins and ends with a 2 and has a_1 1s and a_2 2s, w_b begins with the first 3 and ends with the last 3, and w_c begins with a 2 and ends with the last 2$\}$

Theorem 4.2 (Left Statistics for $R_n(1/2/3/4)$). *If $w \in R_n(1/2/3/4)$, then*

$$ls(w) = \begin{cases} 2h_3 + h_2 & w \in A \\ a_2 + b_2 + 2b_3 + c_2 + 2c_3 & w \in B \ \textit{or} \ C \end{cases}$$

$$lb(w) = \begin{cases} 0 & w \in A \\ x + 2y + 2z + a_1 + 2b_1 + 2c_1 & w \in B \ \textit{or} \ C \end{cases}$$

The rb and rs statistics for $w \in R_n(1/2/3/4)$ are more easily described when we restate the word characterization for this avoidance class as follows:

$R_n(1/2/3/4) = \{w \in R_n :$

(A) $m \leq 3$ and $w = 1^{h_1} 2^{h_2} 3^{h_3}$

(B) $w = w_a w_b 3^c$ where w_a ends with the last 1; w_b ends with the last 2

(C) $w = w_a w_b 3^c$ where w_a ends with the last 2, w_b ends with the last 1

(D) $w = w_a w_b 2^c$ where w_a ends with the last 3, w_b ends with the last 1

(E) $w = w_a w_b 2^c$ where w_a ends with the last 1, w_b ends with the last 3

(F) $w = w_a w_b 1^c$ where w_a ends with the last 2, w_b ends with the last 3

(G) $w = w_a w_b 1^c$ where w_a ends with the last 3, w_b ends with the last 2

(H) $m = 2$ and $w = w_a 2^b$ where w_a ends with the last 1

(I) $m = 2$ and $w = w_a 1^b$ where w_a ends with the last 2$\}$

where w_a contains a_1 1s, a_2 2s, and a_3 3s; w_b contains b_1 1s, b_2 2s and b_3 3s

Theorem 4.3 (Right Statistics for $R_n(1/2/3/4)$). *If $w \in R_n(1/2/3/4)$, then*

$$rb(w) = \begin{cases} (m-1)h_1 + (m-2)h_2 & w \in A \\ 2a_1 + a_2 + b_2 & w \in B \ \textit{or} \ E \\ 2a_1 + a_2 + b_1 & w \in C, D, F \ \textit{or} \ G \\ a_1 & w \in H \ \textit{or} \ I \end{cases}$$

$$rs(w) = \begin{cases} 0 & w \in A \\ a_2 + 2a_3 + b_3 & w \in B, C, E \text{ or } F \\ a_2 + 2a_3 + b_2 & w \in D \text{ or } F \\ a_2 & w \in H \text{ or } I \end{cases}$$

Next we include the statistics for words in $R_n(1/2/34)$. By Theorem 3.4, if $w \in R_n(1/2/34)$, then w is in one of the following subsets of $R_n(1/2/34)$:

(A) $m \leq 2$
(B) $w = u3v$ and $b_1 = 0 = b_2$
(C) $w = u3v$ and $b_1 = 0$, $b_2 = 1$
(D) $w = u3v$ and $b_1 = 1$, $b_2 = 0$
(E) $w = u3v$ and $b_1 = 1 = b_2$ such that $x < y$ where x is the position of the 1 in v, and y is the position of the 2 in v
(F) $w = u3v$ and $b_1 = 1 = b_2$ such that $x > y$

Also let l be the position of the last $1 \in u$, z be the position of the first $2 \in u$, and h be the position of the last $2 \in u$.

Theorem 4.4 (Statistics for $R_n(1/2/34)$)**.** *If $w \in R_n(1/2/34)$, then for the statistics on w when $w \in A$ are given by Theorem 4.1, and for the other cases:*

$$ls = a_2 + b_2 + 2 + \sum_{i=3}^{m-1} i$$

$$rb = \begin{cases} (m-1)a_1 + (m-2)a_2 + |v| - y + \sum_{i=3}^{m}(m-i) & w \in C \\ (h - a_2)(m-1) + (m-2)(|u| - h) + (m-2)a_2 + |v| - x + \sum_{i=3}^{m}(m-i) & w \in B \text{ or } D \\ (m-1)a_1 + (m-2)a_2 + 2|v| - x - y + \sum_{i=3}^{m}(m-i) & w \in E \\ (m-1)a_1 + (m-2)a_2 + 2|v| - x - y - 1 + \sum_{i=3}^{m}(m-i) & w \in F \end{cases}$$

$$lb = a_1 - z + x + y + 1$$

$$rs = \begin{cases} y - 1 & w \in D \\ a_2 + x + y - 1 & w \in E \\ a_2 + x + y & w \in F \\ (l - a_1) + y & w \in B \text{ or } C \end{cases}$$

Similarly when we state the statistics for $w \in R_n(12/3/4)$ we identify the subset of the avoidance class in which w resides. By Theorem 3.3 the possible subsets include:

(A) $m \leq 2$
(B) $w = uv$ where $m \in u$ and v contains two distinct letters
(C) $w = uv$ where $m \in u$ and v is empty or contains one distinct letter
(D) $w = uv$ where $m \in v$ and v contains two distinct letters, where v is of type $c^{b_c}d^{b_d}$ and x is the position of the first $d \in v$ where $d > c$.

Theorem 4.5 (Statistics for $R_n(12/3/4)$)**.** *If $w \in R_n(12/3/4)$, then when $m \leq 2$ refer to Theorem 4.1 for the Wachs and White statistics for w; otherwise*

$$ls = \sum_{i=1}^{|u|-1} i + (c-1)b_1 + (d-1)b_2$$

$$rb = (y - b_2) + \sum_{i=1}^{|u|}(m - i)$$

$$lb = \begin{cases} (m-c)b_c + (m-d)b_d & w \in B \text{ or } C \\ (m-c-1)b_c + b_d - x + 1 & w \in D \end{cases}$$

$$rs = \begin{cases} |u| - c + |u| - d + b_c - x & w \in B \text{ or } D \\ |u| - d + b_c - x & w \in C \end{cases}$$

Proof. There are four results to prove.

(1) Let $w(\sigma)$ be of the form uv. Because every element in u is only found once, the largest element in u will be $|u|$, so the sum from one to $|u| - 1$ is equivalent to $\sum_{i=2}^{|u|}(i-1)a_i$. Now consider v to have b_1 c letters, and b_2 d letters. We know that c must have $(c-1)$ unique letters that are smaller and come before c, and therefore the contribution from the $c's$ will be $(c-1)b_1$. We also know that d must have $(d-1)$ unique elements that are smaller and come before d, and therefore all of the $d's$ will contribute $(d-1)b_2$ to the left and smaller statistic. Therefore the left and smaller statistic is $\sum_{i=1}^{|u|-1} i + (c-1)b_1 + (d-1)b_2$.

(2) Let y be the last occurrence of d where $d > c$ in v, then $(y - b_2)$ will tell us how many $c's$ come before the last d. Now consider the sum, whether m is in u or v it must come after every element in u so the sum from one to $|u|$ of $(m - i)$ will give us the number of larger elements that must occur after each element in u.

(3) When the largest element is found in u

$$lb = (m-c)b_1 + (m-d)b_2$$

Let m be in u. Then there are $(m - c)$ distinct letters that are larger and occur before each c and there are b_1 $c's$, therefore the contribution to the whole statistic based on c is $(m-c)b_1$. Now consider d, there are $m - d$ unique elements that are larger and occur before each d and there are b_2 $d's$, therefore the contribution to the whole statistic based on d is $(m-d)b_2$. These are the only things that are not weakly increasing, therefore there will be nothing that contributes to the left and bigger statistic from u because u is weakly increasing. Therefore $lb = (m-c)b_1 + (m-d)b_2$.

When the largest element is found in v the formula for lb is:

$$lb = (m - c - 1)b_1 + b_1 - x + 1$$

Where x is the position of the first largest element in v.

Let the largest element be found in v. Then the largest element in u is $(m-1)$ and there are $(m-1) - c$ unique elements in u that come before the first c. There are also b_1 $c's$ so this contributes $(m-c-1)b_1$ to the left and

bigger statistic. Now because $d = m$ there is no contribution from any d to the left and bigger statistic, however we must consider the $c's$ that occur after the first d. There will be $|v| - x$ elements that come after the first m. Now of these elements we only want the $c's$ so subtract $b_2 - 1$ because when we subtracted x earlier we accounted for the first m. So

$$|v| - x - (b_2 - 1) = b_1 + b_2 - x - b_2 + 1$$

$$|v| - x - (b_2 - 1) = b_1 - x + 1$$

Therefore $lb = (m - c - 1)b_1 + b_1 - x + 1$.

(4) When b_1 and b_2 are greater than zero.

$$rs = |u| - c + |u| - d + b_1 - x$$

Let b_1 and b_2 be greater than zero. Then there are $|u| - c$ unique elements that come before c that are larger than c, each of these will contribute one to the right and smaller statistic. There are $|u| - d$ unique elements that are larger and come before the first d, these will each also contribute one to the right and smaller statistic. Finally let y be the last occurrence of d, there will be $(y - b_2 + 1)$ $c's$ that come before the last d that will each contribute one to the statistic.

When b_1 is zero remove $|u| - c$ from equation.

$$rs = |u| - d + b_1 - x$$

When b_2 is zero, remove $|u| - d$ from the equation.

$$rs = |u| - c + b_1 - x$$

$\square$

The next avoidance class we will consider is $R_n(13/2/4)$. By Theorem 3.9, when $w \in R_n(13/2/4)$, w will be one of the following forms:

(A) $m \leq 2$

(B) w is weakly increasing and $m \geq 3$

(C) $w = uv$ where $v = m^{b_m} z^{b_z}$ such that $z \leq (m - 1)$

(D) $w = uv$ where v is not weakly increasing and v is of type $m^{b_m} z^{b_z}$ such that $z = (m - 1)$ where x is the position of the last $m \in v$, and y is the position of the last $z \in v$ (v begins with the first m in w)

Theorem 4.6 (Statistics for $R_n(13/2/4)$). *If $w \in R_n(13/2/4)$, then when $m \leq 2$ refer to Theorem 4.1 for the Wachs and White statistics for w; otherwise*

$$lb = \begin{cases} 0 & w \in B \\ (m - z)b_z & w \in C \text{ or } D \end{cases}$$

$$rs = \begin{cases} 0 & w \in B \\ \sum\limits_{i=z+1}^{m} a_i & w \in C \\ (y - b_z) & w \in D \end{cases}$$

$$rb = \begin{cases} l & w \in B \\ \sum\limits_{i=1}^{m-1}(m-i)a_i & w \in C \\ (x-a_m) + \sum\limits_{i=1}^{m-1}(m-i)a_i & w \in D \end{cases}$$

$$ls = (z-1)b_z + \sum_{i=2}^{m}(i-1)a_i$$

The last avoidance class on which we investigated the Wachs and White statistics was $R_n(1/24/3)$. By Theorem 3.10, when $w \in R_n(1/24/3)$ this word could take the following forms:

(A) $m \leq 2$
(B) w is weakly increasing
(C) w is a weakly increasing word, except for a block of b_1 1s inserted between two distinct letters where x is the distinct letter immediately before this block of 1s
(D) $w = uv$ where u is of type $1^{a_1}2^{a_2}$ and $v = 3^{b_3}4^{b-4}...m^{b_m}$ where c is the position of the first 2 in u and d is the position of the last 2 in u

Theorem 4.7 (Statistics for $R_n(1/24/3)$). *If $w \in R_n(1/24/3)$, then when $m \leq 2$ refer to Theorem 4.1 for the Wachs and White statistics for w, otherwise:*

$$lb = \begin{cases} 0 & w \in B \\ (x-1)b_1 & w \in C \\ a_1 - c + 1 & w \in D \end{cases}$$

$$rs = \begin{cases} 0 & w \in B \\ \sum\limits_{i=2}^{x}a_i & w \in C \\ f - a_1 & w \in D \end{cases}$$

$$rb = \begin{cases} l & w \in B \\ \sum\limits_{i=1}^{m}(m-i)a_i + (m-x)b_1 & w \in C \\ (d-a_2)(m-1) + (|u|-d)(m-2) + \sum\limits_{i=2}^{m-1}(m-i)a_i & w \in D \end{cases}$$

$$ls = \sum_{i=1}^{m}(i-1)a_i$$

5. Equidistribution of Wachs and White Statistics

The goal of this work has been to identify distributions of Wachs and White Statistics that are the same for different avoidance classes. Up to this point we have found the cardinalities of the avoidance classes, (because no two avoidance classes can have a statistic equidsitributed unless they are of the same size). We have also characterized the words in each avoidance class we investigated, and found formulas for the Wachs and White statistics for these word. These equations will become very useful in proving our equidistribution results.

5.1. **The Patterns** $1/24/3$ **and** $13/2/4$**.** First, notice that the complement of $1/24/3$ is $13/2/4$. Therefore $1/24/3 \equiv 13/2/4$. We found four equidistribution results on the avoidance classes based on the patterns $1/24/3$ and $13/2/4$. The proof of the first result will be included here, we refer the reader to the appendix for the full proof.

Theorem 5.1 (Equidistribution of Statistics I)**.** *Consider the avoidance classes based on $1/24/3$ and $13/2/4$, then*

(i) $LS(13/2/4) \sim RB(1/24/3)$
(ii) $RB(13/2/4) \sim LS(1/24/3)$
(iii) $LB(13/2/4) \sim LB(1/24/3)$
(iv) $RS(13/2/4) \sim RS(1/24/3)$

Proof. Suppose σ_1 avoids $13/2/4$ and σ_2 avoids $1/24/3$. Also let w denote the RGF for σ_1 and let y denote the RGF for σ_2. Let $A_1, A_2, A_3 \subset R_n(13/2/4)$ such that $w \in A_1$ where w has $m \leq 2$ or w is weakly increasing, $w \in A_2$ where $w = uv$ where v is a block of $z \leq (m-1)$, and $w \in A_3$ where $w = uv$ where v contains m and $(m-1)$ in any order after the first m. Also let $B_1, B_2, B_3 \subset R_n(1/24/3)$ where $y \in B_1$ where y has $m \leq 2$ or y weakly increasing, $y \in B_2$ where y has a block of 1s inserted between two distinct letters in a weakly increasing word, and finally $y \in B_3$ when $y = uv$ where u contains 1s and 2s in any order allowable by the RGF.

To show the equidistribution of the Wachs and White statistics we will identify statistic preserving functions from A_i to B_i. Following this we will show that these functions are bijections.

a. For the proof that $ls(w) = rb(y)$ when $w \in A_1$ and $y \in B_1$ refer to [Dahlberg et al. (2015)].

b. Let $w \in A_2$, $y \in B_2$. Then the following are equations for the statistics of individual words:

$$ls(w) = (z-1)b_z + \sum_{i=2}^{m}(i-1)a_i$$

$$rb(y) = (m-x)b_1 + \sum_{i=1}^{m-1}(m-i)a_i$$

Now let $f : A_2 \to B_2$ where when $w = uv$, $f(w) = y$ such $b_z = b_1$ (where b_z is the number of $z \in v$ and b_1 is the size of the block of inserted 1s in y), the distinct letter $x \in y$ before this block of 1s is $x = m - z + 1$ where z is the last letter in w, and the weakly increasing base of y is formed by taking the RGF of the complement of the partition in a bijection with u. An example would be $f(1234422) = 1123114$. Then $x = 4 - 2 + 1 = 3$ so we insert the block of 1s after the last 3 and there will be $b_2 = 2$ 1s in this block. Also we take the complement of 12344 to get the base of the word to be 11234.

We will prove that f is injective. Consider $w_1, w_2 \in A_2$ where $w_1 \neq w_2$. Then either $u_1 \neq u_2$ or $v_1 \neq v_2$. When $u_1 \neq u_2$ then $u_1^c \neq u_2^c$, so the weakly increasing word in $f(w_1)$ differs from the weakly increasing word in $f(w_2)$ so $f(w_1) \neq f(w_2)$. If $v_1 \neq v_2$ then there will be a different number of zs in w_1 and w_2, so the block of 1s inserted in y will be of different sizes so $f(w_1) \neq f(w_2)$. Therefore when $f(w_1) = f(w_2)$, $w_1 = w_2$, meaning that f is injective.

Let f^{-1} be the inverse of f, where $f^{-1}(y) = w$ is formed by taking the RGF of the complement of the partition corresponding to the weakly increasing base of y. This will be weakly increasing by Lemma 2.6. Then $z \in w$ is determined by $z = m - x + 1$ and $b_1 = b_z$.

Because f is injective and f^{-1} exists, we conclude that f is bijective.

Now consider (notice that c_i denotes the number of the letter i in the weakly increasing base of $f(w)$)

$$rb[f(w)] = (m - x)b_1 + \sum_{i=1}^{m-1}(m - i)c_i$$

$$= [m - (m - z + 1)]b_z + \sum_{i=1}^{m-1}(m - i)a_{m-i+1}$$

$$= (z - 1)b_z + \sum_{i=1}^{m-1}[(m + 1 - i) - 1]a_{m-i+1}$$

$$= (z - 1)b_z + \sum_{i=2}^{m}(i - 1)a_i = ls(w)$$

Therefore because $rb(f(w)) = ls(w)$ and f is bijective, $LS(A_2) \sim RB(B_2)$.

For the rest of the proof see appendix.

$\square$

5.2. Other Equidistribution Results for Single Pattern Avoidance Classes.

The majority of equidistribution results based on single pattern avoidance classes we found focused on the two patterns $13/2/4$ and $1/24/3$. However we have another equidistribution result based on the patterns $124/3$ and $134/2$. From Theorem 2.5 we have $124/3 \equiv 134/2$.

Theorem 5.2 (Equidistribution of Statistics II). *Consider the avoidance classes based on the patterns $124/3$ and $134/2$, then*

$$LS(124/3) \sim LS(134/2)$$

Proof. Let $\phi : R_n(134/2) \to R_n(124/3)$ such that

$$\phi(w) = \begin{cases} w & \text{if } w \text{ is weakly increasing} \\ w* & \text{if } w \text{ is not weakly increasing} \end{cases}$$

where $w*$ is defined to be the word that is formed from flipping each block-singleton letter to a singleton-block letter. We can see that this function is well-defined because there is only one way to flip the block-singleton letter to a singleton-block letter. So $\phi(122233213) = 123222133$ and $\phi(11222133243) = 12113222433$. Now let $w_1, w_2 \in R_n(134/2)$ such that $\phi(w_1) = \phi(w_2)$. There are two options based on what form w_1 takes:

i) Suppose w_1 is weakly increasing. Then $w_1 \in R_n(124/3)$ by Lemma 3.1. Because w_1 is weakly increasing ϕ is the identity function which is clearly bijective.

ii) Suppose w_1 is not weakly increasing. Then w_1 is weakly increasing except for some number of block-singleton letters. There is only one way to flip a block-singleton letter to a singleton block, therefore when $\phi(w_1) = \phi(w_2)$, $w_1 = w_2$. So when w_1 is not weakly increasing ϕ is injective.

Next consider the function $\phi^{-1} : R_n(124/3) \to R_n(134/2)$, where $y \in R_n(124/3)$ and

$$\phi^{-1}(y) = \begin{cases} y & \text{if } y \text{ is weakly increasing} \\ y* & \text{if } y \text{ is not weakly increasing} \end{cases}$$

where we have $y*$ defined as the word that is formed by changing every singleton-block letter to a block-singleton. So $\phi^{-1}(123222133) = 122233213$ and $\phi^{-1}(12113222433) = 11122133243$. So it is clear that ϕ^{-1} reverses ϕ. Therefore because ϕ is injective and there exists ϕ^{-1} which reverses ϕ, ϕ is a bijection between $R_n(124/3)$ and $R_n(134/2)$.

An important thing to note about this bijection is that it preserves the occurrences of each letter that occurs in the input word. Therefore if there are the same occurrences of each letter in w and $\phi(w)$, then $ls(w) = ls(\phi(w))$ by Lemma 2.3. Therefore because there is a bijection between $R_n(134/2)$ and $R_n(124/3)$ that preserves the left and smaller statistic for every word, $LS(134/2) \sim LS(124/3)$. $\square$

6. Avoidance Classes based on Multiple Patterns

The idea of pattern avoidance can be extended to avoidance classes based on multiple patterns. Here in this paper we will investigate both (3,4)-pair avoidance classes and (4,4)-pair avoidance classes. Specifically we will focus on avoidance classes based on at least one pattern of interest from the single pattern results above.

The investigation of these multiple pattern avoidance classes will follow the same structure as the previous results: specifically we will characterize the avoidance classes, find the cardinality of the avoidance classes, and finally investigate the Wachs and White statistics for these avoidance classes.

6.1. **3,4-Pairs.** As we begin to consider avoidance classes determined by pairs of patterns, it is helpful to realize that when we have $R_n(\sigma_1, \sigma_2)$ it will be more interesting when $w(\sigma_2) \in R_n(\sigma_1)$, (where σ_1 is the pattern of length three). Otherwise $R_n(\sigma_1, \sigma_2)$ is already determined by the single pattern avoidance class $R(\sigma_1)$. For example, consider $R_n(1/2/3, 1/2/34)$. Then $w(1/234) = 1222 \in R_n(1/2/3, 1/2/3/4)$ because $w \in R_n(1/2/3)$. And $w(12/3/45) = 11233 \notin R_n(1/2/3, 1/2/34)$ because $w \notin R_n(1/2/3)$. So we have the following property:

Lemma 6.1. *Let $\sigma_1 \in \Pi_k$, $\sigma_2 \in \Pi_l$ where $l < k$, such that σ_2 contains σ_1. Then $R_n(\sigma_1, \sigma_2) = R_n(\sigma_1)$.*

Theorem 6.1. *The word characterization and cardinality of the avoidance class based on the patterns 12/3 and 1/24/3 are given by:*

   (i) $R_n(12/3, 1/24/3) = \{w \in R_n : w = ab$ where a is strictly increasing prefix ending with m, and
   (A) b is a suffix of m's
   (B) b is a suffix of $1s$ where $|b| \geq 1\}$
   (ii) $\#\Pi_n(12/3, 1/24/3) = 2(n-1)$

Proof. Let $w \in R_n(12/3, 1/24/3)$. Then, by Lemma 2.1, because $w \in R_n(12/3)$, the only element repeated is the last element, x, in the word such that x occurs once after $x - 1$ and then in a block at the end of the word. We also know that $w \in R(1/24/3)$. So by Theorem 3.9 w could take three different forms.

a. First let $m \leq 2$. Then if $w \in R_n(12/3)$, m could be 1, or $m = 2$. If $m = 2$ then $w = 12b$ where b is either a block of 1s or a block of 2s by Lemma 2.1.

b. Let w be weakly increasing word with a block of ones of any size inserted between two distinct terms. Then w will contain $12/3$ if we consider the subpartition associated with the subword containing the first $1 \in w$, the first 1 inserted between two distinct letters, and the distinct letter after the block of inserted ones. This subpartition will standardize to $12/3$. So when $w \in R_n(12/3)$ it cannot take this form unless the block of 1s is inserted after the last $m \in w$.

c. Let w begin with 1s and 2s in any order followed by a weakly increasing suffix that begins with the first occurrence of 3. Then if w also avoids $12/3$ w must begin with 12 and the weakly increasing suffix must actually be strictly increasing, except for m which can be a block of any size one or larger by Lemma 2.1.

Therefore $w = ab$ where a is strictly increasing prefix ending with $m - 1$, and b is a suffix of m's or a suffix of 1s where $|b| \geq 1$.

Next consider $\#\Pi(12/3, 1/24/3)$. When $m = 1$ there are $S(n, 1) = 1$ forms w could take. When $m = n$ there are $S(n, n) = 1$ forms w could take. Now consider w where $1 < m < n$. Then there will be one form of a because a is strictly increasing, and two possibilities for the form of b: either b is a block of 1s or a block of ms. Therefore $\#\Pi_n(12/3, 1/24/3) = 2 + 2(n - 2) = 2(n - 1)$. $\square$

While we continue to investigate the word characterization and cardinality for multiple pattern avoidance classes, the proofs for the following results can be found in the appendix.

Theorem 6.2. *The word characterization and cardinality of the avoidance class based on the patterns $1/23$ and $13/2/4$ are given by:*

(i) $R_n(1/23, 13/2/4) = \{w \in R_n :$
   (A) $w = ab$ where a is a prefix of ones of any length and b is strictly increasing beginning with the first occurrence of 2 or b can be empty
   (B) $w = abc$ where a is a prefix of 1s of any length, b is strictly increasing beginning with the first occurrence of 2 and ending with m, and c is a suffix of length one containing a 1}

(ii) $\#\Pi_n(1/23, 13/2/4) = 2(n - 1)$

Therefore $\#\Pi_n(12/3, 1/24/3) = \#\Pi_n(1/23, 13/2/4)$. So these pairs of patterns are Wilf-equivalent which we will write as $(12/3, 1/24/3) \equiv (1/23, 13/2/4)$. Now we will continue to characterize and enumerate $(3, 4)$-pair avoidance classes.

Theorem 6.3. *The word characterization and cardinality of the avoidance class based on the patterns $13/2$ and $123/4$ are given by:*

(i) $R_n(13/2, 123/4) = \{w \in R_n : w$ is weakly increasing and no element except m can be repeated more than twice}

(ii) $\#\Pi_n(13/2, 123/4) = 1 + \sum_{j=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-j}{2} \rfloor} \binom{n-j-i}{i}$

Theorem 6.4. *The word characterization and cardinality of the avoidance class based on the patterns $13/2$ and $1/234$ are given by:*

(i) $R_n(13/2, 1/234) = \{w \in R_n : w = ab$ where a is a block of ones, b is weakly increasing beginning with 2 with no element repeated more than twice}

(ii) $\#\Pi_n(13/2, 1/234) = 1 + \sum_{j=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-j}{2} \rfloor} \binom{n-j-i}{i}$

Remember that the goal of this section is to find $(3, 4)$-pair avoidance classes that are Wilf-equivalent. From Theorems 6.3 and 6.4 we conclude that $(13/2, 123/4) \equiv (13/2, 1/234)$. So these two avoidance classes are candidates for the equidistribution of Wachs and White statistics. Before continuing on to consider these distributions first let us expand our characterization and enumeration of avoidance classes determined by $(4, 4)$-pairs.

6.2. **4,4-Pairs.** For $(4, 4)$-pair avoidance classes, we are no longer concerned about the containment or avoidance of patterns in the pair. Two patterns of the same size cannot contain one another if they are distinct.

Theorem 6.5. *The word characterization and cardinality of the avoidance class based on the patterns $13/2/4$ and $124/3$ are given by:*

(i) $R_n(13/2/4, 124/3) = \{w \in R_n :$
   (A) *w is weakly increasing*
   (B) *$m \leq 2$ and 1 and 2 are singleton-block letters*
   (C) *$m \leq 2$ and 1 is a singleton-block letter and 2 is a block letter*
   (D) *$w = ab$ where a is weakly increasing and b is a block of a letter z such that z is a singleton-block letter in w and $z \leq (m-1)$*
   (E) *$w = ab$ where a weakly increasing from 1 to $m-2$ and b contains the singleton-block letters $(m-1)$ and m such that $b = (m-1)m(m-1)^l m^n l\}$*

(ii) $\#\Pi_n(13/2/4, 124/3) = 2n - 5 + \sum_{m=1}^{n} \binom{n-1}{n-m} + \sum_{|a|=3}^{n-1} \sum_{m=3}^{|a|} \binom{|a|-1}{|a|-m}(m-1) +$

$\sum_{|a|=2}^{n-4} \sum_{m=3}^{|a|+2} \binom{|a|-1}{|a|-(m-2)}(n - |a| - 2)$

Proof. Let $w \in R_n(13/2/4, 124/3)$. Then $w \in R_n(13/2/4)$ and so we know that w could be of three forms:

   a. First let $m \leq 2$. By Theorem 3.9, $w \in R_n(13/2/4)$. However we also want $w \in R_n(124/3)$ so there are some restrictions. By Theorem 3.8 if w is weakly increasing, where $w = 1^{a_1} 2^{a_2}$, or if w consists of only block letters and singleton-block letters, then $w \in R_n(13/2/4, 124/3)$. Therefore 1 can be a singleton-block, or 1 and 2 are singleton-blocks.

   b. Next let $w = ab$ where a is weakly increasing and b is a suffix beginning with m and containing m and $m-1$ in any order after that. By Theorem 3.9, $w \in R_n(13/2/4)$. However by Theorem 3.8, if $w \in R_n(124/3)$ then a must contain only one occurrence of $m-1$, and $b = m(m-1)^l m^n$ or $b = m^{a_m}(m-1)^l$. So either $(m-1)$ and m are singleton-blocks, or only $(m-1)$ is a singleton-block.

   c. Finally let $w = ab$ where a is weakly increasing from 1 to m, and b contains a block of some $z < m-1$. Then by Theorem 3.9, $w \in R_n(13/2/4)$. By Theorem 3.8, if $w \in R_n(124/3)$, then z can only be a letter that occurs exactly once in a. Therefore z is a singleton-block in w.

Therefore the characterization given by Theorem 6.5 describes every possible $w \in R_n(13/2/4, 124/3)$.

Next consider the enumeration of the avoidance class.

(i) Let w be weakly increasing and $1 \leq m \leq n$. Then by Lemma 2.2 there are $\sum_{m=1}^{n} \binom{n-1}{n-m}$ possibilities for w.

(ii) Let $m = 2$ and only 1 be a singleton-block letter. Then there are three assigned positions. We know that the first letter is 1 and the second letter is 2 and the last letter is also 1. Therefore the length of the block of 1s can vary from one to $n - 2$. So there are $n - 2$ possible words for w.

(ii) Let $m = 2$ and 1 and 2 be singleton-block letters. Then $w = 121^l 2^k$ where $l, k > 0$. So l can vary from one to $(n - 3)$ and as l varies, so too will k. So there are $n - 3$ possible words of this form.

(iii) Let $w = ab$ where $m \geq 3$, b is a block of z where z is a singleton-block, and a is weakly increasing. Then $3 \leq m \leq |a|$. Also note that $1 \leq z \leq (m - 1)$. There will be $\sum_{|a|=3}^{n-1} \sum_{m=3}^{|a|} \binom{|a|-1}{|a|-m}(m-1)$ forms of w by Lemma 2.2.

(iv) Let $w = ab$ where $m \geq 3$, a is weakly increasing, and $b = (m-1)m(m-1)^l m^n$. So $2 \leq |a| \leq (n-4)$, $3 \leq m \leq |a| - 2$ and the last two elements $(m-1), m \notin a$, and there are $n - |a| - 2$ possible forms of b. Therefore there are $\sum_{|a|=2}^{n-4} \sum_{m=3}^{|a|+2} \binom{|a|-1}{|a|-(m-2)}(n - |a| - 2)$ ways to form w.

Therefore $\#\Pi_n(13/2/4, 124/3) = 2n - 5 + \sum_{m=1}^{n} \binom{n-1}{n-m} + \sum_{|a|=3}^{n-1} \sum_{m=3}^{|a|} \binom{|a|-1}{|a|-m}(m-1) +$

$\sum_{|a|=1}^{n-4} \sum_{m=3}^{|a|+2} \binom{|a|-1}{|a|-(m-2)}(n - |a| - 2)$. $\qquad\qquad \square$

The proofs for the following results can be found in the appendix.

Theorem 6.6. *The word characterization and cardinality of the avoidance class based on the patterns $13/2/4$ and $134/2$ are given by:*

(i) $R_n(13/2/4, 134/2) = \{w \in R_n :$
    (A) w is weakly increasing
    (B) $m = 2$, 1 a block-singleton
    (C) $m = 2$, 1 and 2 block-singletons
    (D) $w = ab$ where a weakly increasing and b is either the last occurrence of a block-singleton in w, or $(m-1), m \in b$ such that both are block-singleton letters, $(b = (m-1)^l m^k (m-1)m)\}$

(ii) $\#\Pi_n(13/2/4, 134/3) = 2n - 5 + \sum_{m=1}^{n} \binom{n-1}{n-m} + \sum_{|a|=3}^{n-1} \sum_{m=3}^{|a|} \binom{|a|-1}{|a|-m}(m-1) +$

$\sum_{|a|=1}^{n-4} \sum_{m=3}^{|a|+2} \binom{|a|-1}{|a|-(m-2)}(n - |a| - 2)$.

Theorem 6.7. *The word characterization and cardinality of the avoidance class based on the patterns $14/2/3$ and $1/24/3$ are given by:*

(i) $R_n(14/2/3, 1/24/3) = \{w \in R_n :$
    (A) w is weakly increasing
    (B) $m \leq 2$
    (C) $w = ab$ where a is a prefix of 1s and 2s beginning with a 1, and b is a weakly increasing word beginning with the first occurrence of 3}

(ii) $\#\Pi_n(14/2/3, 1/24/3) = 2^{n-1} + \sum\limits_{|b|=1}^{n-2} \sum\limits_{i=1}^{|b|} \binom{|b|-1}{|b|-i}(2^{n-|b|} - 1)$

Theorem 6.8. *The word characterization and cardinality of the avoidance class based on the patterns $14/2/3$ and $13/2/4$ are given by:*

(i) $R_n(14/2/3, 13/2/4) = \{w \in R_n :$
  (A) *w is weakly increasing*
  (B) $m \le 2$
  (C) *$w = ab$ where a is weakly increasing and b begins with the first occurrence of $(m-1)$ and then contains m and $(m-1)$ in any order*$\}$

(ii) $\#\Pi_n(14/2/3, 1/24/3) = 2^{n-1} + \sum\limits_{|a|=1}^{n-2} \sum\limits_{i=1}^{|a|} \binom{|a|-1}{|a|-i}(2^{n-|a|} - 1)$

Theorem 6.9. *The word characterization and cardinality of the avoidance class based on the patterns $1/24/3$ and $134/2$ are given by:*

(i) $R_n(1/24/3, 134/2) = \{w \in R_n :$
  (A) *w is weakly increasing*
  (B) *w is a weakly increasing except for the block-singleton 1*$\}$

(ii) $\#\Pi_n(1/24/3, 134/2) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$

Theorem 6.10. *The word characterization and cardinality of the avoidance class based on the patterns $1/24/3$ and $124/3$ are given by:*

(i) $R_n(1/24/3, 124/3) = \{w \in R_n :$
  (A) *w is weakly increasing*
  (B) *$w = ab$ where a is weakly increasing with only one occurrence of z followed by the suffix, b, where $|b| \ge 1$ and contains $z \le (m-1)$ such that z is a singleton-block in w*$\}$

(ii) $\#\Pi_n(1/24/3, 124/3) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$

Theorem 6.11. *The word characterization and cardinality of the avoidance class based on the patterns $1/24/3$ and $1/23/4$ are given by:*

(i) $R_n(1/24/3, 1/23/4) = \{w \in R_n :$
  (A) $m \le 2$
  (B) *$w = abc$ where a is a prefix of 1s, b is strictly increasing beginning with 2 and ending with $(m-1)$ and c is a suffix of ms or 1s*
  (C) *w is a strictly increasing word with a prefix, a, of 1s where $|a| \ge 1$ and a 1 inserted between two terms*$\}$

(ii) $\#\Pi_n(1/24/3, 1/23/4) = 1 + 2^{n-1} + \sum\limits_{m=3}^{n-1}(nm - m^2)$

Theorem 6.12. *The word characterization and cardinality of the avoidance class based on the patterns $13/2/4$ and $1/23/4$ are given by:*

(i) $R_n(13/2/4, 1/23/4) = \{w \in R_n :$
  (A) $m \le 2$
  (B) *$w = abc$ where a is a prefix of 1s, b is strictly increasing beginning with 2 and ending with $(m-1)$ and c is a block of $z \le m$*$\}$

(ii) $\#\Pi_n(13/2/3, 1/23/4) = 1 + 2^{n-1} + \sum_{m=3}^{n-1}(nm - m^2)$

After this study of $(3, 4)$-pair avoidance classes and $(4, 4)$-pair avoidance classes one finding supported by the cardinalities of these avoidance classes is the following conjecture:

Conjecture 6.1. *Let $\pi_1 \in \Pi_k$ and $\pi_2 \in \Pi_l$ such that $k \leq l$.*

(i) *If $\pi_2, \pi_2^c \in \Pi_l(\pi_1)$, then (π_1, π_2) is Wilf equivalent to (π_1, π_2^c).*
(ii) *If $\pi_2 \in \Pi_l(\pi_1)$ and $\pi_2^c \notin \Pi_l(\pi_1)$, then (π_1, π_2) is Wilf equivalent to (π_1^c, π_2^c).*

7. Statistics across Multiple Pattern Avoidance Classes

The following theorems give formulas for statistics of individual words of some of the avoidance classes based on multiple patterns.

Theorem 7.1. *The Patterns $12/3$ and $1/24/3$*

$$ls(w) = \sum_{i=1}^{m} a_i - m$$

$$rb(w) = \sum_{i=1}^{m-1}(m - i)a_i$$

$$lb = \begin{cases} 0 & \textit{if } w = ab \textit{ where } b \textit{ is a block of } m\textit{'s} \\ (m-1)|b| & \textit{if } w = ab \textit{ where } b \textit{ is a block of } 1\textit{'s} \end{cases}$$

$$rs = \begin{cases} 0 & \textit{if } w = ab \textit{ where } b \textit{ is a block of } m\textit{'s} \\ |a| - 1 & \textit{if } w = ab \textit{ where } b \textit{ is a block of } 1\textit{'s} \end{cases}$$

Proof. The equation for $ls(w)$ is given by Lemma 2.3. The equation for $rb(w)$ is given by Lemma 2.4.

Now consider when $w(\sigma) \in R_n(12/3, 1/24/3)$. Then there are two possibilities for the form of $w(\sigma)$. The first possibility is when $w(\sigma) = ab$ where a is strictly increasing and b is a block of m's of any length by Theorem 6.2. Then $lb(w) = 0$ and $rs(w) = 0$ because w is weakly increasing.

Now consider the other form that w could take, which is $w = ab$ where a is strictly increasing and b is a block of ones of any size. Then a will have no contribution to the left and bigger statistic and every one in b will have $(m-1)$ elements to the left and bigger and there are a total of b ones. Therefore $lb(w) = |b|(m-1)$. Now let $x \in a$ where $x > 0$ then x will be larger than the ones in b. So every element in a other than the first element, the only 1 in a, will have a contribution of one to $rs(w)$. Therefore $rs(w) = |a| - 1$. $\qquad\square$

Theorem 7.2. *The patterns $1/23, 13/2/4$*

$$ls(w) = \sum_{i=1}^{m} a_i - m$$

$$rb(w) = \sum_{i=1}^{m-1}(m - i)a_i$$

$$lb = \begin{cases} 0 & \textit{if the word does not end with a one} \\ m-1 & \textit{if the last letter is a one} \end{cases}$$

$$rs = \begin{cases} 0 & \textit{if the word does not end with a one} \\ \sum\limits_{i=2}^{m} a_i & \textit{if the last letter is a one} \end{cases}$$

8. Multiple Pattern Equidistribution of Statistics

Like the single pattern avoidance classes, the purpose of characterizing and enumerating avoidance classes based on pairs of patterns has been to consider the distribution of Wachs and White statistics across these avoidance classes. Because of the large number of Wilf-equivalent pairs of patterns considered we found a wealth of equidistribution results for avoidance classes based on $(3,4)$-pairs as well as $(4,4)$-pairs.

We have already seen that $(12/3, 1/24/3) \equiv (1/23, 13/2/4)$. Therefore we investigated the distribution of the Wachs and White statistics on these two avoidance classes.

Theorem 8.1. *We have the following equidistribution results for the avoidance classes* $R_n(12/3, 1/24/3)$ *and* $R_n(1/23, 13/2/4)$:

$$RB(12/3, 1/24/3) \sim LS(1/23, 13/2/4)$$
$$RS(12/3, 1/24/3) \sim LB(1/23, 13/2/4)$$

Proof. Let $\phi : R_n(12/3, 1/24/3) \to R_n(1/23, 13/2/4)$ such that for every $w \in R_n(12/3, 1/24/3)$, where $w = ab$ $\phi(w) = b'a'$. Here a is strictly increasing and a' is formed by adding one to every letter in a. Also b (the block of ms in w), becomes b' by ϕ such that $|b'| = |b|$ and b' is a block of 1s. An example is $\phi(1234555) = 1112345$ because $a = 1234$ so $a' = 2345$ and $b = 555$ so $b' = 111$. When $w = abc$ where a is the first 1, b is a strictly increasing subword from 2 to m, and c is a block of 1s, then $\phi(w) = cba$. Then $\phi(1234111) = 1112341$ because $a = 1$, $b = 234$, and $c = 111$.

Now using proof by contradiction we will show that ϕ is injective. Let $w_1, w_2 \in R_n(12/3, 1/24/3)$ such that $w_1 \neq w_2$. Then if $w_1 = a_1 b_1$ and $w_2 = a_2 b_2$ we have $b'_1 a'_1 = \phi(w_1) = \phi(w_2) = b'_2 a'_2$. However $w_1 \neq w_2$; because a is strictly increasing, the only way for this to happen is for $b_1 \neq b_2$. Then $b'_1 \neq b'_2$.

Next consider when $w_1 = a_1 b_1 c_1$ and $w_2 = a_2 b_2 c_2$. Then if $c_1 b_1 a_1 = \phi(w_1) = \phi(w_2) = c_2 b_2 a_2$, the only way for this to occur is when $c_1 = c_2, b_1 = b_2$ and $a_1 = a_2$. Therefore $w_1 = w_2$. Therefore ϕ is injective.

Next consider the inverse of ϕ, $\phi^{-1} : R_n(1/23, 13/2/4) \to R_n(12/3, 1/24/3)$ such that

$$\phi^{-1}(y) = \begin{cases} ab & y = b'a' \\ abc & y = cba \end{cases}$$

Then because ϕ is injective, ϕ^{-1} exists, and ϕ^{-1} reverses ϕ, ϕ a bijection.

Now consider $ls(\phi(w))$, when $w = ab$ then because a' is strictly increasing from 2 to m and b' is a block of 1s, and because a is strictly increasing from 1 to $(m-1)$ and b is a block of ms,

$$ls(\phi(w)) = ls(a') + ls(b') = \sum_{i=2}^{m} a'_i - m + 0 = \sum_{i=1}^{m-1} a_i - m = rb(a) = rb(a) + rb(b) = rb(w)$$

When $w = abc$ then a contributes m to $rb(w)$, and when $y = cba$, a contributes m to $ls(w)$, $rb(c) = 0 = ls(c)$ and $rb(b) = ls(b)$ because b is strictly increasing.

Therefore $RB(12/3, 1/24/3) \sim LS(1/23, 13/2/4)$.

Next consider $rs(w)$ when $w = ab$; then by 7.1 and 7.2 we know that $rs(w) = 0 = lb(\phi(w))$. Also consider when $w = abc$, then $lb(\phi(w)) = m - 1 = |a| + |b| - 1 = rs(w)$. Therefore $RS(12/3, 1/24/3) \sim LB(1/23, 13/2/4)$. $\square$

Theorem 8.2. *The equidistribution of statistics on avoidance classes $R_n(1/24/3, 134/2)$ and $R_n(1/24/3, 124/3)$ are*

$$LS(1/24/3, 134/2) \sim LS(1/24/3, 124/3)$$

$$RS(1/24/3, 134/2) \sim RS(1/24/3, 124/3)$$

Proof. Let $f : R_n(1/24/3, 134/2) \to R_n(1/24/3, 124/3)$ such that

$$f(w) = \begin{cases} w & w \text{ weakly increasing} \\ 12^k 1^l & w = 1^l 2^k 1 \\ 121^l 2^k & w = 1^l 2^k 12 \\ w' & w \text{ weakly increasing except block-singleton 1} \end{cases}$$

where w' is formed by switching the block-singleton to a singleton-block. We know that $f(w_1) = f(w_2)$ because,

    i) When w_1 is weakly increasing, $w_1 = f(w_1) = f(w_2) = w_2$ because we then know that $f(w_2)$ is a weakly increasing word.

    ii) When w_1 contains one or more block-singletons, then $f(w_1)$ is formed by switching the block-singleton to a singleton block without changing anything else in w_1, so the weakly increasing base of w_1 must stay the same, and the length of the block of the block-singleton letters and position of the singleton will become the first position for the block. Therefore if $f(w_1) = f(w_2)$, $w_1 = w_2$.

Therefore f is injective. Next consider f^{-1} where $f^{-1}(y) = y$ when y is strictly increasing or $f^{-1}(y) = y'$ where y' switches every singleton-block to a block-singleton. By the same reasoning as above, f^{-1} is injective. Therefore f is a bijection.

Because f preserves the number of each letter l that occurs in w, by Lemma 2.3, $ls(w) = ls(f(w))$. Therefore $LS(1/24/3, 134/2) \sim LS(1/24/3, 124/3)$. Also by Lemma 2.4, and the fact that 1 contributes nothing to the rs statistic for w, $rs(w) = rs(f(w))$. Therefore $RS(1/24/3, 134/2) \sim LS(1/24/3, 124/3)$. $\square$

Theorem 8.3. *The equidistribution of statistics on avoidance classes $R_n(13/2/4, 134/2)$ and $R_n(13/2/4, 124/3)$ are:*

$$LS(13/2/4, 134/2) \sim LS(13/2/4, 124/3)$$

$$RS(13/2/4, 134/2) \sim RS(13/2/4, 124/3)$$

Proof. Let $\phi : R_n(13/2/4, 134/2) \to R_n(13/2/4, 124/3)$ such that

$$\phi(w) = \begin{cases} w & w \text{ weakly increasing} \\ w' & w \text{ contains at least one block-singleton} \end{cases}$$

where w' is formed by switching each block-singleton to a singleton-block.

Then let $\phi(w_1) = \phi(w_2)$ such that $w_1, w_2 \in R_n(13/2/4, 134/2)$. Consider the following possibilities:

i) If w_1 is weakly increasing, then ϕ is simply the identity. Therefore $w_1 = w_2$.

ii) If w_1 contain block-singletons, then when we "switch" a block-singleton to a singleton-block. This is done by moving the singleton to the position directly after the distinct element originally before the block, and moving the block directly after the distinct element originally before the singleton. There is only one way to do this and it preserves our original weakly increasing base of the word w_1. So if $w'_1 = \phi(w_1) = \phi(w_2) = w'_2$ we know that the only way for $w'_1 = w'_2$ is for $w_1 = w_2$. Therefore ϕ is injective.

By the same reasoning we can show that ϕ^{-1} is injective. Therefore ϕ is a bijection.

Now consider $ls(\phi(w))$. Because ϕ does not change the number of occurrences of each letter, we know that $ls(\phi(w)) = ls(w)$ by Lemma 2.3. Therefore $LS(13/2/4, 134/2) \sim LS(13/2/4, 124/3)$.

Next consider $rb(\phi(w))$. While $\phi(w)$ may or may not be weakly increasing we know that $\phi(w)$ has the same number of descents as w, because we only switched the block-singletons to singleton-blocks and we did not change anything else in the word, therefore $rb(\phi(w)) = rb(w)$. So $RS(13/2/4, 134/2) \sim RS(13/2/4, 124/3)$. $\square$

Theorem 8.4. *The equidistribution of statistics on avoidance classes $R_n(14/2/3, 1/24/3)$ and $R_n(14/2/3, 13/2/4)$ are:*

(1) $LB(14/2/3, 1/24/3) \sim RS(14/2/3, 1/24/3) \sim LB(14/2/3, 13/2/4) \sim RS(14/2/3, 13/2/4)$
(2) $LS(14/2/3, 1/24/3) \sim RB(14/2/3, 13/2/4)$
(3) $RB(14/2/3, 1/24/3) \sim LS(14/2/3, 13/2/4)$

Proof. We refer the reader to the REU paper [Dahlberg et al. (2015)] for the proof of (1). By the proof of Theorem 5.1 we also have proven results (2) and (3). $\square$

9. OPEN PROBLEMS AND FUTURE WORK

After investigating specific avoidance classes for the equidistribution of Wachs and White statistics we have many new ideas for research.

- We would like to generalize our results for the equidistribution of statistics across single pattern avoidance classes to explain why specific avoidance classes have fail to have any equidistribution results of Wachs and White statistics.
- While investigating avoidance classes determined by pairs of patterns we discovered an interesting property of Wilf-equivalence for some of the avoidance classes, Conjecture 6.1, and would like to formally prove this conjecture.
- We found many more equidistribution results for the multiple pattern avoidance classes versus our single pattern avoidance classes in this paper and would like to generalize a theorem about this equidistribution of Wachs and White statistics across avoidance classes based on pairs of patterns.

10. Appendix

Theorem 3.4. *The word characterization and cardinality of the avoidance class based on the pattern* $1/2/34$ *are given by:*

(i) $R_n(1/2/34) = \{w \in R_n :$
    (A) $m \leq 2$
    (B) w is a word of the form $u3v$ where u is a prefix of type $1^{a_1}2^{a_2}$, and v contains b_1 1s, b_2 2s, b_3 4s, and so on such that $b_i \leq 1\}$

(ii) $\#\Pi_n(12/3/4) = 2^{n-1} + n(n-2) - \frac{(n-1)(n-2)}{2} + \sum\limits_{|v|=2}^{n-2} \sum\limits_{j=1}^{i-1} \binom{n-|v|}{2}\binom{|v|}{j} +$
$\sum\limits_{|v|=2}^{n-2} \sum\limits_{j=1}^{i-1} (n-i)\binom{i-1}{j}$

- Claim: If $w(\sigma)$ has at most two blocks, or w is a word of the form $u3v$ where u is a prefix with a_1 one's and a_2 two's in any order allowed by the restricted growth function, and v contains b_1 one's, b_2 two's, b_3 four's, and so on such that $b_i \leq 1$, then σ avoids $1/2/34$.

  Proof.   – i) Let $w(\sigma)$ have $m \leq 2$. Then by the definition of restricted growth function, σ can have no more than two blocks. Then there is no subpartition of σ that will standardize to three blocks, because any subpartition of σ must also only have one or two blocks. Therefore because $1/2/34$ is a pattern with three blocks, there is no way that σ can standardize to the pattern when there are at most two blocks in σ.

  – ii) Let $w(\sigma)$ have the form $u3v$ where u is a prefix with a_1 one's and a_2 two's in any order allowed by the restricted growth function, and v contains b_1 one's, b_2 two's, b_3 four's, and so on such that $b_i \leq 1$. Then by the definition of restricted growth function, every element larger than $a_1 + a_2 + 1$, meaning every element in σ associated with v in the word must be either in a block of its own, or is the only element, after the lone element in the third block, to be in the first or second block because one letter after v can be a one and there can also be a two which would signify that the element relating to the position of the one is in the first block, and similarly with the two. So there are no blocks with two or more elements where there are two smaller elements in σ that are found in two distinct blocks. Therefore no larger element can be repeated twice after two other elements in a subpartition where those two other elements are in separate blocks, so no subpartition of σ will standardize to the pattern. So σ must avoid the pattern $1/2/34$. □

- Claim: If $w(\sigma)$ has more than 2 blocks and is not a word of the form $u3v$ where u is a prefix with a_1 one's and a_2 two's in any order allowed by the restricted growth function, and v contains b_1 one's, b_2 two's, b_3 four's, and so on such that $b_i \leq 1$, then σ contains the pattern $1/2/34$.

  Proof. Let $w(\sigma)$ have more than 2 blocks and not be a word of the form $u3v$ where u is a prefix with a_1 one's and a_2 two's in any order allowed by the restricted growth function, and v contains b_1 one's, b_2 two's, b_3 four's, and so on such that $b_i \leq 1$. That means some element in v is repeated, or

there is more than one 3 because there must be a one and a two found in the word before any larger letters by the definition of the restricted growth function so the u prefix will still be valid no matter the word. Now consider if it is any letter equal to or larger than three that is repeated. If we take the elements in the partition relating to the two repeated letters in w, and the first element in the partition, relating to the first one in the word, along with the smallest element in the second block, relating to the first two in w, then this will give us a subpartition that standardizes to $1/2/34$. If a one is found more than once in v, then if we take the smallest element in the second block, which is associated with the first two in the word, the smallest element in the third block, which is associated with the first three in w, and the elements in σ associated with the first two occurrences of one in v, this subpartition will standardize to $1/2/34$. Finally if a two is found more than once in v, then if we take a subpartition associated with the elements linked to the first two occurrences of two in v, the smallest element, which will be the first one in the word, and the smallest element in the third block, which is associated with the first three in the word, this subpartiton will also standardize to $1/2/34$. So when $w(\sigma)$ has more than two blocks and is not of the form $u3v$, σ will contain $1/2/34$. Therefore for σ to avoid $1/2/34$ it must either have no more than two blocks or be of the form $u3v$.

$\square$

Theorem 3.5. *The word characterization and cardinality of the avoidance class based on the pattern $1/234$ are given by:*

(i) $R_n(1/234) = \{w \in R_n : w$ is of the form uv such that u is of the form $1^{a_1}2$ and v has b_1 $1s$, b_2 $2s$, and so on, such that $b_1, b_3, ..., b_m \leq 2$ and $b_2 \leq 1\}$

(ii) $\#\Pi_n(1/234) = 1 + \sum\limits_{r=1}^{\lfloor \frac{n}{2} \rfloor} \prod\limits_{j=0}^{r-1} \binom{n-2j}{2} + \sum\limits_{i=1}^{n-2} \sum\limits_{r=1}^{\lfloor \frac{n-i}{2} \rfloor} \prod\limits_{j=0}^{r-1} r\binom{n-2j-i}{2}$

- Claim: If $w(\sigma)$ is of the form uv such that u is of the form $1^{a_1}2$ and v has b_1 ones, b_2 twos, and so on, such that $b_1, b_3, ..., b_m \leq 2$ and $b_2 \leq 1$, then σ avoids $1/234$.

  Proof. Let $w(\sigma)$ be of the form uv such that u is of the form $1^{a_1}2$ and v has b_1 ones, b_2 twos, and so on, such that $b_1, b_3, ..., b_m \leq 2$ and $b_2 \leq 1$. Then by the definition of restricted growth function, no block after the first is larger than size three, and there are no more than two elements in the first block that are larger than any element in a different block. So let us consider taking three elements from one block, this must be from the first block because all other blocks cannot have more than two elements. To get a subpartition to standardize we must have some element smaller than these three elements and in a different block. There is no way to do this because while there are at most two elements in our block of there elements with at least one element in a different block, the other element must only have elements from the first block come before it. Therefore σ must avoid $1/234$ because there is no block of three larger elements when $w(\sigma)$ is of this form. $\square$

- Claim: If $w(\sigma)$ is not of the form uv such that u is of the form $1^{a_1}2$ and v has b_1 ones, b_2 twos, and so on, such that $b_1, b_3, ..., b_m \leq 2$ and $b_2 \leq 1$, then σ contains 1/234.

  Proof. Let $w(\sigma)$ not be of the form uv such that u is of the form $1^{a_1}2$ and v has b_1 ones, b_2 twos, and so on, such that $b_1, b_3, ..., b_m \leq 2$ and $b_2 \leq 1$. Then some $b_i > 2$ or $b_2 > 1$. So if we take the first element in the partition along with three elements from the block associated with b_2 and standardize this subpartition we will get 1/234 using the definition of the restricted growth function. Also if we take the first element in the second block, associated with the first two in the word, and then take the first three elements in v associated with the $b_i > 2$, this will standardize to 1/234. Therefore σ contains 1/234. So if σ avoids 1/234 then $w(\sigma)$ is of the form uv such that u is of the form $1^{a_1}2$ and v has b_1 ones, b_2 twos, and so on, such that $b_1, b_3, ..., b_m \leq 2$ and $b_2 \leq 1$. $\square$

Theorem 3.6. *The word characterization and cardinality of the avoidance class based on the pattern 123/4 are given by:*

(i) $R_n(123/4) = \{w \in R_n:$ *w is of the form uv where u has a_1 1s, a_2 2s, ..., a_m ms such that $a_1, ..., a_m \leq 2$ and v can be any size of one letter repeated$\}$*

(ii) $\#\Pi_n(123/4) = 1 + \sum\limits_{r=1}^{\lfloor \frac{n}{2} \rfloor} \prod\limits_{j=0}^{r-1} \binom{n-2j}{2} + \sum\limits_{i=1}^{n-2} \sum\limits_{r=1}^{\lfloor \frac{n-i}{2} \rfloor} \prod\limits_{j=0}^{r-1} r\binom{n-2j-i}{2}$

- Claim: If w is of the form uv where u has a_1 ones, a_2 twos, ..., a_m m's such that $a_1, ..., a_m \leq 2$ and v can be any size of one letter repeated, then σ avoids 123/4.

  Proof. Let w is of the form uv where u has a_1 ones, a_2 twos, ..., a_m m's such that $a_1, ..., a_m \leq 2$ and v can be any size of one letter repeated. Then by the definition of restricted growth function there are no blocks of size three or larger unless it is the block with the largest element in σ. Therefore if you try to take three elements in one block together then there is no element that is larger because the block of three or more that we took these three elements from has the largest element in it so no subpartition will standardize to 123/4. Therefore when w is of the form uv, σ avoids 123/4. $\square$

- Claim: If w is not of the form uv where u has a_1 ones, a_2 twos, ..., a_m m's such that $a_1, ..., a_m \leq 2$ and v can be any size of one letter repeated, then σ contains 123/4.

  Proof. Let w not be of the form uv where u has a_1 ones, a_2 twos, ..., a_m m's such that $a_1, ..., a_m \leq 2$ and v can be any size of one letter repeated. Then v must contain more than one distinct letter because v begins at the third occurrence of any letter in the word. This means that there is one block of size three followed by at least one more larger element that must be in a different block, by the definition of restricted growth function. Therefore if those three elements from the same block and the larger element found in a separate block are in a subpartition, they will standardize to 123/4. Therefore when $w(\sigma)$ does not follow this given form, σ contains 123/4. So when σ avoids 123/4, $w(\sigma)$ must be of the form uv. $\square$

- Cardinality

Proof. From Sagan we know that the cardinality of the avoidance class of a pattern is equal to the cardinality of the avoidance class of the complement of that pattern. Therefore $\#\Pi_n(123/4) = \#\Pi_n(1/234)$. Now consider $\pi = 123/4$. Then any word that avoids this will be of the form uv where u contains a_i i's in any order allowed by the restricted growth function, such that $a_1, ..., a_m \leq 2$ and v can be any size of one element repeated. Then there is one way to have every $a_i = 1$, this is when u is strictly increasing and $|u| = n$. Now when v is empty we can have letters repeated at most once. This means that we can have between one and $\lfloor \frac{n}{2} \rfloor$ pairs of letters which also means by the definition of the restricted growth function that we can have at most two elements in every block of the set partition, so we cannot have more than half the number of elements as the number of pairs. Now there are $\binom{n}{2}$ ways to choose the two elements that will appear in the same block together when there is only one pair. Next we must pick the next two elements to appear in the same block together, however two elements are already assigned so we will have $n-2$ options and 2 elements must be chosen we will multiply our first result with this result to get the total number of ways to have two pairs with all other elements in blocks of size one in the partition. This will continue for every possible number of pairs, so we will get the product of these combinations where we stop the product at each possible number of pairs and add all of these products together. Next consider when v is nonempty. There are one to $\lfloor \frac{n}{2} \rfloor$ possible values for the letter in v. Because v begins at the third repeat of any one letter it must be one of the letters that is repeated in u, v can be any length from one to $n-3$ because the smallest u possible is 122 where v contains twos. So we will have the summation of the product again, this time however let r be the letter in v, we will multiply the combination by r to get the number of ways to have a specific number of pairs with every possible value of the letter in v.

$\square$

Theorem 3.9. *The word characterization and cardinality of the avoidance class based on the pattern* $13/2/4$ *are given by:*

(i) $R_n(13/2/4) = \{w \in R_n$:

    (A) $m \leq 2$, or

    (B) $m > 3$ and $w = uv$ where u is weakly increasing and v begins at the first m and contains m and $(m-1)$ in any order, or

    (C) $m > 3$ and $w = uv$ where u is weakly increasing and v begins at the first m and contains a block of ms followed by a block of zs where $z < (m-1)\}$

(ii) $\#\Pi_n(13/2/4) = 2^{n-1} + \sum\limits_{m=3}^{n} \binom{n-1}{n-m} + \sum\limits_{m=3}^{n-1} \sum\limits_{|u|=m-1}^{n-2} \binom{|u|-1}{|u|-m+1} [(2^{(n-|u|-1)} - 1) + (m-2)(n-|u|-1)]$

- Claim: w has some letters a, b, and c such that these letters appear in the order $abac$ where other letters may come between if and only if w contains $13/2/4$.

    Proof.   – Let w have some letters a, b, and c such that these letters appear in the order $abac$ where other letters may come between. Then

by the restricted growth function if we take the elements associated with these letters as a subpartition, we will have the smallest element along with the third smallest element in one block, the second smallest element in its own block, and the largest element in a separate block. This means that this subpartition will standardize to $13/2/4$.

– Let w contain $13/2/4$. Then a subpartition of σ must standardize to $13/2/4$. Which would mean that there must be two elements in a block together where at least one element is greater than one of these two elements and smaller than the other and is in a different block, and there is an element larger than all other elements so far that is in another different block. Let the first block of this subpartition be the ath block, the next block be the bth block, and the block containing the largest element of the subpartition will be the cth block. Then by the definition of the restricted growth function $abac$ must occur in the word w but other letters can come between them. Therefore when w contains $13/2/4$, w must contains $abac$.

$\square$

• Claim: If $m \leq 2$, w is weakly increasing, w takes on the form uv where u is weakly increasing and v begins at the first m and contains m and $(m-1)$ in any order, or v begins at the first m and contains a block of $m's$ followed by a block of $z's$ where $z \leq (m-1)$, then σ avoids $13/2/4$.

Proof. – i) Let $m \leq 2$. Then this means that $w(\sigma)$ can have no letter larger than two, and σ can have at most two blocks by the definition of the restricted growth function. Then because the pattern $13/2/4$ has three blocks, there will be no subpartition of σ that standardizes to the pattern. Therefore when $w(\sigma)$ has no letter larger than two, σ avoids $13/2/4$ because $13/2/4$ has three blocks.

– ii) Let $w(\sigma)$ be of the form uv where u is weakly increasing and v begins with the first m and contains a block of a_m $m's$ followed by a block of b_z $z's$, where $z \leq (m-1)$. Then while the word allows for one repeated letter to have a different letter come in between them, there will be no distinct letter that occurs afterward, because no different unique letter comes after the first z in v. Therefore w contains no letters which follow $abac$. Therefore when $w(\sigma)$ is of this form, σ avoids $13/2/4$.

– iii) Let $w(\sigma)$ be of the form uv where u is weakly increasing and v begins with the first m, then contains m and $(m-1)$ in any order. Then while there may be two elements with the same letter with a unique letter coming between the two, if an element comes after these letters that is distinct from the repeated letter, it cannot be distinct from the letter that came between these repeated letters. Therefore no part of w contains $abac$, so no subpartition of σ will standardize to $13/2/4$. So when w is of this form, σ avoids $13/2/4$.

$\square$

• Claim: If $w(\sigma)$ has elements larger than two and does not follow the form uv where v begins at the first m and contains a block of m followed by a block of z where $z < (m-1)$, or uv where v begins at the first m and contains m and $(m-1)$ in any order, then σ contains $13/2/4$.

Proof. – i)Let w have more than two letters and be of the form uv where v begins at the first m and contains m and z in any order after this where at least one m occurs after z, where $z \leq (m-1)$. Then if we take the first occurrence of whatever z is in u, the next distinct letter in u which will be have the value $z+1$ cannot be m because $z < (m-1)$, and the first occurrence of m after the first z in v we can see that there are three distinct letters forming $z(z+1)zm$ which follows $abac$ where other letters can come between these letters but they must occur in this order. Therefore this means that σ must contain $13/2/4$.

– ii)Let v contain more than two distinct letters from u. Now consider taking the first occurrence in u of the second distinct letter in v, the first m in v, the first occurrence of the second unique letter in v, and the first occurrence of the third unique letter in v. This will be four letters, three of which are distinct, that follow $abac$, meaning that σ must contain $13/2/4$.

– iii) Let u not be weakly increasing. This means that there must be at least one descent in $w(\sigma)$. So if we take the first letter occurring out of order, the first occurrence of this letter in u, a distinct letter that comes between these two, and another distinct letter that follows after such as m, this will follow $abac$ and therefore σ must contain $13/2/4$.
□

- Cardinality

Proof. – Let $m \leq 2$. Then by the Stirling numbers of the second kind, there are $S(n,1)+S(n,2) = 1+2^{n-1}-1 = 2^{n-1}$ different set partitions.

– Let $w(\sigma)$ be of the form uv where v is empty, then $w(\sigma)$ is weakly increasing and there must be at least one element in each block of the set partition where $3 \leq m \leq n$. Therefore m elements are taken and there are $n-m$ elements left to assign a block. Because we know that w is weakly increasing, the order these elements are assigned does not matter, for we are only interested in how large each block is. Therefore we will use the idea that there are $n-m$ elements and m blocks to choose from where order does not matter and repetition is allowed, so there will be $\binom{n-m+m-1}{n-m} = \binom{n-1}{n-m}$ different set partitions associated with weakly increasing words.

– Let $w(\sigma)$ be of the form uv where there are two distinct letters in v. Then let us begin by focusing on u which can be between $m-1$ and $n-2$ where m is between 3 and $n-1$ because at least one letter in the word is repeated. Then we know that u must contain at least one letter between one and $m-1$. So there will be $|u|-m+1$ other letters in u to assign a name. Because u must be weakly increasing we again do not care which order these letters are assigned names in, all we care is learning how many times each letter is repeated. There are $|u|-m+1$ letters to assign names, $m-1$ possible names to assign, order does not matter and repetition is allowed, therefore $\binom{|u|-m+1+m-1-1}{|u|-m+1} = \binom{|u|-1}{|u|-m+1}$ different $u's$ possible for that specific length of u.

  * Let v begin with the first m and contain a block of $m's$ followed by a block of $z's$ where z is any letter less than $m-1$. Then

         there are $n - |u|$ letters in v and there are $m - 2$ options for z. Because v has two blocks, we are also interested in the size of the blocks. For each size of v there can be a block of $z's$ of size one to $n - |u| - 1$. So there are $(m-2)(n - |u| - 1)$ different $v's$ of this type possible for each length of v.

     ∗ Let v begin with the first m and contain m and $(m-1)$ in any order after this. Then there are $n - |u|$ letters which can either be m or $(m-1)$ where we know the first is m. So by the Stirling numbers of the second kind there are $S(n - |u|, 2) = 2^{n-|u|-1} - 1$ possible $v's$.

$\square$

Theorem 3.10. *The word characterization and cardinality of the avoidance class based on the pattern $1/24/3$ are given by:*

   (i) $R_n(1/24/3) = \{w \in R_n :$

     (A) $m \leq 2$, or

     (B) $m > 3$ and w is a weakly increasing word with a block of $1s$ of any size inserted between two distinct terms, or

     (C) $m > 3$ and w is a word beginning with $1s$ and $2s$ in any order followed with a weakly increasing suffix that begins with the first occurrence of $3\}$

   (ii) $\#\Pi_n(1/24/3) = 2^{n-1} + \sum_{m=3}^{n} \binom{n-1}{n-m} + \sum_{m=3}^{n-1} \sum_{|u|=m-1}^{n-2} \binom{|u|-1}{|u|-m+1}[(2^{(n-|u|-1)} - 1) + (m-2)(n - |u| - 1)]$

- Claim: If $m \leq 2$, a weakly increasing word with a block of ones of any size inserted between two distinct terms, or a word beginning with one's and two's in any order followed with a weakly increasing suffix that begins with the first occurrence of three, then σ avoids $1/24/3$.

   Proof.   – i) Let $m \leq 2$. Then there are no more than two blocks in σ by the definition of the restricted growth function, and no partition with only two blocks can have a subpartition with more than two blocks, meaning that σ must avoid any pattern that has three or more blocks. Therefore σ avoids $1/24/3$ when $w(\sigma)$ has no letter larger than two.

     – ii) Let $w(\sigma)$ be weakly increasing. Then there are no descents in $w(\sigma)$ meaning that every block in σ only contains letters which are consecutive in each respective block. Therefore σ cannot contain $1/24/3$ because there is no possible way to have two elements in one block with another element that smaller than one of these and larger than the other in a separate block.

     – iii) Let $w(\sigma)$ be weakly increasing with a block of ones inserted between two distinct letters. Then the only block that can contain nonconsecutive letters in σ is the first block by the restricted growth function. However, because no element is in a block before the first block, there can be no subpariiton that standardizes to $1/24/3$. To standardize to this pattern there would need to be a distinct letter that occurred before the block with nonconsecutive letters. Therefore σ avoids $1/24/3$

     – iv) Let $w(\sigma)$ contain one's and two's followed by a weakly increasing suffix which begins with the first three. Then because the only descents

can occur with the one's and two's, while an element may be smaller than either elements in a different block, and these elements in this block may not be consecutive, the element that comes between the two in this block is found in the other. Therefore, σ must avoid $1/24/3$

□

- Claim: If $w(\sigma)$ has more than two distinct letters and is not weakly increasing with only one block of ones inserted between two distinct terms and does not contain one's and two's followed by a weakly increasing suffix that begins with the first three, then σ contains $1/24/3$.

  Proof. – i) Let $w(\sigma)$ be weakly increasing with more than one block of ones inserted between distinct letters. Then by the definition of the restricted growth function, if an element from the second block is taken which will be associated with a two in the word, one element from the first block in σ which is associated with the first group of inserted ones so the element must be larger than that associated with the two in the word, one element associated with a letter that comes between the two separate groups of inserted ones, which will be larger than the element from the first block, and finally one element from the first block of the partition that is associated with a letter from the last group of inserted ones which will be the largest element in the subpartition. Then if we standardize this subpariton we will have a small element in its own block, the second smallest and the largest elements in a different block, and the third smallest element in its own block. This standardizes to $1/24/3$. Therefore σ contains $1/24/3$.
  – ii) Let $w(\sigma)$ be weakly increasing with a block of ones inserted between two of the same letter where this letter is larger than two. Then using the definition of the restricted growth function, if one element associated with the distinct letter that occurs directly before this group of ones, the element that is associated with the letter that occurs directly after this group of ones, the element associated with the first two in the word, and the element associated with the first of the inserted ones in the word are taken as a subpartition of σ this will standardize to $1/24/3$. This is because the smallest element is in its own block in the subpartition, this is the element in the second block of σ, then the second smallest element and the largest elements are in a block together, and finally the third smallest element is in a block by itself.
  – iii) Let $w(\sigma)$ have one's and two's followed by a suffix that begins with the first three and is not weakly increasing. Then by the definition of the restricted growth function, consider the subpartition based on the first element that is associated with the letter that occurs out of order in the suffix of the word where this letter is greater than one, the element associated with the first occurrence of this letter in the word, the element associated with the letter that comes directly before the first letter that occurs out of order, and an element associated with the first one in the word. Then by the definition of standardization this subpartition becomes $1/24/3$. Now if the letter that occurs out of order is a one in the suffix, then take the element associated with the first one in the suffix, take the element associated with the first one

after a two in the prefix of the word, also take the element associated with the first two in the word, and the element associated with the first three in the word. This will also standardize to $1/24/3$. Therefore σ contains $1/24/3$.

□

Theorem 4.2 (Left Statistics for $R_n(1/2/3/4)$)**.** *If* $w \in R_n(1/2/3/4)$, *then*

$$ls(w) = \begin{cases} 2h_3 + h_2 & w \in A \\ a_2 + b_2 + 2b_3 + c_2 + 2c_3 & w \in B \text{ or } C \end{cases}$$

$$lb(w) = \begin{cases} 0 & w \in A \\ x + 2y + 2z + a_1 + 2b_1 + 2c_1 & w \in B \text{ or } C \end{cases}$$

There are two clear cases, either the word is weakly increasing, otherwise there is one or more descents, which will cause the word to be of the form $1^{a_1}w_1 1^{a_2}w_2 1^{a_3}w_3 1^{a_4}$ where w_1 has b_1 one's and b_2 two's where w_1 must begin and end with a two, w_2 has c_1 one's, c_2 two's, c_3 three's such that w_2 begins with the first three and ends with the last two or with the last three, whichever comes first, and finally w_3 begins with a three (two) and ends with a three (two) such that w_3 has d_1 one's, (d_2 two's), and d_3 three's.

- Left and Smaller

$$ls = \begin{cases} 2h_3 + h_2 \\ b_2 + c_2 + 2c_3 + d_2 + 2d_3 \end{cases}$$

Proof. Let $m \leq 3$ where w is weakly increasing. Then there must be two elements to the left and smaller than each 3 found in $w(\sigma)$ because a one and a two must occur at least once before any three appears. There must be one elements to the left and smaller than each 2, because a one must appear once before any 2 appears by the definition of the restricted growth function. Therefore the left and smaller statistic will be $2h_3 + h_2$ where h_3 is the number of threes and h_2 is the number of ones. □

Proof. Let $w(\sigma)$ be of the form $1^{a_1}w_1 1^{a_2}w_2 1^{a_3}w_3 1^{a_4}$. There are b_2 two's in w_1, c_2 two's in w_2, and d_2 two's in w_3. There must be at least a one that occurs before any two appears, therefore there will be $b_2 + c_2 + d_2$ elements with a smaller element that occurs before it. There are c_3 three's in w_2 and d_3 three's in w_2. There must be two smaller elements, a one and two must occur before any three appears in the word, therefore the three's contribute $2c_3 + 2d_3$. Therefore the left and smaller statistic will be $b_2 + c_2 + d_2 + 2c_3 + 2d_3$. □

- Left and Bigger

$$lb = \begin{cases} 0 \\ a_2 + 2a_3 + 2a_4 + b_1 + 2c_1 + 2d_1 \end{cases}$$

Proof. – Let $w(\sigma)$ be weakly increasing. Then there are no descents in the word, so no letter in the word will have another letter occur to the left and be larger. Therefore $lb = 0$. Let $w(\sigma)$ be of the form $1^{a_1}w_1 1^{a_2}w_2 1^{a_3}w_3 1^{a_4}$, then for the left and bigger statistic, every one that occurs after the first two contributes to the statistic. There are

$a_2 + a_3 + a_4 + b_1 + c_1 + d_1$ ones that occur after the first two. Every one that occurs after the first three also contributes one to the statistic. There are $a_3 + a_4 + c_1 + d_1$ ones that occur after the first three. Every two that occurs after the first three also contributes one to the statistic. There are $c_2 + d_2$ twos that occur after the first three. Therefore the right and bigger statistic is $a_2 + 2a_3 + 2a_4 + b_1 + 2c_1 + c_2 + 2d_1 + d_2$.
$\qquad\square$

Theorem 4.3 (Right Statistics for $R_n(1/2/3/4)$). *If $w \in R_n(1/2/3/4)$, then*

$$rb(w) = \begin{cases} (m-1)h_1 + (m-2)h_2 & w \in A \\ 2a_1 + a_2 + b_2 & w \in B \text{ or } E \\ 2a_1 + a_2 + b_1 & w \in C, D, F \text{ or } G \\ a_1 & w \in H \text{ or } I \end{cases}$$

$$rs(w) = \begin{cases} 0 & w \in A \\ a_2 + 2a_3 + b_3 & w \in B, C, E \text{ or } F \\ a_2 + 2a_3 + b_2 & w \in D \text{ or } F \\ a_2 & w \in H \text{ or } I \end{cases}$$

- Right and Bigger

  Proof. Let $w(\sigma)$ have no more than three blocks that is weakly increasing. Then there are $(m-1)$ distinct letters larger than each one and there are h_1 ones. Then there are $(m-2)$ distinct letters larger than each two and there are h_2 twos. Therefore the right and bigger statistic is $(m-1)h_1 + (m-2)h_2$. $\qquad\square$

  Proof. Let the last two be found in w_1 and the last three be found in w_2. Then there are $a_1 + b_1$ ones that come before the last two, $a_1 + a_2 + b_1 + c_1$ ones that come before the last three, and $b_2 + c_2$ twos that come before the last three. Therefore the right and bigger statistic will be $2a_1 + a_2 + 2b_1b_2 + c_1 + c_2$. $\qquad\square$

  Proof. Let the last two be found in w_2 and the last three be found in w_3. Then there are $a_1 + a_2 + b_1 + c_1$ ones that occur before the last two, $a_1 + a_2 + a_3 + b_1 + c_1 + d_1$ ones that occur before the last three, and $b_2 + c_2$ twos that occur before the last three. Therefore the right and bigger statistic will be $2a_1 + 2a_2 + a_3 + 2b_1 + b_2 + 2c_1 + c_2 + d_1$. $\qquad\square$

  Proof. Let the last two be found in w_3 and the last three be found in w_2. Then there are $a_1 + a_2 + a_3 + b_1 + c_1 + d_1$ ones that occur before the last two, $a_1 + a_2 + b_1 + c_1$ ones that occur before the last three, and $b_2 + c_2$ ones that occur before the last three. Therefore the right and bigger statistic will be $2a_1 + 2a_2 + a_3 + 2b_1 + b_2 + 2c_1 + c_2 + d_1$. $\qquad\square$

- Right and Smaller
  - Contribution based on the last one:
    * If the last one occurs before w_1, then there will be no contribution to the rs statistic.

      Proof. Let the last one be before w_1. Then there are no ones that occur after any two, and there are no ones that occur after

any three, therefore there will be no ones to the left and smaller than any letter in the word. So there will be no contribution to the rs statistic based on the last one.   □

* If the last one occurs in w_1, then this will cause a contribution of $p_1 - b_1$ where p_1 is the position of the last one in w_1.

  Proof. Let the last one occur in w_1. When the last one is in w_1 there will be the position that last one occurred in relation to w_1, denoted by p_1, minus the number of ones that occurred in w_1, denoted by b_1, $p_1 - b_1$ will give us the number of two's that occurred before the last one, each of these two's will contribute one to the right and smaller statistic. Therefore when the last one is in w_1 there are $p_1 - b_1$ contributions to the right and smaller statistic.   □

* If the last one occurs between w_1 and w_2, then this will cause a contribution of b_1 to the rs statistic for the word.

  Proof. Let the last one occur between w_1 and w_2. There will be b_2 two's that occur before this last one, so there are b_2 objects with something smaller that occurs after them. Therefore each of these two's will contribute one to the right and smaller statistic, so when the last one occurs between w_1 and w_2 this causes b_2 to be contributed to the rs statistic.   □

* If the last one occurs in w_2, then this will cause $b_2 + p_2 - c_1$ to be contributed to the rs statistic.

  Proof. Let the last one occur in w_2, then there will be b_2 two's in w_1 that occur this last one and will each contribute one to the rs statistic. There are also $p_2 - c_1$ two's and three's in w_2 that occur before the last one because there are a total of c_1 ones in w_2 and the last one is in position p_2 so there will be $p_2 - c_1$ other letters that are larger than one and come before the last one in the word, each of these will also contribute one to the right and smaller statistic. Therefore this contributes $b_2 + p_2 - c_1$.   □

* If the last one occurs after w_2 but before w_3, then this will contribute $b_2 + c_2 + c_3$ to the rs statistic.

  Proof. Let the last one occur between w_2 and w_3. Then there are b_2 two's that occur in w_1 and each of these occur before the last one. There are also c_2 two's that occur in w_2 that also occur before the last one, and each of these will contribute one to the right and smaller statistic. Finally there are c_3 three's that occur before the last one and each of these will contribute one to the right and smaller statistic.   □

* If the last one occurs in w_3, then this will contribute $b_2 + c_2 + c_3 + p_3 - d_1$.

Proof. Let the last one occur in w_3. Then there are $b_2 + c_2$ two's that occur in w_1 and w_2 that must occur before the last one, each will contribute one to the right and smaller statistic. There are c_3 three's that occur in w_2 which all occur before the last one, so each of these contribute one to the right and smaller statistic. There are also $p_3 - d_1$ two's and three's that occur before the last one in w_3, where p_3 is the position of the last one in w_3 and there are d_1 ones in w_3 so there are $p_3 - d_1$ two's and three's that occur before the last one. $\qquad\square$

* If the last one occurs after w_3, then this causes a contribution of $b_2 + c_2 + c_3 + d_2 + d_3$ to the right and smaller statistic.

Proof. Let the last one occur after w_3. Then there are $b_2 + c_2 + d_2$ two's that occur before the last one, so each of these will contribute one to the right and smaller statistic. There are also $c_3 + d_3$ three's that occur before the last one and each of these will contribute one to the right and smaller statistic. $\qquad\square$

– Contribution based on the last two:
  * If the last two is in w_1, then this will cause a contribution of 0 to the right and smaller statistic.

  Proof. Let the last two occur in w_1. There are no three's that occur before or in w_1 so there are no letters that are larger than two to contribute to the right and smaller statistic. $\qquad\square$

  * If the last two is in w_2, then this will cause a contribution of c_3 to the right and smaller statistic.

  Proof. Let the last two be in w_2, then the only larger letter could be a three. We know that w_2 will end with the last two by how we defined the word. So there are a total of c_3 three's in w_2 that must occur before the last two. Each of these will contribute one to the right and smaller statistic. Therefore when the last two is in w_2 this contributes c_3 to the rs statistic. $\qquad\square$

  * If the last two is in w_3, then this will cause a contribution of $c_3 + d_3$ to the right and smaller statistic.

  Proof. Let the last two be in w_3. Then there are c_3 three's in w_2 which must occur before it, because two is present in w_3 we know that d_3 must be zero. Therefore there are $c_3 + d_3$ elements that will contribute one each to the right and smaller statistic. $\qquad\square$

Theorem 4.4 (Statistics for $R_n(1/2/34)$). *If $w \in R_n(1/2/34)$, then for the statistics on w when $w \in A$ are given by Theorem 4.1, and for the other cases:*

$$ls = a_2 + b_2 + 2 + \sum_{i=3}^{m-1} i$$

$$rb = \begin{cases} (m-1)a_1 + (m-2)a_2 + |v| - y + \sum_{i=3}^{m}(m-i) & w \in C \\ (h-a_2)(m-1) + (m-2)(|u|-h) + (m-2)a_2 + |v| - x + \sum_{i=3}^{m}(m-i) & w \in B \text{ or } D \\ (m-1)a_1 + (m-2)a_2 + 2|v| - x - y + \sum_{i=3}^{m}(m-i) & w \in E \\ (m-1)a_1 + (m-2)a_2 + 2|v| - x - y - 1 + \sum_{i=3}^{m}(m-i) & w \in F \end{cases}$$

$$lb = a_1 - z + x + y + 1$$

$$rs = \begin{cases} y - 1 & w \in D \\ a_2 + x + y - 1 & w \in E \\ a_2 + x + y & w \in F \\ (l - a_1) + y & w \in B \text{ or } C \end{cases}$$

- Left and Smaller:

$$ls = a_2 + b_2 + 2 + \sum_{i=3}^{m-1} i$$

Proof. Let $w(\sigma)$ be of the form $u3v$. Then there will be $a_2 + b_2$ two's after the first one. There is also a three and a two and a one must come before that three, so this will contribute 2 to the left and smaller statistic. For every letter larger than three there will be one less than that element total distinct letters smaller. Therefore $ls = a_2 + b_2 + 2 + \sum_{i=3}^{m-1} i$. □

- Right and Bigger

$$rb = \begin{cases} (m-1)a_1 + (m-2)a_2 + |v| - y + \sum_{i=3}^{m}(m-i) \\ (h-a_2)(m-1) + (m-2)(|u|-h) + (m-2)a_2 + |v| - x + \sum_{i=3}^{m}(m-i) \\ (m-1)a_1 + (m-2)a_2 + 2|v| - x - y + \sum_{i=3}^{m}(m-i) \\ (m-1)a_1 + (m-2)a_2 + 2|v| - x - y - 1 + \sum_{i=3}^{m}(m-i) \end{cases}$$

Proof. Let w be of the form $u3v$.
  – Let $b_1 = 0$ and $b_2 = 1$. Then there is a two present in v but no one in v. Then there will be $(m-1)$ letters to the right and bigger than each one present in the word, because there is a two present in v we know that a two will be present after every one in the word so the last two in u does not impact this contribution to our statistic, there are a total of a_1 ones. There are also $(m-2)$ letters to the right and bigger than each two present in u and a_2 two's in u. Because there is a two in v let y denote the placement of this 2. Then there are $|v| - y$ distinct letters that come after the two in v that must be larger than two. Also every letter three or larger has $m-i$ letters larger than it where i is the letter value. Therefore $lb = (m-1)a_1 + (m-2)a_2 + |v| - y + \sum_{i=3}^{m}(m-i)$.

– Let $b_2 = 0$ and let h be the position of the last 2 in u. Then there are $h - a_2$ ones that occur before the last two in u. Each of these one's will have $(m - 1)$ letters to the right and bigger than 1 in the word. There will be $(|u| - h)$ ones that occur in u after the last two, these ones will have the letters three to m larger than them and occurring to the right, so each one has $(m - 2)$ letters to the right and bigger. Now there will be a_2 two's in the word and there are a total of $m - 2$ distinct letters to the right and bigger. Next there are a total of $|v| - x$ letters that occur after the one in v, all of these must be distinct and none can be one, therefore each is to the right and bigger than this one. Finally all other letters between three and m will have $m - i$ letters larger than them where i is the value of the letter and each letter between three and m can only occur once in the word, therefore this contributes $\sum_{i=3}^{m}(m - i)$ to the right and bigger statistic.

So $rb = (h - a_2)(m - 1) + (m - 2)(|u| - h) + (m - 2)a_2 + |v| - x + \sum_{i=3}^{m}(m - i)$.

– Let $x < y$ where $b_1 = b_2 = 1$. Then there are $(m - 1)$ letters that are larger and come after every one in u because there is a two in v every one in u will have the letters two through m after it. There are a_1 ones in u. Now there are $(m - 2)$ letters in w that are distinct and are larger than two that come after every two in u, specifically the letters three through m. There are a total of a_2 two's in u. When $x < y$ that means that the one in v comes before the 2 in v. Then every letter that comes after the one in v is larger than one, so there are $|v| - x$ letters that are distinct and to the right of this one. Every letter that comes after the two in v will larger than two because the one is found earlier in v. Therefore there will be $|v| - y$ letters to the right and bigger than the two in v. Therefore $rb = (m - 1)a_1 + (m - 2)a_2 + 2|v| - x - y$.

– Let $x > y$ where $b_1 = b_2 = 1$. Then there are $(m - 1)$ letters that are larger and come after every one in u because there is a two in v every one in u will have the letters two through m after it. There are a_1 ones in u. Now there are $(m - 2)$ letters in w that are distinct and are larger than two that come after every two in u, specifically the letters three through m. There are a total of a_2 two's in u. When $x > y$ that means that the one in v occurs after the two in v. So there will be $|v| - x$ letters to the right and bigger than the one in v. There will be $|v| - y - 1$ letters to the right and bigger than the two in v because there are $|v| - 1$ letters which come after, one of which is a one which is less than two. Therefore $rb = (m - 1)a_1 + (m - 2)a_2 + 2|v| - x - y - 1$.
□

- Left and Bigger
$$lb = a_1 - z + x + y + 1$$

Proof. Let z be the position of the first 2. Then there are $|u| - z$ letters in u that come after the first two. Of these letters we must remove all other two's to get the number of ones that come after the fist two, so this would be $|u| - z - (a_2 - 1) = a_1 + a_2 - z - a_2 + 1 = a_1 - z + 1$. So $a_1 - z + 1$ ones come after the first two and each of these will have one distinct letter to

the left and bigger than them. The next thing to consider is the one in v. Again let x be the position of the one in v. Then there will be $x + 1$ letters to the left and bigger, the two's present in u, the three and all of the letters that occur before one in v. And let y be the position of the two in v. This two will have three to the left and bigger, it will also have every letter from the beginning of v until the occurrence of the letter larger than it if $x > y$, this would give us v letters to the left and bigger than the two present in v. Now consider if $x > y$. Then that two in v is not distinct from the two in u for the lb contribution of the one in v, every element in $3v$ that comes before 1 is larger than it, so this will have an lb contribution of x. Now when considering the two in v, this comes before the one, so every element in v that comes before this one is also to the left and bigger, and each of these letters in v must be distinct. Therefore $lb = a_1 - z + x + y + 1$ $\square$

- Right and Smaller

$$rs = \begin{cases} a_2 + x + y - 1 \\ a_2 + x + y \\ (l - a_1) + y \end{cases}$$

Proof.   – Let one and two both be present in v, and let the position of the one, also referred to as x, be less than the position of the two, referred to as y. Then all of the two's found in u will have a one to the right of them. There are a_2 two's of this type and each will contribute one to the right and smaller statistic. The next possible contribution to the right and smaller statistic is based on the placement of the one in v. When one is in the xth position of v then there are $x - 1$ letters in v that come before the one and are larger than one. Also there is the three which comes before v and is larger than one. Each of these letters will then contribute one to the right and smaller statistic, so this will be $x - 1 + 1 = x$. Finally consider the contribution based on the two in v. When $x < y$ there are $y - 1$ letters that come before two in v, however one of these is a one, which is less than two, so there are $y - 2$ letters larger than two that come before. There is also a three which must come before this two. Because each of these cases will contribute one to the rs statistic, the whole statistic becomes $a_2 + x + y - 1$. No other contributions are possible because these are the last two possible descents in the word, and you must have a descent present after a letter for that letter to possibly contribute to the rs statistic.
    – Let one and two both be present in v where $y < x$. Then because there is a one in v, each two in u has one distinct letter to the right and smaller, contributing a_2 to the rs statistic. Next because the two comes before the one in v there will be $y - 1$ letters to the left of two that are larger from v, and there is also the three, causing a contribution of y to the rs statistic. Finally consider the one, there are $x - 1$ letters that come before one and are larger, because two is also larger, and there is the one three found before v, so this will contribute x to the rs statistic. Therefore $rs = a_2 + x + y$.
    – Let $b_1 = 0$ and let l denote the last one present in u. There will be $l - a_1$ two's that appear before the last one, each of these must

contribute one to the rs statistic. Then when there is a two present in v there are $y - 1$ letters that come before it and are larger in v, and there is one three that occurs before v. Therefore $rs = (l - a_1) + y$. If there is no two present in v, then $y = 0$ and the rs statistic is still $(l - a_1) + y$.

$\square$

11. THE PATTERN 1/2/34

- Left and Smaller:

$$ls = a_2 + b_2 + 2 + \sum_{i=3}^{m-1} i$$

Proof. Let $w(\sigma)$ be of the form $u3v$. Then there will be $a_2 + b_2$ two's after the first one. There is also a three and a two and a one must come before that three, so this will contribute 2 to the left and smaller statistic. For every letter larger than three there will be one less than that element total distinct letters smaller. Therefore $ls = a_2 + b_2 + 2 + \sum_{i=3}^{m-1} i$. $\square$

- Right and Bigger

$$rb = \begin{cases} (m-1)a_1 + (m-2)a_2 + |v| - y + \sum_{i=3}^{m}(m-i) \\ (h - a_2)(m-1) + (m-2)(|u| - h) + (m-2)a_2 + |v| - x + \sum_{i=3}^{m}(m-i) \\ (m-1)a_1 + (m-2)a_2 + 2|v| - x - y + \sum_{i=3}^{m}(m-i) \\ (m-1)a_1 + (m-2)a_2 + 2|v| - x - y - 1 + \sum_{i=3}^{m}(m-i) \end{cases}$$

Proof. Let w be of the form $u3v$.

  - Let $b_1 = 0$ and $b_2 = 1$. Then there is a two present in v but no one in v. Then there will be $(m-1)$ letters to the right and bigger than each one present in the word, because there is a two present in v we know that a two will be present after every one in the word so the last two in u does not impact this contribution to our statistic, there are a total of a_1 ones. There are also $(m-2)$ letters to the right and bigger than each two present in u and a_2 two's in u. Because there is a two in v let y denote the placement of this 2. Then there are $|v| - y$ distinct letters that come after the two in v that must be larger than two. Also every letter three or larger has $m - i$ letters larger than it where i is the letter value. Therefore $lb = (m-1)a_1 + (m-2)a_2 + |v| - y + \sum_{i=3}^{m}(m-i)$.
  - Let $b_2 = 0$ and let h be the position of the last 2 in u. Then there are $h - a_2$ ones that occur before the last two in u. Each of these one's will have $(m-1)$ letters to the right and bigger than 1 in the word. There will be $(|u| - h)$ ones that occur in u after the last two, these ones will have the letters three to m larger than them and occurring to the right, so each one has $(m-2)$ letters to the right and bigger. Now there will be a_2 two's in the word and there are a total of $m - 2$ distinct letters to the right and bigger. Next there are a total

of $|v| - x$ letters that occur after the one in v, all of these must be distinct and none can be one, therefore each is to the right and bigger than this one. Finally all other letters between three and m will have $m - i$ letters larger than them where i is the value of the letter and each letter between three and m can only occur once in the word, therefore this contributes $\sum\limits_{i=3}^{m}(m - i)$ to the right and bigger statistic.

So $rb = (h - a_2)(m - 1) + (m - 2)(|u| - h) + (m - 2)a_2 + |v| - x + \sum\limits_{i=3}^{m}(m - i)$.

- Let $x < y$ where $b_1 = b_2 = 1$. Then there are $(m - 1)$ letters that are larger and come after every one in u because there is a two in v every one in u will have the letters two through m after it. There are a_1 ones in u. Now there are $(m - 2)$ letters in w that are distinct and are larger than two that come after every two in u, specifically the letters three through m. There are a total of a_2 two's in u. When $x < y$ that means that the one in v comes before the 2 in v. Then every letter that comes after the one in v is larger than one, so there are $|v| - x$ letters that are distinct and to the right of this one. Every letter that comes after the two in v will larger than two because the one is found earlier in v. Therefore there will be $|v| - y$ letters to the right and bigger than the two in v. Therefore $rb = (m - 1)a_1 + (m - 2)a_2 + 2|v| - x - y$.

- Let $x > y$ where $b_1 = b_2 = 1$. Then there are $(m - 1)$ letters that are larger and come after every one in u because there is a two in v every one in u will have the letters two through m after it. There are a_1 ones in u. Now there are $(m - 2)$ letters in w that are distinct and are larger than two that come after every two in u, specifically the letters three through m. There are a total of a_2 two's in u. When $x > y$ that means that the one in v occurs after the two in v. So there will be $|v| - x$ letters to the right and bigger than the one in v. There will be $|v| - y - 1$ letters to the right and bigger than the two in v because there are $|v| - 1$ letters which come after, one of which is a one which is less than two. Therefore $rb = (m - 1)a_1 + (m - 2)a_2 + 2|v| - x - y - 1$.
  $\square$

- Left and Bigger
$$lb = a_1 - z + x + y + 1$$

Proof. Let z be the position of the first 2. Then there are $|u| - z$ letters in u that come after the first two. Of these letters we must remove all other two's to get the number of ones that come after the fist two, so this would be $|u| - z - (a_2 - 1) = a_1 + a_2 - z - a_2 + 1 = a_1 - z + 1$. So $a_1 - z + 1$ ones come after the first two and each of these will have one distinct letter to the left and bigger than them. The next thing to consider is the one in v. Again let x be the position of the one in v. Then there will be $x + 1$ letters to the left and bigger, the two's present in u, the three and all of the letters that occur before one in v. And let y be the position of the two in v. This two will have three to the left and bigger, it will also have every letter from the beginning of v until the occurrence of the letter larger than it if $x > y$, this would give us v letters to the left and bigger than the two present in v. Now consider if $x > y$. Then that two in v is not distinct from the two

in u for the lb contribution of the one in v, every element in $3v$ that comes before 1 is larger than it, so this will have an lb contribution of x. Now when considering the two in v, this comes before the one, so every element in v that comes before this one is also to the left and bigger, and each of these letters in v must be distinct. Therefore $lb = a_1 - z + x + y + 1$ □

- Right and Smaller

$$rs = \begin{cases} a_2 + x + y - 1 \\ a_2 + x + y \\ (l - a_1) + y \end{cases}$$

Proof. – Let one and two both be present in v, and let the position of the one, also referred to as x, be less than the position of the two, referred to as y. Then all of the two's found in u will have a one to the right of them. There are a_2 two's of this type and each will contribute one to the right and smaller statistic. The next possible contribution to the right and smaller statistic is based on the placement of the one in v. When one is in the xth position of v then there are $x - 1$ letters in v that come before the one and are larger than one. Also there is the three which comes before v and is larger than one. Each of these letters will then contribute one to the right and smaller statistic, so this will be $x - 1 + 1 = x$. Finally consider the contribution based on the two in v. When $x < y$ there are $y - 1$ letters that come before two in v, however one of these is a one, which is less than two, so there are $y - 2$ letters larger than two that come before. There is also a three which must come before this two. Because each of these cases will contribute one to the rs statistic, the whole statistic becomes $a_2 + x + y - 1$. No other contributions are possible because these are the last two possible descents in the word, and you must have a descent present after a letter for that letter to possibly contribute to the rs statistic.
  – Let one and two both be present in v where $y < x$. Then because there is a one in v, each two in u has one distinct letter to the right and smaller, contributing a_2 to the rs statistic. Next because the two comes before the one in v there will be $y - 1$ letters to the left of two that are larger from v, and there is also the three, causing a contribution of y to the rs statistic. Finally consider the one, there are $x - 1$ letters that come before one and are larger, because two is also larger, and there is the one three found before v, so this will contribute x to the rs statistic. Therefore $rs = a_2 + x + y$.
  – Let $b_1 = 0$ and let l denote the last one present in u. There will be $l - a_1$ two's that appear before the last one, each of these must contribute one to the rs statistic. Then when there is a two present in v there are $y - 1$ letters that come before it and are larger in v, and there is one three that occurs before v. Therefore $rs = (l - a_1) + y$. If there is no two present in v, then $y = 0$ and the rs statistic is still $(l - a_1) + y$.
  □

Theorem 4.6 (Statistics for $R_n(13/2/4)$)**.** *If $w \in R_n(13/2/4)$, then when $m \leq 2$ refer to Theorem 4.1 for the Wachs and White statistics for w; otherwise*

$$lb = \begin{cases} 0 & w \in B \\ (m-z)b_z & w \in C \text{ or } D \end{cases}$$

$$rs = \begin{cases} 0 & w \in B \\ \sum\limits_{i=z+1}^{m} a_i & w \in C \\ (y - b_z) & w \in D \end{cases}$$

$$rb = \begin{cases} l & w \in B \\ \sum\limits_{i=1}^{m-1} (m-i)a_i & w \in C \\ (x - a_m) + \sum\limits_{i=1}^{m-1} (m-i)a_i & w \in D \end{cases}$$

$$ls = (z-1)b_z + \sum_{i=2}^{m} (i-1)a_i$$

There are again three distinct cases where σ avoids the pattern $13/2/4$.

- Case 1: When no letter in $w(\sigma)$ is larger than two. (Proven in REU Paper). Refer to section 10 for formulas.
- Case 2: When w is of the form uv such that $z \leq m-1$.
  - Left and Bigger Statistic

  $$lb = (m-z)b_z$$

  Proof. Let w be of the form uv such that $z \leq m-1$ and let b_z be the number of $z's$ in v. Then there will be a block of $z's$ at the end of the word, however the rest of the word is weakly increasing. Each letter in this block will have $m-z$ letters that come before it in the word and that are larger, no other letter in the word will have a larger letter come before it. Therefore there are b_z such letters that will each have $m-z$ distinct letters to the left and bigger so $lb = (m-z)b_z$. □
  - Left and Smaller Statistic

  $$ls = (z-1)b_z + \sum_{i=2}^{m} (i-1)a_i$$

  Proof. Let w be of the form uv such that $z \leq m-1$ and let b_z be the number of $z's$ in v. Then there will be a total of $z-1$ letters to the left and smaller than each z found in v and because there are b_z $z's$ this will contribute $(z-1)b_z$ to the left and smaller statistic. Every other letter will have one less than the value of that specific letter to the left and smaller than it, and will have a_i occurrences of the letter, where i is the value of the letter. In this case one will not have any letters to the left and smaller because one is the smallest possible letter in w. So i can be between 2 and m. Therefore the left and smaller statistic is $ls = (z-1)b_z + \sum\limits_{i=2}^{m} (i-1)a_i$. □

– Right and Bigger Statistic

$$rb = \sum_{i=1}^{m-1} (m - i)a_i$$

Proof. Let w be of the form uv where $z \leq m - 1$. Because u must be weakly increasing by our word characterization earlier, there must be $m - i$ letters larger than each letter of value i to the right of that letter in the word. In w i can be any value between one and $m - 1$ because m will not have any letters larger than it. There will be a_i $i's$, therefore $rb = \sum_{i=1}^{m-1} (m - i)a_i$. □

– Right and Smaller Statistic

$$rs = \sum_{i=z+1}^{m} a_i$$

Proof. Let w be of the form uv where $z \leq m - 1$. Because u must be weakly increasing by the definition of our word characterization earlier, the only descent in the word can be from the block of $z's$ at the end of v. Then every element larger than z will have one distinct letter to the right and smaller. This will include every letter between $z + 1$ and m, so we must count the total occurrences, which gives us $rs = \sum_{i=z+1}^{m} a_i$. □

- Case 3: When w is of the form uv such that $z = m - 1$.
  – Left and Bigger Statistic

$$lb = (m - z)b_z$$

Proof. Let w be of the form uv where $z = m - 1$. Then by the word characterization earlier, we know that m and z can occur in any order in v as long as m occurs first. There will be $m - z$ letters to the left and bigger than every z in v and there are b_z $z's$, so this contributes $(m - z)b_z$ to the left and bigger statistic. □

  – Left and Smaller Statistic

$$ls = (z - 1)b_z + \sum_{i=2}^{m} (i - 1)a_i$$

Proof. Let w be of the form uv such that $z = m - 1$ and let b_z be the number of $z's$ in v. Then there will be a total of $z - 1$ letters to the left and smaller than each z found in v and because there are b_z $z's$ this will contribute $(z - 1)b_z$ to the left and smaller statistic. Every other letter will have one less than the value of that specific letter to the left and smaller than it, and will have a_i occurrences of the letter, where i is the value of the letter. In this case one will not have any letters to the left and smaller because one is the smallest possible letter in w. So i can be between 2 and m. Therefore the left and smaller statistic is $ls = (z - 1)b_z + \sum_{i=2}^{m} (i - 1)a_i$. □

– Right and Bigger Statistic

$$rb = (x - a_m) + \sum_{i=1}^{m-1}(m - i)a_i$$

Proof. Let w be of the form uv where $z = m - 1$ and let x be the position of the last m in v. Then there will be $(x - a_m)$ $z's$ that occur before the last m and each of these contributes one to the right and bigger statistic. Now because u must be weakly increasing, every other letter in the word has $m - i$ distinct letters larger than it where i is the value of the letter and there are a_i of each ith letter where i can be anything between one and $m - 1$ because m has no letter larger than it in the word. Therefore $rb = (x - a_m) + \sum_{i=1}^{m-1}(m - i)a_i$. $\square$

– Right and Smaller Statistic

$$rs = (y - b_z)$$

Proof. Let w be of the form uv where $z = m - 1$ and let y be the last occurrence of $m-1$ in v. Then there will $(y - b_z)$ $m's$ that come before the last $m - 1$ and therefore each of these $m's$ will have one distinct letter to the right and smaller. Therefore $rs = (y - b_z)$. $\square$

Theorem 4.7 (Statistics for $R_n(1/24/3)$). *If $w \in R_n(1/24/3)$, then when $m \leq 2$ refer to Theorem 4.1 for the Wachs and White statistics for w, otherwise:*

$$lb = \begin{cases} 0 & w \in B \\ (x - 1)b_1 & w \in C \\ a_1 - c + 1 & w \in D \end{cases}$$

$$rs = \begin{cases} 0 & w \in B \\ \sum_{i=2}^{x} a_i & w \in C \\ f - a_1 & w \in D \end{cases}$$

$$rb = \begin{cases} l & w \in B \\ \sum_{i=1}^{m}(m - i)a_i + (m - x)b_1 & w \in C \\ (d - a_2)(m - 1) + (|u| - d)(m - 2) + \sum_{i=2}^{m-1}(m - i)a_i & w \in D \end{cases}$$

$$ls = \sum_{i=1}^{m}(i - 1)a_i$$

There are three cases.

- Case 1: Same as the $13/2/4$ case 1 where the largest letter is at most two [Dahlberg et al. (2015)].
- Case 2: When w is weakly increasing except for a block of ones inserted between two distinct letters.
  - Left and Bigger Statistic

$$lb = (x - 1)b_1$$

Proof. Let w be weakly increasing except for a block of ones inserted between two distinct letters. Also let x denote the distinct letter immediately before the first inserted one, and b_1 denote the number of ones inserted into w. There will be $x - 1$ distinct letters that come before this block of ones that are larger than one and there are b_1 one's inserted, this is the only possible descent. Therefore $lb = (x-1)b_1$. $\square$

– Left and Smaller Statistic

$$ls = \sum_{i=1}^{m}(i-1)a_i$$

Proof. Every letter will have one less than the value of that specific letter to the left and smaller than it, and will have a_i occurrences of the letter, where i is the value of the letter. So $ls = \sum_{i=1}^{m}(i-1)a_i$. $\square$

– Right and Bigger Statistic

$$rb = \sum_{i=1}^{m}(m-i)a_i + (m-x)b_1$$

Proof. Let w be weakly increasing except for a block of ones inserted between two distinct letters. Also let x denote the distinct letter immediately before the first inserted one, and b_1 denote the number of ones inserted into w. Then there are $m - x$ letters that are larger than one and are found after the block of inserted ones, so each inserted one will have $m - x$ letters to the right and bigger. Because there are b_1 such letters this will contribute $(m - x)b_1$ to the rb statistic. For every other letter in the word there will be $m - i$ where i is the value of the letter, distinct letters larger than i, and there are a_i occurrences of each letter i in the word, where i can be anything between one and $m - 1$ because m cannot have any letter larger than it. Therefore $rb = \sum_{i=1}^{m}(m-i)a_i + (m-x)b_1$. $\square$

– Right and Smaller Statistic

$$rs = \sum_{i=2}^{x}a_i$$

Proof. Let w be weakly increasing except for a block of ones inserted between two distinct letters. Also let x denote the distinct letter immediately before the first inserted one, and b_1 denote the number of ones inserted into w. Every letter larger than one that occurs before the block of inserted ones will contribute one to the right and smaller statistic. Therefore every occurrence of the letters two through x will contribute one to the right and smaller statistic. So $rs = \sum_{i=2}^{x}a_i$. $\square$

- Case 3: When w contains a prefix u of ones and twos in any order followed by a weakly increasing suffix.
    - Left and Bigger Statistic

$$lb = a_1 - c + 1$$

Proof. Let w contain a prefix u of ones and two's in any order followed by a weakly increasing suffix. Then let c be the position of the first two in w. Then $|u| = a_1 + a_2$. We want the number of ones that occur after the first two because each of these will contribute one to the left and bigger statistic. Nothing else will contribute because every thing after u is weakly increasing and contains letters three or greater. So to find the number of ones occurring after the first two, we will subtract the position of the first two from the length of u and then will subtract $a_2 - 1$ to remove the rest of the twos present in the word. So

$$|u| - c - (a_2 - 1) = a_1 + a_2 - a_2 - c + 1 = a_1 - c + 1$$

Therefore $lb = a_1 - c + 1$. $\qquad\square$
– Left and Smaller Statistic

$$ls = \sum_{i=1}^{m} (i-1)a_i$$

See case 1 for proof.
– Right and Bigger Statistic

$$rb = (d - a_2)(m - 1) + (|u| - d)(m - 2) + \sum_{i=2}^{m-1} (m - i)a_i$$

Proof. Let w contain a prefix u of ones and two's in any order followed by a weakly increasing suffix. Also let d be the position of the last 2 in u. Then there will be $d - a_2$ ones that come before the last two, each of these ones will have $m - 1$ letters that are larger and to the right. Then there are $(|u| - d)$ ones that come after that last two. These ones will have $m - 2$ letters that are larger and occur to the right. Then every letter between two and $m - 1$ will have $m - i$ letters that are to the right and bigger of each letter i. Therefore
$$rb = (d - a_2)(m - 1) + (|u| - d)(m - 2) + \sum_{i=2}^{m-1} (m - i)a_i. \qquad\square$$
– Right and Smaller Statistic

$$rs = f - a_1$$

Proof. Let w contain a prefix u of ones and two's in any order followed by a weakly increasing suffix. Also let f be the position of the last one in u. Then there will be $f - a_1$ two's that occur before the last one, meaning that each of these two's will have one distinct letter to the right and smaller. Therefore $rs = f - a_1$. $\qquad\square$

Theorem 5.1 (Equidistribution of Statistics I). *Consider the avoidance classes based on $1/24/3$ and $13/2/4$, then*
  (i) $LS(13/2/4) \sim RB(1/24/3)$
 (ii) $RB(13/2/4) \sim LS(1/24/3)$
(iii) $LB(13/2/4) \sim LB(1/24/3)$
(iv) $RS(13/2/4) \sim RS(1/24/3)$

Proof. Let σ_1 avoid $13/2/4$ and let σ_2 avoid $1/24/3$. Also let w denote the restricted growth function for σ_1 and let y denote the restricted growth function for σ_2. Let $A_1, A_2, A_3 \subset R_n(13/2/4)$ such that A_1 contains all w where $m \leq 2$ or w weakly

increasing, A_2 contains all $w = uv$ where v is a block of m followed by a block of $z \leq (m-1)$, and A_3 contains all $w = uv$ where v contains m and $(m-1)$ in any order after the first m. Also let $B_1, B_2, B_3 \subset R_n(1/24/3)$ where B_1 contains all w where $m \leq 2$ or w weakly increasing, B_2 contains all w with a block of 1s inserted between two distinct letters in a weakly increasing word, and finally B_3 contains all $w = uv$ where u contains 1s and 2s in any order allowable by the RGF.

(1) Let $m \leq 2$ for both w and y. Refer to REU paper and the distribution of statistics for the avoidance class based on $1/2/3$.

(2) Let $w = uv$ where u is weakly increasing and v begins with the first m in w and then contains a block of m's followed by a block of z's where $z \leq (m-1)$. When y is a weakly increasing word with a block of ones inserted between two distinct terms.

- Left and Smaller of w with Right and Bigger of y

$$ls(w) = (z-1)b_z + \sum_{i=2}^{m}(i-1)a_i$$

$$rb(y) = (m-x)b_1 + \sum_{i=1}^{m-1}(m-i)a_i$$

Let $w \in A_2$, $y \in B_2$. Then the following are equations for the statistics of individual words:

$$ls(w) = (z-1)b_z + \sum_{i=2}^{m}(i-1)a_i$$

$$rb(y) = (m-x)b_1 + \sum_{i=1}^{m-1}(m-i)a_i$$

Now let the function $f : A_2 \to B_2$ where when $w = uv$, $f(w) = y$ such $b_z = b_1$ (where b_z is the number of $z \in v$ and b_1 is the size of the block of inserted 1s in y), the distinct letter $x \in y$ before this block of 1s is $x = m - z + 1$ where z is the last letter in w, and the weakly increasing base of y is formed by taking the RGF of the complement of the partition in a bijection with u. An example would be $f(1234422) = 1123114$.

We will prove that f is injective. Consider $w_1, w_2 \in A_2$ where $w_1 \neq w_2$. If $a_1 \neq a_2$ then $a_1^c \neq a_2^c$ so the weakly increasing word in $f(w_1)$ differs from the weakly increasing word in $f(w_2)$ so $f(w_1) \neq f(w_2)$. If $b_{z_1} \neq b_{z_2}$ then $b_{1_{w_1}} \neq b_{2_{w_2}}$ so $f(w_1) \neq f(w_2)$. Therefore when $f(w_1) = f(w_2)$, $w_1 = w_2$, meaning that f is injective.

Let f^{-1} be the inverse of f such that $f^{-1} : B_2 \to A_2$, where $f(y) = w$ such that w is formed by taking the RGF of the complement of the partition corresponding to the weakly increasing base of y. This will be weakly increasing by Lemma 2.6. The function will define $z \in w$ by $z = m - x + 1$ and $b_1 = b_z$.

We will show that f^{-1} is injective. Consider $y_1, y_2 \in B_2$ where $y_1 \neq y_2$. If the weakly increasing word in y_1 differs from the weakly increasing word in y_2 then $a_{1_{y_1}} \neq a_{1_{y_2}}$ so $f^{-1}(y_1) \neq f^{-1}(y_2)$. If $b_{1_{y_1}} \neq b_{1_{y_2}}$, then

$f^{-1}(y_1) \neq f^{-1}(y_2)$. Therefore when $f^{-1}(y_1) = f^{-1}(y_2)$, $y_1 = y_2$, so f^{-1} is injective. Therefore f is bijective.
Now consider

$$rb[f(w)] = (m - x)b_1 + \sum_{i=1}^{m-1}(m - i)c_i$$

$$= [m - (m - z + 1)]b_z + \sum_{i=1}^{m-1}(m - i)a_{m-i+1}$$

$$= (z - 1)b_z + \sum_{i=1}^{m-1}[(m + 1 - i) - 1]a_{m-i+1}$$

$$= (z - 1)b_z + \sum_{i=2}^{m}(i - 1)a_i = ls(w)$$

Therefore because $rb(f(w)) = ls(w)$ and f is a bijective function, $LS(A_2) \sim RB(B_2)$.

- Right and Bigger of w with Left and Smaller of y. Now consider, (using the equations for statistics from Theorem 4.5 and Theorem 4.6)

$$ls(f(w)) = \sum_{i=1}^{m}(i - 1)c_i$$

$$= \sum_{i=1}^{m}(i - 1)a_{m-i+1}$$

$$= \sum_{i=1}^{m}[m - (m - i + 1)]a_{m-i+1}$$

$$= \sum_{i=1}^{m-1}(m - i)a_i = rb(w)$$

Therefore because f is a bijection from A_2 to B_2 and $rb(w) = ls(f(w))$, $RB(A_2) \sim LS(B_2)$.

- Left and Bigger. Now consider, $lb(f(w))$. From Theorems 4.5 and 4.6 we have,

$$lb(f(w)) = (x - 1)b_1$$
$$= [(m - z + 1) - 1]b_z$$
$$= (m - z)b_z = lb(w)$$

Therefore $LB(A_2) \sim LB(B_2)$ because f is a bijective function from A_2 to B_2 and $lb(f(w)) = lb(w)$.

- Right and Smaller

$$rs(w) = \sum_{i=z+1}^{m}b_i$$

$$rs(y) = \sum_{i=2}^{x}\tilde{a}_i$$

Let $w \in A_2$, $y \in B_2$. Now let the function $h : A_2 \rightarrow B_2$ where h takes some $w = abc$ of length n and with m distinct letters and a

is the weakly increasing prefix that ends with the last occurrence of $z - 1$ where a corresponds to $\tilde{c} \in y$ such that $\tilde{c}$ is the complement of a, because a is weakly increasing $\tilde{c}$ is weakly increasing by the lemma. The first occurrence of z in w marks the beginning of $b \in w$, where the subword b ends with the last occurrence of m where b corresponds to $\tilde{a}$ which is the prefix of the word y where z becomes one, $z + 1$ becomes two and so on such that each element from b becomes $i - z + 1$ in y such that a ends with $m - z + 1$. The suffix of w, c, is the block of z that occur after the last m in w. The function h will cause the size of c to correspond to the size of $\tilde{b}$ where $\tilde{b}$ is the block of inserted ones after the last occurrence of $m - z + 1$ in y.

Now consider $w_1, w_2 \in A_2$ where $w_1 \neq w_2$. If $a_1 \neq a_2$, then $a_1^c \neq a_2^c$ meaning that $\tilde{c}_1 \neq \tilde{c}_2$, therefore $h(w_1) \neq h(w_2)$. If $b_1 \neq b_2$, then the number of $i - z + 1$ will be different between $\tilde{a}_1$ and $\tilde{a}_2$ where i has a different number of occurrences in b_1 than in b_2, therefore If $h(w_1) \neq h(w_2)$. If $c_1 \neq c_2$, then $\tilde{b}_1 \neq \tilde{b}_2$ because they will be of different sizes, therefore $h(w_1) \neq h(w_2)$. Therefore if $h(w_1) = h(w_2)$, then $w_1 = w_2$. So h is one-one.

Let $h^{-1} : B_2 \rightarrow A_2$ where h^{-1} is the inverse of h. Where $h^{-1}(y) = w$ where $y = \tilde{a}\tilde{b}\tilde{c}$ such that $\tilde{a}$, the weakly increasing prefix of y that ends with the last occurrence of x the last letter before a block of inserted ones, corresponds to $b \in w$ where 1 becomes z and every element $i \in \tilde{a}$ becomes $i + z - 1$ in b, so the last element in $\tilde{a}$, $m - z + 1$ becomes $(m - z + 1) + z - 1 = m$ in b. Then $\tilde{b}$, the block of inserted ones, corresponds to $c \in w$ where c is the same length as $\tilde{b}$ however composed of a block of z where z is determined to be $m - x + 1$. Finally $\tilde{c}$ the weakly increasing suffix of y corresponds to the weakly increasing prefix of w, a, where $\tilde{c}^c = a$.

Now consider $y_1, y_2 \in B_2$ where $y_1 \neq y_2$. If $\tilde{a}_1 \neq \tilde{a}_2$, then for each i where there are a different number of occurrences of i in $\tilde{a}_1$ versus $\tilde{a}_2$ there will be a different number of occurrences of $i + z - 1$ in b_1 versus b_2, therefore $h^{-1}(y_1) \neq h^{-1}(y_2)$. If $\tilde{b}_1 \neq \tilde{b}_2$ then the lengths must not be the same, so the length of c_1 and c_2 will also differ, meaning that $h^{-1}(y_1) \neq h^{-1}(y_2)$. If $\tilde{c}_1 \neq \tilde{c}_2$ then $\tilde{c}_1^c \neq \tilde{c}_2^c$ meaning that $a_1 \neq a_2$, therefore $h^{-1}(y_1) \neq h^{-1}(y_2)$. So when $h(w_1) = h(w_2)$, then $w_1 = w_2$. So h^{-1} is one-one.

Because h and h^{-1} are both one-one, then h is a bijective function between A_2 and B_2. Therefore $x = m - z + 1$ and has the same number of occurrences as a_m the number of m in w. The word y ends such that the weakly increasing suffix of y is created from the complement of the weakly increasing prefix of w that occurs before the first z. (An example of the function would be $h(12234522) = 11234115$). Now the same number of elements larger than z will occur before z as the number of elements larger than one that occur before the block of inserted ones. This function h is reversible, and the inverse $h^{-1} : A_2 \rightarrow B_2$ takes some word $y \in B_2$ such that $h^{-1}(y)$ will be the word created by $z = m - x + 1$, all letters after the block of ones inserted will start the word, such that m in y becomes 1 in $h^{-1}(y)$ and

so on until $m - x$ followed by z, $z + 1$ and so on until the block of m's followed by a block of z's corresponding in length to the block of ones in y. Because h is reversible, h is a bijective function between A_2 and B_2. Now consider,

$$rs(h(w)) = \sum_{i=2}^{x} \tilde{a}_i$$

$$= \sum_{i=z+1}^{m} b_i = rs(w)$$

Therefore because h is bijective, and $rs(h(w)) = rs(w)$, $RS(A_2) \sim RS(B_2)$.

(3) Let w be of the form uv where u is weakly increasing and v begins with the first m and then contains m and $(m - 1)$ in any order and let y be of the form uv where u contains 1 and 2 in any reasonable order allowed by the definition of the restricted growth function, followed by v which begins with the first occurrence of 3 and is a weakly increasing suffix.

- Left and Smaller of w with Right and Bigger of y

  Let $w \in A_3$ where A_3 is the set of words in case 3 of the avoidance class for the pattern $13/2/4$. Let $y \in B_3$ where B_3 is the set of all words in case 3 of the avoidance class for the pattern $1/24/3$. Let the function $g : A_3 \mapsto B_3$. The function $g(w)$ will take a word $w = abc$ where a is weakly increasing from one to $m - 2$, b is the subword of the $m - 1$'s that occur after a, followed by c which begins with the first occurrence of m followed by m and $m - 1$ in any order. Then let $g(w) = y$ where $y = \tilde{a}\tilde{b}\tilde{c}$ such that $\tilde{a}$ begins with one followed by one's and two's in any order ending one position before the last occurrence of two, $\tilde{a}$ is found by the function g such that $c \in w$ corresponds to $\tilde{a} \in y$ such that $m \in c$ becomes $1 \in \tilde{a}$ and $(m - 1) \in c$ becomes $2 \in \tilde{a}$. The next subword $\tilde{b} \in y$ is formed from $b \in y$. The first $(m - 1)$ in b will become 2, which is found in the first position of b' and is the last two in y. All other $(m - 1) \in b$ will become ones $\in \tilde{b}$ which occur after the last two in y. Therefore there are $|b| - 1$ one's that occur after the last two. Finally the last subword $\tilde{c} \in y$ is formed by taking the complement of the weakly increasing subword $a \in w$, by the lemma the complement of a weakly increasing word is weakly increasing, and by the definition of a where a ends with the last occurrence of $m - 2$, the complement of $m - 2$ is $m - (m - 2) + 1 = 3$ so $\tilde{c}$ is weakly increasing beginning with the first occurrence of 3. Because y will have a prefix of ones and twos in any order which composed of the two subwords $\tilde{a}$ and $\tilde{b}$ followed by a weakly increasing suffix $\tilde{c}$ which begins with the first occurrence of 3, y is a word in B_3.

  Now consider two words $w_1, w_2 \in A_3$ where $w_1 \neq w_2$. If $a_1 \neq a_2$ then $a_1' = \tilde{c}_1$ and $a_1' = \tilde{c}_2$, and $a_1' \neq a_2'$ so $g(w_1) \neq g(w_2)$. If, however, $b_1 \neq b_2$ then $|b_1| \neq |b_2|$ and because $|b_1| = |\tilde{b}|$ and $|b_2| = |\tilde{b}_2|$, then $|\tilde{b}_1| \neq |\tilde{b}_2|$ so $\tilde{b}_1 \neq \tilde{b}_2$, therefore $g(w_1) \neq g(w_2)$. Finally consider if $c_1 \neq c_2$. Then by the definition of the function g the ones and twos in $\tilde{a}_1$ will not correspond to the position of the ones and twos in $\tilde{a}_2$,

so $g(w_1) \neq g(w_2)$. So when $g(w_1) = g(w_2)$, $w_1 = w_2$. Therefore g is one-one.

Now consider $g^{-1} : B_3 \mapsto A_3$. The function $g^{-1}(y) = w$ where $y = \tilde{a}\tilde{b}\tilde{c}$ will result in some $w \in A_3$ where $w = abc$ such that by taking the complement of the weakly increasing suffix $\tilde{c} \in y$ we create $a \in w$. By the lemma, the complement of a weakly increasing word is weakly increasing, so a is weakly increasing, and because $\tilde{c}$ begins with the first occurrence of 3, a will end in the last occurrence of $m - 3 + 1 = m - 2$. The subword $\tilde{b} \in y$ where $\tilde{b}$ begins with the last two and then contains a block of ones ending with the last one in the word will correspond to the subword $b \in w$ such that every element in $\tilde{b}$ corresponds to an $(m - 1)$ in b where $|\tilde{b}| = |b|$. Finally the subword $\tilde{a} \in y$ where $\tilde{a}$ begins with one and then contains ones and twos in any order ending before the last occurrence of two which corresponds to the subword $c \in w$ where each one in $\tilde{a}$ corresponds to an m in c and each two in $\tilde{a}$ corresponds to an $(m - 1)$ in c. Therefore $g^{-1}(y)$ has a weakly increasing prefix composed of a and b followed by m and $(m-1)$ in any order after the first occurence of m which is the subword c, meaning that $g^{-1}(y)$ is in A_3.

Now consider two words $y_1, y_2 \in B_3$ where $y_1 \neq y_2$. If $\tilde{a_1} \neq \tilde{a_2}$ then $c_1 \neq c_2$ because of the definition of g^{-1} the m's and $(m-1)$'s in $g^{-1}(y_1)$ and $g^{-1}(y_2)$ will not be found in the same locations and there may be different quantities of each. Therefore $g^{-1}(y_1) \neq g^{-1}(y_2)$. If $\tilde{b_1} \neq \tilde{b_2}$ then $b_1 \neq b_2$ because they will not be of the same length. Therefore $g^{-1}(y_1) \neq g^{-1}(y_2)$. Finally consider if $\tilde{c_1} \neq \tilde{c_2}$, then $\tilde{c_1}' \neq \tilde{c_2}'$ meaning $a_1 \neq a_2$. Therefore $g^{-1}(y_1) \neq g^{-1}(y_2)$!!! So when $g^{-1}(y_1) = g^{-1}(y_2)$, then $y_1 = y_2$. Therefore g^{-1} is one-one. Because both g and g^{-1} are one-one, g is a bijective function between A_3 and B_3.

Now consider,

$$rb(g(w)) = (m - 1)(\tilde{a_1}) + (m - 2)(\tilde{a_2} + |\tilde{b}|) + \sum_{i=3}^{m-1}(m - i)\tilde{c_i}$$

Then by the definition of g,

$$rb(g(w)) = (m - 1)a_m + (m - 2)(a_{m-1} + b_{m-1}) + \sum_{i=3}^{m-1}[m - (m - i + 1)]\tilde{c}_{m-i+1}$$

$$= (m - 1)a_m + (m - 2)(a_{m-1} + b_{m-1}) + \sum_{i=3}^{m}(i - 1)a_i = ls(w)$$

Because g is a bijective function and $rb(g(w)) = ls(w)$, $LS(A_3) \sim RB(B_3)$.

- Right and Bigger of w with Left and Smaller of y.

  Let $w \in A_3$ such that $w = abc$, a is the weakly increasing prefix of w which ends at the last occurrence of $m-2$, b begins with the first $m-1$ and ends with the last occurrence of m in the word, and finally, c is a block of $m-1$ that occur after the last m in the word. Also let $y \in B_3$ such that $y = \tilde{a}\tilde{b}\tilde{c}$ where $\tilde{a}$ is the prefix of one's and two's beginning with one and ending with the last occurrence of two, $\tilde{b}$ is a block of

ones after the last two in the word, and $\tilde{c}$ begins with the first three and is a weakly increasing suffix for w. Now let $h : A_3 \to B_3$ where $h(w) = y$ such that the complement of a becomes $\tilde{c}$ and by the lemma this will ensure that $\tilde{c}$ is a weakly increasing suffix because a is weakly increasing also $\tilde{c}$ will begin with a three because $m - (m - 2) + 1 = 3$, h will cause the $m - 1$'s in c to become ones in $\tilde{b}$, and h takes b where m becomes one and each $m - 1$ becomes a two and they are found in reverse order in the prefix of y, $\tilde{a}$ so $\tilde{a}$ begins with one and ends with the last two in y because b contains the first occurrence of $m - 1$ which corresponds to the last two in y.

Now consider two words $w_1, w_2 \in A_3$ such that $w_1 \neq w_2$. If $a_1 \neq a_2$ then $a'_1 \neq a'_2$ by the definition of the complement, so $\tilde{c}_1 \neq \tilde{c}_1$, meaning that $h(w_1) \neq h(w_2)$. If $b_1 \neq b_2$ then the number, order, or both of the one's and two's in $\tilde{a}_1$ will differ from that of $\tilde{a}_2$ because the number, order, or both of m's and $(m-1)$'s differ in b_1 and b_2, therefore $h(w_1) \neq h(w_2)$. If $c_1 \neq c_2$, then $|c_1| \neq |c_2|$, meaning that $|\tilde{b}_1| \neq |\tilde{b}_2|$, therefore $h(w_1) \neq h(w_2)$!!! Therefore when $h(w_1) = h(w_2)$, then $w_1 = w_2$. This means that h is a one-one function.

Let h^{-1} be the inverse function of h, such that $h^{-1} : B_2 \to A_2$. This function will cause $\tilde{a} \in y$ to correspond to $b \in w$ where the ones in $\tilde{a}$ correspond to the m's in b, and the two's in $\tilde{a}$ correspond to $(m - 1)$'s in b such that each element is found in reverse order from that of $\tilde{a}$ in b. Then $\tilde{b}$, the block of ones after the last two, corresponds to $c \in w$ the block of $(m - 1)$'s after the last m. Finally $\tilde{c}$, the weakly increasing suffix of w which begins with a three, corresponds to the weakly increasing prefix $a \in w$ where $\tilde{c}^c = a$.

Now consider two words $y_1, y_2 \in B_3$ where $y_1 \neq y_2$. If $\tilde{a}_1 \neq \tilde{a}_2$, then the number and/or order of one's and two's differs between these two subwords, because the number and order of m and $m - 1$ in b_1 and b_2 will depend on the order and number of $\tilde{a}_1$ and $\tilde{a}_2$ respectively, $b_1 \neq b_2$, therefore $h^{-1}(y_1) \neq h^{-1}(y_2)$. If $\tilde{b}_1 \neq \tilde{b}_2$, $|\tilde{b}_1| \neq |\tilde{b}_2|$ so $|c_1| \neq |c_2|$, therefore $h^{-1}(y_1) \neq h^{-1}(y_2)$. If $\tilde{c}_1 \neq \tilde{c}_2$, then $\tilde{c}_1^c \neq \tilde{c}_2^c$ so $a_1 \neq a_2$, therefore $h^{-1}(y_1) \neq h^{-1}(y_2)$!!! Therefore when $h^{-1}(y_1) = h^{-1}(y_2)$, $y_1 = y_2$. So h^{-1} is one-one. Because h and h^{-1} are both one-one, the function h is bijective between A_3 and B_3. The m terms in w correspond to ones in the prefix of y before the last two, all z's before the last m in w correspond to twos in the word y, and finally all $z's$ after x the position of the last m in w correspond to ones after the last two in y. All z that occur before the last mA will be equal to $x - a_m$ where x is the position of the last m in the suffix of w beginning with the first z. This corresponds to the total number of two's in y. The weakly increasing suffix of y is the complement of the prefix of w. (An example is $h(122345544) = 112113445$.) This function is reversible, the inverse, $h^{-1} : B_3 \to A_3$ will take some $y \in B_3$ such that $g^{-1}(y) = w$ where w is the word created by taking the complement of the weakly increasing suffix of y as the prefix in w, the one's in the prefix of y before the last two become m in w, all two's in y correspond to z's before the last m in w, the number of one's after the last two in y

corresponds to the z's that occur after the last m in w. Now consider,

$$ls(h(w)) = \tilde{a}_2 + \sum_{i=3}^{m}(i-1)\tilde{c}_i$$

$$= b_{m-1} + \sum_{i=1}^{m-2}[(m-i+1)-1]\tilde{c}_{m-i+1}$$

$$= b_{m-1} + \sum_{i=1}^{m-2}(m-i)a_i = rb(w)$$

Therefore because h is a bijective function between A_3 and B_3 and $ls(h(w)) = rb(w)$, $RB(A_3) \sim LS(B_3)$.

- Left and Bigger.

  Let $w \in A_3$ and $y \in B_3$ where $w = ab$ such that a is the weakly increasing prefix ending with the last occurrence of $m-2$, and b is the suffix of $m-1$ and m in any order beginning with the first occurrence of $m-1$ in w and $y = \tilde{a}\tilde{b}$ such that $\tilde{a}$ is the prefix of ones and twos, followed by $\tilde{b}$ the weakly increasing suffix beginning with the first three in y. Let $g : A_3 \mapsto B_3$ where $g(w)$ will cause $a \in w$ to correspond to $\tilde{b}$ where $a^c = \tilde{b}$ and by the lemma $\tilde{b}$ will be weakly increasing. The suffix b in w corresponds to $\tilde{a} \in y$ where each $m-1$ in b becomes 1 found in the same position of b as in $\tilde{a}$, $m \in b$ becomes 2 in the prefix. So the position of the first two in $\tilde{a}$, c, is equal to the position of the first m in the suffix of w, b.

  Now consider $w_1, w_2 \in A_3$ where $w_1 \neq w_2$. If $a_1 \neq a_2$ then $a_1^c \neq a_2^c$ so $\tilde{b}_1 \neq \tilde{b}_2$, therefore $g(w_1) \neq g(w_2)$. If $b_1 \neq b_2$ then the number and order of m's and $(m-1)$'s will differ, and the corresponding $\tilde{a}_1$ and $\tilde{a}_2$ will differ in the number and order of one's and two's, therefore $g(w_1) \neq g(w_2)$. Therefore if $g(w_1) = g(w_2)$, $w_1 = w_2$ meaning that g is one-one.

  Let g^{-1} be the inverse function where $g^{-1} : B_3 \to A_3$. The subword $\tilde{a}$ corresponds to $b \in w$ where one's become $m-1$ in b, and twos become m's. The subword $\tilde{b}$ corresponds to $a \in w$ where $\tilde{b}^c = a$ which will be weakly increasing by the lemma.

  Now consider $y_1, y_2 \in B_3$ where $y_1 \neq y_2$. If $\tilde{a}_1 \neq \tilde{a}_2$ then the ones and twos in these two subwords differ in position and potentially the quantity, so the position and potentially the quantity of $(m-1)$'s and m's will differ between b_1 and b_2, therefore $g^{-1}(y_1) \neq g^{-1}(y_2)$. If $\tilde{b}_1 \neq \tilde{b}_2$ then $\tilde{b}_1^c \neq \tilde{b}_2^c$ meaning that $a_1 \neq a_2$ so $g^{-1}(y_1) \neq g^{-1}(y_2)$. So when $g^{-1}(y_1) = g^{-1}(y_2)$, $y_1 = y_2$. Therefore g^{-1} is one-one. Because g and its inverse g^{-1} are one-one, g is a bijection between A_3 and B_3. Now let d be the position of the first m in b, and c the position of the first two in $\tilde{a}$. Then

  $$lb(g(w)) = |\tilde{a}| - c - \tilde{a}_2 + 1 = |b| - d - a_m + 1 = lb(w)$$

  Because g is a bijective function between A_3 and B_3 and $lb(g(w)) = lb(w)$, $LB(A_3) \sim LB(B_3)$.

- Right and Smaller

$$rs(w) = t - b_{m-1}$$

$$rs(y) = f - \tilde{a}_1$$

Where f is the position of the last one in u. Let g be the function defined in the last case (the one for $LB(A_3) \sim LB(B_3)$). Let t be the position of the last occurrence of $m - 1$ in $b \in w$ and let f be the position of the last occurrence of one in $\tilde{a} \in y$

$$rs(g(w)) = f - a_1$$

$$= t - b_{m-1} = rs(w)$$

Therefore because g is a bijective function between A_3 and B_3 and $rs(g(w)) = rs(w)$, $RS(A_3) \sim RS(B_3)$.

Because $A_1 \cup A_2 \cup A_3 = R_n(13/2/4)$ and $B_1 \cup B_2 \cup B_3 \cup = R_n(1/24/3)$ and $LS(A_1) \sim RB(B_1)$, $RB(A_1) \sim LS(B_1)$, $RS(A_1) \sim RS(B_1)$, $LB(A_1) \sim LB(B_1)$, $LS(A_2) \sim RB(B_2)$, $RB(A_2) \sim LS(B_2)$, $RS(A_2) \sim RS(B_2)$, $LB(A_2) \sim LB(B_2)$, $LS(A_3) \sim RB(B_3)$, $RB(A_3) \sim LS(B_3)$, $RS(A_3) \sim RS(B_3)$, $LB(A_3) \sim LB(B_3)$,

$$LS(13/2/4) \sim RB(1/24/3)$$

$$RB(13/2/4) \sim LS(1/24/3)$$

$$RS(13/2/4) \sim RS(1/24/3)$$

$$LB(13/2/4) \sim LB(1/24/3)$$

$$\square$$

Theorem 6.2. *The word characterization and cardinality of the avoidance class based on the patterns $1/23$ and $13/2/4$ are given by:*

(i) $R_n(1/23, 13/2/4) = \{w \in R_n :$
   (A) $w = ab$ where a is a prefix of ones of any length and b is strictly increasing beginning with the first occurrence of 2 or b can be empty
   (B) $w = abc$ where a is a prefix of 1s of any length, b is strictly increasing beginning with the first occurrence of 2 and ending with m, and c is a suffix of length one containing a $1\}$

(ii) $\#\Pi_n(1/23, 13/2/4) = 2(n - 1)$

Proof. Let $w \in R_n(1/23, 13/2/4)$. Then $w \in R_n(1/23)$ so by Sagan in Lemma 2.1, we know that $m \geq 1$ and that w is formed by inserting a 1 into a word of the form $1^l 23...m$ [Sagan (2010)]. We also know that $w \in R_n(13/2/4)$ so w can take the following forms by Theorem 3.9:

a. We can have $m \leq 2$. Then either $m = 1$ or when $m = 2$ by Lemma 2.1 $w = 1^l 21$ or $w = 1^l 2$.

b. Because $w \in R_n(13/2/4)$ we could have $w = ab$ where a is weakly increasing and b begins with the first m and then contains m and $m - 1$ in any order. By Theorem 3.9 we know w avoids $13/2/4$ however by Lemma 2.1 we know that w contains $1/23$. Therefore w cannot take this form.

   c. Finally let $w = ab$ where a is weakly increasing and b begins with the first
       m and contains a block of ms followed by a block of zs where $z/leq(m-1)$.
       By Theorem 3.9, $w \in R_n(13/2/4)$, however by Lemma 2.1 we know $z = 1$
       and there can only be one occurrence of 1 in b. Also by Lemma 2.1, a
       begins with a block of 1s followed by a strictly increasing subword.

Therefore by Lemma 2.1 and Theorem 3.9, when $w \in R_n(1/23, 13/2/4)$, $w = abc$
where a is a prefix of 1s, b is a strictly increasing subword beginning with 2 and
ending with m where b can be empty, followed by c which is either empty or contains
a 1. $\square$

$$\#\Pi_n(1/23, 13/2/4) = 2(n-1)$$

Proof. First let $w = ab$ where a is a prefix of 1s and b is strictly increasing beginning
with the first 2 where b can be empty. Then $|a|$ can be anything from one to n, and
because b is strictly increasing, there are a total of n ways to form w.

   Next let $w = abc$ where a is a prefix of 1s, b is strictly increasing from 2 to
m, and c is one occurrence of 1. Then $|a|$ can vary from one to $n - 2$. Because b is
strictly increasing and c is a suffix containing exactly one 1, there is only one way
to form bc. So there are $n - 2$ ways to form w when $w = abc$.

   Therefore $\#\Pi_n(1/23, 13/2/4) = n + n - 2 = 2(n-1)$ $\square$

Theorem 6.3. *The word characterization and cardinality of the avoidance class
based on the patterns $13/2$ and $123/4$ are given by:*

   *(i) $R_n(13/2, 123/4) = \{w \in R_n : w$ is weakly increasing and no element except
   m can be repeated more than twice\}*

   (ii) $\#\Pi_n(13/2, 123/4) = 1 + \sum_{j=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-j}{2} \rfloor} \binom{n-j-i}{i}$

Proof. Let $w \in R_n(13/2, 123/4)$. Then $w \in R_n(13/2)$ and $w \in R_n(123/4)$. Then
because $w \in R_n(13/2)$, w must be weakly increasing by Sagan [Sagan (2010)]. Also
w must be of the form ab where a is an RGF ending with m where no element is
repeated more than twice and b is a block of some element x of any size by the
fact that $w \in R_n(123/4)$ from Theorem 3.6. Because w must be weakly increasing,
$x = m$. Then $w = ab$ where a is weakly increasing with no element repeated more
than twice and b is a block of m's of any length. $\square$

$$\#\Pi_n(13/2, 123/4) = 1 + \sum_{j=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-j}{2} \rfloor} \binom{n-j-i}{i}$$

Proof. Consider the enumeration of the avoidance class. We have just shown that
$w = ab$. When $m = 1$ then there is one way to form w by the Sterling numbers
of the second kind. Next consider when $1 < m \leq n$. Because $1 < m \leq n$, the
block of ms, beginning with the first occurrence of m can vary between the length
of one and $n-1$. Next consider how many of every other element can occur. Every
element 1 through $m - 1$ must occur at least once in a. There are $\binom{n-a_m-i}{i}$ ways
for us to choose i letters that get to be repeated in the word where a_m is the
total number of $m \in w$, because when we want to repeat i letters we will have
$m - 1 = n - a_m - i$, so there are $n - a_m - i$ choices for these i letters. Our choice
of i can vary from 0 to half of $n - a_m$ when $n - a_m$ is even because every element

from 1 to $m-1$ could be repeated, or 0 to half of $n - a_m - 1$ when $n - a_m$ is odd, because then every element except one between 1 and $m-1$ could be repeated in this case. Therefore our choice of i can vary from 0 to $\lfloor \frac{n-a_m}{2} \rfloor$. Therefore $\#\Pi_n(13/2, 123/4) = 1 + \sum_{j=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-j}{2} \rfloor} \binom{n-j-i}{i}$ where $j = a_m$. $\qquad\square$

Theorem 6.4. *The word characterization and cardinality of the avoidance class based on the patterns $13/2$ and $1/234$ are given by:*

  (i) $R_n(13/2, 1/234) = \{w \in R_n : w = ab$ *where a is a block of ones, b is weakly increasing beginning with 2 with no element repeated more than twice*$\}$

  (ii) $\#\Pi_n(13/2, 1/234) = 1 + \sum_{j=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-j}{2} \rfloor} \binom{n-j-i}{i}$

Proof. Let $w \in R_n(13/2, 1/234)$. Then $w \in R_n(13/2)$ and $w \in R_n(1/234)$. By Sagan we know that because $w \in R_n(13/2)$ then w is weakly increasing [Sagan (2010)]. By Jonathan Bloom $w = ab$ where a is a block of 1's of any length, and b begins with 2 and is in any reasonable order for an RGF and no element in b is repeated more than twice. Because $w \in R_n(13/2, 1/234)$ w must be weakly increasing where no element except one can be repeated more than twice. $\qquad\square$

$$\#\Pi_n(13/2, 1/234) = 1 + \sum_{j=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-j}{2} \rfloor} \binom{n-j-i}{i}$$

Proof. Consider the enumeration of the avoidance class. When $m = 1$ there is one way to order w. When $1 < m \le n$, a_1, or the number of 1s present in w will vary between one and $n-1$ when w is strictly increasing. Next let i denote the number of letters between 2 and m that are repeated. Because w is a weakly increasing word, as we find the cardinality order will not be a concern. We know that the length of w following the block of 1s is $n - a_1$ and that if there are i letters repeated, that there are $n - a_m - i$ choices for those repeated letters. Therefore there are $\binom{n-a_i-i}{i}$ ways to form the weakly increasing suffix of w for every i where i can vary from 0, when no letters are repeated, to $\frac{n-a_1}{2}$ when $n - a_m$ is even and every letter in the suffix is repeated, or the upper limit for i will be $\frac{n-a_1-1}{2}$ when $n - a_m$ is odd and every letter except one is repeated in the suffix. Therefore $0 \le i \le \lfloor \frac{n-a_1}{2} \rfloor$. Therefore $\#\Pi_n(13/2, 1/234) = 1 + \sum_{j=0}^{n-1} \sum_{i=0}^{\lfloor \frac{n-j}{2} \rfloor} \binom{n-j-i}{i}$ where $j = a_1$. $\qquad\square$

Theorem 6.6. *The word characterization and cardinality of the avoidance class based on the patterns $13/2/4$ and $134/2$ are given by:*

  (i) $R_n(13/2/4, 134/2) = \{w \in R_n :$
    (A) *w is weakly increasing*
    (B) *$m = 2$, 1 a block-singleton*
    (C) *$m = 2$, 1 and 2 block-singletons*
    (D) *$w = ab$ where a weakly increasing and b is either the last occurrence of a block-singleton in w, or $(m-1), m \in b$ such that both are block-singleton letters, $(b = (m-1)^l m^k (m-1) m)$*$\}$

  (ii) $\#\Pi_n(13/2/4, 134/3) = 2n - 5 + \sum_{m=1}^{n} \binom{n-1}{n-m} + \sum_{|a|=3}^{n-1} \sum_{m=3}^{|a|} \binom{|a|-1}{|a|-m}(m-1) +$
  $\sum_{|a|=1}^{n-4} \sum_{m=3}^{|a|+2} \binom{|a|-1}{|a|-(m-2)}(n - |a| - 2).$

Proof. Let $w \in R_n(13/2/4, 134/2)$. Then $w \in R_n(13/2/4)$ and $w \in R_n(134/2)$. Then by Theorem 3.9 w must be of one of these forms:

1. $m \leq 2$
2. $w = ab$ where a is weakly increasing ending with the last m in the first block of m's in w and b begins with $m - 1$ and contains m and $m - 1$ in any order.
3. $w = ab$ where a is weakly increasing, b is a block of some $x < (m - 1)$.

By theorem 3.7, because $w \in R_n(134/2)$ then any element $x \in w$ where x repeated more than twice will have a block of x's followed later in the word with a maximum of one more occurrence of x. Then there are three possibilities for $w \in R_n$.

a) Let w have $m \leq 2$. Then for w to be in $R_n(134/2)$, w could just contain all $1s$, w could be weakly increasing, or when w is not weakly increasing, then 1 could be the only block-singleton, or 1 and 2 are block-singletons.
b) Let $w = ab$ where b begins with $m - 1$ and contains m and $m - 1$ in any order. Then because $w \in R_n(134/2)$, $|b| = 1$ such that w is a weakly increasing word followed by one occurrence of $m - 1$, or $(m - 1)$ followed by m.
c) Let $w = ab$ where b is a block of some element $x < m - 1$. Then by theorem 3.7, $|b| = 1$ where the only occurrence in b is the element $x < m - 1$.

$\square$

$$\#\Pi_n(13/2/4, 134/3) = 2n - 5 + \sum_{m=1}^{n} \binom{n-1}{n-m} + \sum_{|a|=3}^{n-1} \sum_{m=3}^{|a|} \binom{|a|-1}{|a|-m}(m-1) + \sum_{|a|=1}^{n-4} \sum_{m=3}^{|a|+2} \binom{|a|-1}{|a|-(m-2)}(n-$$
$|a| - 2)$.

Proof. Let $w \in R_n(13/2/4, 134/2)$. Then w can take the following forms:

(i) Let w be weakly increasing and $1 \leq m \leq n$ by Lemma 2.2, there are $\sum_{m=1}^{n} \binom{n-1}{n-m}$ possibilities for w.
(ii) Let $m = 2$ and only 1 be a block-singleton letter. Then there are three assigned positions. We know that the first letter is 1 and the second letter is 2 and the last letter is also 1. Therefore the length of the block of 1s can vary from one to $n - 2$. So there are $n - 2$ possible words for w.
(ii) Let $m = 2$ and 1 and 2 are block-singleton letters. Then there are $n - 3$ possible words of this form.
(iii) Let $w = ab$ where $m \geq 3$, b is a block of z where z is a block-singleton, and a is weakly increasing. Then $3 \leq m \leq |a|$, and $3 \leq |a| \leq (n - 1)$. Also note that $1 \leq z \leq (m - 1)$ so there are $(m - 1)$ options for z. Then there will be $\sum_{|a|=2}^{n-1} \sum_{m=3}^{|a|} \binom{|a|-1}{|a|-m}(m - 1)$ possible words of this form.
- (iv) Finally let $w = ab$ where $b = (m - 1)^l m^k (m - 1)m$ and a is weakly increasing. Then we know that $1 \leq |a| \leq (n - 4)$, and so $3 \leq m \leq |a| + 2$ and there will be $(n - |a| - 2)$ ways to form b. Therefore there are a total of $\sum_{|a|=1}^{n-4} \sum_{m=3}^{|a|+2} \binom{|a|-1}{|a|-(m-2)}(n - |a| - 2)$ possible words of this form.

Therefore $\#\Pi_n(13/2/4, 134/3) = 2n - 5 + \sum\limits_{m=1}^{n} \binom{n-1}{n-m} + \sum\limits_{|a|=3}^{n-1} \sum\limits_{m=3}^{|a|} \binom{|a|-1}{|a|-m}(m-1) +$

$\sum\limits_{|a|=1}^{n-4} \sum\limits_{m=3}^{|a|+2} \binom{|a|-1}{|a|-(m-2)}(n - |a| - 2).$ $\qquad\square$

Theorem 6.7. *The word characterization and cardinality of the avoidance class based on the patterns $14/2/3$ and $1/24/3$ are given by:*

(i) $R_n(14/2/3, 1/24/3) = \{w \in R_n :$
  (A) *w is weakly increasing*
  (B) *$m \leq 2$*
  (C) *$w = ab$ where a is a prefix of 1s and 2s beginning with a 1, and b is a weakly increasing word beginning with the first occurrence of 3$\}$*

(ii) $\#\Pi_n(14/2/3, 1/24/3) = 2^{n-1} + \sum\limits_{|b|=1}^{n-2} \sum\limits_{i=1}^{|b|} \binom{|b|-1}{|b|-i}(2^{n-|b|} - 1)$

Proof. Let $w \in R_n(14/2/3, 1/24/3)$. Then by Theorem 3.10 we know that,

i) w could have $m \leq 2$. Because $1/24/3$ has three blocks, and any w of this form will be associated with a partition of only two blocks, $w \in R_n(1/24/3)$.

ii) Next consider when $w = ab$ where a is a prefix of 1s and 2s in any legal RGF order followed by a weakly increasing suffix b which begins with the first 3. Then because there are never two or more distinct letters between two of the same letters in w, $w \in R_n(14/2/3)$.

iii) Finally w could be a weakly increasing word with a block of 1s inserted between two distinct letters. Then $w \notin R_n(14/2/3)$ because the the subword of the first 1, 2, and 3, and first 1 of the second block of 1s will be associated with the standardized subpartition $14/2/3$. Therefore w cannot take this form.

$\qquad\square$

$$\#\Pi_n(14/2/3, 1/24/3) = 2^{n-1} + \sum\limits_{|b|=1}^{n-2} \sum\limits_{i=1}^{|b|} \binom{|b|-1}{|b|-i}(2^{n-|b|} - 1)$$

Proof. Let $w \in R_n(14/2/3, 1/24/3)$. Then by the word characterization we have:

i) When w has $m \leq 2$ we know that there are 2^{n-1} possible words of this form.

ii) When $w = ab$ where a is a prefix of 1s and 2s in any order and b is weakly increasing beginning with the first occurrence of 3, we know that w could be weakly increasing, and that there are a total of $2^{n-|b|} - 1$ possible forms for a. Then by Lemma 2.2 we know that there are a total of $\sum\limits_{|b|=1}^{n-2} \sum\limits_{i=1}^{|b|} \binom{|b|-1}{|b|-i}(2^{n-|b|} - 1)$ possible w where $i = m - 1$.

Therefore $\#\Pi_n(14/2/3, 1/24/3) = 2^{n-1} + \sum\limits_{|b|=1}^{n-2} \sum\limits_{i=1}^{|b|} \binom{|b|-1}{|b|-i}(2^{n-|b|} - 1).$ $\qquad\square$

Theorem 6.8. *The word characterization and cardinality of the avoidance class based on the patterns $14/2/3$ and $13/2/4$ are given by:*

(i) $R_n(14/2/3, 13/2/4) = \{w \in R_n :$
    (A) w is weakly increasing
    (B) $m \leq 2$
    (C) $w = ab$ where a is weakly increasing and b begins with the first occurrence of $(m-1)$ and then contains m and $(m-1)$ in any order$\}$

(ii) $\#\Pi_n(14/2/3, 1/24/3) = 2^{n-1} + \sum\limits_{|a|=1}^{n-2} \sum\limits_{i=1}^{|a|} \binom{|a|-1}{|a|-i}(2^{n-|a|} - 1)$

Proof. Let $w \in R_n(14/2/4, 13/2/4)$. Then by Theorem 3.9 we know the following,

  i) The first form w could take is when $m \leq 2$, then from part (i) of the previous proof $w \in R_n(14/2/3)$.
  ii) Next consider when $w = ab$ where a is weakly increasing and b contains a block of $z < (m-1)$. This $w \notin R_n(14/2/3)$ because if we consider the subword associated with the first z, the first $(m-1)$, and the first m along with the first occurrence of $z \in b$, this is associated with the standardized subpartition $14/2/3$.
  iii) Finally consider when $w = ab$ where a is weakly increasing and b contains m and $m-1$ in any legal RGF order. Then there are never two or more distinct letters between two of the same letter so $w \in R_n(14/2/3)$.

$\square$

$$\#\Pi_n(14/2/3, 13/2/4) = 2^{n-1} + \sum_{|a|=1}^{n-2} \sum_{i=1}^{|a|} \binom{|a|-1}{|a|-i}(2^{n-|a|} - 1)$$

Proof. Let $w \in R_n(14/2/3, 13/2/4)$. Then by the word characterization we have:

  i) When w has $m \leq 2$ we know that there are 2^{n-1} possible words of this form.
  ii) When $w = ab$ where a is a weakly increasing prefix and b contains $(m-1)$ and m in any order, we know that w could be weakly increasing, and that there are a total of $2^{n-|a|} - 1$ possible forms for b. Then by Lemma 2.2 we know that there are a total of $\sum\limits_{|a|=1}^{n-2} \sum\limits_{i=1}^{|a|} \binom{|a|-1}{|a|-i}(2^{n-|a|} - 1)$ possible w where $i = m - 1$.

Therefore $\#\Pi_n(14/2/3, 13/2/4) = 2^{n-1} + \sum\limits_{|a|=1}^{n-2} \sum\limits_{i=1}^{|a|} \binom{|a|-1}{|a|-i}(2^{n-|a|} - 1)$. $\square$

Theorem 6.9. *The word characterization and cardinality of the avoidance class based on the patterns $1/24/3$ and $134/2$ are given by:*

(i) $R_n(1/24/3, 134/2) = \{w \in R_n :$
    (A) w is weakly increasing
    (B) w is a weakly increasing except for the block-singleton $1\}$

(ii) $\#\Pi_n(1/24/3, 134/2) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$

Proof. By Theorem 3.7 and Theorem 3.10. $\square$

$$\#\Pi_n(1/24/3, 134/2) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$$

Proof. Let us consider the cardinality of the avoidance class. As we have shown there are two cases we need to consider.

(i) The first case is when w is weakly increasing. Then as shown in previous proofs (find for reference here), there will be $\binom{n-1}{n-m}$ different words for each possible m where we know that $1 \leq m \leq n$. Therefore when w is weakly increasing there are $\sum_{m=1}^{n} \binom{n-1}{n-m} = 2 + \sum_{m=2}^{n-1} \binom{n-1}{n-m}$.

(ii) The second, and only other option, is that $w = ab$ where a is weakly increasing, followed by b, where $|b| = 1$ and b contains only the letter $z \leq (m-1)$. First consider the different possibilities for the weakly increasing prefix. We know that $|a| = n-1$ because $|b| = 1$, and we know that when we have a weakly increasing word there are $\binom{n-2}{n-m-1}$ for some fixed m where $2 \leq m \leq n-1$. Then there are also $(m-1)$ options for z because $z \leq (m-1)$. Therefore there are $\sum_{m=2}^{n-1} \binom{n-2}{n-m-1}(m-1)$ possible words of the form $w = ab$.

Therefore $\#\Pi_n(1/24/3, 134/2) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$. $\qquad\square$

Theorem 6.10. *The word characterization and cardinality of the avoidance class based on the patterns $1/24/3$ and $124/3$ are given by:*

(i) $R_n(1/24/3, 124/3) = \{w \in R_n :$
   (A) w *is weakly increasing*
   (B) $w = ab$ *where a is weakly increasing with only one occurrence of z followed by the suffix, b, where $|b| \geq 1$ and contains $z \leq (m-1)$ such that z is a singleton-block in $w\}$*
(ii) $\#\Pi_n(1/24/3, 124/3) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$

Proof. Let $w \in R_n(1/24/3)$, then by Theorem 3.10 we know that

i) We have $m \leq 2$, then $w \in R_n(1/23/4)$ also because w is associated with a partition of only two blocks.

ii) Next $w = ab$ where a is a prefix of 1s and 2s in any order followed by the weakly increasing suffix b. However for $w \in R_n(1/23/4)$ we must have b strictly increasing except for m, and there must only be one occurrence of $2 \in a$, and either one 1 after 2 or the entire word is increasing.

iii) Finally w is a weakly increasing word with a block of 1s inserted. This block of 1s must be of length one unless the block occurs at the end of the word for $w \in R_n(1/23/4)$ and the only other letter than can be repeated is 1, or if the block of inserted 1s occurs somewhere else, m can also be repeated.

$\qquad\square$

$$\#\Pi_n(1/24/3, 124/3) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$$

Proof. Let $w \in R_n(1/24/3, 124/3)$. Then there are several possibilities.

i) We have w where w is weakly increasing. Then by Lemma 2.2 there are
$$\sum_{m=1}^{n} \binom{n-1}{n-m} = 2 + \sum_{m=2}^{n-1} \binom{n-1}{n-m}.$$

ii) Otherwise $w = ab$ where b contains a block of z where z a singleton-block in w and a is weakly increasing. Then $1 \leq z \leq (m-1)$ and there are m blocks to assign $n - m - 1$ letters. So by multisets we have that there are $\sum_{m=2}^{n-1} \binom{n-2}{n-m-1}(m-1)$ ways to order w.

Therefore $\#\Pi_n(1/24/3, 124/3) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$. $\qquad\square$

Theorem 6.12. *The word characterization and cardinality of the avoidance class based on the patterns $13/2/4$ and $1/23/4$ are given by:*

(i) $R_n(13/2/4, 1/23/4) = \{w \in R_n :$
   (A) $m \leq 2$
   (B) $w = abc$ where a is a prefix of 1s, b is strictly increasing beginning with 2 and ending with $(m-1)$ and c is a block of $z \leq m\}$

(ii) $\#\Pi_n(13/2/3, 1/23/4) = 1 + 2^{n-1} + \sum_{m=3}^{n-1}(nm - m^2)$

Proof. Let $w \in R_n(13/2/4)$. Then by Theorem 3.9,

i) First we can consider when $m \leq 2$ which also indicates that $w \in R_n(13/2/4)$.
ii) Next we can consider when $w = ab$ where a is weakly increasing and b is some letter z that $z \leq m$, then for $w \in R_n(1/23/4)$ we also know that only 1 and z can be repeated.
iii) Finally consider when $w = ab$ such that a is weakly increasing and b contains m and $m-1$ in any order (such that b is not the same as b from part (ii)). Then $w \notin R_n(1/23/4)$ because we can take the subword of the first 1, the first $m-1$, the next $m-1$ and the m that must occur after these two (m-1) terms. The standardized subpartition associated with this subword is $1/23/4$.

$\qquad\square$

$$\#\Pi_n(13/2/4, 1/23/4) = 1 + 2^{n-1} + \sum_{m=3}^{n-1}(nm - m^2)$$

Proof. Let $w \in R_n(13/2/4, 1/23/4)$ where $m \leq 2$. There are 2^{n-1} possible w of this form.

Next let $w = abc$ where a is a block of 1s, b a strictly increasing word from 2 to m and c a potentially empty block of some letter $z \leq m$. Then when $z = m$ we know that $3 \leq m \leq n$ and that there are $n - m + 1$ options for the length of the block of 1s which also defines the length of the block of ms. So when $z = m$ there are $\sum_{m=3}^{n}(n - m + 1) = 1 + \sum_{m=3}^{n-1}(n - m + 1)$ possible w of this form. Next consider when $z \leq (m-1)$. Then $3 \leq m \leq (n-1)$, there are $m - 1$ possible z, and the block of 1s can vary in length from one to $n - m$. Therefore there are $\sum_{m=3}^{n-1}(n-m)(m-1)$ possible w of this form.

Therefore $\#\Pi_n(13/2/4, 1/23/4) = 1 + 2^{n-1} + \sum_{m=3}^{n-1}(n - m + 1 + nm - m^2 - n + m) = 1 + 2^{n-1} + \sum_{m=3}^{n-1}(nm - m^2)$ $\qquad\square$

Theorem 6.9. *The word characterization and cardinality of the avoidance class based on the patterns 1/24/3 and 134/2 are given by:*

(i) $R_n(1/24/3, 134/2) = \{w \in R_n :$
    (A) w is weakly increasing
    (B) w is a weakly increasing except for the block-singleton 1}

(ii) $\#\Pi_n(1/24/3, 134/2) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$

Proof. First let $w \in R_n(1/24/3, 134/2)$. Then $w \in R_n(1/24/3)$ and $w \in R_n(134/2)$. By Theorem 3.7 and Theorem 3.10, w can either be weakly increasing, or if there is a block-singleton present it must be 1.

Next let us consider the cardinality of the avoidance class. As we have shown there are two cases we need to consider.

    (i) The first case is when w is weakly increasing. Then by Lemma 2.2 there are $\sum_{m=1}^{n} \binom{n-1}{n-m} = 2 + \sum_{m=2}^{n-1} \binom{n-1}{n-m}$ possible forms of w.

    (ii) The second, and only other option, is is weakly increasing except for a block-singleton 1, however the only difference here is that there are $(m-1)$ options for the distinct letter that occurs before the singleton 1 and there are only $n-1$ weakly increasing letters. Therefore there are $\sum_{m=2}^{n-1} \binom{n-2}{n-m-1}(m-1)$ possible words.

Therefore $\#\Pi_n(1/24/3, 134/2) = 2 + \sum_{m=2}^{n-1}(\binom{n-1}{n-m} + \binom{n-2}{n-m-1}(m-1))$.   □

Theorem 7.2. *The patterns 1/23,13/2/4*

$$ls(w) = \sum_{i=1}^{m} a_i - m$$

$$rb(w) = \sum_{i=1}^{m-1} (m-i)a_i$$

$$lb = \begin{cases} 0 & \text{if the word does not end with a one} \\ m-1 & \text{if the last letter is a one} \end{cases}$$

$$rs = \begin{cases} 0 & \text{if the word does not end with a one} \\ \sum_{i=2}^{m} a_i & \text{if the last letter is a one} \end{cases}$$

Proof. The first two formulas for $ls(w)$ and $rb(w)$ are given by Lemma 2.1, 2.2. Now let $w \in R_n(1/23, 13/2/4)$. Then by Theorem 6.3 there are two possibilities for the form of w. The first is when $w = ab$ where a is a block of ones and b is strictly increasing. Then w is weakly increasing so $lb(w) = 0$ and $rs(w) = 0$. The other possibility is that $w = abc$ where a is a block of 1's, b is strictly increasing, and c is a 1 at the end of the word. Then because ab this portion of the word will not contribute to $lb(w)$. Now w will have a one after ab so there will be $m-1$ elements that are larger and occur before this 1. Therefore when $w = abc$, $lb(w) = m-1$. Now consider the right and smaller statistic. Every element in b will be larger than the 1 in c, so each will have exactly one contribution to $rs(w)$. Therefore when $w = abc$, $rs(w) = |b|$.   □

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