VIBRATION UNDER VARIABLE MAGNITUDE MOVING DISTRIBUTED MASSES OF NON-UNIFORM BERNOULLI–EULER BEAM RESTING ON PASTERNAK ELASTIC FOUNDATION

T. O. Awodola∗**, S. A. Jimoh, B. B. Awe**
Federal University of Technology, Akure, Nigeria
∗E-mail: toawodola@futa.edu.ng

Abstract. The dynamic response to variable magnitude moving distributed masses of simply supported non-uniform Bernoulli–Euler beam resting on Pasternak elastic foundation is investigated in this paper. The problem is governed by fourth order partial differential equation with variable and singular coefficients. The main objective of this work is to obtain closed form solution to this class of dynamical problem. In order to obtain the solution, a technique based on the method of Galerkin with the series representation of Heaviside function is first used to reduce the equation to second order ordinary differential equations with variable coefficients. Thereafter the transformed equations are simplified using (i) The Laplace transformation technique in conjunction with convolution theory to obtain the solution for moving force problem and (ii) finite element analysis in conjunction with Newmark method to solve the analytically unsolvable moving mass problem because of the harmonic nature of the moving load. The finite element method is first used to solve the moving force problem and the solution is compared with the analytical solution of the moving force problem in order to validate the accuracy of the finite element method in solving the analytically unsolvable moving mass problem. The numerical solution using the finite element method is shown to compare favorably with the analytical solution of the moving force problem. The displacement response for moving distributed force and moving distributed mass models for the dynamical problem are calculated for various time t and presented in plotted curves.

Keywords: moving mass; finite element; Newmark method; Pasternak elastic foundation; Galerkin's method; resonance.

1. INTRODUCTION

Force vibration of elastic bodies (stretched string, spring mass system, rods, etc.) have been extensively studied by several authors [1–11]. The vibrations may be due to (i) a force (load) which is a function of the space coordinates only or (ii) a force which

varies in both space and time. Such forces can either be of constant magnitude or variable magnitude. The present work concerns the effects of a force of variable magnitude moving at constant speed on an elastic body. In particular, the elastic body under consideration is the beam. It should, however, be mentioned from the onset, that such an elastic body (long and thin or stubby) is normally considered as a one-dimensional body [9–11] whose physical properties (stiffness, mass, length) are described with reference to a single dimension, the position along the elastic axis. Consequently, the partial differential equation describing the motion of such an elastic body is made up of only two independent variables, distance along the axis and time. Limiting the discussions to that of a moving force on a finite beam, Timoshenko [2], Inglis [12] and Muscolino [13] considered the problem of transverse oscillations of a beam subjected to a harmonic force moving with a uniform velocity. They assumed that the beam is simply supported. An analysis of the effect of such a moving force on the beam is given. Recently, Steele [14] investigated the effect of this moving force on beams to a unit force moving at a uniform velocity. The effect of this moving force on beams with and without an elastic foundation are analyzed.

Some of the previous works involving non-uniform beams include that of Wu [15] studied the dynamic responses of multi-span non-uniform beams under moving loads using the transfer matrix method. Dogush [16] also studied dynamic behavior of multi-span non-uniform beams traversed by a moving load at constant and variable velocities using both modal analysis and direct methods. Ahmadian et al. [17] investigated the analysis of a variable cross-section beam subjected to a moving concentrated force and mass using finite element method. However, the above research works on both uniform and non-uniform beams are impactful, but moving loads have been idealized as moving concentrated loads which act at certain points on the structure and along a single line in space. That is, the moving load is modelled as a lumped load. In practice, it is known that loads are actually distributed over a small segment or over the entire length of the structural member as they traverse the structure. Such moving loads are termed uniform distributed loads. Concentrated forces are mere mathematical idealization, which cannot be found in the real world, where surface forces act over an area and body forces act within volume. We also remark at this juncture, that only long thin uniform beam (called Euler's beam) resting on one parametric foundation or bi-parametric foundation that is not harmonic in nature were considered.

Thus, the present investigation is concerned with the vibration under variable magnitude moving distributed masses of non-uniform Bernoulli–Euler beam resting on Pasternak elastic foundation. The vital aspects of inertia terms are considered. Specifically, the elastic properties of the beam such as the flexural rigidity and the mass per unit length of the beam are assumed not constant. That is, the beam is of non-uniform cross-section and mass contains negligible damping.

2. THEORY AND FORMULATION OF THE PROBLEM

In this study, the problem of a non-uniform Bernoulli–Euler beam and carrying a mass M is investigated. The beam's properties such as moment of inertia I and the mass per unit length μ of the beam remained changing along the span length L. The transverse displacement $V(x,t)$ of the beam travelling at a uniform velocity as shown in Fig. 1 is
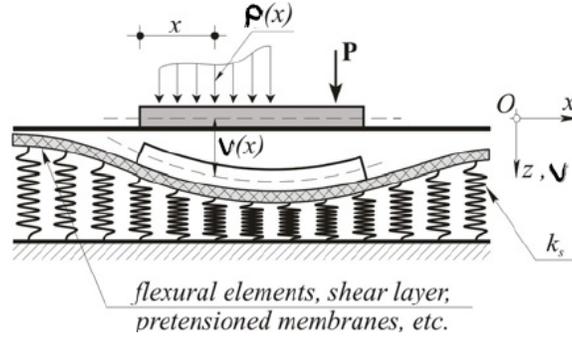
Fig. 1. Geometry of a variable moving masses of non-uniform Bernoulli–Euler beam resting on Pasternak foundation

given as

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right]+\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2}-N_0\frac{\partial^2 V(x,t)}{\partial x^2}+K_0 V(x,t)-G_0\frac{\partial^2 V(x,t)}{\partial x^2}=P(x,t), \quad (1)$$

where t is the time coordinate, $\mu(x)$ is the variable mass per unit length of the beam, $EI(x)$ is the variable flexural stiffness, x is the spatial coordinate, K_0 is the foundation stiffness, G_0 is the shear modulus, N_0 is the axial force and $P(x,t)$ is the uniform distributed load acting on the beam. In this problem, the distributed load moving on the beam under consideration has mass commensurable with the mass of the beam. Consequently, the load inertia is not negligible but significantly affects the behaviour of the dynamical system. Thus, the distributed load $P(x,t)$ takes the form

$$P(x,t) = \cos(\omega t)\sum_{i=1}^{j} M_i gH\left[x - f(t)\right]\left[1-\frac{1}{g}\frac{\mathrm{d}^2 V(x,t)}{\mathrm{d}t^2}\right],$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\partial^2}{\partial t^2}+2\frac{\mathrm{d}f(t)}{\mathrm{d}t}\frac{\partial^2}{\partial x\partial t}+\left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)^2\frac{\partial^2}{\partial x^2}+\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\frac{\partial}{\partial x}, \quad (2)$$

where g denotes the acceleration due to gravity, $\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}$ is a convective acceleration operator, $\dfrac{\partial^2}{\partial t^2}$ is the support beam's acceleration at the point of contact with the moving mass, $\dfrac{\mathrm{d}f(t)}{\mathrm{d}t}\dfrac{\partial^2}{\partial x\partial t}$ is the well-known Coriolis acceleration, $\left(\dfrac{\mathrm{d}f(t)}{\mathrm{d}t}\right)^2\dfrac{\partial^2}{\partial x^2}$ is the centripetal acceleration of the moving mass and $\dfrac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}\dfrac{\partial}{\partial x}$ is the acceleration component in the vertical direction when the moving load is not constant.

In the same vein, for constant velocity c the direction/distance travelled by the load on the beam at any given instance of the time t is given as

$$f(t) = c_i t, \quad (3)$$

where x_0 represent the point of application of force $P(x,t)$ at any instant time $t=0$.

Moreover, the moving load is assumed to be of mass, M and time t is assumed to be limited to that interval of time within which the mass M is on the beam. i.e.

$$0 \leq f(t) \leq L, \tag{4}$$

and $H\left[x - f(t)\right]$ is the Heaviside function, which is a typical engineering function made to measure engineering application involving function that are either "on" or "off" and it is defined as

$$H(x) = \begin{cases} 1, & x > ct. \\ 0, & x \leq ct. \end{cases} \qquad H\left[x - f(t)\right] = \begin{cases} 1, & x \geq f(t). \\ 0, & x < f(t). \end{cases} \tag{5}$$

As an example, let the variable moment of inertia I and the mass per unit length of the beam be defined, respectively, as [18]

$$I(x) = I_0\left(1 + \sin\frac{\pi x}{L}\right)^3, \quad \mu(x) = \mu_0\left(1 + \sin\frac{\pi x}{L}\right), \tag{6}$$

where I_0 and μ_0 are constant moment of inertia and constant mass per unit length of the corresponding uniform beam respectively. To this end, substituting Eqs. (2), (3) and (6) into (1), after some simplification and rearrangement yields

$$\frac{EI_0}{4}\left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right)\frac{\partial^4 V(x,t)}{\partial x^4} + \frac{6\pi EI_0}{4L}\left(5\cos\frac{\pi x}{L} + 4\sin\frac{2\pi x}{L}\right.$$

$$\left. - \cos\frac{3\pi x}{L}\right)\frac{\partial^3 V(x,t)}{\partial x^3} + \frac{3\pi^2 EI_0}{4L^2}\left(3\sin\frac{3\pi x}{L} + 8\cos\frac{2\pi x}{L} - 5\sin\frac{\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial x^2}$$

$$+ \mu_0\left(1 + \sin\frac{\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial t^2} - N_0\frac{\partial^2 V(x,t)}{\partial x^2} + K_0 V(x,t) - G_0\frac{\partial^2 V(x,t)}{\partial x^2}$$

$$+ \cos(\omega t)\sum_{i=1}^{j} M_i H(x - c_i t)\left[\frac{\partial^2 V(x,t)}{\partial t^2} + 2c_i\frac{\partial^2 V(x,t)}{\partial x \partial t} + c_i^2\frac{\partial^2 V(x,t)}{\partial x^2}\right] = \sum_{i=1}^{j} M_i g \cos\omega t H(x - c_i t). \tag{7}$$

The boundary conditions of the above problem are assumed to be arbitrary, that is, it can take any form of the classical boundary conditions. The initial conditions however without any loss of generality is given by

$$V(x,0) = \frac{\partial V(x,0)}{\partial t} = 0. \tag{8}$$

Eq. (7) forms the fundamental equation of the dynamic problem.

2.1. Solution procedure

Eq. (7) is a non-homogeneous partial differential equation with variable coefficients. Evidently, the method of separation of variables is inapplicable as a difficulty arises in getting separate equations whose functions are function of a single variable. Thus, we resort to a modification of the approximate method best suited for solving diverse problem in dynamics of structures generally referred to as Galerkin's Method. Therefore, we use the Galerkin's method described in Oni and Awodola [19,20] to reduce the fourth order

partial differential equation to a sequence of second order ordinary differential equation. Thus, yield a solution of the form

$$V(x,t) = \sum_{m=1}^{n} Y_m(t)U_m(x), \tag{9}$$

where $U_m(x)$ is chosen as a suitable kernel of the Galerkin's method in (9) such that the boundary conditions given are satisfied. It is remarked at this juncture that the beam under consideration is assumed to have general boundary conditions at the edges $x = 0$ and $x = L$, therefore, we choose an appropriate selection of function for beam problem i.e beam shapes. Thus, the mth normal mode of vibration of the beam

$$U_m(x) = \sin\frac{\lambda_m x}{L} + A_m \cos\frac{\lambda_m x}{L} + B_m \sinh\frac{\lambda_m x}{L} + C_m \cosh\frac{\lambda_m x}{L}, \tag{10}$$

is chosen such that the boundary conditions are satisfied. The kernel is chosen as

$$U_k(x) = \sin\frac{\lambda_k x}{L} + A_k \cos\frac{\lambda_k x}{L} + B_k \sinh\frac{\lambda_k x}{L} + C_k \cosh\frac{\lambda_k x}{L}, \tag{11}$$

where in (10) and (11), λ_m and λ_k are the mode frequency. A_m, B_m, C_m, A_k, B_k and C_k are constants which are obtained by substituting (6) and (7) into appropriate boundary condition. Therefore, substituting Eqs. (9) into (7), yields

$$
\begin{aligned}
\sum_{m=1}^{n} & \left[\left(1 + \sin\frac{\pi x}{L}\right) U_m(x)\ddot{Y}_m(t) + \frac{EI_0}{4\mu_0}\left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right) U_m''(x)Y_m(t) \right. \\
& + \frac{6\pi EI_0}{4\mu_0 L}\left(5\cos\frac{\pi x}{L} + 4\sin\frac{2\pi x}{L} - \cos\frac{3\pi x}{L}\right)U_m'''(x)Y_m(t)\frac{3\pi^2 EI_0}{4\mu L^2}\left(3\sin\frac{3\pi x}{L} + 8\cos\frac{2\pi x}{L}\right. \\
& \left. - 5\sin\frac{\pi x}{L}\right)U_m''(x)Y_m(t) - \frac{N_0}{\mu}U_m''(x)Y_m(t) + \frac{K_0}{\mu}U_m(x)Y_m(t) - \frac{G_0}{\mu}U_m''(x)Y_m(t) \\
& + \sum_{i=1}^{j} M_i \cos\omega t \left[H(x - c_i t)U_m(x)\ddot{Y}_m(t) + 2c_i H(x - c_i t)U_m'(x)\dot{Y}_m(t) \right. \\
& \left. + c_i^2 H(x - c_i t)U_m''(x) \right] - \sum_{i=1}^{j} M_i g \cos\omega t H(x - c_i t) \right] = 0.
\end{aligned} \tag{12}
$$

In order to determine an expression for $Y_m(t)$, we shall consider a mass M travelling with uniform velocity c along the x-coordinate. The solution for any arbitrary umber of moving masses can be obtained by superposition of the individual solution since the governing equation is linear. Therefore, for the single mass M_1, it is required that the expression on the left hand side of Eq. (12) is orthogonal to the function $U_k(x)$. Thus, using Eqs. (10) and (11) in (12), yields

$$I_0^*\ddot{Y}_m(t) + I_1^*Y_m(t) + \frac{\cos\omega t}{\mu_0}M\left[I_2^*\ddot{Y}_m(t) + 2cI_3^*\dot{Y}_m(t) + c^2I_4^*Y_m(t)\right] = \frac{g\cos\omega t}{\mu_0}MI_5^0, \tag{13}$$

where

$$I_0^* = \sum_{m=1}^{n} \int_0^L \left(1 + \sin\frac{\pi x}{L}\right) U_m(x) U_k(x) \mathrm{d}x, \quad I_1^* = I_{1A} + I_{1B} + I_{1C} - I_{1D} + I_{1E} - I_F, \quad (14)$$

$$I_{1A} = \frac{EI_0}{4\mu_0} \sum_{m=1}^{n} \int_0^L \left(10 - 6\cos\frac{2\pi x}{L} + 15\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}\right) U_m^{iv}(x) U_k(x) \mathrm{d}x, \quad (15)$$

$$I_{1B} = \frac{6\pi EI_0}{4\mu_0 L} \sum_{m=1}^{n} \int_0^L \left(5\cos\frac{\pi x}{L} + 4\sin\frac{2\pi x}{L} - \cos\frac{3\pi x}{L}\right) U_m'''(x) U_k(x) \mathrm{d}x, \quad (16)$$

$$I_{1C} = \frac{3\pi^2 EI_0}{4\mu_0 L^2} \sum_{m=1}^{n} \int_0^L \left(3\sin\frac{3\pi x}{L} + 8\cos\frac{2\pi x}{L} - 5\sin\frac{\pi x}{L}\right) U_m'' U_k(x) \mathrm{d}x, \quad (17)$$

$$I_{1D} = \frac{N}{\mu_0} \sum_{m=1}^{n} \int_0^L U_m'' U_k(x) \mathrm{d}x, \; I_{1E} = \frac{K_0}{\mu_0} \sum_{m=1}^{n} \int_0^L U_m U_k(x) \mathrm{d}x, \; I_{1F} = \frac{G_0}{\mu_0} \sum_{m=1}^{n} \int_0^L U_m'' U_k(x) \mathrm{d}x,$$
$$(18)$$

$$I_2^* = \sum_{m=1}^{n} \int_0^L H(x - ct) U_m U_k(x) \mathrm{d}x, \quad I_3^* = \sum_{m=1}^{n} \int_0^L H(x - ct) U_m' U_k(x) \mathrm{d}x, \quad (19)$$

$$I_4^* = \sum_{m=1}^{n} \int_0^L H(x - ct) U_m'' U_k(x) \mathrm{d}x, \quad I_5^0 = \int_0^L H(x - ct) U_k(x) \mathrm{d}x. \quad (20)$$

Using the property of Heaviside function, it can be expressed in series form given by [13] i.e.

$$H(x - ct) = \frac{1}{4} + \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi x \cos(2n+1)\pi ct}{2n+1} - \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi x \sin(2n+1)\pi ct}{2n+1}.$$
$$(21)$$

Thus, in view of (14)–(20) and (21), it can be shown that

$$\ddot{Y}_m(t) + \frac{I_1^*(m,k)}{I_0^*(m,k)} Y_m(t) + \frac{\varepsilon_0 \cos\omega t}{I_0^*(m,k)} \left\{ \left[L\psi_{1A}(m,k) + \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_5^*(m,k) \right. \right.$$

$$\left. - \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} \times I_6^*(m,k) \right] \ddot{Y}_m(t) + 2c \left[L\psi_{2A}(m,k) + \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_7^*(m,k) \right.$$

$$\left. - \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} I_8^*(m,k) \right] \dot{Y}_m(t) + c^2 \left[L\psi_{3A}(m,k) + \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)\pi ct}{2n+1} I_9^*(m,k) \right.$$

$$\left. \left. - \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi ct}{2n+1} I_{10}^*(m,k) \right] Y_m(t) \right] \right\} = \frac{MgL\cos\omega t}{\mu\lambda_k I_0^*(m,k)} \times \left[-\cos\lambda_k x + A_k \sin\lambda_k x \right.$$

$$\left. + B_k \cosh\lambda_k x + C_k \sinh\lambda_k x + \cos\frac{\lambda_k ct}{L} - A_k \sin\frac{\lambda_k ct}{L} - B_k \cosh\frac{\lambda_k ct}{L} - C_k \sinh\frac{\lambda_k ct}{L} \right],$$
$$(22)$$

where

$$\varepsilon_0 = \frac{M}{\mu L}.$$ (23)

Eqs. (22) is now the fundamental equation governing the dynamic problem. This coupled non-homogeneous second order ordinary differential equation holds for all variants of classical boundary conditions. It follows that two special cases of Eq. (22) arise, namely, the moving force and moving mass problems.

2.2. Non-uniform Bernoulli–Euler beam traversed by moving distributed force for simply supported end condition

In this section, an approximate model of the differential equation describing the response of the elastic structure is obtained by neglecting inertia terms. i.e. $\Gamma_0 = 0$ and we shall limit our example on simply supported end condition. In this case, the displacement and the bending moment vanish. Thus

$$V_m(0,t) = 0 = V_m(L,t), \quad \frac{\partial^2 V_m(0,t)}{\partial x^2} = 0 = \frac{\partial^2 V_m(L,t)}{\partial x^2},$$ (24)

and hence for normal modes

$$U_m(0) = 0 = U_m(L), \quad \frac{\partial^2 U_m(0)}{\partial x^2} = 0 = \frac{\partial^2 U_m(L)}{\partial x^2},$$ (25)

which implies that

$$U_k(0) = 0 = U_k(L), \quad \frac{\partial^2 U_k(0)}{\partial x^2} = 0 = \frac{\partial^2 U_k(L)}{\partial x^2}.$$ (26)

Applying (25) and (26) on (10), we have

$$A_m = A_k = B_m = B_k = C_m = C_k = 0, \quad \lambda_m = m\pi, \quad \lambda_k = k\pi,$$ (27)

as the mode frequencies, and

$$U_m(x) = \sin\frac{m\pi x}{L}, \quad U_k(x) = \sin\frac{k\pi x}{L},$$ (28)

as the mode functions. Using (24)–(28) in (22), yields

$$\ddot{Y}_m(t) + \omega_f^2 Y_m(t) = P_k \cos\omega t \left[\cos\theta_k t + R_k\right],$$ (29)

where

$$\omega_f^2 = \frac{I_1^*(m,k)}{I_0^*(m,k)}, \quad P_k = \frac{MgL}{\mu k\pi I_0^*(m,k)}, \quad \theta_k = \frac{k\pi c}{L}, \quad R_k = -(-1)^k,$$ (30)

$$I_0^*(m,k) = \begin{cases} \dfrac{-4mkL}{\pi[1-(m+k)^2][1-(m-k)^2]}, & m \neq k \\ \dfrac{L}{2} - \dfrac{4m^2 L}{\pi(1-4m^2)}, & m \equiv k \end{cases}$$ (31)

$$I_1^*(m,k) = I_{1A} - I_{1B} - I_{1C} + I_{1D} + I_{1E} + I_{1F},$$

$$I_{1A} = \begin{cases} \dfrac{m^4\pi^4 EI_0}{4\mu_0 L^4}\left[\dfrac{-60mkL}{\pi[1-(m+k)^2][1-(m-k)^2]} + \dfrac{12mkL}{\pi[9-(m+k)^2][9-(m-k)^2]}\right], & m \neq k \\[3mm] \dfrac{m^4\pi^4 EI_0}{4\mu_0 L^4}\left[5L - \dfrac{4m^2 L}{\pi(1-4m^2)} + \dfrac{4m^2 L}{3\pi(9-4m^2)}\right], & m \equiv k. \end{cases}$$

(32)

$$I_{1B} = \begin{cases} \dfrac{3m^3\pi^3 EI_0}{8\mu_0 L^3}\left[\dfrac{m+k}{9-(m+k)^2} - \dfrac{m-k}{9-(m-k)^2} - 5\left(\dfrac{m+k}{1-(m+k)^2} + \dfrac{m-k}{1-(m-k)^2}\right)\right], & m \neq k \\[3mm] \dfrac{3m^3\pi^3 EI_0}{8\mu_0 L^3}\left[\dfrac{2m}{9-4m^2} - \dfrac{10m}{1-4m^2}\right], & m \equiv k. \end{cases}$$

(33)

$$I_{1C} = \begin{cases} \dfrac{3m^2\pi^4 EI_0}{4\mu_0 L^4}\left[\dfrac{-36mkL}{\pi[9-(m+k)^2][9-(m-k)^2]} + \dfrac{20mkL}{\pi[1-(m+k)^2][1-(m-k)^2]}\right], & m \neq k \\[3mm] \dfrac{3m^2\pi^4 EI_0}{4\mu_0 L^4}\left[\dfrac{-4mkL}{\pi(9-4m^2)} + \dfrac{20m^2 L}{\pi(1-4m^2)}\right], & m \equiv k. \end{cases}$$

(34)

$$I_{1D} = \frac{m^2\pi^2 N}{2\mu_0 L}, \quad I_{1E} = \frac{K_0 L}{2\mu_0}, \quad I_{1F} = \frac{m^2\pi^2 G_0}{2\mu_0 L}.$$

(35)

Therefore, solving (29) using the method of Laplace transforms and convolution with the initial conditions (8), we have

$$V(x,t) = \sum_{m=1}^{n}\frac{P_k}{2\omega_f}\left\{\frac{\omega_f}{(\omega_f^2-\omega^2)(\omega_f^2-\Omega_1^2)(\omega_f^2-\Omega_2^2)}\left[2R_k(\omega_f^2-\Omega_1^2)(\omega_f^2-\Omega_2^2)(\cos\omega t - \cos\omega_f t)\right.\right.$$
$$\left.\left. +(\omega_f^2-\omega^2)(\omega_f^2-\Omega_2^2)(\cos\Omega_1 t - \cos\omega_f t) + (\omega_f^2-\omega^2)(\omega_f^2-\Omega_1^2)(\cos\Omega_2 t - \cos\omega_f t)\right]\sin\frac{m\pi x}{L}.\right.$$

(36)

(36) above represents the transverse displacement response to a distributed force moving at constant velocity of a non-uniform simply supported Bernoulli–Euler beam resting on Pasternak elastic foundation.

2.3. Non-uniform Bernoulli–Euler beam traversed by moving distributed mass

In this section, the solution to the entire equation (22) is sought when no terms of the coupled differential equation is neglected. Evidently, an exact solution to this equation is not possible. All conventional methods break down, even the popular Struble's technique [21] could not handle it because of the variability of the magnitude of the moving load. Hence we resort to using finite element method (FEM) to model the structure. Then Newmark numerical method of integration is used to solve the resulting semi-discrete time-dependent equation to obtain the desired responses.

2.3.1. *Finite Element Method (FEM)*

The finite element technique assumes that the unknown transverse deflection of the non-uniform beam, $V(x,t)$, can be represented approximately by a set of piecewise continuous functions which are defined over a finite number of sub-regions called elements

and composed of the numerical values of the unknown deflection within the region. Thus, the first step involved in the technique, consist of dividing the spatial solution domain of the non-uniform beam, which happened to be the length of the beam in this case, into a number of sub-domains known as finite elements. These elements are joined to each other at selected points called nodes. Next, the weak or variational form of the governing equation (1) is constructed as follows:

Consider a typical element of length L so that its domain $\lambda_e = (0, L)$. Substituting (2) and (3) into (1), we have

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right] + \mu(x)\frac{\partial^2 V(x,t)}{\partial t^2} - N_0\frac{\partial^2 V(x,t)}{\partial x^2} + K_0 V(x,t) - G_0\frac{\partial^2 V(x,t)}{\partial x^2}$$
$$= \cos(\omega t)\sum_{i=1}^{j} M_i H(x - c_i t)\left[g - \left(\frac{\partial^2 V(x,t)}{\partial t^2} + 2c_i\frac{\partial^2 V(x,t)}{\partial x \partial t} + c_i^2\frac{\partial^2 V(x,t)}{\partial x^2}\right)\right]. \tag{37}$$

In order to solve Eq. (37), we shall consider a mass M travelling with uniform velocity c along the x-coordinate. The solution for any arbitrary number of moving masses can be obtained by superposition of the individual solution since the governing equation is linear. Therefore, for the single mass M_1, let $W(x)$ be Galerkin's weight function. Multiplying Eq. (37) by the weight function and integrate over the domain λ_e, after some simplification and rearrangement yields

$$\int_0^{L^e} EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\frac{\partial^2 W(x)}{\partial x^2}dx + \int_0^{L^e}\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2}W(x)dx - \int_0^{L^e}(N_0 + G_0)\frac{\partial^2 V(x,t)}{\partial x^2}W(x)dx$$
$$+ \int_0^{L^e}K_0 V(x,t)W(x)dx - Mg\cos(\omega t)\int_0^{L^e}H(x-ct)W(x)dx + M\cos(\omega t)\int_0^{L^e}H(x-ct)\frac{\partial^2 V(x,t)}{\partial t^2}W(x)dx$$
$$+ 2Mc\cos(\omega t)\int_0^{L^e}H(x-ct)\frac{\partial^2 V(x,t)}{\partial x \partial t}W(x)dx + Mc^2\cos(\omega t)\int_0^{L^e}H(x-ct)\frac{\partial^2 V(x,t)}{\partial x^2}W(x)dx$$
$$+ W(L^e)B_3^e - W(0)B_1^e - \frac{\partial W}{\partial x}\bigg|_{x=L^e}B_4^e + \frac{\partial W}{\partial x}\bigg|_{x=0}B_2^e = 0, \tag{38}$$

where

$$\lambda = \frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right], \quad \phi = EI(x)\left[\frac{\partial^2 V(x,t)}{\partial x^2}\right], \quad B_k^e = \left[\lambda W(x)\right]\bigg|_0^{L^e} - \left[\phi\frac{\partial W(x)}{\partial x}\right]\bigg|_0^{L^e}, \tag{39}$$

λ is the shear force, ϕ is the bending moment and $B_k^e, (k = 1, 2, \ldots, 4)$ are the extremely important and necessary four boundary terms, two at each of the end nodes of the element. Furthermore, it can be readily shown that

$$\int_0^{L^e}H(x-ct)f(x)dx = \int_{ct}^{L^e}f(x). \tag{40}$$

Thus, Eq. (38) becomes

$$
\int_0^{L^e} EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\frac{\partial^2 W(x)}{\partial x^2}\mathrm{d}x + \int_0^{L^e}\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2}W(x)\mathrm{d}x - \int_0^{L^e}(N_0+G_0)\frac{\partial^2 V(x,t)}{\partial x^2}W(x)\mathrm{d}x
$$

$$
+ \int_0^{L^e} K_0 V(x,t)W(x)\mathrm{d}x - Mg\cos(\omega t)\int_{ct}^{L^e} W(x)\mathrm{d}x + M\cos(\omega t)\int_{ct}^{L^e}\frac{\partial^2 V(x,t)}{\partial t^2}W(x)\mathrm{d}x
$$

$$
+ 2Mc\cos(\omega t)\int_{ct}^{L^e}\frac{\partial^2 V(x,t)}{\partial x\partial t}W(x)\mathrm{d}x + Mc^2\cos(\omega t)\int_{ct}^{L^e}\frac{\partial^2 V(x,t)}{\partial x^2}W(x)\mathrm{d}x \tag{41}
$$

$$
+ W(L^e)B_3^e - W(0)B_1^e - \left.\frac{\partial W}{\partial x}\right|_{x=L^e}B_4^e + \left.\frac{\partial W}{\partial x}\right|_{x=0}B_2^e = 0.
$$

Eq. (41) is the desired weak form of the variable magnitude moving distributed masses of non-uniform Bernoulli–Euler beam resting on elastic foundation. Therefore, we seek an approximate solution over the element under consideration and thereby construct the corresponding shape function. To this end, it is assumed that the unknown deflection $V(x,t)$ could be expressed approximately as

$$
V(x,t) \approx V_n(x,t) = H_1(x)V_1(t) + H_2(x)V_2(t) + H_3(x)V_3(t) + H_4(x)V_4(t)
$$

$$
= \sum_{k=1}^{4} H_k(x)V_k(t) = \{H\}\{V(t)\}, \qquad j = 1,2,3,4 \tag{42}
$$

where $H_j(x)$ are called Hermite cubic shape functions and $V_k(t)$ are the modal deflection functions and H is a row vector defined as

$$
[H] = \big[H_1(x), H_2(x), H_3(x), H_4(x)\big]. \tag{43}
$$

Using the procedures involved in constructing the Hermit-cubic interpolation functions in [22], yields

$$
H_1 = 1 - \frac{3x^2}{h^2} + \frac{2x^3}{h^3}, \quad H_2 = x - \frac{x^2}{h} + \frac{x^3}{h^2}, \quad H_3 = \frac{3x^2}{h^2} - \frac{2x^3}{h^3}, \quad H_4 = -\frac{x^2}{h} + \frac{x^3}{h^2}, \tag{44}
$$

where x is the spatial coordinate. Now substituting Eqs. (42)–(44) into the weak form (41), after some simplification and rearrangement gives

$$
\big[K^e\big]\{V(t)\} + \big[C^e\big]\{\dot V(t)\} + \big[M^e\big]\{\ddot V(t)\} + \{f^e\} + \{Q^e\} = 0. \tag{45}
$$

The matrix equation (45) is the governing equation describing the behavior of a typical finite element of the non-uniform beam traversed by a harmonic moving load. $\big[K^e\big]$ is the element stiffness matrix, $\big[M^e\big]$ is the element mass matrix, $\big[C^e\big]$ is the element centripetal matrix, $\{f^e\}$ is the force vector and $\{Q^e\}$ is the element boundary term vector.

The next step is assembling of the element equations. The procedure for assembling various matrices and vectors for several beam elements which constitute a mesh is well discussed in [23, 24]. Hence the governing assembled equation of motion describing the dynamic behavior of the moving load problem with Pasternak foundation is

$$
[K]\{V(t)\} + [C]\{\dot V(t)\} + [M]\{\ddot V(t)\} = \{F\}, \tag{46}
$$

where $[K], [M], [C]$ and $[F]$ are the assembled (global or overall) stiffness, mass, centripetal and load vector.

In order to obtain a complete and unique solution (46), the prescribed boundary conditions must be imposed on both the deflection/slopes and the shear force/bending moments, respectively. Finally, for a harmonic free vibration system without the centripetal matrix, (46) reduces to

$$\left([K] - \omega_i^2 [M] \right) \{V(t)\} = 0, \tag{47}$$

where ω^2 denotes the natural frequency and $V(t)$ is the corresponding mode shape of the system. Various methods can be used to solve for the eigenvalue ω_i^2 and the corresponding $\{V(t)\}$. The dynamic response of the non-uniform beam under a partially distributed moving load are obtained by solving the equation of motion (46) directly by Newmark method.

2.3.2. *Newmark beta method algorithm*

(a) Initial Computation
(1) Form stiffness $[K]$, mass $[M]$, and damping $[C]$ matrices
(2) Initialize $\{V_0\}$, $\{\dot{V}_0\}$ and $\{\ddot{V}_0\}$
(3) Select time step Δt, parameters α and β, and calculate integration constants.
$\beta \geq 0.5; \alpha \geq 0.25(0.5 + \beta)^2$
$a_0 = \dfrac{1}{\beta(\Delta t)^2}; a_1 = \dfrac{\alpha}{\beta \Delta t}; a_2 = \dfrac{1}{\beta \Delta t}; a_3 = \dfrac{1}{2\beta} - 1; a_5 = \dfrac{\Delta t}{2}\left(\dfrac{\alpha}{\beta} - 2\right);$
$a_6 = \Delta t(1 - \beta); a_7 = \beta \Delta t$
(4) Form effective stiffness matrix:
$[\bar{K}] = [K] + a_0[M] + a_1[C]$
(5) Triangularize $[\bar{K}]$: $[\bar{K}] = [L][D][L]^T$

(b) For each time step:
(1) Calculate effective force vector at time $t + \Delta t$
$\{\bar{F}_{t+\Delta t}\} = \{F_{t+\Delta t}\} + [M]\left(a_0\{V_t\} + a_2\{\dot{V}_t\} + a_3\{\ddot{V}_t\}\right)$
$+ [C]\left(a_1\{V_t\} + a_4\{\dot{V}_t\} + a_5\{\ddot{V}_t\}\right)$
(2) solve displacement at time $t + \Delta t$
$[\bar{K}]\{V_{t+\Delta t}\} = \{\bar{F}_{t+\Delta t}\}$
(3) calculate $\{\dot{V}\}$ and $\{\ddot{V}\}$ at time $t + \Delta t$

- $\{\ddot{V}_t\} = a_0\left(\{V_{t+\Delta t}\} - \{V_t\}\right) - a_2\{\dot{V}_t\} - a_3\{\ddot{V}_t\}$
- $\{\dot{V}_t\} = a_1\left(\{V_{t+\Delta t}\} - \{V_t\}\right) - a_4\{\dot{V}_t\} - a_5\{\ddot{V}_t\}$

3. COMMENTS ON THE CLOSED FORM SOLUTIONS

Theoretically speaking, the deflections of the non-uniform Bernoulli–Euler beam may increase beyond bounds. Practically, this means that the beam is in a state of resonance. The speed of the load which brings about resonance effect in the system is termed the critical speed. (36) clearly shows that the simply supported non-uniform Bernoulli–Euler beam resting on a Pasternak foundation and traversed by a moving distributed force reaches a state of resonance whenever

$$\omega_f = \omega \quad \text{or} \quad \omega_f = \Omega_1 \quad \text{or} \quad \omega_f = \Omega_2. \tag{48}$$

Eq. (30) shows that, the dynamic system will attain the state of resonance whenever velocity is

$$c = \frac{L}{m\pi}(\omega - \omega_f) \quad \text{or} \quad c = \frac{L}{m\pi}(\omega_f - \omega). \tag{49}$$

Thus, Eq. (49) is critical speed for the dynamic problem.

4. ANALYSIS OF RESULT AND DISCUSSION

In order to illustrate the analysis presented in this work, non-uniform beam of length $L = 5$ m is considered. The load velocity $c = 50$ ms^{-1}, Young modulus $E = 2.10924 * 10^9$ Nm^{-2}, moment of inertia $I = 0.00287698$ m^4, $\pi = 22/7$, mass per unit length of the beam $\mu = 2758.291$ Kgm^{-1} and ratio of the mass of the load to the mass of the beam is 0.25. The transverse deflection of the beam is calculated and plotted against time for various values of axial force N, foundation stiffness K and shear modulus G. Values of N are varied between 4E+03 and 4E+09 while the values of K varies from 4E+03 to 4E+09. The values of G are varied from 4E+03 to 4E+09 N/m^3. The results are as shown on the various graphs in Figs 2–7.
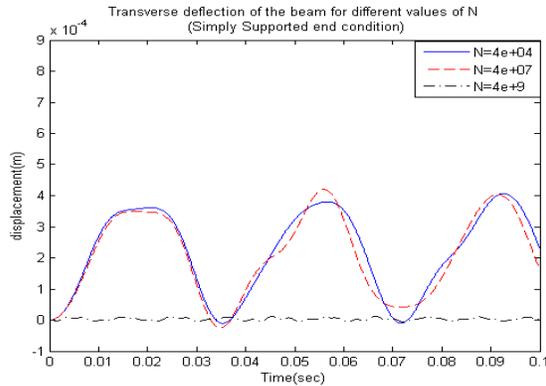

Fig. 2. Transverse displacement of the non-uniform simply supported Bernoulli–Euler beam for various values of axial force N and fixed values of K(40000) and G(40000) that traversed by moving distributed force
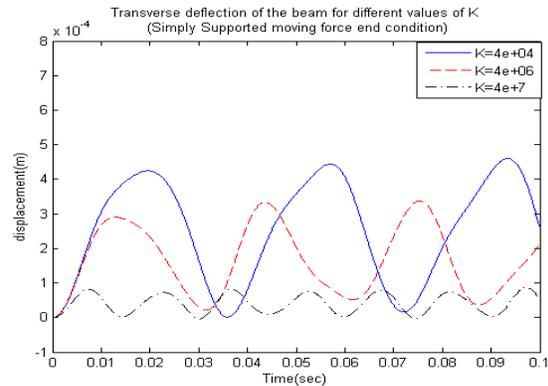
Fig. 3. Transverse displacement of the non-uniform simply supported Bernoulli–Euler beam for various values of foundation stiffness K and fixed values of G(40000) and N(40000) that traversed by moving distributed force
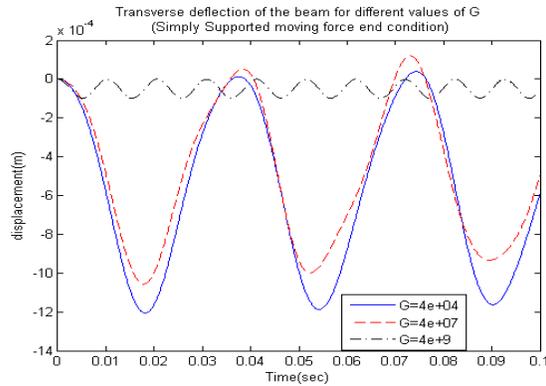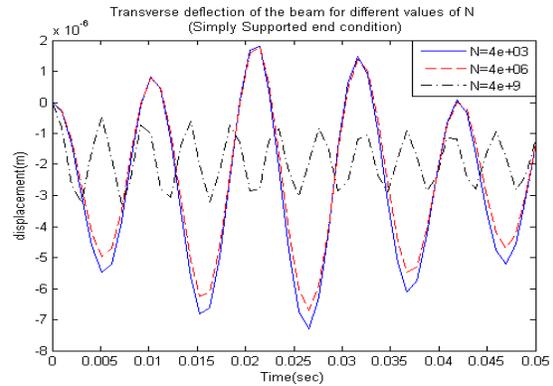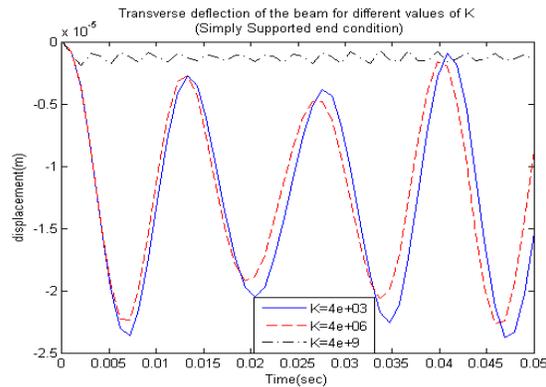
Fig. 4. Transverse displacement of the non-uniform simply supported Bernoulli–Euler beam for various values of shear modus *G* and fixed values of *N*(40000) and *K*(40000) that traversed by moving distributed force

Fig. 5. Transverse displacement of the non-uniform simply supported Bernoulli–Euler beam for various values of axial force N and fixed values of *K*(4000) and *G*(4000) that traversed by moving distributed mass

Fig. 6. Transverse displacement of the non-uniform simply supported Bernoulli–Euler beam for various values of foundation stiffness *K* and fixed values of *G*(4000) and *N*(4000) that traversed by moving distributed mass
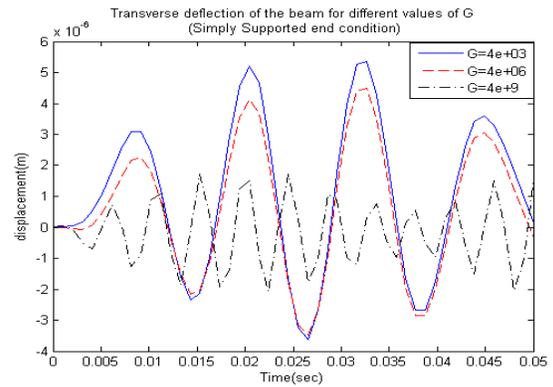

Fig. 7. Transverse displacement of the non-uniform simply supported Bernoulli–Euler beam for various values of shear modus *G* and fixed values of *N*(4000) and *K*(4000) that traversed by moving distributed mass

Figs. 2–4 show the transverse displacement responses of a non-uniform simply supported Bernoulli–Euler beam under distributed moving load travelling at constant velocity under the action of moving distributed force for various values of (i) axial force *N* and for fixed values of other parameters; (ii) various values of foundation stiffness *K* and for fixed values of other parameters and (iii) various values of shear modulus *G* and for fixed values of other parameters. The result shows that as *N*, *K* and *G* increases, the deflection of the beam decreases. Similar results are obtained when the beam is subjected to moving mass as shown in Figs. 5–7.
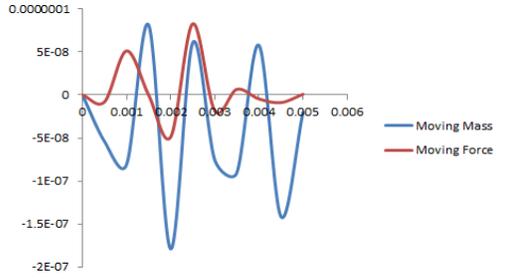
Fig. 8. Comparison of the transverse displacement of the moving distributed mass and moving distributed force for the non-uniform simply supported Bernoulli–Euler beam
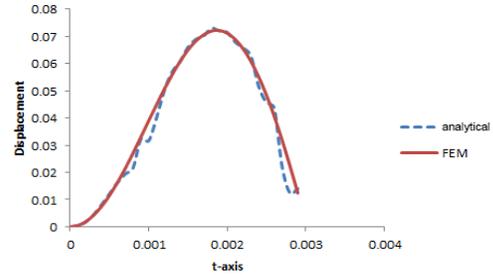

Fig. 9. Comparison of the transverse displacement of the analytical and FEM solutions for the non-uniform simply supported Bernoulli–Euler beam
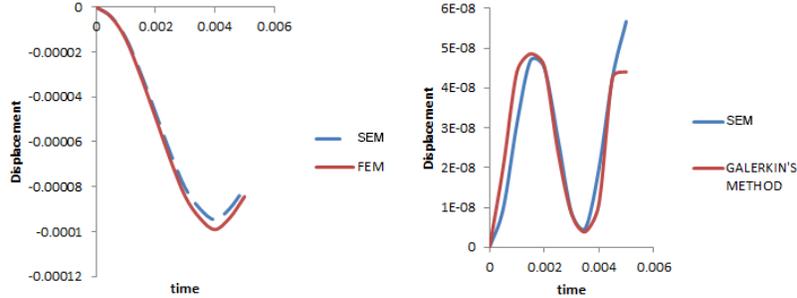

Fig. 10. Comparison of the transverse displacement of the moving distributed mass and moving distributed with SEM force for the non-uniform simply supported Bernoulli–Euler beam

Different comparisons of the transverse displacements are shown in Figs. 8–10. In order to verify the accuracy of the present method, the vibration under variable magnitude moving distributed masses of a simply supported non-uniform Bernoulli–Euler beam resting on Pasternak elastic foundation obtained by the present method and the frequency-domain spectral element method (SEM) are compared at two different velocities in Fig. 10. The results show that the dynamic responses obtained by the present method are almost identical to those obtained by using the SEM.

5. CONCLUSION

The vibration under variable magnitude moving distributed masses of simply supported non-uniform Bernoulli–Euler beam resting on Pasternak elastic foundation is investigated. The problem is governed by fourth-order partial differential equations with variable and singular coefficients. The main objective of this work is to obtain closed-form solution to this class of dynamical problem. In particular, the simply supported non-uniform Bernoulli–Euler beam is considered. In this dynamical problem, the beam is not uniform but varied along the span of the beam and as such, the solution to the governing equation are generally not obtainable by finite integral transform. Consequently, an approach generally used in solving dynamical problem called Galerkin's

method is used to transform the governing equation with singular and variable coefficients. The resulting Galerkin's equations are thereafter solved using (i) The method of Laplace transformation and convolution theory to obtain the analytical solutions of the one-dimensional dynamical problem for moving force problem and (ii) finite element analysis in conjunction with Newmark method for the case of the moving mass problem which is analytically unsolvable because of the harmonic nature of the moving load. To verify the accuracy of the present method used in (i), the dynamic responses of a simply supported non-uniform Bernoulli–Euler beam obtained by the finite element method (FEM) are compared in Fig. 9 and by frequency-domain spectral element method (SEM) in Fig. 10. The analytical solutions obtained are analyzed and resonance conditions for the problems started. Numerical analysis are carried out and the study exhibits the following interesting features:

1. As the values of axial force increases, the displacement amplitude of the simply supported non-uniform Bernoulli–Euler beam under the action of moving uniformly distributed force decreases for fixed shear modulus G and foundation stiffness K. The same results and analyses obtained for moving mass case.

2. When the axial force N and shear modulus G are fixed, the displacement of a simply supported non-uniform Bernoulli–Euler beam resting on Pasternak foundation and traversed by moving distributed force decreases as the foundation stiffness increase in the dynamical problem. Similar results and analyses are obtained for the moving mass case.

3. For fixed axial force N, foundation stiffness K, the response amplitude of the simply supported non-uniform Bernoulli–Euler beam under the action of moving force decreases as the shear modulus G is increased. Similar results and analyses are obtained for moving mass case. Finally, this research work has suggested valuable methods of solutions to this class of dynamical problem involving simply supported non-uniform Bernoulli–Euler beam under variable magnitude moving distributed masses.

REFERENCES

[1] A. N. Krylov. *Mathematical collection of the papers of the Academy of Sciences*, Vol. 61. Piterburg, (1905).

[2] S. P. Timoshenko. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, **41**, (245), (1921), pp. 744–746. https://doi.org/10.1080/14786442108636264.

[3] J. T. Kenney. Steady-state vibrations of beam on elastic foundation for moving load. *Journal of Applied Mechanics*, **21**, (1954), pp. 359–364.

[4] M. M. Stanisic, J. C. Hardin, and Y. C. Lou. On the response of the plate to a multi-masses moving system. *Acta Mechanica*, **5**, (1), (1968), pp. 37–53. https://doi.org/10.1007/bf01624442.

[5] S. Sadiku and H. H. E. Leipholz. On the dynamics of elastic systems with moving concentrated masses. *Ingenieur-Archiv*, **57**, (3), (1987), pp. 223–242. https://doi.org/10.1007/bf02570609.

[6] S. T. Oni and S. N. Ogunyebi. Dynamical analysis of a prestressed elastic beam with general boundary conditions under the action of uniform distributed masses. *Journal of the Nigerian Association of Mathematical Physics*, **12**, (1), (2008), pp. 103–120. https://doi.org/10.4314/jonamp.v12i1.45492.

[7] S. T. Oni and B. Omolofe. Flexural motions under accelerating loads of structurally pre-stressed beams with general boundary conditions. *Latin American Journal of Solids and Structures*, **7**, (3), (2010), pp. 285–306. https://doi.org/10.1590/s1679-78252010000300004.

[8] E. Esmailzadeh and M. Ghorashi. Vibration analysis of beams traversed by uniform partially distributed moving masses. *Journal of Sound and Vibration*, **184**, (1), (1995), pp. 9–17. https://doi.org/10.1006/jsvi.1995.0301.

[9] S. T. Oni. On thick beams under the action of a variable traveling transverse load, Abacus. *Journal of Mathematical Association of Nigeria*, **2**, (1997), pp. 531–546.

[10] M. M. Stanisic and J. C. Hardin. On the response of beams to an arbitrary number of concentrated moving masses. *Journal of the Franklin Institute*, **287**, (2), (1969), pp. 115–123. https://doi.org/10.1016/0016-0032(69)90120-3.

[11] M. Stanisic, J. A. Euler, and S. T. Montgomery. On a theory concerning the dynamical behavior of structures carrying moving masses. *Ingenieur-Archiv*, **43**, (5), (1974), pp. 295–305. https://doi.org/10.1007/bf00537218.

[12] C. E. Sir Inglis. *A mathematical treatise on vibrations in railway bridges*. Cambridge University Press, (1934).

[13] G. Muscolino and A. Palmeri. Response of beams resting on viscoelastically damped foundation to moving oscillators. *International Journal of Solids and Structures*, **44**, (5), (2007), pp. 1317–1336. https://doi.org/10.1016/j.ijsolstr.2006.06.013.

[14] C. R. Steele. Beams and shells with moving loads. *International Journal of Solids and Structures*, **7**, (9), (1971), pp. 1171–1198. https://doi.org/10.1016/0020-7683(71)90060-6.

[15] J. S. Wu and C. W. Dai. Dynamic responses of multispan nonuniform beam due to moving loads. *Journal of Structural Engineering*, **113**, (3), (1987), pp. 458–474. https://doi.org/10.1061/(asce)0733-9445(1987)113:3(458).

[16] Y. Dugush and M. Eisenberger. Vibrations of non-uniform continuous beams under moving loads. *Journal of Sound and Vibration*, **254**, (5), (2002), pp. 911–926. https://doi.org/10.1006/jsvi.2001.4135.

[17] M. Ahmadian, E. Esmailzadeh, and M. Asgari. Dynamic analysis of non-uniform cross sections beam under moving mass using finite element method. In *Proceedings of the 4th CSME Forum, Calgari, Alberta, Canada, May 21st-23rd*, (2006), pp. 384–395.

[18] S. T. Oni. *On the dynamic response of elastic structure to moving multi-mass system*. PhD thesis, University of Ilorin, Nigeria, (1991).

[19] S. T. Oni and T. O. Awodola. Vibrations under a moving load of a non-uniform Rayleigh beam on variable elastic foundation. *Journal of Nigerian Association of Mathematical Physics*, **7**, (2003), pp. 191–206.

[20] S. T. Oni and T. O. Awodola. Dynamics behavior under moving concentrated masses of elastically supported finite Bernoulli–Euler beam on Winkler foundation. *Latin American Journal of Solid and Structures*, **7**, (1), (2010), pp. 2–20.

[21] R. Struble. *Nonlinear differential equation*. New York, San-Francisco, Toronto, London, Sydney, (1962).

[22] J. L. Junkins and Y. Kim. *Introduction to dynamics and control of Flexible structure*. American Institute of Aeronautic and Astronautic, Inc. Washington, DC, (1993).

[23] J. J. Wu. Vibration analyses of a portal frame under the action of a moving distributed mass using moving mass element. *International Journal for Numerical Methods in Engineering*, **62**, (14), (2005), pp. 2028–2052. https://doi.org/10.1002/nme.1268.

[24] T. Irvine. *Transverse vibration of a beam via the finite element method*. Revision E, Vibration data, Citeseer, (2010).