Communications in Physics, Vol. 26, No. 4 (2016), pp. 301-323
DOI:10.15625/0868-3166/26/4/9184

Invited Paper

PROPAGATION OF ULTRASHORT PULSES IN NONLINEAR MEDIA

CAO LONG VAN[†]

*Institute of Physics, University of Zielona Góra, ul. Prof. Szafrana 4a,
65-516 Zielona Góra, Poland*

[†]*E-mail:* caolongvanuz@gmail.com

Received 01 January 2017

Accepted for publication 24 February 2017

Abstract. *In this paper, a general propagation equation of ultrashort pulses in an arbitrary dispersive nonlinear medium has been used for the case of Kerr media. This equation which is called Generalized Nonlinear Schroedinger Equation usually has very complicated form and looking for its solutions is usually a very difficult task. Theoretical methods reviewed in this paper to solve this equation are effective only for some special cases. As an example we describe the method of developed elliptic Jacobi function expansion and its expended form: F-expansion Method. Several numerical methods of finding approximate solutions are briefly discussed. We concentrate mainly on the methods: Split-Step, Runge-Kutta and Imaginary-time algorithms. Some numerical experiments are implemented for soliton propagation and interacting high order solitons.*

Keywords: ultrashort pulses, Kerr media, generalized nonlinear Schroedinger equation, solitons.

Classification numbers: 05.45.Yv, 41.65.Tg, 42.65.Re, 42.65. Hw.

I. INTRODUCTION

Propagation of ultrashort laser pulses (in fs) in a medium has been intensively considered both theoretically and experimentally during the last few decades [1–4] because of their potential applications in technology. Modern lasers can generate very short pulses with durations on the order of 10^{-15} seconds. This allows us to look at very fast events, such as molecules vibrating, or charge transfer in biological systems. By manipulating the shape of the pulse and using it to control precisely the quantum phenomena, one can observe the formation of molecules from cold atoms (noncrystalline structure), or the initiation of a quantum phase transition in a solid. The ultrashort pulse could be used as a photonic reagent in different chemical reactions. Short pulse with a large energy focused by lens creates a very high peak intensity which has several potential applications as in creation of unusual states of matter (plasmas) by reaching very high temperatures. It can be used also as an energy source for x-ray lasers etc. In the last time, the considered problem of this paper is a subject of particular interest in nonlinear optics in relation to so-called supercontinuum generation (SG), which is a flagship application of photonic crystal fibers (PCF). An injected monochromatic pulse may be dramatically broadened (spectrally), which creates a coherent beam generation of high brightness, even comparable to that of monochromatic lasers. Then a supercontinuum is generated when a collection of nonlinear processes act together upon a pump beam which leads to severe spectral broadening of the incoming pump beam, for example using a PCF. Such supercontinuum beams have wide applications, for example in trace gas sensing, in classical absorption spectroscopy, cavity-enhanced absorption spectroscopy, cavity ring-down spectroscopy, and in a diverse range of fields as optical coherence tomography, frequency metrology, fluorescence lifetime imaging, optical communications… [5].

During the propagation of ultrashort pulses in the medium, several new effects have been appeared in the comparison with the propagation of short pulses (in ps), in particular the effects of dispersion and nonlinear effects of higher orders. The influence of these effects leads to dramatic changes both in amplitude and spectrum of the pulse. It splits into different components and its spectrum also evolves into several bands known as optical shock and self-frequency shift phenomena [1, 2, 6, 7]. These effects should be studied in detail for future concrete applications of ultrashort pulses, especially in the domain of optical soliton communication. One can also explain the appearance of supercontinuum mentioned above [8].

A powerful method of deriving a general equation for short-duration intense pulses has been developed in the last time [9–11]. This method is constructed on basis of a consistent and mathematically rigorous expansion of the nonlinear wave equation with the assumption that the nonlinear processes involved in the problem are perturbations. For the Kerr medium with the delayed nonlinear response of the medium, induced by the stimulated Raman scattering and the characteristic features of both the spectrum and the intensity of the pulse, one can obtain in Sec. II an approximate equation in the most condensed form describing the propagation of the ultrashort pulses, usually called in literature as the generalized nonlinear Schroedinger equation (GNLS). Generally it is very difficult to find analytic solutions for this equation. Some analytic methods are reviewed in [12] and analyzed again in Sec. III, in which we introduce also a very important concept: the solitons. A normalized form of GNLS will be demonstrated with its general features. We will consider in detail the third-order dispersion (TOD), the self-steepening and the self-shift frequency for the ultrashort pulses in some special cases. It will be shown that when the higher-order terms are included, the pulse propagation equation becomes very complicated [8, 13]. Under some conditions its solutions in the form of dark and bright solitons are obtained [14]. We will use the so-called developed Jacobi expansion formalism for finding analytic solutions in the case when the fourth-order dispersion (FOD) is also taken into account. However in general we should use different numerical methods to solve GNLS. In Sec. IV we present three useful numerical methods, namely the Split-Step Fourier, the fourth order Runge-Kutta and the imaginary-time methods. We describe in Subsection IV.4 so-called Variational Method (VM), which is a powerful tool in finding multidimensional soliton solutions of nonlinear differential partial equations. In the some problems considered here, we present the good agreement between predictions of variational method and direct numerical calculations. It is interesting to note that the estimate obtained from a simple variational model can be in good agreement with numerical results even for complicated systems. Section V contains conclusions.

II. PROPAGATION EQUATION FOR ULTRASHORT PULSES

II.1. General propagation equation in the nonlinear dispersion media

Starting from Maxwell's equations one can obtain the wave equation for propagating electric field [1, 11, 15, 16] in the form

$$\nabla^2\vec{E}(\vec{r},t)-\nabla(\nabla.\vec{E}(\vec{r},t))-\frac{1}{c^2}\frac{\partial^2\vec{E}(\vec{r},t)}{\partial t^2}=\mu_0\frac{\partial^2\vec{P}_l(\vec{r},t)}{\partial t^2}+\mu_0\frac{\partial^2\vec{P}_{nl}(\vec{r},t)}{\partial t^2}, \tag{1}$$

where $\vec{P}_l(\vec{r},t)$ and $\vec{P}_{nl}(\vec{r},t)$ are respectively the linear and nonlinear polarization.

Usually the electric field $\vec{E}$ is expressed in the form of a superposition of monochromatic components with different frequencies and wavevectors centered at their central values ω_0 and $\vec{k}_0$. We restrict ourselves only to consider the propagation of the electric field in an arbitrary direction, say Oz which is usually chosen as the direction of $\vec{k}_0$. Then we can write

$$\vec{E}(r,t)=\vec{x}.E(z,t)=\frac{1}{2}\vec{x}\left[A(z,t)e^{-i\omega_0 t+ik_0 z}+c.c\right], \tag{2}$$

where $\vec{x}$ is the unit vector of the x axis perpendicular to the propagation direction, $A(z,t)$ denotes complex envelope function,$c.c$ is the complex conjugate of the first term.

Assuming that the medium is homogeneous and isotropic, the linear polarization vector of the medium can be written as follows

$$\vec{P}_l(\vec{r},t)\equiv\vec{P}_l(\vec{z},t)=\vec{x}P_l(z,t)=\vec{x}\varepsilon_0\int\limits_{-\infty}^{\infty}\chi^{(1)}(t-t')E(z,t')dt'=\vec{x}\varepsilon_0\tilde{\chi}^{(1)}*E, \tag{3}$$

where $*$ is the convolution product which displays the causality: the response of the medium in the time$t$ is caused by the action of the electric field in all previous times t'. The quantity $\chi^{(1)}$ denotes the scalar susceptibility of the medium.

The nonlinear polarization vector is generally presented as

$$\begin{aligned}\vec{P}_{nl}(\vec{r},t)=\varepsilon_0\Bigg[&\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\chi^{(2)}(t-t_1,t-t_2):\vec{E}(\vec{r},t_1)\vec{E}(\vec{r},t_2)dt_1dt_2\\&+\int\int\limits_{-\infty}^{\infty}\int\chi^{(3)}(t-t_1,t-t_2,t-t_3)\vdots\vec{E}(\vec{r},t_1)\vec{E}(\vec{r},t_2)\vec{E}(\vec{r},t_3)dt_1dt_2dt_3+...\Bigg],\end{aligned} \tag{4}$$

where $\chi^{(n)}(t-t_1,t-t_2,\ldots,t-t_n)$ denotes the n-order nonlinear susceptibility. It is well-known that for the homogeneous isotropic medium, the elements of the even-order nonlinear susceptibility $\chi^{(2k)}(t-t_1,\ldots,t-t_{2k})$ vanish [1, 15, 16] because of the spatial inversion symmetry, as a result only the nonlinear polarizations of odd orders remain in the expression (5). We focus only on the third-order nonlinear susceptibility, so we have the Kerr medium. Then the tensor $\chi^{(3)}$ has $3^4=81$

elements as a matrix with 3 lines and 27 columns, but only 21 of its elements are different from zero and three of them are independent [1]. Therefore

$$\begin{aligned}\vec{P}_{nl}(\vec{r},t) \equiv& \vec{P}_{nl}(z,t) = \vec{x}P_{nl}(z,t) \\ =&\vec{x}.\varepsilon_0 \int \int_{-\infty}^{\infty} \int \chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3)E(z,t_1)E(z,t_2)E(z,t_3)dt_1 dt_2 dt_3.\end{aligned} \tag{5}$$

Because of the complexity considered problem, several simplifications are made. First of all, usually the nonlinear polarization is much smaller than the electric field and the linear polarization $\left|\vec{P}_{nl}(z,t)\right| << \left|\vec{P}_{l}(z,t)\right|, \left|\vec{P}_{nl}(z,t)\right| << \varepsilon_0 \left|\vec{E}(z,t)\right|$, so it can be treated as a perturbation. Furthermore we have the approximate formula [11]: $\nabla.\vec{E}(z,t) \approx 0$. Then from (1) we obtain the following scalar wave equation

$$\Delta E(z,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(E(z,t) + \tilde{\chi}^{(1)} * E\right) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 P_{nl}(z,t)}{\partial t^2}. \tag{6}$$

Implementing the method presented in [9] we get the following equation:

$$\begin{aligned}&\left[i\frac{\partial}{\partial z} + i\beta'(\omega_0)\frac{\partial}{\partial t} - \frac{\beta''(\omega_0)}{2}\frac{\partial^2}{\partial t^2} + \sum_{p=3}^{\infty}\frac{i^p}{p!}\left(\frac{\partial^p \beta(\omega)}{\partial \omega^p}\right)_{\omega_0}\frac{\partial^p}{\partial t^p}\right]E(z,t)e^{ik\omega_0 t - ik_0 z} \\ &\quad + \left[1 + i\left(\frac{1}{\omega_0} - \frac{n'(\omega_0)}{n(\omega_0)}\right)\frac{\partial}{\partial t} + \left(\frac{n'(\omega_0)}{n(\omega_0)\omega_0} - \left(\frac{n'(\omega_0)}{n(\omega_0)}\right)^2 + \frac{n''(\omega_0)}{2n(\omega_0)}\right)\frac{\partial^2}{\partial t^2}\right. \\ &\quad \left. + \sum_{q=3}^{\infty}\frac{i^q}{q!}\frac{\beta(\omega_0)}{(\omega_0/c)^2}\left(\frac{\partial^q}{\partial \omega^q}\frac{(\omega/c)^2}{\beta(\omega)}\right)_{\omega_0}\frac{\partial^q}{\partial t^q}\right]\frac{(\omega_0/c)^2}{2\beta(\omega_0)\varepsilon_0}P_{nl}(z,t)e^{ik\omega_0 t - ik_0 z} \\ &+ \sum_{q=0}^{\infty}\sum_{m=2}^{\infty}\left\{\frac{i^q(2m-3)!!}{(-1)^{m-1}\varepsilon_0^m q!(2m)!!}\left(\frac{\partial^q}{\partial \omega^q}\frac{(\omega/c)^{2m}}{\beta^{2m-1}(\omega)}\right)_{\omega_0}\frac{\partial^q}{\partial t^q}\left(\phi^m(z,t)e^{ik\omega_0 t - ik_0 z}\right)\right\} = 0.\end{aligned} \tag{7}$$

The quantities

$$\phi^m(z,t) = F^{-1}\left\{\frac{P^m_{nl}(k+k_0,\omega+\omega_0)}{E^{m-1}(k+k_0,\omega+\omega_0)}\right\} = F^{-1}\left\{\frac{\left[F\left\{P_{nl}(z,t)\right\}\right]^m}{F\left[\left\{E(z,t)\right\}\right]^{m-1}}\right\} \tag{8}$$

denote higher-order perturbations,F and F^{-1} are the Fourier and the inverse Fourier Transforms, whereas the notations $\beta'(\omega_0);\beta''(\omega_0);n'(\omega_0);n''(\omega_0);\ldots$ are first-order and second-order derivatives of the respective functions, calculated at the value ω_0.

In subsequent sections we will use the Eq. (7) to describing different optical phenomena. Using specific properties both of spectrum and intensity of ultrashort pulses, one can simplify the Eq. (7) through neglecting the higher-order nonlinear perturbations and preserving only the linear and the nonlinear terms with their lower-order derivatives. For doing this, first of all we should consider in more detail the nonlinear polarization of the medium in the next subsection.

II.2. Raman response function

The expression for nonlinear polarization of the Kerr medium has the form (5), where the quantity $\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3)$ characterizes its optical property. It depends not only on the microscopic structure of the molecules and their ordering in the medium, but also on the characteristics of the propagating pulses. Usually involved microscopic processes have the characteristic time of femtoseconds (in particular the characteristic time for the electron response is of the order 0.1 fs, whereas for the nuclei and lattice 10 fs [3]). Therefore for the picosecond pulses, the nonlinear response of the medium can be treated as instantaneous. Then the nonlinear susceptibility can be presented as [2, 3, 15]

$$\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3)=\chi^{(3)}\delta(t-t_1)\delta(t-t_2)\delta(t-t_3). \tag{9}$$

In this formula $\chi^{(3)}$ is a real constant of the order 10^{-22} m/V^2, and $\delta(t-t_i)$ $(i=1,2,3)$ are the Dirac functions. The reduced equation obtained from (7) for this case is the well-known NLS equation [1, 6, 15, 16], which describes pretty good the experimental observations for the propagation process of picosecond pulses.

However when input pulses are shorter than 4-5 ps (for example tens or hundreds fs), the time width of the propagating pulses is comparable with the characteristic times of the microscopic processes. As a result the simplified assumption of the instantaneous medium response is no longer valid. Then some additional terms describing the delayed response of the medium should be included in the expression (9). This delayed response, among other things, is related to the reduced Raman scattering on the molecules of the medium [4, 11]. In the framework of the Lorentz atomic model in the adiabatic approximation [1, 3, 11], we can express the nonlinear susceptibility of the Kerr medium in the form [3, 15]:

$$\chi^{(3)}_{xxxx}(t-t_1,t-t_2,t-t_3)=\chi^{(3)}\left[(1-f_R)\delta(t-t_1)+f_R h_R(t-t_1)\right]\delta(t-t_2)\delta(t-t_3). \tag{10}$$

In this expression we have two contributions, namely one of the electron layer and one of the nuclei plus the crystal lattice. The electron part is treated as instantaneous, whereas the delayed response of the nuclei and the lattice is given by the function $h_R(t)$ called the **Raman response function**. It is assumed to have the following form [3, 11, 15]:

$$h_R(t)=\frac{\tau_1^2+\tau_2^2}{\tau_1\tau_2^2}e^{-t/\tau_2}\sin(t/\tau_1). \tag{11}$$

The Raman response function fulfills the normalization condition $\int\limits_0^\infty h_R(t)dt=1$, where the involved constants f_R, τ_1 and τ_2 are characteristics of the medium. The Fourier Transform of the $h_R(t)$ which is also called the Raman response function, but at the frequency ω, has the following form

$$g_R(\omega)=\frac{1/\tau_1^2+1/\tau_2^2}{-\omega^2-2i\omega/\tau_2+\left(1/\tau_1^2+1/\tau_2^2\right)}. \tag{12}$$

The imaginary part of $g(\omega)$ is called the Raman amplification function [3, 4, 17].

II.3. Generalized Nonlinear Schroedinger Equation

Now we substitute the expression (10) into (5) and expand the terms containing the powers of the intensity of the electric field. One can neglect the high-order harmonics because the phase-matching condition is not fulfilled. Then we get the following expression for the nonlinear polarization:

$$P_{nl}(z,t) = \frac{3\varepsilon_0\chi^{(3)}}{8}\left[(1-f_R)\,|A(z,t)|^2 A(z,t) + f_R A(z,t)\int\limits_{-\infty}^{t} h_R(t-t_1)\,|A(z,t)|^2\,dt_1 + c.c\right]. \quad (13)$$

Because the physical properties of the medium do not depend on the choice of the beginning of the time scale, we can rewrite the second term in (13) as:

$$\int\limits_{-\infty}^{t} h_R(t-t_1)\,|A(z,t)|^2\,dt_1 = \int\limits_{0}^{\infty} h_R(t_1)\,|A(z,t-t_1)|^2\,dt_1. \quad (14)$$

We expand now to the first order of the square of the module of the envelope under the integral sign in (14) and apply the normalization condition for the function $h_R(t)$Then we obtain

$$\int\limits_{0}^{\infty} h_R(t_1)\,|A(z,t-t_1)|^2\,dt_1 \approx |A(z,t)|^2 - \frac{T_R}{f_R}\frac{\partial\,|A(z,t)|^2}{\partial t}, \quad (15)$$

where T is the characteristic time for the Raman scattering effect:

$$T_R = f_R\int\limits_{0}^{\infty} t h_R(t)dt. \quad (16)$$

Finally we can have the nonlinear polarization in the form:

$$P_{nl}(z,t) = \frac{3\varepsilon_0\chi^{(3)}}{8}\left[A(z,t)\,|A(z,t)|^2 + T_R A(z,t)\frac{\partial\,|A(z,t)|^2}{\partial t} + c.c\right]. \quad (17)$$

As it has been emphasized above, the general equation (7) is very complicated, so we should simplify it into an approximate form. It is worth to note that the time and intensity characters of the ultrashort pulses lead to the fact that their spectrum is much broader and the pulse power is larger in comparison to that of the short pulses. Therefore we should consider in Eq. (7) the third-order dispersion terms [2, 6, 15] and the first-order term of the Kerr nonlinearity [1, 16].

Now we substitute the expression (17) for the nonlinear polarization into (7) and perform some further simplifications, namely we omit the fast oscillating terms and neglect the high-order derivatives of the nonlinear term. Then we obtain the following simplest approximate pulse propagation equation:

$$\begin{aligned} &i\frac{\partial A(z,t)}{\partial z} + i\beta'(\omega_0)\frac{\partial A(z,t)}{\partial t} - \frac{\beta''(\omega_0)}{2}\frac{\partial^2 A(z,t)}{\partial t^2} - \frac{i\beta'''(\omega_0)}{6}\frac{\partial^3 A(z,t)}{\partial t^3} \\ &+\gamma\left[|A(z,t)|^2 A(z,t) + i\tau_s\frac{\partial\,|A(z,t)|^2 A(z,t)}{\partial t} - T_R A(z,t)\frac{\partial\,|A(z,t)|^2}{\partial t}\right] = 0 \end{aligned} \quad (18)$$

where

$$\gamma = \frac{3\chi^{(3)}\omega_0}{8n(\omega_0)c}, \quad \tau_s = \frac{1}{\omega_0}\frac{n'(\omega_0)}{n(\omega_0)} \approx \frac{1}{\omega_0}. \quad (19)$$

Introducing the new parameters and variables

$$L_D = \frac{\tau_0^2}{|\beta''(\omega_0)|}, \quad L_{NL} = \frac{1}{\gamma P_0}, \quad N^2 = \frac{L_D}{L_{NL}}, \quad \delta_3 = \frac{\beta'''(\omega_0)}{6|\beta''(\omega_0)|\,\tau_0},$$
$$S = \frac{\tau_s}{\tau_0}, \quad \tau_R = \frac{T_R}{\tau_0}, \quad \tau = \frac{t-\beta'(\omega_0)z}{\tau_0}, \xi = \frac{z}{L_D}, \quad U(\xi,\tau) = \frac{1}{\sqrt{P_0}}A(z,t), \tag{20}$$

where τ_0 and P_0 denote respectively the time width and the maximal power at the top of the envelope function, we can find Eq. (18) in the normalized form:

$$\frac{\partial U}{\partial \xi} = -sign\left(\beta''(\omega_0)\right)\frac{i}{2}\frac{\partial^2 U}{\partial \tau^2} + \delta_3\frac{\partial^3 U}{\partial \tau^3} + iN^2\left(|U|^2 U + iS\frac{\partial}{\partial \tau}\left(|U|^2 U\right) - \tau_R U\frac{\partial |U|^2}{\partial \tau}\right). \tag{21}$$

The equation (21) represents the lowest-order approximate form when the higher-order dispersion and nonlinearity effects in the general propagation equation (7) are taken into account. It is one of the most proper approximate forms in describing the propagation process of the ultrashort pulses, called the generalized nonlinear Schroedinger equation (GNLS) [2,3,6,7]. It has evidently a more complicated form than the nonlinear Schroedinger equation describing the propagation of the short pulses [1,6,15,16], namely it contains the higher-order dispersive and nonlinear terms. The parameters characterizing these effects: δ_3, S, τ_R govern respectively the effects of TOD, self-steepening and the self-shift frequency. It follows from the formulas (20) that when τ_0 decreases (i.e. the pulse is shorter) and the magnitude of these parameters increases, the higher-order effects should be taken into account (see subsection III.5). In the following we will investigate physical consequences of these effects in changing the pulse shape and the pulse spectrum.

Under the influence of TOD, when the propagation distance is larger the oscillation of the envelope function is stronger, creating a long trailing edge to the later time. The spectrum is broadened to two sides and splits into several peaks [6,15].

Self-steepening of the pulse leads to the formation of a steep front in the trailing edge of the pulse, what is similar to the usual shock wave formation, so this effect is called the optical shock. The pulse becomes more asymmetric during the propagation and finally its tail breaks up [1,6,7,16].

It is well-known that in the stimulated Raman scattering the Stokes process is more effective than the anti-Stokes process [4,15]. This fact leads to the so-called self-shift frequency of the pulse, in which the spectrum is shifted down to the low-frequency region. We say that the medium “amplifies” the long wavelength parts of the pulse. As a result the pulse loses its energy and undergoes a complex evolution when it enters deeply into medium.

For the concrete ultrashort pulses with the width $\tau_0 \approx 50$ fs and the carrier wavelength $\lambda_0 = 1.55\ \mu$m, the higher-order parameters in (20) during their propagation in the medium SiO_2 have the corresponding values of $\delta_3 \approx 0.03$, $S \approx 0.03$, $\tau_R \approx 0.1$. These values are smaller than one, so the higher-order effects are treated as the perturbations in comparison with the Kerr effect. Therefore for the pulse propagating in a silica optical fiber, the self-shift frequency effect dominates over the TOD and the self-steepening for the pulses with the width of hundreds and tens femtoseconds. The self-steepening becomes important only for the pulses of nearly 3 fs [6,15].

When t has the value of picoseconds or larger, the values of δ_3, S and τ_R are very small so they can be neglected. Then Eq. (21) reduces to the well-known NLS equation for the short pulses [1,15,16]. As it has been emphasized above, NLS can be solved by the Inverse Scattering

Method [12], but this Method cannot be applied to the Equation (21) any more. The problem of fixing a general analytic method for this equation is a very difficult task except some special cases, when some specific conditions should be satisfied. A review of some analytic methods is given in [12]. They will also briefly described in Sec. III. Demonstration of several numerical methods for obtaining approximate solutions of Eq. (21) is the subject of Sec. IV.

III. ANALYTICAL METHODS TO SOLVE THE PULSE PROPAGATION EQUATION

III.1. Solitons

Equation (7) with the particular form for the nonlinear polarization (5) and the initial condition for the input pulse allow us to investigate the pulse propagation in the medium. It is perhaps the most general form for the one-dimensional case which contains all orders of the dispersion and the nonlinearity. As is has been recognized above, this equation is very complicated and finding a general analytic method (given for example in [12]) for this equation is a difficult task, even when it is reduced into a more simple approximate form. It may be useful to look at some special solutions to the general nonlinear partial differential equation (NPDE) before activating the big machine of any general analytic or numerical scheme for solving it. Without detailed study of symmetries, we may expect that, among others, our integrable equations will have solutions in the form of so called travelling wave. The travelling wave is a solution of the form

$$u(x,t) = U(z), \quad z = x - Vt. \tag{22}$$

At first we restrict ourselves to the case of equations in with a variable which is a real wave function. When the wave consists of a single travelling bump (a displacement from an unperturbed state) or a travelling shock (kink) we call it **solitary wave**. This name is extended to the solutions of equations like the nonlinear Schroedinger (NLS), which describe evolution of an envelope of fast oscillations and it is only the envelope of an (usually complex) wave function which has the form of a travelling wave (22). Generally two or more solitary waves may travel at the same time with different velocities; if they travel towards each other, they would “collide” sooner or later. If they get through the collision unchanged (except for a possible shift in their positions and phases), they are called **solitons**. A great number of physical processes involved in a given nonlinear problem can be analyzed in terms of the formation of these (spatial, temporal or spatiotemporal) localized structures.

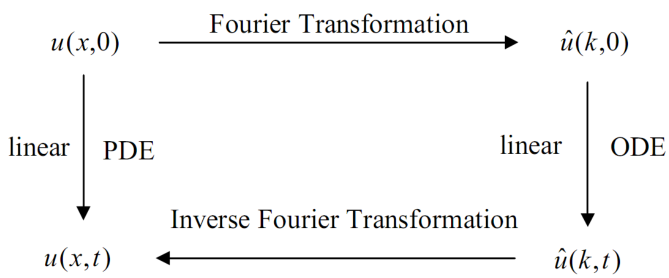
For the special case when the medium is an isotropic Kerr medium, we obtain the cubic nonlinear Schroedinger equation (NLS) describing the propagation of light pulses in fibers [9,16]. Optical soliton in fibers is created by the exact balancing between the group velocity dispersion (GDV) and its counterpart self-phase modulation (SPM). SPM is the nonlinear effect due to the lowest dominant nonlinear susceptibility in silica fibers. One of the most famous physicists working in this domain wrote that the parameters of fiber are a gift from God and that it is a sin not to use solitons in telecommunication! But generally we should take into account the higher order contributions [9,10,18].

The search for spatiotemporal solitons in optical media, alias “light bullets” (LBs) [19], is a challenge to fundamental and applied research in nonlinear optics. A review of this is given in [20,52].

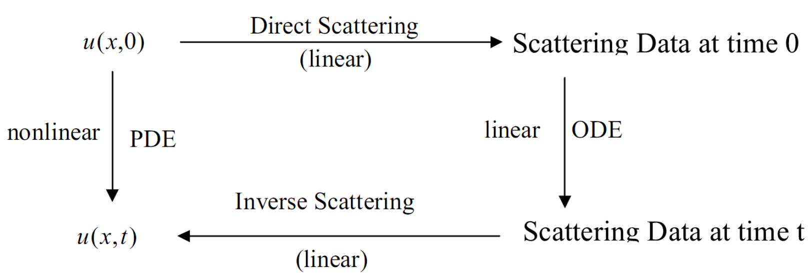
The combination of the scattering and inverse scattering transformations (ST and IST) is the most successful analytical approach to the initial value problem for integrable nonlinear partial differential equations. This approach called Inverse Scattering Method is first derived for the KdV equation by Gardner, Green, Kruskal and Miura [21]. It was later extended to many other integrable equations and remains the basic tool till now. It will be described in the following Subsection.

III.2. Inverse Scattering Method

The method may be treated as a generalization of the Fourier transformation applied to any initial value problem with an unknown function $u(x,t)$:



Here some notations are introduced: ODE for ordinary differential equation(s) and PDE for partial differential equation(s). While the IS method may be represented by the following commutative diagram



However the ST/IST method is much more complex than the Fourier transformation: firstly the ST is not just calculation of a single (or multiple) integral, it requires solving a linear differential equation, which usually is nontrivial. Secondly the IS transformation is again not just calculation of an integral. In fact this step is the most demanding one: in order to retrieve the potential from the scattering data we have to solve a linear integral equation. All steps of the diagram given above are described in detail in [12, Chapter 7].

III.3. Hirota's Method

The IS method described above is very important as it revealed the possibility of solving nonlinear equations by converting them to a sequence of linear problems. On the other hand, its use for obtaining special solutions (e.g. solitons) is limited because of its complexity.

A less universal but very effective method was invented by Ryogo Hirota [22, 23], see also [24–26]. Hirota's method provides just special soliton solutions, but it is very practical. Hirota noticed that most of the soliton solutions are rational combination of exponential functions. His approach has its roots in Pade's method of approximating transcendent functions with rational functions [27].

Hirota's method consists of several stages, each of which requires some invention and attention. These are: proper substitution in order to express the equation in the bilinear variables (1), reduction of the excess degrees of freedom which transforms the equations into their bilinear form (2), the perturbation scheme (3) and finally solution of the system of equations at its successive orders of magnitude (4). These steps are described in detail in [12, Chapter 8].

In our works on the equations describing interaction of higher harmonics with the fundamental mode in a laser beam [28, 29] we used the Hirota formalism to the systems of equations

$$\begin{aligned} iU_{,z} + U_{,tt} + U * W = 0, \\ iW_{,z} + PW_{,tt} \pm U^2 = 0, \end{aligned} \tag{23}$$

for the 2^{nd} harmonic and

$$\begin{aligned} iu_z + u_{xx} - u + \left[(1/9)\,|u|^2 + 2\,|\omega|^2\right] u + (1/3)\, u^{*2}\omega = 0, \\ i\sigma\omega_z + \omega_{xx} - \alpha u + \left(9\,|\omega|^2 + 2\,|u|^2\right)\omega + (1/9)\,u^3 = 0 \end{aligned} \tag{24}$$

for the 3^{rd}, where the U and u are the amplitudes of the fundamental frequency modes, while the W and w are the amplitudes of the 2^{nd} and 3^{rd} harmonics respectively (all of them rescaled to reduce the number of coefficients). The equations describe propagation of these nonlinearly interacting modes along a waveguide.

We have found that the Hirota scheme worked merely for the exact resonance cases, i.e. not only had the frequencies of the higher harmonics found to be multiplies of the fundamental one, but also the ratio of the dispersion coefficients had to be equal to the ratio of frequencies. Moreover, the only solitary wave solutions of that type were single travelling waves. For the amplitudes of 2^{nd} harmonic we found a new equation of the NLS type which they satisfy, namely

$$iU_z + U_{tt} \pm \sqrt{2}|U|U = 0. \tag{25}$$

III.4. The Painlevé Property

In his Nobel-Prize lecture H. David Politzer remarked: "It is amazing how much easier it can be to solve a problem once you are assured that a solution exists!". How can we learn that a given nonlinear partial differential equation has an integral before going to solve it? We just need a simple straightforward criterion of **integrability**.

A mathematical model may reveal two fundamental kinds of behavior: regular or chaotic. Closed phase orbits, stable solitary waves, are the typical properties of the first kind. We usually link the word integrability with such a behavior. Dense filling of the phase space with the orbits, sensitivity to initial conditions, fractional dimension of the attractors characterizes the second. The term nonitegrability is often used in this context.

If we go into more details the word "integrability" may be understood in many different ways. However solutions of a large class of PDE all the properties of either class in common. From the point of view of applications, it is of utmost importance to be able to distinguish between the above two classes. It is also very convenient to know those properties of the solution before we try to solve a given PDE. One of the simple, flexible and therefore universal ways of checking to which of the classes an equation belongs is testing it for the Painlevé property.

The Painlevé property is defined as a property of solutions of ODE lub PDE for functions of a complex variable, or in its generalization of many complex variables. If the variables are real, it requires extension to the complex plane or a Cartesian product of such planes. The Painlevé property is defined as the *absence of movable critical points. Critical* means that the solution is not unique in a neighborhood of such a point, it splits at that point into a multivalued entity. *Movable* means that position of such a point is unpredictable until solving the equations.

The definition of the Painlevé property and a summary of the method called "*Painlevé test*" (which should rather be named after Kovalevskaya [30]) are given in [12, Chapter 9]. Its version extended for partial differential equations is presented in [31,32]. A comprehensive review of its interpretations, methods, and validity was given in [33] for ODE and [34] for PDE. A pedestrian's approach may be found in [35].

The Painlevé test has been used in [36] for finding soliton solutions for GNLS. Even in the case, when a considered system is non-integrable in the sense of Painlevé, it can be partially integrable for all [28] or some [29] values of its parameters.

III.5. Developed Jacobi elliptic function expansion

We consider now a nonlinear partial differential equation in a general form given in [14]:

$$N\left(F, |F|, \frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial t^2}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial x \partial t} \cdots\right) = 0. \tag{26}$$

We try to look for the traveling wave solutions in the form

$$F = u(\xi)\, e^{i(kx-\omega t)},\ \xi = cx - \lambda t + x_0, \tag{27}$$

where u(ξ) is a real function, λ is a constant parameter and $k$and ω stand for the wave number and the frequency, respectively. Substitution of (27) into (26) leads to an ordinary differential equation

$$N\left(u, \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \frac{d^3u}{d\xi^3}, \cdots\right) = 0. \tag{28}$$

We assume the *ansatz* of the solution in the form of a finite series of Jacobi elliptic functions *cn*(ξ,*m*) (or *sn*(ξ,*m*)), i.e.

$$u(\xi) = \sum_{j=0}^{n} a_j cn^j(\xi). \tag{29}$$

Here a_j are constants which will be calculated later, and the highest degree of the function u is

$$O(u(\xi)) = n. \tag{30}$$

From properties of Jacobi elliptic functions one can conclude that the highest degree of derivatives is fixed as

$$O\left(d^p u(\xi)/d\xi^p\right) = n + p. \tag{31}$$

n in (29) is selected in such a way that the highest degree of derivatives is equal to the degree of the nonlinear term. Now we substitute (29) into (28). Equating the coefficients of all power of cn(ξ). sn(ξ), dn(ξ) to zero we obtain a set of algebraic equations for a_j. By solving these equations, we get the final result for u in the form (29). We will apply below this method to find the soliton solutions for Eq. (25) given above and GNSE in [37].

III.5.1. ***Eq. (25) for the second harmonic***

We seek the soliton solutions for Eq. (25) given above:

$$iU_z + U_{tt} \pm \sqrt{2}|U|U = 0. \tag{32}$$

Now we perform the transformation

$$U = V(\xi)\exp[t(kz - \omega t)]\,; \quad \xi = ct - \lambda z + z_0. \tag{33}$$

For the sake of simplicity we assume $|V| = V$. Then using the formalism presented above we obtain (for the modulus number m = 1, see [46]) the solution

$$V = -2\sqrt{2}c + 3c^2\sqrt{2}\sec h^2(ct - kz + z_0)\,.$$

III.5.2. ***GNLS with the four-order dispersion***

As it has been emphasized above, in the case of ultrashort light pulses, higher-order terms should be added to the nonlinear Schroedinger equation (NLS). Then instead of NLS we consider the GNLS in the form

$$\begin{aligned} l\frac{\partial E}{\partial z} =& i\left(\alpha_1\frac{\partial^2 E}{\partial t^2} + \alpha_2\frac{\partial^4 E}{\partial t^4} + \alpha_3|E|^2 E\right) + \alpha_4\frac{\partial^3 E}{\partial t^3} \\ &+ \alpha_5\frac{\partial\left(|E|^2 E\right)}{\partial t} + \alpha_6 E\frac{\partial|E|^2}{\partial t}, \end{aligned} \tag{34}$$

where the real parameters α_i (i=1,...,6) have the following physical interpretations: $\alpha_1$corresponds to the group velocity dispersion (GVD), α_2 to the four-order dispersion (FOD), α_3 to self-phase modulation (SPM), α_4 to third-order dispersion (TOD), $\alpha_5$to self-steepening (SS) and α_6 to the self frequency shift (SFS) arising from stimulated Raman scattering (SRS). Thus in comparison with Eq. (21), the FOD is included. In order to obtain traveling wave solutions of Eq. (34), we apply the developed Jacobi elliptic function expansion method described above. At first we seek the electric field in the form

$$E(z,t) = u(\xi)\exp[i.(kz - \omega.t)]\,, \ \xi = ct - \lambda z + z_0. \tag{35}$$

After considerations we concluded that if the term FOD is included, the traveling wave solutions do not exist. We conclude also that for the existence of solutions in this type, the orders of dispersion higher than three can not be taken into account. By using proper forms of *ansatz* for u we obtained both a bright soliton solution

$$E(z,t) = \sqrt{\frac{6\alpha_4}{2\alpha_6 + 3\alpha_5}}c.\ \exp[i.(kz - \omega t)].\sec h\left[ct - \left(-c^2\alpha_4 - 2\alpha_1\omega + 3\alpha_4\omega^2\right)z + z_0\right]. \tag{36}$$

and a dark soliton solution in the form:

$$E(z,t) = \sqrt{\frac{-6\alpha_4}{2\alpha_6 + 3\alpha_5}}c.\tanh\left[ct - \left(2c^2\alpha_4 - 2\alpha_1\omega + 3\alpha_4\omega^2\right)z + z_0\right].\exp[i.(kz - \omega t)]\,. \tag{37}$$

The obtained expressions (36) and (37) are just the results previously given by several authors (e.g. the formulas (9), (12) in [12] and (56), (58) in [38]).

III.6. F-expansion Method

F-expansion method [39–41] is a generalization of Jacobi elliptic function expansion method described above. F-expansion method was extended later in different ways [42]. Recently Zhang *et al.* [42–44] introduced a generalized F-expansion method to obtain new and more general (even non-traveling) exact solutions in which the restriction on $\xi(x,y,z,\ldots,t)$ as merely a linear function and the restriction on the coefficients being constants (as in the case of the traveling wave) are removed. As a result these authors obtained many new exact non-traveling waves [42]. In the following we use the procedure described in [42] and restrict ourselves only to the case of traveling wave solutions (22). As it has been emphasized in [45], these waves are of crucial importance from the physical point of view due to comparatively simple experimental arrangements necessary to generate, detect and observe them, so one can easily verify experimentally different proposed theoretical models.

We consider now a general nonlinear partial differential equation of the form

$$N\left(u,\frac{\partial u}{\partial t},\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial x\partial t}\cdots\right)=0. \tag{38}$$

Using the traveling wave transformation $\xi = k(x-ct)$, where k and c are constants, we can rewrite Eq. (38) as the following nonlinear ordinary differential equation:

$$N\left(u,\frac{du}{d\xi},\frac{d^2u}{d\xi^2},\frac{d^3u}{d\xi^3},\cdots\right)=0. \tag{39}$$

Now we seek the formal travelling wave solutions in the form

$$u = a_0+\sum_{i=1}^{n}\left[a_iF^i(\xi)+b_i\frac{1}{F^i(\xi)}\right], \tag{40}$$

where k, c are both constants (k, $c \neq 0$). Here k denotes the value of wave number, c denotes speed of wave. These constants will be determined later. To determine the value of n, we balance the linear term of highest order in Eq. (39) with the highest order nonlinear term. $F(\xi)$ and $F'(\xi)$ in (40) satisfy the following equation

$$F'^2(\xi) = PF^4(\xi)+QF^2(\xi)+R, \tag{41}$$

where P, Q and R are fixed parameters. Eq. (41) has the different Jacobi elliptic function solutions for different chosen values of these parameters. They can be found in [42]. From Eq. (41), we get

$$\begin{cases} F''(\xi) = 2PF^3(\xi)+QF(\xi), \\ F'''(\xi) = \left[6PF^2(\xi)+Q\right]F'(\xi), \\ F''''(\xi) = 24P^2F^5(\xi)+20PQF^3(\xi)+\left(Q^2+12PR\right)F(\xi), \\ \cdots \end{cases} \tag{42}$$

We substitute now the expansion (40) into Eq. (39). Using the relations (41), (42) and equating to zero the coefficients of $F'^i(\xi)F^j(\xi)$ $(i=0,1;\ j=0,\pm1,\pm2,\cdots)$ we obtain a system of algebraic equations for a_0, a_i, b_i, k and c. Solving these equations yields the final result for u in the form (40). In [46] we apply this procedure to solve analytically different nonlinear PDEs.

IV. NUMERICAL METHODS TO SOLVE GNLS

IV.1. Split-Step Algorithm of second order

Firstly we describe Split-Step Algorithm of second order for finding approximate solutions of GNLS. Eq. (21) can be presented in the form

$$\frac{\partial U}{\partial \xi} = \left(\hat{L} + \hat{N}(U)\right) U, \tag{43}$$

where $\hat{L}$ and $\hat{N}$ are the linear and nonlinear operator acting on the envelope function, respectively. Some calculations performed in [18] yield the following formula describing Split-Step algorithm for the problem (43):

$$U(\xi + \Delta\xi, \tau) \approx \exp\left(\frac{\Delta\xi}{2}\hat{L}\right) \exp\left(\Delta\xi \hat{N}\left(U(\xi,\tau)\right)\right) \exp\left(\frac{\Delta\xi}{2}\hat{L}\right) U(\xi,\tau). \tag{44}$$

This expression allows us to find the approximate value of the envelope function in the location $\xi + \Delta\xi$ from its value in the ξ. For this purpose we should know how the action of the linear and nonlinear operators on the envelope function is defined. One can calculate them just by Fourier Transform, because these operators contain the time partial derivatives.

The value of the time variable belongs to the finite interval $[a,b]$ which is usually so large that its borders do not have any influence on the final results of the calculations. We put now the periodic condition on borders that $U(\xi,a) = U(\xi,b)$ for $\xi \in [0,\xi_0]$. For sake of simplicity we rescale the variable in (44) in such a way that the interval $[a,b]$ is normalized into the interval $[0,2\pi]$. In the next we divide this interval into$N$points with distance between them $\Delta\tau = 2\pi/N$ and denote these points as $\tau_j = 2\pi j/N, j = 0,1,2,...,N$. Then we have the so called Discrete Fourier Transform

$$U(\xi,\omega_k) = F_k\left[U(\xi,\tau_j)\right] = \frac{1}{N}\sum_{j=0}^{N-1} U(\xi,\tau_j)\exp\left(-i\omega_k\tau_j\right), -\frac{N}{2} \le \omega_k \le \frac{N}{2} - 1. \tag{45}$$

The Inverse Discrete Fourier Transform is defined as:

$$U(\xi,\tau_j) = F_j^{-1}\left[U(\xi,\omega_k)\right] = \sum_{k=-N/2}^{N/2-1} U(\xi,\omega_k)\exp\left(i\omega_k\tau_j\right), j = 0,1,2,\ldots,N. \tag{46}$$

As usual F here denotes Fourier Transform and F^{-1} denotes its inverse transform. Calculations in (45) and (46) are performed effectively by the fast algorithm FFT [47]. The time partial derivatives of the envelope function in both the linear and nonlinear operator $\hat{L}$ and $\hat{N}$can be easily checked by multiplying the Fourier coefficients $U(\xi,\omega_k)$ by powers of $-i\omega_k$ corresponding to the order of derivative and then performing the Inverse Fourier Transform. As an example, second-order derivative of the envelope function in the point (ξ,τ_j) can be calculated as $F_j^{-1}\left[-\omega_k^2 F_k\left[U(\xi,\tau_j)\right]\right]$.

IV.2. Runge-Kutta algorithm of the fourth order

Equation (21) can be also solved by using the Runge-Kutta algorithm. In this method the time discretization and calculations of time partial derivatives are performed in the same manner as in the previous subsection, but the spatial derivatives are calculated by Runge-Kutta algorithm of the fourth order, very popular for solving the differential equations [7,47–49].

Applying Fourier Transform for calculating the time partial derivatives equation we have

$$\frac{d}{d\xi}\left(F\left[U\right]\right)=\left(\left(-i\omega\right)^{2}\frac{i}{2}+\left(-i\omega\right)^{3}\delta_{3}\right)F\left[U\right]$$
$$+iN^{2}\left[\left(1+iS\left(-i\omega\right)\right)F\left[\left|U\right|^{2}U\right]-\tau_{R}F\left[UF^{-1}\left[\left(-i\omega\right)F\left[\left|U\right|^{2}\right]\right]\right]\right]. \qquad (47)$$

Introducing the following notation

$$V=\exp\left(\left(\frac{i\omega^{2}}{2}-i\omega^{3}\delta_{3}\right)\xi\right)F\left[U\right], \qquad (48)$$

after some calculations [13] we obtain the value of the envelope function in the location $\xi+\Delta\xi$:

$$U\left(\xi+\Delta\xi\right)=F^{-1}\left[V\left(\xi+\Delta\xi\right)\exp\left(\left(-\frac{i\omega^{2}}{2}+i\omega^{3}\delta_{3}\right)\left(\xi+\Delta\xi\right)\right)\right]. \qquad (49)$$

Errors in using (49) are of orders $(\Delta\xi)^5$. Formula (49) has a higher accuracy in comparison to calculations performed by (4). A price paying for this is longer computational time, because the number of calculation steps is very large.

In the simulations performed below, we have applied both of the methods described above and compared the obtained results. When the interval $\Delta\xi$ is relatively small, they are practically the same.

Firstly we compare the numerical results obtained by using algorithms introduced above with analytical results for some special cases. By this way we test the accuracy of these numerical simulations. We will compare our results with the results for the NLS equation describing the propagation of picosecond pulses [6, 15].

As a result of Inverse Scattering Transform Method presented in III.2, for the case when the higher-order parameters δ_3, S and τ_R in Eq. (21) vanish and the initial shape of the pulses is taken as the pulse of secant hyperbolic form, the considered equation will have the soliton solutions [50, 51]. These solitons exhibit the periodic feature with a characteristic period. The order of soliton is determined by the parameter N in (21). Except for the first-order (temporal) soliton (usually called the fundamental soliton), when the amplitude of the envelope function remains unchanged during propagation, the shape and spectrum of higher order solitons change in a complicated manner, but their shape follows a periodic pattern so that the input shape is recovered at the period $\xi=\pi/2$. When the value of N is larger, the envelope changes in a more complicated way over one soliton period.

For example we simulated the propagation for the first-order and tenth-order ($N = 10$) solitons over one soliton period for the input pulse with the following initial amplitude [15]:

$$U\left(0,\tau\right)=N\operatorname{sec}h\left(\tau\right)U(0,t). \qquad (50)$$

The pulse intensity $\left|U\left(\xi,\tau\right)\right|^2$ is plotted in Fig. 1. We see from Fig. 1(a) that the envelope function of the pulse has an unchanged shape as the shape of the initial form (50). In Figure 1(b) the envelope function has a complex evolution during propagation, but in the end of the period it comes back to the initial shape. This process repeats in the next periods. Obtained numerical results are in good agreement with analytical predictions concerning the periodic feature in the evolution of the envelope function. Analytical expressions for the higher-order solitons are very complicated. They are explicitly given in literature only in the case of the second- and third-order [6, 50, 51]. For the tenth-order soliton only numerical results are demonstrated.

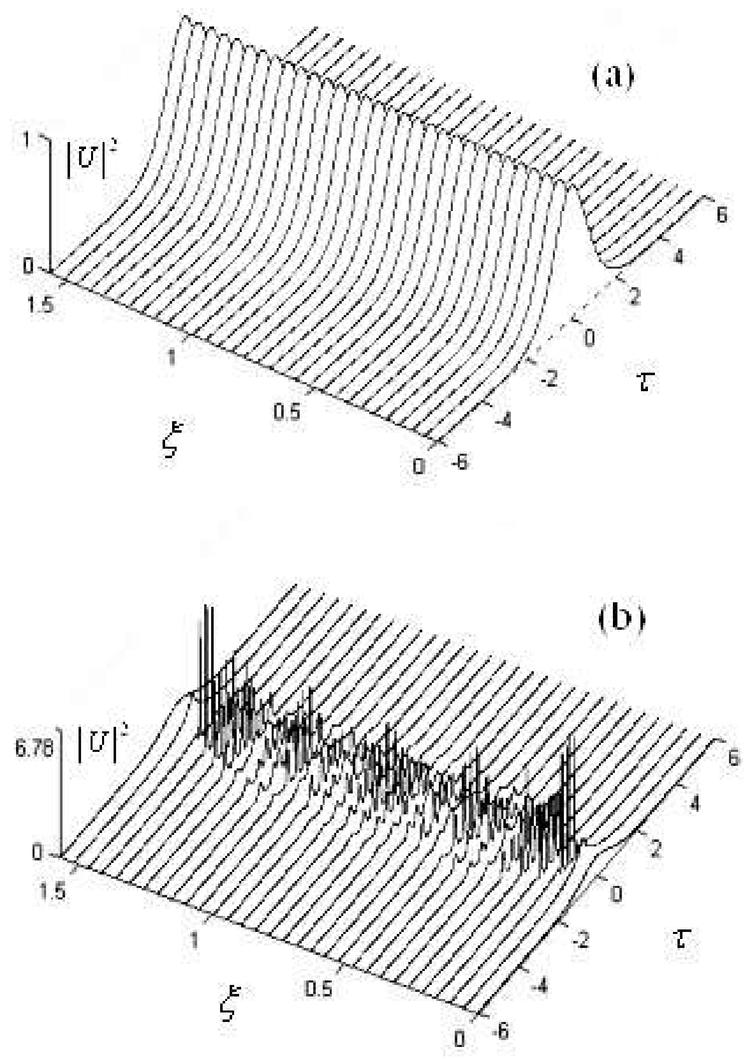


Fig. 1. Change of the pulse intensity in the propagation for the case of fundamental (a) and tenth-order solitons (b) over one soliton period $\xi = \pi/22$ [52].

Now we consider the case of multiple soliton propagation. The input amplitude for a soliton pair entering the medium is assumed as

$$U(0,\tau) = \operatorname{sech}(\tau - \tau_1) + r \operatorname{sech}[r(\tau + \tau_2)] \exp(i\theta), \tag{51}$$

where r is the relative amplitude of the two solitons, whereas θ is the relative phase between them [6, 15, 48, 53]. It follows from analytical results [50, 51] that neighboring solitons either come closer or move apart because of the nonlinear interaction between them. The collision time depends on both the relative phase θ and the amplitude ratio r. Solitons collide in a periodical way along the distance of propagation; usually with collision period much greater than the soliton period. After the collision, the shape of the wave amplitudes remain unchanged and stable. This effect is similar to the collision of the rigid particles. The name "soliton" reflects the particle feature of the nonlinear waves [6, 15].

In the next we consider the collision between the fundamental solitons and the higher-order solitons. The parameters in (70) are chosen as$r = 1, \theta = 0$(equal-amplitude and in-phase case) and $\tau_1 = \tau_2$ (initial spacing). Numerical results are presented in Fig. 2. Figure 2(a) displays collision between two fundamental solitons, where $\tau_1 = \tau_2 = 3.5$ and the propagation distance $\xi = 90$. Our results are in good agreement with the results given in [6, 53].

As it has been emphasized above, in considering the propagation of ultrashort pulses, the higher order effects in nonlinear media should be added to NLS, so instead of this equation we should consider GNLS (21) derived above, where the terms related to TOD and Raman scattering are included. The effect of TOD is significant for fs pulses when the GVD is close to zero. The Raman scattering leads to self-steepening effect which causes the breakup of higher-order solitons, usually called soliton fission [8]. This effect has an important contribution in the supercontinuum generation (SG) which is recently a flagship application of photonic crystal fibers [5]. Injected monochromatic pulse may be dramatically broadened (spectrally), which creates a coherent beam generation of high brightness comparable to that of monochromatic lasers. Generally speaking, a supercontinuum is formed when a collection of nonlinear processes act together upon a pump beam in order to cause severe spectral broadening of the original pump beam. SG has several important applications. Therefore it is recently a subject of intensive study in nonlinear optics.

SG can be qualitatively explained as follows. After soliton fission, new generated solitons have higher peak power and shorter temporal duration. They propagate with a higher group velocity and are red-shifted [5]. Simultaneously, non-solitonic radiation emitted in the blue wavelength domain of spectrum generates a chanel to lose energy. As a result, the phase-matching conditions for the red-shifting solitons are fulfilled which stimulate several other nonlinear effects as four-wave mixing and cross-phase modulation. Combination of these nonlinear effects is able to fill the remaining gaps in the spectrum. As a consequence, a flat and broad spectrum can be obtained as white light callled supercontinuum.

IV.3. Imaginary-time Method

Imaginary time method (ITM) is a effective tool used to generate stationary states of quantum systems. At first we describe background of this method for the linear regime and then extend it to the nonlinear one. Until now we have no rigorous proof for this extension but in fact the ITM works very efficiently. In this subsection, we will take notations related to the quantum theory of Bose-Einstein condensates (BECs), but all results could be transferred from atom optics to nonlinear optics by analogy between the propagation equation in the Kerr medium derived above and the Gross-Pitajevski equation for BECs [20].

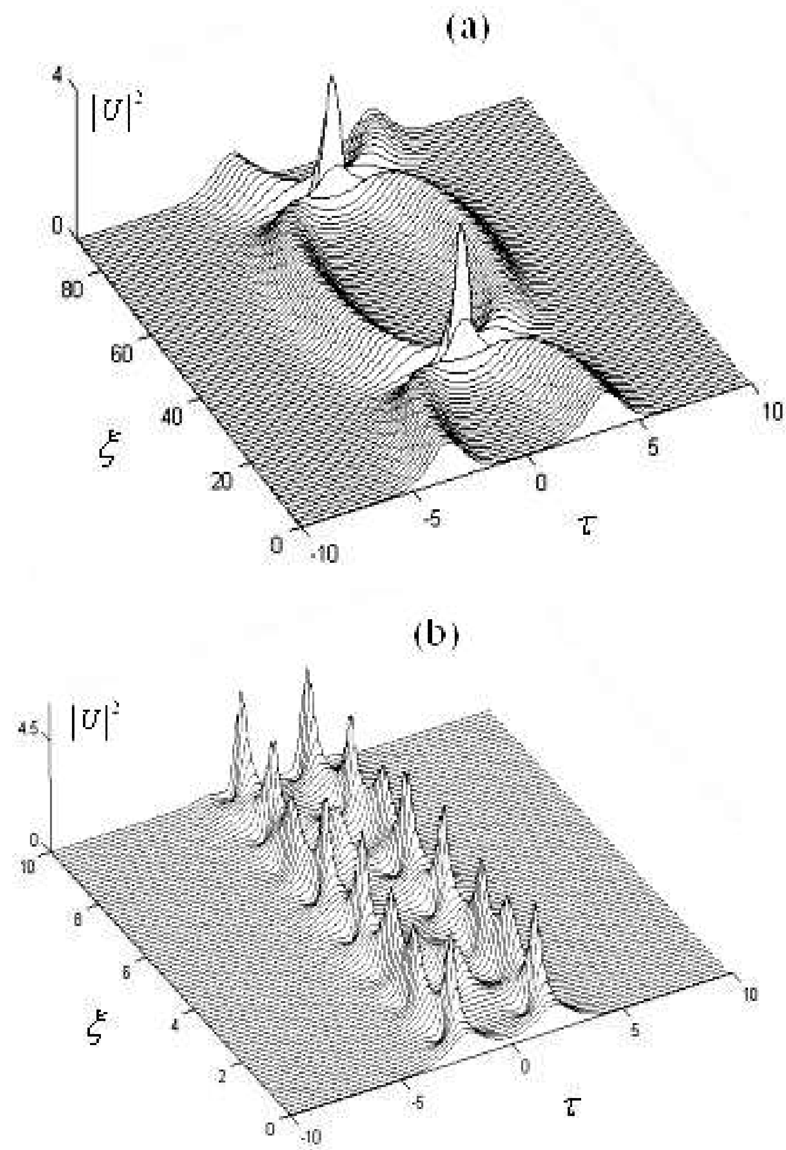


Fig. 2. Collision between two fundamental solitons over the propagation distance $\xi = 90$ (a) and between two second-order solitons over the propagation distance $\xi = 10$ (b) [52].

We assume that Hamiltonian $\hat{H}$ of considered system is bounded from below. The eigenvalue problem is written as follows

$$\hat{H}\phi_j(x,y) = E_j\phi_j(x,y) \tag{52}$$

Using the ITM for such Hamiltonian, for the beginning we introduce an initial wave-function Ψ_0(x,y). The algorithm described below drives this wave function into the ground state $\phi_0(x,y)$. We formally represent Ψ_0(x, y) as a combination in the basis $\{\phi_j$(x; y)$\}$:

$$\Psi_0(x,y) = \sum_{j=0}^{\infty} a_j \phi_j(x,y). \tag{53}$$

Assuming $\Psi_0(x,y)$ as wave-function at time t = 0, the time evolution is described by an unitary operator $\hat{U}$:

$$\begin{aligned}\Psi(x,y,t) =&\hat{U}(t)\Psi_0(x,y) = e^{-\frac{i}{\hbar}\hat{H}t}\sum_{j=0}^{\infty} a_j\phi_j(x,y)\\ =&\sum_{j=0}^{\infty} a_j e^{-\frac{i}{\hbar}E_j t}\phi_j(x,y).\end{aligned} \tag{54}$$

The name of the method has the origin from the changing ”time evolution” into the imaginary regime $it \Rightarrow \tau$ (t is real time)

$$\Psi(x,y,t) = \sum_{j=0}^{\infty} a_j e^{-\frac{E_j\tau}{\hbar}}\phi_j(x,y). \tag{55}$$

Unlike real time, the ”imaginary time” leads to exponential damping factors: $E_j\tau/\hbar$. Then terms correspond to high energies are damped faster than low energy ones, in particular the ground state is damped least. For $\tau \to \infty$ all components tend to zero. one has to re-normalize the wave function To guarantee the unitary evolution in every step of calculations after each time step, we renormalize the wave function . In this manner, the wave-function after n time steps has the form

$$\Psi(x,y,n\Delta\tau) = \sum_{j=0}^{\infty} \frac{a_j e^{-E_j n\Delta\tau/\hbar}}{\sqrt{\sum_{k=0}^{\infty}|a_k|^2 e^{-2E_k n\Delta\tau/\hbar}}}\phi_j(x,y). \tag{56}$$

Because $E_0 = \min\{E_k\}$, the denominator of the above expression behaves like $\sqrt{|a_0|^2 e^{-2E_0 n\Delta\tau/\hbar}} = |a_0| e^{-E_0 n\Delta\tau/\hbar}$ at the limit $n \to \infty$. It follows that

$$\lim_{n\to\infty}\Psi(x,y,n\Delta\tau) = \frac{a_0}{|a_0|}\phi_0(x,y). \tag{57}$$

Thus, this algorithm converges any initial wave function $\Psi_0(x,y)$ to the ground state $\phi_0(x,y)$.

Now we turn to the nonlinear regime. We suppose that evolution equation of quantum system has the form

$$i\Psi_t = -\frac{1}{2}(\Psi_{xx} + \Psi_{yy}) + U(x,y)\Psi + g|\Psi|^2\Psi, \tag{58}$$

which can be rewritten as

$$\Psi_t = -i\left(\hat{D} + \hat{N}\left[|\Psi|^2\right]\right).\Psi. \tag{59}$$

This equation has the same form as Eq. (43), where $\hat{D}$ and $\hat{N}$ are linear and nonlinear operators respectively given by

$$\hat{D} = -\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{60}$$

$$\hat{N}\left[|\Psi|^2\right]^{=} U(x,y) + g|\Psi|^2. \quad (61)$$

Using the ITM for Eq. (59), we introduce an trial wave-function Ψ_0(x, y) with given norm

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Psi_0(x,y)|^2 \, dxdy = N. \quad (62)$$

Evolution of the system in small interval time Δt is approximated to

$$\Psi(x,y,\Delta t) \approx e^{-\Delta t\left(\hat{D}+\hat{N}\right)}\Psi_0(x,y) \approx e^{-i\Delta t\hat{D}}e^{-i\Delta t\hat{N}}\Psi_0(x,y). \quad (63)$$

The basic idea of this approximation is that over sufficiently small interval Δt, the linear and nonlinear terms can be assumed to act independently.

Similar to the linear regime, if we change ”time evolution” in the imaginary regime: $i\Delta t \Rightarrow \Delta\tau$, the result is

$$\Psi(x,y,\Delta\tau) \approx e^{-\Delta\tau\hat{D}}e^{-\Delta\tau\hat{N}}\Psi_0(x,y). \quad (64)$$

We observe again the exponential damping of amplitude of the wave-function. For avoiding this fact, we renormalize the wave-function as follows

$$\tilde{\Psi}(x,y,\Delta\tau) = \sqrt{\frac{N}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Psi_0(x,y,\Delta\tau)|^2 \, dxdy}}\Psi(x,y,\Delta\tau). \quad (65)$$

This new wave-function is implemented as Ψ_0(x, y) for evolution in next interval time $\Delta\tau$. In practice, we repeate the calculations (64) and (65) until convergence is reached. Notice that in nonlinear regime the superposition (53) is invalid. Therefore we can not treat the obtained wave-function $\Phi(x,y)$ as ground state of the system. Generally one can proof that the Hamiltonian describing Eq. (58) is not bounded from below. It means that the wave-function $\Phi(x;y)$ is a stationary state of considered system. Mathematically speaking, it corresponds to a fixed point of the Hamiltonian in functional space.

We have used this numerical method in [56] to consider very interesting phenomenon, namely the collapse of the pulse in the Kerr medium. The formalism of ITM have been applied there as a test for the variational approach described in the next subsection.

IV.4. Variational Method

The main purpose of variational method is to replace the nonlinear PDE with a system of ODEs - which are much easier in consideration. For the sake of simplicity we illustrate this method on the example of the Nonlinear Schroedinger equation – NLS which is derived above and becomes one of the basic equations of modern mathematical physics. In the case of multidimensional system, the original NLS is even very hard to solve numerically, so that the VM is applied for giving us useful information about the considered system and a good hint for performing numerical simulation. The VM used for finding optical solitons has been initiated by the papers of Anderson *et al.* [54] and reviewed perfectly by Malomed [55]. In the framework of Lagrangian formalism, the evolution equation is derived from minimal principle of the *action functional* S:

$$S = \int L dt \quad (66)$$

where

$$L = \int \Im \, dxdy. \quad (67)$$

L is known as Lagrangian while $\Im$ is called *Lagrangian density*, t is evolution variable and x, y are spatial variables. Here we restrict ourselves to the two-dimensional case. The Lagrangian density is functional of the wave-function (or the slowly varying amplitude in nonlinear optics) of the system, its partial derivatives with respect to above variables and their corresponding complex conjugates:

$$\Im = \Im\left(\psi, \psi^*, \psi_t, \psi_t^*, \psi_x, \psi_x^*, \psi_y, \psi_y^*\right). \tag{68}$$

The condition of extremum of the action $\delta S/\delta\psi* = 0$ yields Euler – Lagrange equation in the form:

$$\frac{\partial}{\partial t}\frac{\partial \Im}{\partial \psi_t^*} + \frac{\partial}{\partial x}\frac{\partial \Im}{\partial \psi_x^*} + \frac{\partial}{\partial y}\frac{\partial \Im}{\partial \psi_y^*} - \frac{\partial \Im}{\partial \psi^*} = 0. \tag{69}$$

The VM concentrates on direct manipulation of the Lagrangian L instead of solving equation (69). The variational formalism begins with postulating a trial function (*ansatz*). The ansatz contains a set of variational parameters $X_i(t)$ which are functions of the evolution variable t.

Substituting the postulated *ansatz* into the Lagrangian density and integrating it we will obtain the effective Lagrangian $L_{eff}(X_i(t))$. Substituting the effective Lagrangian into equation (69) we get finally a set of ODEs:

$$\frac{d}{dt}\frac{\partial L_{eff}}{\partial X_{it}} - \frac{\partial L_{eff}}{\partial X_i} = 0. \tag{70}$$

Therefore the VM reduces complex dynamics described by NPDE to a relatively simple system of ODEs governing evolution of the variational parameters.

VA provides a convenient framework to study stationary solutions which correspond to the *fixed points* of the equations (70). The fixed points can be found by setting $dX_i/dt = 0$ and reducing the ODEs to a system of algebraic equations. Stability of fixed points against small pertubations can be checked by linearization of equations (70) around the fixed points. It provides an indication about the stability of the corresponding stationary solutions.

We illustrated this general formalism in [57] by showing variational calculations for one dimensional NLS equation, which plays the role as a background for another calculations in considering several phenomena in nonlinear optics. The NLS equation written in "non-optical" notation with the interchange between temporal and spatial variables describes BEC with the attractive interaction [20].

V. CONCLUSIONS

In this paper the generalized nonlinear Schrodinger (GNLS) equation for the propagation process of the ultrashort pulses in the Kerr medium has been derived. The influence of the higher-order dispersive and nonlinear effects, especially the nonlinear effect induced by the stimulated Raman scattering, have been considered in detail.

Because the GNLS equation is strongly nonlinear, the problem of solving it is a difficult task. Several analytical and approximate methods of solving it are used. They are reviewed in this paper. In particular we find an exact analytical solution for this equation in the general case by using the developed Jacobi elliptic function expansion. We presented the powerful variational method and three useful numerical methods. Our results calculated by these methods are in good agreement with those obtained before by several authors. .

It has been shown in [8] that self-steepening can lead to the breakup of higher-order solitons into *N* fundamental solitons. This phenomenon is called soliton fission which plays important role in SG. Using numerical methods described earlier to GNLS we can qualitatively explain SG in some models of photonic crystal fibers [8].

Finally, the GNLS for the pulse propagation with proper substitution of variable becomes the Gross-Pitaevski equation describing dynamics of coherent waves generated from Bose-Einstein condensates (BEC). Therefore one can use the methods presented here to consider different important phenomena in atom optics [20]. This would be the subject of our future review paper.

ACKNOWLEDGEMENTS

I would like to express my deep gratitude to Dr. Bui Dinh Thuan, Dr. Piotr Goldstein, Dr. Nguyen Viet Hung and Prof. Marek Trippenbach for pleasant and fruitful collaborations.

REFERENCES

[1] R. W. Boyd, *Nonlinear Optics*, Academic Press Inc., 2003.
[2] U. Bandelow, A. Demircan and M. Kesting, *Simulation of Pulse Propagation in Nonlinear Optical Fibers*, WIAS, 2003.
[3] J. H. B. Nijhof, H. A. Ferwerda, and B. J. Ho enders, *Pure Appl. Opt.* **4** (1994) 199.
[4] C. Headley III and G. P. Agrawal, *J. Opt. Soc. Am. B* **13** (1996) 2170.
[5] J. M. Dudley and J. R. Taylor, *Supercontinuum Generation in optical fibers*, Cambridge, 2010.
[6] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals*, Academic Press, San Diego, 2003.
[7] P. L. Francois, *J. Opt. Soc. Am. B* **8** (1991) 276.
[8] T. Do Thanh, V. Nguyen Thanh, T. Bui Dinh and V. Cao Long, *Computational Methods in Science and Technology* **22** (2016) 239.
[9] V. Cao Long, H. Nguyen Viet, M. Trippenbach, K. Dinh Xuan, *Computational Methods in Science and Technology* **14** (2008) 5.
[10] Cao Long Van and Marek Trippenbach, *New Optical Solitary waves in the higher order nonlinear Schrodinger Equation*, Scientific Biuletin 126, Technical University of Zielona Gora, Poland 2001, p.19 (Cao Long Van ed.).
[11] R. S. Tasgal and Y. B. Band, *Private Communication*, 2003.
[12] Cao Long Van and Piotr P. Goldstein, *A Concise Course in Nonlinear Partial Differential Equations*, University of Zielona Gora, 2008.
[13] V. Cao Long, H. Nguyen Viet, M. Trippenbach and K. Dinh Xuan, *Computational Methods in Science and Technology* **14** (2008) 13.
[14] Yang Qin, Dai Chao-qin, Zhang Jie-fang, Higher-Order Effects Induced Optical Solitons in Fiber, *International Journal of Theoretical Physics* **44** (2005) 1117.
[15] G. P. Agrawal, *Nonlinear Fiber Optics*, Academic, San Diego, 2003.
[16] Cao Long Van, Dinh Xuan Khoa, Marek Trippenbach, *Introduction to Nonlinear Optics*, National Education Press, Vietnam 2010.
[17] R. H. Stolen and W. J. Tomlinson, *J. Opt. Soc. Am. B* **6** (1992) 565.
[18] Cao Long Van, Marek Trippenbach, Dinh Xuan Khoa, Nguyen Viet Hung, Phan Xuan Anh, *Conference on Theoretical Physics*, Sam Son, Vietnam, 12-14 August 2003; *Journal of Science, Vinh University* **1A** (2003) 50.
[19] V. Cao Long, H. Nguyen Viet, M. Trippenbach and K. Dinh Xuan, *Computational Methods in Science and Technology* **14** (2008) 21.

[20] Cao Long Van, Dinh Xuan Khoa, Marek Trippenbach, *Computational Methods in Science and Technology*, Special Issue **(2)** (2010) 73.
[21] C. S. Gardner, J. M. Green, M. D. Kruskal, and R. M. Miura, *Phys. Rev. Lett.* **19** (1967) 1095.
[22] R. Hirota, *Phys. Rev. Lett.* **27** (1971) 1192.
[23] R. Hirota, *Solitons*, ed. R. K. Bullough and P. J. Caudrey, Springer, Berlin, 1980, p.157.
[24] R. Hirota, *J. Phys. Soc. Japan* **33** (1972) 1456.
[25] R. Hirota, *J. Phys. Soc. Japan* **33** (1972) 1459.
[26] R. Hirota, *J. Math. Phys.* **14** (1972) 805.
[27] G. A. Baker Jr., P. Graves-Morris, Padé approximants, vol. 1, Basic theory; (Encyclopedia of Mathematics and its Applications, vol. 13), Addison-Wesley, London 1981, chapter 1.
[28] V. Cao Long, P. P. Goldstein, and M. Trippenbach, *Acta Phys. Polonica A*, **105** (2004) 437.
[29] V. Cao Long, P. P. Goldstein, and S. Vu Ngoc, *Acta Phys. Polonica A* **106** (2004) 843.
[30] S. Kovalevskaya, *Acta Math.* **12** (1889) 177.
[31] J. Weiss, *J. Math. Phys.* **25** (1984) 2226.
[32] J. Weiss, M. Tabor, G. Carnevale, *J. Math. Phys.* **24** (1983) 522.
[33] R. Conte, *The Painlevé Property One Century Later*, ed. R. Conte, Springer Verlag, New York 1999, p.77.
[34] M. Musette, *The Painlevé Property One Century Later*, ed. R. Conte, Springer Verlag, New York 1999, p. 517.
[35] P. P. Goldstein, *Proc. First Non-Orthodox School on Nonlinearity and Geometry*, Warsaw, September 1995, Warsaw, Eds. D. Wójcik, J. Cieliñski, Polish Scientific Publishers, Warsaw 1998, p.207.
[36] M. Gedalin, T. C. Scott and Y. B. Band, *Phys. Rev. Lett.* **78** (1997) 448.
[37] T. Bui Dinh, V. Cao Long, B. Nguyen Huy, S. Vu Ngoc, *Photonics Lett. of Poland* **2** (2010) 97.
[38] Cao Long Van, Dinh Xuan Khoa, Nguyen Viet Hung, Marek Trippenbach, Reduction of 2D problem of Bose - Einstein condensate trapped in a harmonic potential to the 1D problem, paper presented on the 32th National Theoretical Symposium, Da Nang, Vietnam 2008.
[39] Y. B. Zhou, M. L. Wang and Y. M. Wang, *Phys. Lett. A* 308 (2003) 31.
[40] M. L. Yang, Y. B. Zhou, *Phys. Lett. A* 318 (2003) 84.
[41] M. L. Wang, Y. M. Wang, J. L. Zhang, *Chin. Phys.* **12** (2003) 13412.
[42] S. Zhang, W. Wang and J. L. Tong, *Electronic Journal of Theoretical Physics (EJTP)* **5** (2008) 177.
[43] S. Zhang, *Phys. Lett. A* **358** (2006) 414.
[44] S. Zhang, T. C. Xia, *Appl. Math. Comput.* **189** (2007) 836.
[45] A. M. Samsonov, *Strain Solitons in Solids and How to Construct Them*, Chapman&Hall/CRC, New York 2001.
[46] Thuan Bui Dinh, Van Cao Long and K. W. Wojciechowski, *Physica Status Solidi B* **252** (2015) 1587.
[47] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, *Numerical Recipes in Fortran 77 - The Art of Scientific Computing*, Cambridge University Press, 1992.
[48] G. M. Muslu and H. A. Erbay, *Mathematics and Computers in Simulation* **67** (2005) 581.
[49] J. D. Hoffman, *Numerical Methods for Engineers and Scientists*, Marcel Dekker, 2001.
[50] L. Debnath, *Nonlinear partial differential equations for scientists and engineers*, Birkhauser, 1997.
[51] A. L. Maimistov, A. M. Basharov, *Nonlinear optical waves*, Kluwer Academic, 1999.
[52] Van Cao Long, *Rev. Adv. Mater. Sci.* **23** (2010) 14.
[53] M. S. Ozyaici, *J. Optoelectronics and Advanced Materials* **6** (2004) 71.
[54] D. Anderson, *Phys. Rev. A* **27** (1983) 3135; D. Anderson *et al.*, *J. Opt. Soc. Am. B* **5** (1988) 207.
[55] B. A. Malomed, *Progress in Optics* **43**, Ed. by E. Wolf, Elsevier Science B. V., 2002, Chapter 2, p. 69.
[56] V. Cao Long, W. Leoński and H. Nguyen Viet, *Journal of Computational Methods in Sciences and Engineering* **10** (2010) 451.
[57] Cao Long V., Thuan Bui Dinh, Goldstein Piotr P., Hung Nguyen Viet, Son Doan Hoai, *Computational Methods in Science and Technology*, Special Issue **(2)** (2010) 77.