TENSOR TWO-LOOP SELF-ENERGY INTEGRALS

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Abstract. *Based on the technique for evaluating tensor two-loop self-energy integrals in parallel and orthogonal spaces [1], we present a new UV subtraction procedure for a subset of integrals which contains overall UV-divergence. The numerical and analytical results of the sun-rise integral are discussed in comparison with other methods.*

I. INTRODUCTION

High precision measurements at LHC and future colliders like ILC require an equivalent accuracy in theoretical predictions. To check the validity of the Standard Model, complex calculations dealing with huge numbers of Feynman graphs up to two-loop level within the perturbative theory are necessary. In the last few years, there have had many proposed techniques to calculate massive two-loop selfenergy tensor integrals. It is clear that, up to now, there are no perfect solutions and perfect techniques for the problem. Indeed, this is a very hard and active topic of multi-loop and multi-leg problem, a branch of computational high energy physics. A complete review can be found in [2,3].

In this article, we present a technique to calculate two-loop selfenergy tensor integrals with arbitrary masses in parallel/orthogonal spaces. Kreimer [1] has provided a general and constructive method to express massive two-loop integrals in terms of finite integrals suitable to numerical evaluations plus a set of products of one-loop integrals contains the UV singular part. The separation of the UV-divergent part from its counter UV-finite part can be obtained by a subtraction procedure in which subtraction terms are provided explicitly. From the computational point of view, the method provides a simple recipe to develop a computer program for calculating arbitrary tensor massive two-loop integrals.

However, detailed studies on the structure of tensor two-loop self-energy integrals which contain *overall* UV-divergences [4,5] show that in order to isolate the overall UV-divergence, the method needs to be revised to avoid the intervention of spurious IR-divergences from the subtraction terms.

In this paper, we present in detail a new subtraction procedure which completes the Kreimer's method and is applicable for a subset of degenerate tensor two-loop self-energy integrals. The new subtraction procedure is then implemented into the XLOOPS-GiNaC program package [6] which provided an independent and a strong tool to the physics community. To test the accuracy and the performance of the program and of the method we present results that using XLOOPS-GiNaC to evaluate sunrise two-loop integrals in comparison with various results of other authors when it is possible.

Even though this paper is dedicated to the two-loop self-energy integrals, the technique is also applicable to an arbitrary two-loop tensor integral [7].

II. TENSOR TWO-LOOP SELF-ENERGY INTEGRALS AND THE REDUCTION

A general tensor two-loop integral has the following representation

$$
T^{(2)\,\mu_1\dots\mu_p,\nu_1\dots\nu_r}_{t_1\dots t_{n_l};\dots t_{n_l+n_m};\dots t_{n_l+n_m+n_k}} = \int d^D l \int d^D k \, \frac{l^{\mu_1}\dots l^{\mu_p}}{\prod\limits_{s=1}^{n_l}\left[(l+q_s)^2 - m_s^2 + i\rho\right]^{t_s}}
$$

$$
\times \frac{1}{\prod\limits_{s=n_l+1}^{n_l+n_m}\left[(l+k+q_s)^2 - m_s^2 + i\rho\right]^{t_s}}
$$

$$
\times \frac{k^{\nu_1}\dots k^{\nu_r}}{\prod\limits_{s=n_n+n_m+1}^{n_l+n_m+n_k}\left[(k+q_s)^2 - m_s^2 + i\rho\right]^{t_s}}
\qquad (1)
$$

where n_l, n_k, n_m are the numbers of propagators which carry the loop momenta l, k or $l+k$ respectively. q_s and m_s are external momenta and masses of particles involved.

It is believed that, beyond the one-loop level, the result of a general massive integral may not be expressible in terms of polylogarithms [8,9]. To integrate an integral such as the one in Eq. (1), one introduces a set of proper subtraction terms to separate the singular parts (UV or IR singularities) into analytically solvable integrals and finite integrals which then can be integrated numerically.

For two-loop self-energy integrals, there are four non-trivial topologies shown in Fig.(1), which are not able to be reduced to another topologies or to products of one-loop ones

Different to the Passarino-Veltman's reduction technique, we are going to evaluate Feynman integrals which are associated with diagrams in Fig.(1) by working in a specific frame of reference where the momentum of the incoming particle has only one none vanishing component [1]. In such a frame of reference, for example a space-like particle with $q^2 > 0$, one can parameterize

$$
q^\mu = (q_0, \vec{0}) \qquad (2)
$$

with $\vec{0}$ is the null-vector in $D-1$ dimension space.

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[1]Note that, we can always find down such frame of references if the external momentum q^μ is space-like or time-like. If the external momenta is light-like, $q^2 = 0$, the integral cannot be directly calculated using parallel and orthogonal technique. For this special case, the integral can be evaluated by reducing to scalar integrals [10] that in turn can be evaluated using various techniques such as in [11–14].
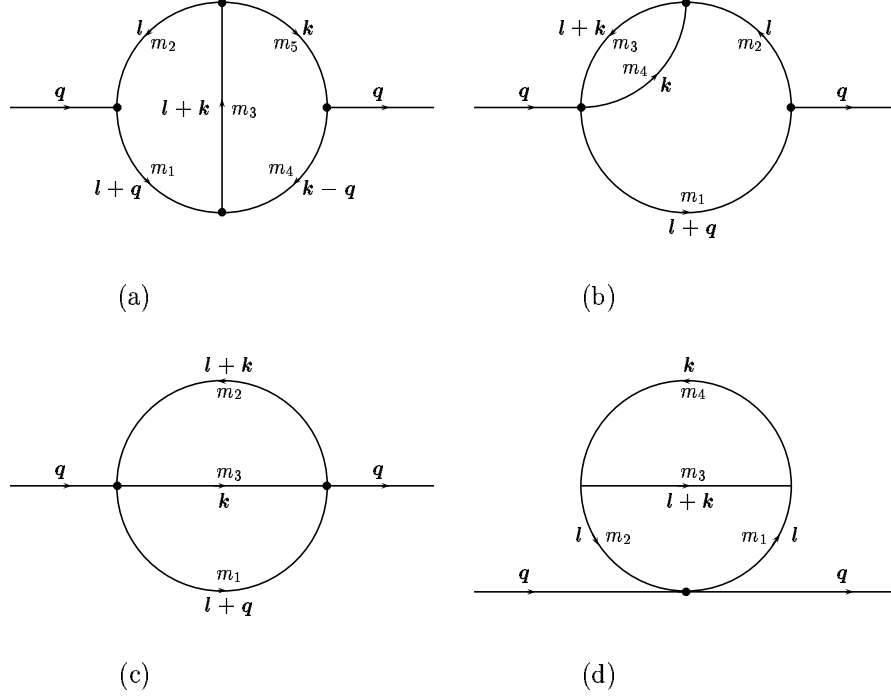
Fig. 1. The non-trivial two-loop self-energy topologies.

In this frame of reference, the components of loop momenta in the parallel and orthogonal spaces spanned by the external momentum q^μ can be defined by

$$
\begin{aligned}
l_0 &= \frac{l \cdot q}{\sqrt{q^2}}, \qquad k_0 = \frac{k \cdot q}{\sqrt{q^2}}, \\
l_\perp^\mu &= l^\mu - \frac{l \cdot q}{q^2} q^\mu = (0, \vec{l}_\perp); \\
k_\perp^\mu &= k^\mu - \frac{k \cdot q}{q^2} q^\mu = (0, \vec{k}_\perp)
\end{aligned}
\tag{3}
$$

with $\vec{l}_\perp$ and $\vec{k}_\perp$ are $D - 1$ dimension vectors in the perpendicular sub-space to q^μ. [2]

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[2]For the time-like cases where $q^2 < 0$ we can process in a similar manner after rearranging the indices.

In terms of the new variables, the integrals which are associated with topologies in Fig.(1) can be rewritten respectively in the form

$$T^a = \int d^D l \int d^D k \frac{l_0^{p_0} l_\perp^{p_\perp} k_0^{r_0} k_\perp^{r_\perp} z^{p_z}}{[(l+q)^2 - m_1^2]^{t_1} [l^2 - m_2^2]^{t_2}} \times \tag{4}$$

$$\times \frac{1}{[(l+k)^2 - m_3^2]^{t_3} [(k-q)^2 - m_4^2]^{t_4} [k^2 - m_5^2]^{t_5}},$$

$$T^{(b)} = \int d^D l \int d^D k \ \frac{l_0^{p_0} l_\perp^{p_\perp} k_0^{r_0} k_\perp^{r_\perp} z^{p_z}}{[(l+q)^2 - m_1^2]^{t_1} [l^2 - m_2^2]^{t_2}[(l+k)^2 - m_3^2]^{t_3} [k^2 - m_4^2]^{t_4}}, \tag{5}$$

$$T^{(c)} = \int d^D l \int d^D k \ \frac{l_0^{p_0} l_\perp^{p_\perp} k_0^{r_0} k_\perp^{r_\perp} z^{p_z}}{[(l+q)^2 - m_1^2]^{t_1} [(l+k)^2 - m_2^2]^{t_2} [k^2 - m_3^2]^{t_3}}, \tag{6}$$

$$T^{(d)} = \int d^D l \int d^D k \ \frac{l_0^{p_0} l_\perp^{p_\perp} k_0^{r_0} k_\perp^{r_\perp} z^{p_z}}{[l^2 - m_1^2]^{t_1} [l^2 - m_2^2]^{t_2}} \frac{1}{[(l+k)^2 - m_3^2]^{t_3} [k^2 - m_4^2]^{t_4}} \tag{7}$$

where z is the cosine of the angle between $\vec{l}_\perp$ and $\vec{k}_\perp$. The variables $p_0, p_\perp, r_0, r_\perp, p_z$ are non-negative integers and t_i are positive integers. Detailed discussions on the transformation of the standard tensor integrals, as in Eq. (1), to a parallel and orthogonal space representation can be found in [4, 15]. Here and to the rest of the paper, we will call the integrals in Eq. (4) to Eq. (7) *tensor integrals in the parallel and orthogonal representation* or simply *tensor integrals* if there is any confuse.

Working in this representation has a big advantage in performing tensor reduction [4,16–18]. The other advantage is that we can find out a two-dimensional representation for the finite part of the integral directly by using the residuum theorem without performing the Feynman parametrisation.

It is worth to note here that the non-negative integers $p_0, p_\perp, r_0, r_\perp, p_z$ are not Lorentz indices. They are exponents of the real variables l_0, $l_\perp$, k_0, $k_\perp$ and z respectively. In general, $p_\perp$ and $r_\perp$ can be odd or even due to the fact that the denominators of the integrands in Eq. (4) to Eq. (7) are not symmetric either in $l_\perp$ or in $k_\perp$ separately. However, since $T^{(2)\ \mu_1...\mu_p,\nu_1...\nu_r}_{t_1...t_{n_l};...t_{n_l+n_m};...t_{n_l+n_m+n_k}}$ is even when the signs of l and k are changed at the same time, thus one has other constraints:

- The integrals with odd $(p_\perp + r_\perp)$ vanish.
- p_z must be smaller than or equal to $\min\{p_\perp, r_\perp\}$.
- Both $(p_\perp - p_z)$ and $(r_\perp - p_z)$ are even.

In general, the integrals in Eq. (4) to Eq. (7) can be degenerate. However, by a simple reduction, we will prove that all the integrals will be reduced to a sum of none-degenerate integrals and degenerate integrals of the form $T^{(c)}$. It means that the degenerate problem stays only in the *sunrise* integral, the Eq. (6).

To have a closer look at this reduction, let us consider the reduction of the integral in Eq. (4).

II.1. Tensor reduction

We first rewrite the integral in Eq. (4) in the following form

$$T^{(a)}(q^2) = \int d^D l \int d^D k \frac{l_0^{p_0} \, l_\perp^{p_\perp} \, k_0^{r_0} \, k_\perp^{r_\perp} \, z^{p_z}}{P_1^{t_1}(l+q) \, P_2^{t_2}(l) \, P_3^{t_3}(l+k) \, P_4^{t_4}(k-q) \, P_5^{t_5}(k)} \tag{8}$$

with

$$
\begin{aligned}
P_1(l+q) &= (l+q)^2 - m_1^2 + i\,\rho, \\
P_2(l) &= l^2 - m_2^2 + i\,\rho, \\
P_3(l+k) &= (l+k)^2 - m_3^2 + i\,\rho, \\
P_4(k-q) &= (k-q)^2 - m_4^2 + i\,\rho, \\
P_5(k) &= k^2 - m_5^2 + i\,\rho.
\end{aligned}
\tag{9}
$$

We can always remove the variable z in the numerator by inserting

$$l_\perp \, k_\perp \, z = \frac{1}{2} \left[-P_3 + (l_0 + k_0)^2 - l_\perp^2 - k_\perp^2 - m_3^2 - i\,\rho \right] \tag{10}$$

into the numerator of the integral in Eq. (8). After expanding recursively, one obtains only three basics integrals of the form

$$J_0 = \int d^D l \int d^D k \frac{1}{P_1^{t_1}(l+q) \, P_2^{t_2}(l) \, P_3^{t_3}(l+k) \, P_4^{t_4}(k-q) \, P_5^{t_5}(k)} \tag{11}$$

$$J_1 = \int d^D l \int d^D k \frac{k_0^{r_0}}{P_1^{t_1}(l+q) \, P_2^{t_2}(l) \, P_3^{t_3}(l+k) \, P_4^{t_4}(k)} \tag{12}$$

$$J_2 = \int d^D l \int d^D k \frac{l_0^{p_0} \, k_0^{r_0}}{P_1^{t_1}(l+q) \, P_2^{t_2}(l+k) \, P_3^{t_3}(k)} \tag{13}$$

The integral J_0 is finite so that one can perform a numerical integration directly. A two-fold representation for the numerical integration is discussed in [16]. The topology of two-point diagrams with three propagators is called *sunrise topology*. Its associated integrals, such as $T^{(c)}$ or J_2, are called sunrise integrals. By the same procedure, the two other tensor integrals which are associated with topologies (b) and (c) can also be reduced to expressions containing J_1, J_2 and products of one-loop functions.

II.2. Power counting and degenerate integrals

In order to construct the subtraction terms for the integral J_1, J_2, let us define some notations.

Let w denotes the overall degree of UV-divergence and w_l, w_k, w_{lk} are degrees of UV-sub-divergences. By power counting, for J_1 we define

$$
\begin{aligned}
w_l &= 2(t_1 + t_2 + t_3) - D > 0, \\
w_k &= 2(t_3 + t_4) - D - r_0, \\
w_{lk} &= 2(t_1 + t_2 + t_4) - D - r_0 \\
w &= \frac{1}{2}(w_l + w_k + w_{lk} - D).
\end{aligned}
\tag{14}
$$

and for J_2

$$\begin{aligned}
w_l &= 2(t_1 + t_2) - D - p_0, \\
w_k &= 2(t_2 + t_3) - D - r_0, \\
w_{lk} &= 2(t_1 + t_3) - D - p_0 - r_0 \\
w &= \frac{1}{2}(w_l + w_k + w_{lk} - D).
\end{aligned} \tag{15}$$

The condition for UV-finite of an integral is that all of its degrees of UV-divergence are positive. We call an UV-divergent integral *degenerate* if it is truly overall UV-divergent. By power counting, an integral is degenerate if

$$w_l + w_k + w_{lk} \leq D. \tag{16}$$

Otherwise, the integral is said *non-degenerate*. The method for evaluating a non-degenerate integral is presented in detailed in [1] which we will not review in this paper. From now on, we will consider only the degenerate integrals. It is clear that from the power counting condition, J_0 is finite and can then be integrated numerically in two-dimension space and was discussed in detailed in [5, 16].

In this paper, we consider only the case where $w_{lk} > 0$. In this condition, J_1 is non-degenerate with any positive value of r_0 and the only degenerate integral is of the type J_2. The integral J_1 is also be integrated with a suitable subtraction terms as in [4, 5, 16]. To complete the calculation, we have to consider the case where $w_{lk} \leq 0$. However this problem will be discussed separately in another paper.

To close this section, we conclude that only integrals of type J_2 can be degenerate. This type of integrals is the main interest of this paper. In the next section, we start to construct the subtraction terms for the degenerate integral J_2 with a condition $w_{lk} > 0$.

II.3. The degenerate integral J_2 and its subtraction terms

In this section, we consider the integral J_2 introduced in Eq. (13) which appeared as a result of the tensor reduction

$$J_2 = \int d^D l \, d^D k \, \mathcal{J}_{2\,\{t_i\}}^{p_0, r_0} \qquad (i = 1, 2, 3)$$

with the condition

$$w_{lk} = 2(t_1 + t_3) - D - p_0 - r_0 > 0. \tag{17}$$

Let us rewrite the integrand

$$\mathcal{J}_{2\,\{t_i\}}^{p_0, r_0} = \frac{l_0^{p_0} \, k_0^{r_0}}{P_1^{t_1}(l+q) \, P_2^{t_2}(l+k) \, P_3^{t_3}(k)} \tag{18}$$

where

$$\begin{aligned}
P_1(l+q) &= (l+q)^2 - m_1^2 + i\rho, \\
P_2(l+k) &= (l+k)^2 - m_2^2 + i\rho, \\
P_3(k) &= k^2 - m_3^2 + i\rho.
\end{aligned} \tag{19}$$

By power counting, if $w_l \geq 2t_2$ or $w_k \geq 2t_2$ the integral is non-degenerate and one can evaluate by the same procedure as described in [1].

We now suppose that $w_l \leq 0$, $w_k \leq 0$ and $w \leq 0$ [3]. In order to remove all divergences, one needs subtraction terms to remove the overall-, the l- and the k-sub-divergences. The trick is to use a massive subtraction term to increase the degree of l-sub-divergence to $2t_2$ and isolate the overall divergence in a scalar two-loop vacuum integral $T^{(d)}$ (see Eq. (7)) which is integrable [11–14]. The k-sub-divergence is still subtracted off by the massless subtraction term as in [1]. We define subtraction terms

$$L^{x_l} := \left(1 - \frac{P_1^{t_1}}{\overline{P}_1^{t_1}}\right)^{x_l} = 1 + L_0^{x_l}, \tag{20}$$

$$K^{x_k} := \left(1 - \frac{P_2^{t_2} P_3 t_3}{\widetilde{P}_2^{t_2} \widetilde{P}_3^{t_3}}\right)^{x_k} = 1 + K_0^{x_k}$$

where

$$\overline{P}_1 := l^2 - m_1^2 + i\rho, \qquad \widetilde{P}_2 := (l+k)^2, \qquad \widetilde{P}_3 := k^2,$$

$$L_0^{x_l} = \sum_{n=1}^{x_l} \binom{x_l}{n} (-1)^n \left(\frac{P_1^{t_1}}{\overline{P}_1^{t_1}}\right)^n,$$

$$K_0^{x_k} = \sum_{n=1}^{x_k} \binom{x_k}{n} (-1)^n \left(\frac{P_2^{t_2} P_3^{t_3}}{\widetilde{P}_2^{t_2} \widetilde{P}_3^{t_3}}\right)^n.$$

x_l and x_k are chosen such that

$$x_l + w_l = 2t_2, \qquad x_k + w_k = 1.$$

In contrast to K, the subtraction term L is constructed by a massive propagator $1/\overline{P}_1$ that allows one to choose x_l big enough to make the whole integral finite without introducing an IR-divergence. This is the main difference to [1]. Using subtraction terms L and K, the integral can be rewritten as follows

$$J_2 = \int d^4l \, d^4k \, \mathcal{J}_{2\,\{t_1\}}^{p_0,r_0} L^{x_l} K^{x_k} - \int d^D l \, d^D k \, \mathcal{J}_{2\,\{t_i\}}^{p_0,r_0} L_0^{x_l} \tag{21}$$

$$- \int d^D l \, d^D k \, \mathcal{J}_{2\,\{t_i\}}^{p_0,r_0} L_0^{x_l} K_0^{x_k} - \int d^D l \, d^D k \, \mathcal{J}_{2\,\{t_i\}}^{p_0,r_0} K_0^{x_k}.$$

The first term on the right-hand side of Eq. (21) is finite; so it can be integrated numerically in $D = 4$ dimensions. The last two terms are massless k-integrals which can be reduced to the integral

$$\widetilde{J}_1 = \int d^D l \, d^D k \, \frac{l_0^{p_0} k_0^{r_0}}{P_1^{t_1}(l+q)\widetilde{P}_2^{t_2}(l+k)\widetilde{P}_3^{t_3}(k)} \tag{22}$$

where $t_i \in \mathbb{N}$ and p_0 and r_0 are non-negative integers. The integral $\widetilde{J}_1$ will be solved in the next section.

_____

[3]If $w_k > 0$ (or $w_l > 0$) we need only massive subtraction terms for the l- (or the k-) integral.

The second term of Eq. (21) is the tensor vacuum two-loop integral and has the form

$$\int d^D l\, d^D k \frac{l_0^{p_0}\, k_0^{r_0}}{\left[l^2 - m_1^2 + i\rho\right]^{t_1} \left[(l+k)^2 - m_2^2 + i\rho\right]^{t_2} \left[k^2 - m_3^2 + i\rho\right]^{t_3}} \tag{23}$$

$$= \int d^D l\, d^D k \frac{l^{\mu_1}\ldots l^{\mu_{p_0}} k^{\nu_1}\ldots k^{\nu_{r_0}}}{\left[l^2 - m_1^2 + i\rho\right]^{t_1} \left[(l+k)^2 - m_2^2 + i\rho\right]^{t_2} \left[k^2 - m_3^2 + i\rho\right]^{t_3}} \times$$

$$\times \frac{q_{\mu_1}\ldots q_{\mu_{p_0}} q_{\nu_1}\ldots q_{\nu_{r_0}}}{q^{r_0+p_0}}.$$

An analytical solution of this integral with $t_1 = t_2 = t_3 = 1$ can be found in [12]. A closed analytical solution of a vacuum two-loop integral with arbitrary t_1, t_2 and t_3 can be obtained by integration by part and can be found in [13, 14]. In this paper, we present another approach in calculating this integral. By using the decomposition in Lorentz covariant terms and then contracting with the external momenta q_{μ_i}, the above integral is decomposed into massive vacuum integrals of the form

$$\int d^D l\, d^D k \frac{(l^2)^n (l\cdot k)^m (k^2)^p}{\left[l^2 - m_1^2 + i\rho\right]^{t_1} \left[(l+k)^2 - m_2^2 + i\rho\right]^{t_2} \left[k^2 - m_3^2 + i\rho\right]^{t_3}} \tag{24}$$

where n, m, p are non-negative integer numbers. The integral is then reduced to products of one-loop integrals and (two-loop) scalar massive vacuum integrals of the form

$$S_{\{t_1,t_2,t_3\}} = \tag{25}$$
$$\int d^D l\, d^D k \frac{1}{\left[l^2 - m_1^2 + i\rho\right]^{t_1} \left[(l+k)^2 - m_2^2 + i\rho\right]^{t_2} \left[k^2 - m_3^2 + i\rho\right]^{t_3}}.$$

An efficient recursion relation to reduce Eq. (23) to Eq. (25) and the analytical solution of the massive vacuum integrals $S_{\{t_1,t_2,t_3\}}$ is presented in [11–13]

In the section **II.5**, we present another approach to solve the massive vacuum integral, which is more flexible from the computational point of view. But let us come back to the solution of the integral $\widetilde{J}_1$.

II.4. The integral $\widetilde{J}_1$

This section is dedicated to the analytical solution of the integrals $\widetilde{J}_1$ which appears in Eq. (22). We first write the integral more explicitly

$$\widetilde{J}_1 = \int d^D l\, d^D k \frac{l_0^{p_0}\, k_0^{r_0}}{P_1^{t_1}(l+q)\, [(l+k)^2]^{t_2}\, [k^2]^{t_3}} \tag{26}$$

We can think of the integral over k as a one-loop two-point integral with massless particles and "*external*" momentum l with explicite sulution

$$\int d^D k \frac{k_0^{r_0}}{\left[(l+k)^2\right]^{t_3} [k^2]^{t_4}} = \tag{27}$$

$$= \frac{i\,(-1)^{D/2}(\pi)^{D/2}}{\Gamma(t_3)\Gamma(t_4)} \sum_{j=0}^{[r_0]/2} \frac{r_0!\,(-l_0)^{r_0-2j}\,(l^2)^{j+D/2-t_3-t_4}}{(r_0-2j)!\,j!\,2^{2j}} \times$$

$$\times\,\Gamma(t_4+t_4-D/2-j)\,\mathcal{B}(D/2-t_4+r_0-j, D/2-t_3+j)$$

with $[r_0] = r_0$ if r_0 is even, and $[r_0] = r_0 - 1$ if r_0 is odd and $\mathcal{B}$ is the usual beta function defined by

$$\mathcal{B}(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}. \tag{28}$$

Substituting the right-hand side of Eq. (27) into Eq. (26), one ends up with three-point one-loop integrals of the form

$$\mathcal{K} \int d^D l \frac{l_0^{p_0'}}{(l^2)^\alpha\, P_1^{t_1}(l+q)} \tag{29}$$

where α is an integer when $D = 4$, and $\mathcal{K}$ is a coefficient which depends on D, t_3, t_4 and r_0.

Because the l-integral is still UV-divergent, one can not set $D = 4$, so α is in general a non-integer number. To calculate this kind of integral, we will use another approach rather than in [1] by employing the hypergeometric functions. Using the same reduction procedure as in the previous sections, the integral in Eq. (29) can be reduced to the form

$$\mathfrak{L} = \int d^D l \frac{(l^2)^n}{[(l+q)^2 - m_1^2 + i\rho]^t [l^2]^\alpha} \tag{30}$$

where α is a non-integer number. Now, using Feynman parametrisation and after shifting the momentum, one obtains

$$\mathfrak{L} = \sum_{j_1=0}^{n} \sum_{j_2=0}^{j_1} \mathfrak{C} \int_0^1 dx\, x^{a-1}(1-x)^{a'-1}\, [(z-i\rho)-x]^{-b_1} \tag{31}$$

with j_2 is even, $z = \frac{m^2}{q^2}$, $a = \alpha$ and

$$a' = \left(j_1 + n - \alpha + \frac{D-j_2}{2}\right), \qquad b_1 = -\left(n - j_1 - t - \alpha + \frac{D+j_2}{2}\right) \tag{32}$$

and

$$\mathfrak{C} \;=\; i \begin{pmatrix} n \\ j_1 \end{pmatrix} \begin{pmatrix} j_1 \\ j_2 \end{pmatrix} \, (2)^{j_2} \, \pi^{D/2} \, (-1)^{n - j_1 + \frac{3 j_2}{2} - t - \alpha} \, (q^2)^{n - t - \alpha + D/2} \tag{33}$$

$$\times \; \frac{\mathcal{B}(t + \alpha - n + j_1 - \frac{j_2 + D}{2}, n - j_1 + \frac{j_2 + D}{2})}{\mathcal{B}(t, \alpha)} \frac{\Gamma(D/2) \Gamma(j_2/2 + 1/2)}{\Gamma(D/2 + j_2/2)}.$$

With the help of a good mathematical handbook, the integral over x in Eq. (31) can be integrated in terms of hypergeometric functions

$$\mathfrak{L} = \sum_{j_1 = 0}^{n} \sum_{j_2 = 0}^{j_1} \mathfrak{C} \; (z - i\rho)^{-b_1} \, \mathcal{B}(a, a') \, _2F_1 \left(a, b_1, a + a', \frac{1}{z - i\rho} \right). \tag{34}$$

For the scalar sunrise integral, this result is in agreement with [19]. For the general tensor case, an expansion method (at the pole $D = 4$) for the hypergeometric functions $_2F_1$ is presented in [20]. From the computational point of view, the expression Eq. (34) is ready to be evaluated with the help of nested sums as introduced in [20].

II.5. The massive vacuum integral $S_{\{t_1, t_2, t_3\}}$

To calculate the massive vacuum integral $S_{\{t_1, t_2, t_3\}}$, it is sufficient to consider the special case $t_1 = t_2 = t_3 = 1$. If $t_1 > 1$ (or t_2 or $t_3 > 1$) then the integral can be evaluated as in a non-degenerate case. Let us consider the integral

$$S_{\{1,1,1\}} \;=\; \int d^D l \, d^D k \frac{1}{P_1 \, P_2 \, P_3}$$

with

$$P_1 \;=\; l^2 - m_1^2 + i\rho, \qquad P_2 = (l + k)^2 - m_2^2 + i\rho, \qquad P_3 = k^2 - m_3^2 + i\rho.$$

By a simple transformation one obtains

$$S_{\{1,1,1\}} \;=\; \int d^D l \, d^D k \; \frac{m_1^2}{P_1 \, \widetilde{P}_1 \, P_2 \, P_3} + \int d^D l \, d^D k \; \frac{m_2^2}{\widetilde{P}_1 \, P_2 \, \widetilde{P}_2 \, P_3}$$
$$+ \int d^D l \, d^D k \; \frac{1}{\widetilde{P}_1 \, \widetilde{P}_2 \, P_3}$$

with

$$\widetilde{P}_1 \;=\; l^2, \qquad \widetilde{P}_2 = (l + k)^2. \tag{35}$$

The first two terms of Eq. (35) are actually non-degenerate integrals which can be evaluated by the method in [1]. The last term which contains the overall divergence is now isolated in a simple integral $\widetilde{J}_1$ (see section **II.4**) which can be integrated analytically.

Having discussed the analytical solution of the divergent terms, in order to complete the calculation, one needs to find numerical solutions of the integrals which appear in the finite term of Eq. (21). The method of the calculation is presented in detail in [16]. In the next paragraphs, we will present a two-dimension representation of the finite part which can be evaluated numerically by computer programs such as VEGAS [21] or ParInt [22].

A two-dimension integral representation can be obtained by integrate over the variables $l_\perp$, $k_\perp$ and z and finally results in the following form

$$
\int d^4 l \int d^4 k \ \mathcal{J}_{2\,\{t_i\}}^{p_0\,r_0} \ L^{x_l} \ K^{x_k} \quad = \quad \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dk_0 \sum_{n_1=0}^{x_k} \sum_{n_2=0}^{x_l} \binom{x_k}{n_1} \binom{x_l}{n_2} \times
$$
$$
\times \ (-1)^{n_1+n_2} \ l_0^{p_0} \ k_0^{r_0} \ \mathfrak{J}_2^{n_1,n_2} \tag{36}
$$

with

$$
\mathfrak{J}_2^{n_1,n_2} = 8\pi^2 \int_0^{\infty} dl_\perp \ dk_\perp \int_{-1}^{1} dz \ l_\perp^2 \ k_\perp^2 \ \frac{\left(P_1^{t_1}\right)^{n_2-1} \left(P_2^{t_2} P_3^{t_3}\right)^{n_1-1}}{P_1^{t_1} \ P_2^{t_2} \ P_3^{t_3} \ (\overline{P}_1^{t_1})^{n_2} \left(\widetilde{P}_2^{t_2} \widetilde{P}_3^{t_3}\right)^{n_1}}. \tag{37}
$$

Here, we write the integrals in terms of variables in the parallel and orthogonal space explicitly. The integrals $\mathfrak{J}_2$ are well defined. Further, $\mathfrak{J}_2$ can be decomposed into a sum of integrals of the form

$$
\widehat{J}_2 \quad = \quad 8\pi^2 \int_0^{\infty} dl_\perp dk_\perp \int_{-1}^{1} dz \ \frac{l_\perp^2 \ k_\perp^2}{P_1^{t_1}(l+q) \ P_2^{t_2}(l+k) \ P_3^{t_3}(k)}. \tag{38}
$$

Integrating this integral by using the residuum theorem one obtains [4, 16].

$$
\widehat{J}_2 = -4\pi^4 \ \ln(w_1 + w_2 + w_3) \tag{39}
$$

where

$$
\begin{aligned}
w_1^2 &= (l_0 + q_0)^2 - m_1^2 + i\rho, \\
w_2^2 &= (l_0 + k_0)^2 - m_2^2 + i\rho, \\
w_3^2 &= k_0^2 - m_3^2 + i\rho.
\end{aligned} \tag{40}
$$

This result completes the method of integration for a degenerate tensor two-loop self-energy integral.

III. SUN-RISE INTEGRAL AND NUMERICAL DISCUSSIONS

The method of of integration for a degenerate tensor two-loop self-energy integral was implemented successfully in the program package XLOOPS-GiNaC.[4] In this section, we present the results of calculating the sunrise integral [5] using the program package XLOOPS-GiNaC. The results are then compared with the results of the existing results [14, 19, 23, 24].

The result of XLOOPS-GiNaC consists of two parts. One part is evaluated analytically in terms of the one-loop and/or hypergeometric functions. The other one is evaluated numerically by using Monte Carlo or Quasi-Monte Carlo methods with the help of the routines Vegas [21] or ParInt [22].

In the following, the sum of the analytical value and of the average of the result of the numerical integration will be called the *main value*. The error of Vegas (or ParInt)

---

[4] http://wwwthep.physik.uni-mainz.de/~xloops
[5] Also called sunset integral

will be called the *error*.

We first calculate the finite integral T_{123N} and compare our results with the results of [24]. The finite integral T_{123N} is defined by

$$T_{123N}(p^2; m_1^2, m_2^2, m_3^2) := T_{123}(p^2; m_1^2, m_2^2, m_3^2) - T_{123}(p^2; m_1^2, 0, m_3^2)$$
$$- T_{123}(p^2; 0, m_2^2, m_3^2) + T_{123}(p^2; 0, 0, m_3^2) \tag{41}$$

with

$$T_{123}(p^2; m_1^2, m_2^2, m_3^2) = \tag{42}$$

$$\int d^D l\, d^D k \, \frac{1}{[(l+p)^2 - m_1^2 + i\rho][(l+k)^2 - m_2^2 + i\rho][k^2 - m_3^2 + i\rho]}.$$

In Tables 1 and 2 we show the results of XLOOPS-GiNaC and the results of [24] for small and large p^2 with different values for the masses. For the p^2-values above the threshold (large p^2), we also have a non-zero imaginary part, which is shown in the last two columns of Table 2. Each result in the table is calculated in less than one minute by XLOOPS-GiNaC on a computer with Athlon 1.6 GHz CPU and 512 MB RAM. As seen in the tables, the differences are of order 10^{-3}. The accuracy can be improved by increasing the number of iterations of Vegas and so the time of calculation also increases.

Table 1. Comparison for small p^2 of the real part of the subtracted sunrise integral T_{123N}. In each box, in the first entry we show the value of Table 1 of [24] (small p^2 series expansion). The second entry is our result with its numerical error in the last column.

p^2	m_1	m_2	m_3	Re T_{123N}	Error
100	3	4	20	-6.01715	
				-6.02022	± 0.00167
150	3	4	20	-6.39036	
				-6.38877	± 0.00017
150	5	5	25	-14.5339	
				-14.5780	± 0.00326
200	5	5	25	-15.0523	
				-15.0542	± 0.00192

For a complete calculation of a sunrise integral, we compare the result of XLOOPS-GiNaC with the result of [14, 23]. We will follow the conventions of [14] where the sunrise integral is defined by

$$F_0(D, m_1^2, m_2^2, m_3^2, p^2) = \tag{43}$$

$$\frac{\mu^{8-2D}}{((2\pi)^{D-2})^2} \int d^D k_1 \int d^D k_2 \, \frac{1}{(k_1^2 + m_1^2)(k_2^2 + m_2^2)((p - k_1 - k_2)^2 + m_3^2)}$$

Table 2. Comparison for large p^2 of the real and imaginary part of the subtracted sunrise integral T_{123N}. In each box, in the first entry we show the value of Table 2 of [24] (large p^2 series expansion). The second entry is our result with its numerical error in the next column.

p^2	m_1	m_2	m_3	Re T_{123N}	Vegas Error	Im T_{123N}	Vegas Error
80	2	3	2	0.58743		-11.2628	
				0.58585	$\pm\, 0.00048$	-11.2632	$\pm\, 0.0002$
100	2	3	4	-0.32864		-11.8459	
				-0.32028	$\pm\, 0.00118$	-11.8460	$\pm\, 0.0002$
150	3	4	4	-1.26795		-26.4912	
				-1.21248	$\pm\, 0.00140$	-26.4912	$\pm\, 0.0005$
200	2	3	4	1.69608		-6.02417	
				1.69549	$\pm\, 0.00022$	-6.02412	$\pm\, 0.0001$
250	4	4	4	2.64395		-27.6090	
				2.65248	$\pm\, 0.00893$	-27.6098	$\pm\, 0.0006$

where

$$\mu = m_1 + m_2 + m_3. \tag{44}$$

The integral $F_j(D, m_1^2, m_2^2, m_3^2, p^2)$ can be parametrized by

$$F_0(D, m_1^2, m_2^2, m_3^2, p^2) = \tag{45}$$

$$C^2(D) \left\{ \frac{1}{(D-4)^2} F_0^{(-2)} + \frac{1}{(D-4)} F_0^{(-1)} + F_0^{(0)} \right\}$$

with

$$C(D) = (2\sqrt{\pi})^{4-D} \Gamma(3 - D/2). \tag{46}$$

For the divergent terms we obtain

$$F_0^{(-2)} = -\frac{1}{8} \sum_{i=1}^{3} m_i^2, \tag{47}$$

$$F_0^{(-1)} = \frac{1}{8} \left\{ -\frac{p^2}{4} + \frac{3}{2} \sum_{i=1}^{3} m_i^2 - \sum_{i=1}^{3} m_i^2 \ln\left(\frac{m_i^2}{\mu^2}\right) \right\} \tag{48}$$

which is in agreement with the results of [14,23]. Note that our result differs with the one of [14] by a minus sign in the term $\dfrac{p^2}{4}$ of $F_0^{(-1)}$. This is due to the fact that in [14] the integral is evaluated in Euclidian space.

Table 3. Comparison for the sunrise integral. The masses (in GeV) are $m_1 = 1$, $m_2 = 9$, $m_3 = 200$ and $\mu = m_1 + m_2 + m_3$. Results in columns A are reproduced from [14]. The results of XLOOPS-GiNaC and their errors are shown in column B and C, respectively.

p_r^2	A	B	C
-1	-0.279454902855371	-0.279819	$\pm6.6\text{E-}4$
-0.99	$-0.2798928396415(5)$	-0.280351	$\pm0.9\text{E-}4$
$p_{ps3,r}^2$	-0.280281633048667	-0.280442	$\pm4.4\text{E-}4$
-0.9	$-0.2836780811878(5)$	-0.283169	$\pm5.6\text{E-}4$
$p_{ps2,r}^2$	-0.286233415451605	-0.285265	$\pm16.3\text{E-}4$
-0.825	$-0.2866587055221(5)$	-0.281268	$\pm4.2\text{E-}4$
$p_{ps1,r}^2$	-0.286906928933491	-0.286141	$\pm5.7\text{E-}4$
-0.8	$-0.2876221116285(5)$	-0.288281	$\pm5.4\text{E-}4$
-0.1	$-0.3101507246241(3)$	-0.310516	$\pm1.9\text{E-}4$
0	-0.312816604092084	-0.312800	$\pm0.1\text{E-}4$

In table 3 we present the numerical values of $\frac{F_0^{(0)}}{\mu^2}$ as a function of $\frac{p^2}{\mu^2}$. The abbreviations which are used in the table are

$$
\begin{aligned}
p_{ps1,r}^2 &= \frac{(m_1 - m_2 - m_3)^2}{\mu^2}, \\
p_{ps2,r}^2 &= \frac{(m_1 - m_2 + m_3)^2}{\mu^2}, \\
p_{ps3,r}^2 &= \frac{(m_1 + m_2 - m_3)^2}{\mu^2}.
\end{aligned}
\tag{49}
$$

In column A we show the results of [14] using differential equation techniques. The results of XLOOPS-GiNaC and their errors are shown in columns B and C. The difference of the results is of order 10^{-3} where we use $10^5 \times 10$ iterations per integration for Vegas.

IV. CONCLUSIONS

We have shown that with an extension of the method in [1], a subset of degenerate two-loop self-energy integral can be reduced to finite integral representations and UV-divergent analytic expressions in a similar manner as in non-degenerate cases. The complete method provides a general algorithm of evaluating an arbitrary two-loop self-energy integral which can be easily implemented into a computer program.

It is worth to note that to obtain a stable numerical result, our method needs not to consider different kinematic regions of the integrals such as below or above the threshold or around the pseudo-thresholds.

Even though the finite part of integrals is evaluated numerically in terms of two-dimensions integrals, we are still able to obtain a good accuracy in a reasonable computation time.

ACKNOWLEDGEMENTS

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Received 26 October 2009.