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Analysis of Transient Growth in Iterative Learning Control Using Pseudospectra

Douglas A Bristow and John R. Singler

Abstract—In this paper we examine the problem of transient growth in Iterative Learning Control (ILC). Transient growth is generally avoided in design by using robust monotonic convergence (RMC) criteria. However, RMC leads to fundamental performance limitations. We consider the possibility of allowing safe transient growth in ILC algorithms as a means to circumvent these limitations. Here the pseudospectra is used for the first time to study transient growth in ILC. Basic properties of the pseudospectra that are relevant to the ILC problem are presented. Two ILC design problems are considered and examined using pseudospectra. The pseudospectra provides new results for these problems and illuminates the oft-misunderstood problem of transient growth.

I. INTRODUCTION

ITERATIVE learning control (ILC) [1-3] is used to improve the performance of systems that repeat the same operation many times. ILC uses the tracking errors from previous iterations of the repeated motion to generate a feedforward control signal for subsequent iterations. Convergence of the learning process results in a feedforward control signal that is customized for the repeated motion, yielding very low or zero tracking error.

ILC is a performance-improving control algorithm, rather than a stabilizing algorithm, and thus the emphasis of much of the ILC literature focuses on behavior at convergence. Of course, convergence of the algorithm is typically demonstrated, but comparatively little attention is given to the nature of the convergence. The transient behavior of the learning process, however, is critically important in many practical applications. For example, in robotics and manufacturing applications, slow convergence leads to delays in process startup and possibly costly material waste. Perhaps of greater concern to the ILC designer is the problem of large transient growth [4], whereby the error may grow rapidly and with little warning, potentially damaging hardware.

The problem of large transient growth has been studied extensively by Longman and colleagues [4-8]. Although these works examine the mechanism by which large transient growth occurs, the tools they develop deal primarily with designing algorithms to avoid all transient growth entirely. These algorithms, developed in the frequency domain, can be said to satisfy a robust monotonic convergence (RMC) condition because the control signal

converges monotonically under some suitable norm [8]. Recently, similar results have been obtained using Norm-Optimal ILC [9,10]. In both cases, converged performance is the trade off for RMC (equivalently, more model uncertainty means worse tracking).

In another approach using an exponentially decaying learning filter [11] monotonic convergence is also demonstrated. Although, not discussed explicitly in that work, it is straightforward to extend the approach to achieve RMC (for example using interval uncertainty [12]). In this case convergence to zero tracking error occurs, but the tradeoff for RMC is convergence rate. It may be interesting to combine the above approaches to achieve RMC with some new combination of tradeoffs in converged performance and convergence rate. Although such an approach may provide a better tradeoff, it is not expected to eliminate these tradeoffs.

In order to extend beyond the limitations of RMC, it is necessary to revisit the problem of transient growth. Indeed, some transient growth may not be problematic, provided it is not so large as to damage equipment or so long lasting as to significantly delay convergence. Therefore, we might to develop robust transient convergence conditions and algorithms that have better performance and convergence rate tradeoffs, as compared to RMC. This paper presents initial progress toward this goal. In particular, this paper will 1) introduce the pseudospectra mathematical tool [13] to the ILC community for use in transient analysis and 2) illustrate the utility of this tool in understanding and designing for (safe) transient growth through two ILC design examples.

The remainder of this paper is organized as follows. In Section II we set up the problem of transient growth in ILC. The Pseudospectra is introduced in Section III. The following two sections present two ILC design problems. An example is given for each problem and the pseudospectra is used to provide new insight into the transient behavior in these algorithms. The insight can be used to improve the performance of the algorithms. Finally, concluding remarks are given in Section V.

II. TRANSIENT ANALYSIS PROBLEM SETUP

For simplicity of presentation, we will consider single-input, single-output time-invariant systems (SISO), although extension to analysis of multi-input, multi-output (MIMO) or time-varying systems is straightforward. For brevity, we begin with the well-known lifted system description [3] of a discrete-time (SISO) linear time-invariant (LTI) dynamic system,

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$$\mathbf{e}_j = \mathbf{P}\mathbf{u}_j + \mathbf{e}_0, \quad (1)$$

where,

$$\mathbf{P} = \begin{bmatrix} p_1 & 0 & 0 & \cdots & 0 \\ p_2 & p_1 & 0 & \ddots & \vdots \\ p_3 & p_2 & p_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ p_N & \cdots & p_3 & p_2 & p_1 \end{bmatrix}, \quad (2)$$

is the matrix of the system's Markov parameters and

$$\begin{aligned} \mathbf{e}_j &= [e_j(1) \ e_j(2) \ \cdots \ e_j(N)]^T, \\ \mathbf{u}_j &= [u_j(0) \ u_j(1) \ \cdots \ u_j(N-1)]^T, \\ \mathbf{e}_0 &= [e_0(1) \ e_0(2) \ \cdots \ e_0(N)]^T. \end{aligned}$$

are the vector representations of the error, control, and initial error, respectively, in an N -step learning process.

A linear ILC algorithm written in lifted form is given by,

$$\mathbf{u}_{j+1} = \mathcal{Q}(\mathbf{u}_j + \mathcal{L}\mathbf{e}_j), \quad (3)$$

where \mathcal{Q} and \mathcal{L} are $N \times N$. Combining (1), (3), closed-loop dynamics are given by,

$$\mathbf{u}_{j+1} = \mathbf{T}\mathbf{u}_j + \mathbf{f}_0, \quad (4)$$

where $\mathbf{T} \triangleq \mathcal{Q}(\mathbf{I} - \mathcal{L}\mathbf{P})$ and $\mathbf{f}_0 \triangleq \mathcal{Q}\mathcal{L}\mathbf{e}_0$.

Clearly the ILC system is **exponentially convergent** if $\rho(\mathbf{T}) < 1$, where $\rho(\bullet)$ is the spectral radius, or largest eigenvalue, of (\bullet) . If the system is exponentially convergent, we define $\mathbf{u}_\infty \triangleq \lim_{j \rightarrow \infty} \mathbf{u}_j$, and rewrite (4) as [3],

$$\mathbf{u}_\infty - \mathbf{u}_{j+1} = \mathbf{T}(\mathbf{u}_\infty - \mathbf{u}_j),$$

or equivalently,

$$\mathbf{u}_\infty - \mathbf{u}_j = \mathbf{T}^j(\mathbf{u}_\infty - \mathbf{u}_0), \quad (5)$$

Thus, we have that $\|\mathbf{u}_\infty - \mathbf{u}_j\| \leq \|\mathbf{T}^j\| \cdot \|\mathbf{u}_\infty - \mathbf{u}_0\|$, where $\|\bullet\|$ is 2-norm, or the largest singular value of (\bullet) . Therefore, the transient response of the learning process is bounded by the sequence,

$$\|\mathbf{T}\|, \|\mathbf{T}^2\|, \|\mathbf{T}^3\|, \dots, \|\mathbf{T}^j\|, \dots \quad (6)$$

If \mathbf{T} is known, one may numerically compute the sequence (6), at least for some finite number of iterations. However, such an approach is numerically expensive when N is large and does not provide meaningful design insight. Therefore, we require tools that describe the sequence (6) using only properties of \mathbf{T} , without explicitly calculating the sequence \mathbf{T}^j . Some results, based on eigen- and singular-values, are well known and summarized in Table 1. They are shown graphically in Figure 1.

Table 1. Well known transient response bounds.

| | |
|-----------------------------|---|
| Spectral radius decay rate: | $\ \mathbf{T}^j\ \leq \kappa(\mathbf{V})\rho(\mathbf{T})^k$ ¹ |
|-----------------------------|---|

| | |
|-------------------------------------|--|
| Singular value decay (growth) rate: | $\ \mathbf{T}^{j+1}\ \leq \ \mathbf{T}\ ^{j+1}$ |
|-------------------------------------|--|

| | |
|----------------|------------------|
| Initial slope: | $\ \mathbf{T}\ $ |
|----------------|------------------|

| | |
|-----------------|--|
| Limiting slope: | $\lim_{j \rightarrow \infty} \ \mathbf{T}^j\ ^{1/j} = \rho(\mathbf{T})$ ² |
|-----------------|--|

¹ \mathbf{V} is the matrix of eigenvectors of \mathbf{T} , and $\kappa(\mathbf{V}) \triangleq \bar{\sigma}(\mathbf{V})/\underline{\sigma}(\mathbf{V})$. [13], page 19.

² Proof on page 159, [13].

Remark 1: It is interesting to note that bounds on the transient response, such as $\kappa(\mathbf{V})$, are not related to the eigenvalues. This holds true in ILC analysis where it has been shown that it is easy to set up exponentially stable ILC with small eigenvalues, but very large response [3,14]. Therefore, one can conclude that, generally, eigenvalues and eigenvalue analysis have little practical meaning in ILC. ■

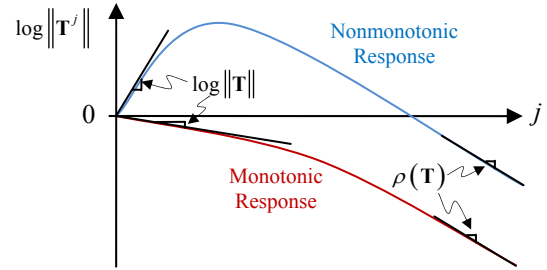


Figure 1. Transient response of the learning process with well-known bounds.

The most widely used transient bounding constraint in ILC is the so called monotonic convergence condition [3,10,15]. The ILC system is said to be **monotonically convergent** if $\|\mathbf{T}\| < 1$. The appeal of monotonic convergence is apparent from Figure 1: convergence rate is known and the largest response is trivially the initial condition. However, as discussed in the Introduction, using monotonic convergence as a design requirement may be artificially restrictive.

In the nonmonotonic case, $\|\mathbf{T}\| \geq 1$, eigen- and singular-value analysis only describe initial and final behavior. Specifically, Table 1 shows that initial slope is based on singular values, whereas final slope is based on eigenvalues. The pseudospectra, discussed in the following section, provides new insight into the critically important transient region in between.

III. THE PSEUDOSPECTRA

In this section we briefly introduce the pseudospectra mathematical tool along with some of the most relevant pseudospectra results. For a complete treatment of the pseudospectra tool, the reader is referred to [13]. The following definition of the pseudospectra is given on page 13 of [13].

Definition: Let $A \in \mathbb{C}^{N \times N}$ and $\varepsilon > 0$ be arbitrary. The ε -pseudospectrum $\sigma_\varepsilon(A)$ of A is the set of $z \in \mathbb{C}$ such that

$$\|(z-A)^{-1}\| > \varepsilon^{-1}. \quad (7)$$

The pseudospectrum is a generalization of the spectrum, or set of eigenvalues. Note that when z is an eigenvalue of A , then $\|(z-A)^{-1}\|$ is unbounded, so $\sigma_\varepsilon(A)$ always contains the eigenvalues of A . More generally, we can think $\sigma_\varepsilon(A)$ as a set of approximate eigenvalues, where the quality of such an approximation is determined by ε (smaller ε gives a better approximation). It turns out that transient growth is related to the extent to which $\sigma_\varepsilon(A)$ extends outside the unit circle. The relationship will be made explicit at the end of this section. First, however, we introduce an example to illustrate the pseudospectrum.

Consider the following two matrices,

$$A_1 = \begin{bmatrix} 0.8 & 0 \\ 0.1 & 0.8 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0.8 & 0 \\ 100 & 0.8 \end{bmatrix}, \quad (8)$$

Although A_1 and A_2 share the same eigenvalues, their transient responses $\|A_i^j\|$, $i=1,2$, as shown in Figure 2, behave quite differently. While the powers of A_1 converge monotonically, the powers of A_2 experience a transient growth. This behavior can be predicted by the pseudospectra. Shown in Figure 3 are several ε -level sets of the A_1 and A_2 pseudospectra. Eigenvalues, located at 0.8 for both systems, are at the center of the level sets. However, whereas $\sigma_\varepsilon(A_1)$ are clustered closely around its eigenvalues, $\sigma_\varepsilon(A_2)$ are much larger. Since level sets $\sigma_\varepsilon(A_2)$ extend well outside of the unit circle, even for small ε , transient growth is expected. An explicit relationship between the pseudospectra and the magnitude of expected transient growth is presented next.

Let the pseudospectrum radius be given by, $\rho_\varepsilon(A) \triangleq \{\max\|z\|: s.t. z \in \sigma_\varepsilon(A)\}$. That is, $\rho_\varepsilon(A)$ is the farthest distance from the origin of all the points contained in a level set $\sigma_\varepsilon(A)$. Define the Kreiss constant as,

$$\mathcal{K}(A) = \sup_{\varepsilon > 0} (\rho_\varepsilon(A) - 1) / \varepsilon.$$

Theorem 1 (page 177 [13]): For any $A \in \mathbb{C}^{N \times N}$, the largest transient is bounded by,

$$\mathcal{K}(A) \leq \sup_{j \geq 0} \|A^j\| \leq eN\mathcal{K}(A). \quad (9)$$

Remark 2: Bounding the transient response, as in Theorem 1, is the necessary first step towards designing ILC algorithms that operate safely in the nonmonotonic convergence regime. Unfortunately, (9) is clearly loose for large N , and thus it may be too imprecise to be immediately useful for many ILC applications. However, Theorem 1 provides a very general result, assuming no structure on A , whereas \mathbf{T} in the ILC problem is structured. One may reasonably expect that such structure can be leveraged to yield tighter bounds. Such efforts will be the subject of

future work, which is addressed in Section VI.

Although the bound (9) is too imprecise to be immediately useful in a rigorous design procedure, there may still be immediate value in using the pseudospectrum in analysis and design. That is, the pseudospectrum can still be a useful tool by providing insight into the behavior and tradeoffs in ILC design. In Sections IV and V, we demonstrate how such this tool may be used on two ILC problems.

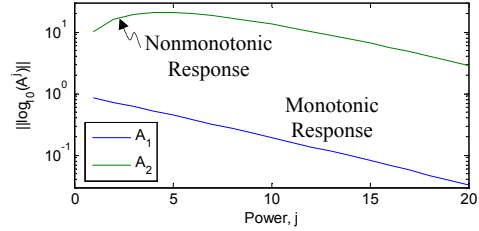


Figure 2. Transient response of $\|A_i^j\|$, $i=1,2$.

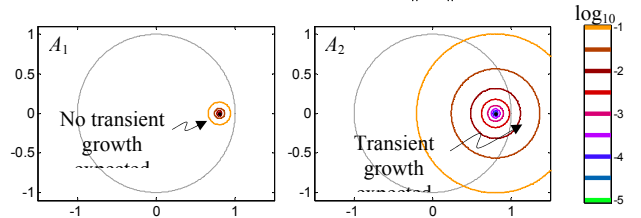


Figure 3. Pseudospectrum of A_1 and A_2 . The colored rings represent constant level set contour lines. ε is given by the colored legend.

IV. PROBLEM 1: MODEL UNCERTAINTY IN NORM-OPTIMAL ILC

The norm-optimal ILC algorithm seeks to optimize the cost,

$$\mathcal{J} = \mathbf{e}_{j+1}^T \mathbf{Q} \mathbf{e}_{j+1} + (\mathbf{u}_{j+1} - \mathbf{u}_j)^T \mathbf{R} (\mathbf{u}_{j+1} - \mathbf{u}_j) + \mathbf{u}_{j+1}^T \mathbf{S} \mathbf{u}_{j+1}, \quad (10)$$

where $\mathbf{Q} = \mathbf{Q}^T > 0$, $\mathbf{R} = \mathbf{R}^T \geq 0$, and $\mathbf{S} = \mathbf{S}^T > 0$. The solution yields learning filters of the form,

$$\begin{aligned} \mathbf{Q} &= (\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R} + \mathbf{S})^{-1} (\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R}), \\ \mathcal{L} &= (\mathbf{P}^T \mathbf{Q} \mathbf{P} + \mathbf{R})^{-1} \mathbf{P}^T \mathbf{Q}. \end{aligned} \quad (11)$$

Previous analysis [9,10,16] shows that the weighting \mathbf{R} , which controls convergence rate, has very little effect on RMC. The implication [16] is that, in practice, one should not bother to tune \mathbf{R} until convergence is established through \mathbf{Q} and \mathbf{S} . \mathbf{R} 's role is thus relegated solely to noise sensitivity [16]. Contrary to this result, many practitioners find that slowing convergence rate is a useful method to achieve convergence. How do we explain this apparent contradiction? One possible explanation is that practitioners are operating in the transient growth regime, for which the RMC theory does not apply. Note, that although the learning system may be operating in a transient growth regime, it is not necessarily the case that transient growth will be observed. The appearance of transient growth in the

response can depend on the particular initial conditions (or in the case of ILC, trajectory) that is used.

The following example supports the explanation that practitioners are operating in the transient growth regime when they note that slower convergence rate helps to achieve robustness. As we will show, \mathbf{R} can have a significant effect on reducing learning transient growth. The implications are twofold. First, \mathbf{R} should be a part of convergence turning procedures in practice. Second, practitioners may already be using nonmonotonic ILC algorithms on real systems (likely this is the inadvertent result of increasing performance weightings beyond the limitations governed by RMC). Therefore, it is imperative that ILC theoreticians develop the missing theoretical and design tools to support ILC design in the transient growth regime.

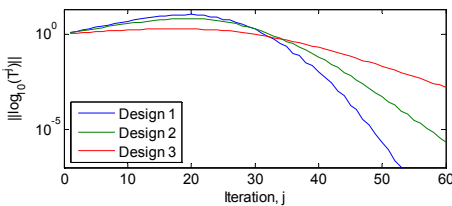


Figure 4. Learning transient bound for norm-optimal ILC designs.

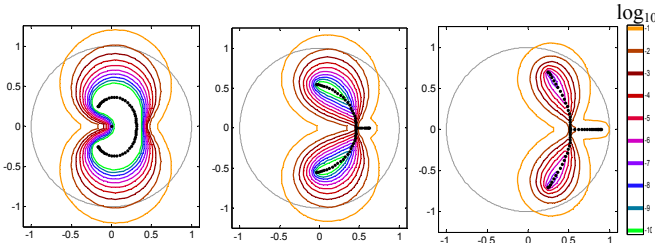


Figure 5. Pseudospectrum of norm-optimal ILC designs. The colored rings represent constant level set contour lines. ϵ is given by the colored legend.

Consider the nominal and perturbed systems,

$$\hat{P}(z) = \frac{0.4(z+0.8)}{(z^2 - 0.8z + 0.5)}, \quad P(z) = \frac{0.4(z+0.8)}{(z^2 - 0.2z + 0.5)},$$

respectively. Three norm-optimal ILCs are designed with different \mathbf{R} weightings (Table 2). A check of the RMC condition [10] will show that all three designs are not RMC. Numerical calculation of the learning transient bound, shown in Figure 4, shows that transient growth is reduced by increasing \mathbf{R} . From the pseudospectra, shown in Figure 5, we can see why this occurs. Increasing \mathbf{R} pulls the level sets in closer to the eigenvalues, while also moving some of the eigenvalues farther from the origin. Tighter grouping of the $\sigma_\epsilon(\mathbf{T})$ levels reduces the magnitude of the transient growth, while shifting eigenvalues away from the origin accounts for the slower convergence rate observed at large iterations. Calculations using several other system perturbations by the authors has yielded the same trend demonstrated here.

Table 2. Weighting matrices for norm-optimal ILC example.

| Design | \mathbf{Q} | \mathbf{R} | \mathbf{S} |
|--------|--------------|--------------|--------------|
|--------|--------------|--------------|--------------|

| | | | |
|---|------------------------|-----------------------|----------------------|
| 1 | $100 \cdot \mathbf{I}$ | $0 \cdot \mathbf{I}$ | $1 \cdot \mathbf{I}$ |
| 2 | $100 \cdot \mathbf{I}$ | $2 \cdot \mathbf{I}$ | $1 \cdot \mathbf{I}$ |
| 3 | $100 \cdot \mathbf{I}$ | $10 \cdot \mathbf{I}$ | $1 \cdot \mathbf{I}$ |

V. PROBLEM 2: TIME-VARYING LOWPASS FILTERING

In classical frequency domain analysis of ILC systems, \mathcal{Q} is an LTI lowpass filter [8]. The filter bandwidth is comparable to converged performance, with higher bandwidth yielding higher performance. However, RMC conditions result in an upper limit to the bandwidth, and thus an upper limit to the performance. Time-varying lowpass filtering [17-19] seeks to circumvent this limitation. In this approach [18,19], the filter \mathcal{Q} is designed to behave like a lowpass filter whose bandwidth varies in time, along the iteration. The goal is to raise the bandwidth above the LTI “upper limit” for short periods that coincide with rapid changes in the reference trajectory. Robustness is recovered by lowering the bandwidth elsewhere along the trajectory. This approach significantly improves performance when tracking aggressive trajectories [17]. Previous work has shown that this approach may be most effective in the nonmonotonic convergence regime [18]. The following example uses the pseudospectra to give new insight into understanding why the time-varying lowpass filter is so effective outside of the RMC regime.

Consider again the nominal system $\hat{P}(z)$ from Section IV. For this example it is not necessary to consider the model perturbation $P(z)$. A P-type learning filter, or $\mathcal{L} = 0.9 \cdot \mathbf{I}$ is used. Three lowpass first-order Butterworth filters are designed and lifted into a \mathcal{Q} for analysis. The first two are LTI filters while the third is a LTV filter. Bandwidths of the three filters are shown in Figure 6.

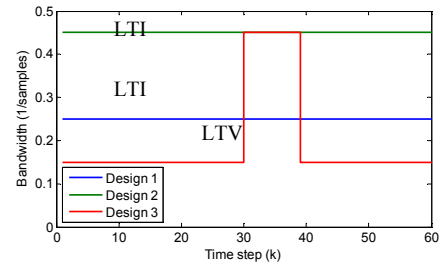


Figure 6. \mathcal{Q} filter bandwidths used for time-varying lowpass filter problem.

Pseudospectra for the three ILC designs are shown in Figure 7. The learning transients are shown in Figure 8. Interestingly, Design 1 and Design 3 have similar transient responses, although very different bandwidth profiles. The pseudospectra provides a new explanation for this behavior. From Figure 7 we can see that the higher and lower bandwidth sections of the LTV filter have an averaging effect on the pseudospectra. Although some eigenvalues (those associated with the higher bandwidth segment) are moved closer to the unit circle, they are offset by moving other eigenvalues (those associated with the lower

bandwidth segment) closer to the origin. The overall effect averages the profile of the level sets outside of the unit circle, which govern the transient properties.

Although transient behavior is similar in Design 1 and Design 3, Design 3 has the performance advantage when its high bandwidth peak is aligned with the aggressive portion of a desired trajectory. Of course, proper alignment of the high bandwidth segment is necessary, and thus time-varying bandwidths are always designed in conjunction with the trajectories [17,18]. Design 2 is presented to illustrate the need for a time-varying bandwidth as opposed to simply increasing the LTI bandwidth. While Design 2 and Design 3 share the same bandwidth during time steps 30 to 40, the transient response in Design 2 is significantly worse. Therefore, the time-varying design is necessary when the trajectory is aggressive enough to necessitate the use of very high bandwidth.

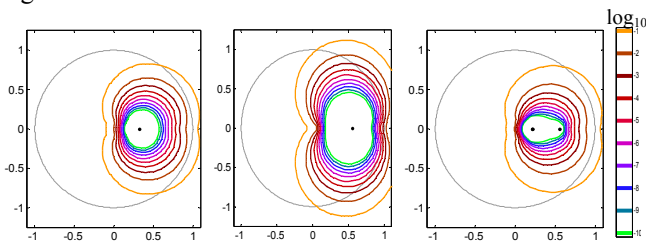


Figure 7. Pseudospectrum of time-varying lowpass ILC. The colored rings represent constant level set contour lines. ϵ is given by the colored legend.

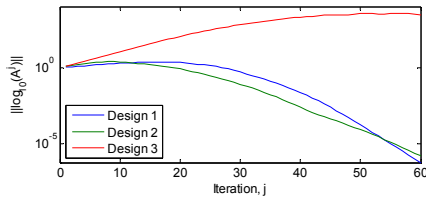


Figure 8. Learning transient bound for time-varying lowpass ILC designs.

VI. CONCLUSIONS

This work revisited the topic of transient growth in ILC. Whereas previous efforts in this area focus primarily on avoiding transient growth, our efforts are directed at laying groundwork for ILC algorithms that can operate safely and predictably in the transient growth regime. We have shown that the pseudospectra can be used to predict and bound transient growth. As importantly, the pseudospectra provides a mathematical framework that illuminates the often misunderstood topic of transient growth in ILC.

Two design problems were considered. The pseudospectra was used to provide new insights into these problems. In the case of norm-optimal ILC, it was found that the convergence rate weighting plays an important role in reducing transients when the model does not accurately capture the plant dynamics. In the other case, it was found that time-varying bandwidth filters can be used to selectively allocate performance along the iteration without significantly changing learning transients. These results cannot be obtained using eigenvalue or singular value analysis, but instead are only evident using pseudospectral analysis.

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