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On classifying finite edge colored graphs with two transitive automorphism groups

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with Two Transitive Automorphism Groups

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Proposed running head:
2-transitive edge colored graphs

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Abstract: This paper classifies all finite edge colored graphs with doubly transitive automorphism groups. This result generalizes the classification of doubly transitive balanced incomplete block designs with $\lambda = 1$ and doubly transitive one-factorizations of complete graphs. It also provides a classification of all doubly transitive symmetric association schemes.

The classification of finite simple groups in the 1980s has led to classification theorems concerning a variety of designs and geometric structures. Edge colored graphs generalize balanced incomplete block designs with $\lambda = 1$ and one-factorizations of complete graphs and provide a reinterpretation of metric spaces. This paper classifies the doubly transitive edge colored graphs (abbreviated 2-t ec-graphs), extending results of Kantor [14] and Cameron and Korchmaros [8]. The doubly transitive symmetric graph designs of Cameron [7] when $\lambda = 1$ match the 2-t ec-graphs for which the number of colors equals the number of vertices. Edge colored graphs, which in this article are always colorings of complete graphs, are closely related to the rainbows Aschbacher defined in [2].

Definitions. An *edge colored graph* (V,C) is a finite set of vertices V and a function C from the set of all undirected edges ab, where $a \neq b$, onto a non-empty set of *edge colors.* We assume that $|V| \geq 2$, where $|V|$ is the number of elements in V. (When needed we denote the set of all edges by E and the set of colors by $C(E)$.) An *automorphism* α of (V,C) is a bijection of V such that for all edges ab and cd, $C(ab) = C(cd)$ iff $C(\alpha(a)\alpha(b)) = C(\alpha(c)\alpha(d))$. An edge colored graph (V,C) is *doubly transitive* iff its group of automorphisms, denoted $A(V,C)$, is doubly transitive on V.

Example 1. For any V the 2-t ec-graph obtained by setting $C(a,b) = 1$ for all edges ab is called the *monochromatic* ec-graph on V. The 2-t ec-graph obtained by setting $C(ab) = ab$ is called the *trivial* ec-graph. We call the monochromatic coloring function C_M and the trivial coloring function C_T. Note that $A(V,C_M) = A(V,C_T) = S_V$, the symmetric group on V.

Example 2. Balanced incomplete block designs (BIBD) generalize many geometric structures. A BIBD is a set V of vertices (also called varieties) and a set B of blocks (subsets of V, such as lines or planes, etc.) such that all blocks contain the same number of vertices, all vertices are in the same number of blocks and each pair of vertices is contained in the same number λ of blocks. When $\lambda = 1$, this last condition corresponds to the geometric property "two points determine a unique line," so we will also use line for block. We denote by $B(a,b)$ the unique line containing a and b. Let V be the set of vertices of a BIBD with $\lambda = 1$ and B be the set of blocks. We convert this BIBD into an edge colored graph (V,C_B) by defining $C_B(ab) = B(a,b)$. We say that (V,C_B) is *derived from* this BIBD. The most important examples of BIBDs with $\lambda = 1$ are the affine spaces $AG(n,p^k)$ and the projective spaces $PG(n,p^k)$ over the field with p^k elements. These have p^k and p^k+1 points per block (line), respectively. Kantor [14] classified all finite doubly transitive BIBDs with $\lambda = 1$, which include the spaces $AG(n,p^k)$ and $PG(n,p^k)$.

Example 3. A one-factorization (or proper edge coloring) of a set with an even number of vertices is an edge colored graph where the edges of each color determine a regular graph of degree one. Cameron and Korchmaros [8] classified all finite doubly transitive one-factorizations.

If we require the automorphism group to be triply transitive, rather than doubly transitive, we can readily classify the corresponding "triply transitive edge colored graphs," given in Theorem 1 below.

Theorem 1. If (V,C) is a finite doubly transitive edge colored graph and G is a group acting triply transitively on V with $G \leq A(V,C)$, then (V,C) is either

(i) the monochromatic edge colored graph and $|V| \geq 2$,

(ii) the trivial edge colored graph and $|V| \geq 2$,

(iii) the doubly transitive one-factorization based on the affine space $AG(n,2)$, where parallel edges are the same color and $|V| = 2^n$, or

(iv) the doubly transitive edge colored graph in Figure 1 and $|V| = 6$.

Proof. The case $|V| = 2$ is trivial. For $|V| \geq 3$ suppose first that there are adjacent edges ab and ax such that $C(ab) = C(ax)$. By triple transitivity, for each $y \in V$ distinct from a and b, there is an automorphism fixing a and b and moving x to y. So for all $y \neq a$, $C(ay) = C(ab)$. In turn, for all $z \neq y$, $C(yz) = C(ya) = C(ab)$, and (V,C) is monochromatic.

Thus, if (V,C) is not monochromatic, adjacent edges are different colors. If all edges have different colors, then (V,C) is the trivial 2-t ec-graph. Suppose that there are distinct a, b, x and y

such that $C(ab) = C(xy)$. By triple transitivity, for each $z \in V$ distinct from a and b, there is an automorphism fixing a and b and taking x to z. This automorphism takes y to some w, so $C(ab) = C(zw)$ and (V,C) is a one-factorization. Cameron [6] showed that the only triply transitive one-factorizations are those in the family in (iii) and the individual one in (iv).

Example 4. Let V be a two-dimensional vector space over a field F and let $\langle \bullet, \bullet \rangle$ be a nondegenerate, symmetric bilinear form on V. For any edge ab let $C(ab) = \langle a-b, a-b \rangle$. Then (V,C) is a 2-t ec-graph. The color $C(ab)$ can be interpreted as the "distance" between a and b. This construction cannot be generalized to higher dimensional finite spaces because there will always be isotropic elements. (See Artin [4].) Any metric space (X,d) becomes an edge colored graph by setting $C(ab) = d(a,b)$ and conversely any finite edge colored graph can become a metric space by assigning to each color a positive real number.

This paper is divided into three sections, the first two of which classify 2-t ec-graphs based on what type of group of automorphism each has. From the classification of finite simple groups the finite doubly transitive groups split into two large collection and one other group. The first collection consists of those doubly transitive groups that have a finite simple subgroup that is doubly transitive. The second collection consists of doubly transitive subgroups of some affine group. The remaining group is $P\Gamma L(2,8) = {}^2G_2(3)$ acting on a set with 28 elements. (See Kantor [14].) Section 1 primarily considers 2-t ec-graphs whose groups of automorphisms contain some simple two-transitive group, culminating in their complete classification in Theorem 6. In essence Theorem 6 says that all such 2-t ec-graphs are found in Examples 1 through 3 above and 5 through 11 below. The 2-t ec-graphs related to the group $P\Gamma L(2,8) = {}^2G_2(3)$ acting on a set with 28 elements follow the same analysis as those relating to finite simple groups and so are considered in this first section. Theorem 7 classifies the 2-t ec-graphs whose automorphism groups contain $P\Gamma L(2,8) = {}^2G_2(3)$. Section 2 classifies, to the extent practical, the 2-t ec-graphs whose groups of automorphisms contain a doubly transitive subgroup of some affine group. This collection of graphs, which includes those of Example 4, has a far more extensive and complicated structure than the collection of graphs in Section 1, making an explicit counterpart to Theorems 6 and 7 infeasible. Examples 12 and 13 give general constructions of such 2-t ec-graphs. Section 3 classifies regular 2-t ec-graphs, as in Examples 3 and 4, where the edges of each color form a regular graph on V. Section 3 classifies other related structures as well.

Section 1. Non-Affine Automorphism Groups

In this section we restrict our attention to 2-t ec-graphs whose groups of automorphisms are subgroups of a finite simple group or of $P\Gamma L(2,8) = {}^2G_2(3)$ acting on a set of 28 elements. Then Lemmas 2 and 3 below provide a means of determining all of the possibilities. Lemma 2 matches possible 2-t ec-graphs with appropriate subgroups of a two transitive group. Lemma 3 implies that in this section we can limit our attention to the finite simple groups and $P\Gamma L(2,8) = {}^2G_2(3)$, rather than all of their two transitive subgroups. First we present the remaining examples of 2-t ec-graphs whose automorphism groups are subgroups of a finite simple group.

Example 5. For $V = PG(n,2)$, each line has three points incident with it. Define $C(E) = V$ and $C(ab) = c$ iff c is the third point on the line incident with a and b. Then $A(V,C) = PGL(n+1,2)$ and (V,C) is a 2-t ec-graph. See Figure 2.

Definition. Let (V,C) and (V,C') be edge colored graphs. We say that (V,C) is *weaker than* (V,C'), written $(V,C) \preceq (V,C')$, iff there is a surjection $\gamma : C(E) \to C'(E)$ such that for every edge ab, $\gamma(C(ab)) = C'(ab)$.

Remarks. If (V,C) is any edge colored graph on V, then $(V,C_T) \preceq (V,C) \preceq (V,C_M)$, where C_T and C_M are, respectively, the trivial and monochromatic colorings of Example 1. If $(V,C) \preceq (V,C')$, then edges with the same color in (V, C) will have the same color in (V, C').

Example 6. Let $V = PG(n, 3)$ and B the set of all lines (blocks) in $PG(n, 3)$. Let (V, C_B) be the 2-t ec-graph derived from this BIBD with $\lambda = 1$. Each line has four points incident with it. We define a weaker 2-t ec-graph (V, C) by splitting each color from B into three so that the six edges defined from the four points of a line are colored in (V, C) as in Figure 3. More precisely, define $C(E) = B \times \{1, 2, 3\}$ and for each line l in B choose any labeling l_i of its points, for $i \in \{0, 1, 2, 3\}$. Define $C(l_0 l_i) = (l, i)$ and $C(l_i l_j) = (l, k)$, where i, j and k are distinct elements in $\{1, 2, 3\}$. Any permutation of $\{l_0, l_1, l_2, l_3\}$ is an automorphism of (l, C_l), where C_l is the restriction of the color function to the points of l. Hence, $A(V, C) = A(V, C_B)$, which is doubly transitive.

Definition. Two edge colored graphs (V, C) and (V', C') are *isomorphic* iff there are bijections $\beta : V \to V'$ and $\gamma : C(E) \to C'(E')$ such that for every edge ab, $\gamma(C(ab)) = C'(\beta(a)\beta(b))$.

Example 7. Let $V = PG(n, 5)$, B the set of all lines (blocks) in $PG(n, 5)$ and (V, C_B) the 2-t ec-graph derived from this BIBD with $\lambda = 1$. Each line has six points on it. Analogously to Example 6 we split each color of C_B into five colors with each line colored in (V, C) as in Figure 1. The double transitivity of $PGL(n + 1, 5)$ allows us to define C more precisely by starting with the coloring of one fixed line k and coloring the others using automorphisms. Define $C(E) = B \times \{1, 2, 3, 4, 5\}$ and for a fixed line k in B label its points k_i, where $i \in \{0, 1, 2, 3, 4, 5\}$. Then for each line l in B, there is some $\alpha \in PGL(n + 1, 5)$ that maps k to l. Label the points of l as l_i, provided l_i is the image of $\alpha(k_i)$. (The triple transitivity of $PGL(2, 5)$ ensures that the choice of α is immaterial up to isomorphism.) Define $C(l_i l_j) = (l, i * j)$, where, for $i \neq j$, $i * j$ is given by the table below.

Table I.

$*$	0	1	2	3	4	5
0		1	2	3	4	5
1	1		5	4	2	3
2	2	5		1	3	4
3	3	4	1		5	2
4	4	2	3	5		1
5	5	3	4	2	1	

As Cameron [6] has noted, the edge colored graph in Figure 1 has $PGL(2, 5)$ for its automorphism group, which is triply transitive. Further, in $PGL(n + 1, 5)$ the stabilizer of a line is isomorphic to $PGL(2, 5)$. Hence, $A(V, C) = A(V, C_B)$, and (V, C) is a 2-t ec-graph.

Example 8. The constructions of Examples 6 and 7 apply to any BIBD with $\lambda = 1$ and 4 or 6 vertices per block, respectively. In particular, they apply to two BIBDs of unitals over the field Z_3 and one BIBD over the field Z_5 yielding three more 2-t ec-graphs, whose automorphism groups are $U_3(3)$, ${}^2G_2(3)$ and $U_3(5)$, respectively. Note that ${}^2G_2(3)$ is not a simple group, but is considered in Theorem 7.

Example 9. Let X be the $2m$-dimensional vector space over Z_2 and $G = PSp(2m, 2)$, the projective symplectic group, for $m > 2$. For the rest of this example we follow the notation and terminology in Dixon and Mortimer [11, 245-248]. It is well known that G acts doubly transitively on the subsets Ω^+ and Ω^- of X. For Ω' either Ω^+ or Ω^- and $\theta_a, \theta_b \in \Omega'$ with $\theta_a \neq \theta_b$, define $C(\theta_a \theta_b) = a + b$. Now t_{a+b}, the unique transvection switching θ_a and θ_b, is an automorphism for (Ω', C). Indeed, all transvections are automorphisms. Further, G is generated by the transvections, so $A(V, C)$ is doubly transitive and (Ω^+, C) and (Ω^-, C) are 2-t ec-graphs.

Definition. For a color c, a *c-chromomorphism* κ is an automorphism such that for every edge ab, if $C(ab) = c$, then $C(\kappa(a)\kappa(b)) = c$; that is, κ preserves the color c, although not necessarily other colors. The subgroup of c-chromomorphisms is denoted $K(V, C, c)$, or $K(V, C, C(ab))$ if we are focusing on the chromomorphisms preserving the color of a particular edge ab. If no confusion will arise, we shorten these names to just $K(c)$ or $K(C(ab))$.

The monochromatic 2-t ec-graph, (V, C_M), has $K(V, C_M, 1) = S_V$, the entire symmetric group. Clearly this is the largest c-chromomorphism subgroup possible on the set V. The c-chromomorphism subgroup for any color $c = C(ab)$ of the trivial coloring C_T on a set V is the smallest such subgroup, which is the stabilizer subgroup $S_{V_{\{a,b\}}}$ of the single edge ab of that color. Similarly, $A(V, C)_{\{a,b\}} \leq K(V, C, C(ab)) \leq A(V, C)$ for any 2-t ec-graph (V, C). Lemma 2 matches the 2-t ec-graphs on V having a given automorphism group with their corresponding c-chromomorphism subgroup. By itself this correspondence gives an unenlightening classification of 2-t ec-graphs. However, it does form a key tool for the more detailed classification that follows. In the lemma we may use any particular edge ab because the group of automorphisms is assumed to be doubly transitive on the vertices and so transitive on the edges.

Lemma 2. Let V be a finite set and G a doubly transitive group acting on V. Fix any edge ab. Then for each subgroup K such that $G_{\{a,b\}} \leq K \leq G$ there is a coloring map C_K such that, up to isomorphism, the doubly transitive edge colored graph (V, C_K) is the unique doubly transitive edge colored graph satisfying $K(V, C_K, C(ab)) = K$ and $G \leq A(V, C_K)$. Further, if a subgroup K' also satisfies $G_{\{a,b\}} \leq K' \leq G$, then $(V, C_K) \preceq (V, C_{K'})$ iff $K \leq K'$.

Proof. Given the original edge ab and any edge $xy \in E$, let $\alpha \in G$ satisfy $\alpha(a) = x$ and $\alpha(b) = y$. Define $C_K(xy)$ to be the orbit of the edge xy under $\alpha K \alpha^{-1}$. Because $G_{\{a,b\}} \leq K$, the subgroup $\alpha K \alpha^{-1}$ is independent of the choice of α and $G_{\{x,y\}} \leq \alpha K \alpha^{-1}$. This implies that each color $C_K(xy)$ is a block in the group theoretic sense of Dixon and Mortimer [11, 12]. The transitivity of G on the edges implies that G acts on the blocks $C_K(xy)$, so $G \leq A(V, C_K)$. Because G is doubly transitive, (V, C_K) is a 2-t ec-graph. Theorem 1.5A in Dixon and Mortimer [11, 13] gives a isomorphism between the partial orderings of the subgroups K and the 2-t ec-graphs (V, C_K), which shows the uniqueness of the 2-t ec-graphs. Because the subgroups form a lattice, the 2-t ec-graphs also form a lattice.

Remarks. Because the subgroups form a lattice, the 2-t ec-graphs also form a lattice. The automorphism group $A(V, C_K)$ can strictly contain G. In particular, if $K = G$ or $K = G_{\{a,b\}}$, then C_K is monochromatic or trivial, respectively, and $A(V, C_K) = S_V$. Thus, given any doubly transitive group G acting on a set V, the monochromatic and trivial ec-graphs are the maximal and minimal elements, respectively, of the lattice of 2-t ec-graphs.

Example 10. Let $G = U_3(3)$ act on the set V of 28 unitals and a and b be distinct elements of V. Then G has two maximal subgroups of size 96 containing $G_{\{a,b\}}$. (See Conway et al [10].) One is the automorphism group of the 2-t ec-graph derived from the BIBD with $\lambda = 1$. By Lemma 2, the other subgroup gives a non-isomorphic 2-t ec-graph (V, C) with six edges of each color. By Theorem 5 part (ix), below, no two of these edges are adjacent. An examination of the orbits of this subgroup shows that if $B(a, b) = \{a, b, c, d\}$, then $C(ab) = C(cd)$.

Example 11. If $G = PSL(2, 9)$ and $V = \{\infty\} \cup Z_3[i]$, then $G_{\{0,\infty\}}$ is generated by the matrices $\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$ and $\begin{bmatrix} 0 & ai \\ a^{-1}i & 0 \end{bmatrix}$, for $a \in Z_3[i]$ and $a \neq 0$. Let K be the subgroup generated by $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and the set $G_{\{0,\infty\}}$. By Lemma 2, there is a unique 2-t ec-graph on V with $A(V, C) = G$ and $K(V, C, C(0\infty)) = K$.

Remark. If in Example 9 we used $m = 2$, then the non-simple group $PSp(4, 2)$ acting on Ω^+, a set with 10 elements, would give an ec-graph isomorphic to the ec-graph of Example 11.

The classification of 2-t ec-graphs in the non-affine case we consider in this section depends on the classification of finite simple groups. This classification ensures that any finite non-affine

2-transitive group has a minimal 2-transitive subgroup (which is either simple or $P\Gamma L(2,8) = {}^2G_2(3)$). The following lemma ensures that the 2-t ec-graphs we obtain from these minimal 2-transitive subgroups include all such graphs we can obtain from any groups containing these minimal subgroups.

Lemma 3. Suppose that H is a subgroup of G and both act doubly transitively on a set V. Then for any edge ab and K_G satisfying $G_{\{a,b\}} \leq K_G \leq G$, the mapping λ given by $\lambda(K_G) = K_G \cap H$ is an injective lattice homomorphism of the subgroups of G containing $G_{\{a,b\}}$ into the lattice of subgroups K_H such that $H_{\{a,b\}} \leq K_H \leq H$.

Proof. Clearly, λ takes a subgroup K_G of G to the subgroup $K_G \cap H$ of H, and if $K_G \leq K'_G$, then $\lambda(K_G) \leq \lambda(K'_G)$. Further, $H_{\{a,b\}} = G_{\{a,b\}} \cap H \leq \lambda(K_G)$, so λ is a lattice homomorphism.

To show that λ is one-to-one, suppose that $K_G \neq K'_G$ and $\alpha \in K_G$ but $\alpha \notin K'_G$. Let $\alpha(a) = x$ and $\alpha(b) = y$. Then $\alpha G_{\{a,b\}} \leq K_G$, but $\alpha G_{\{a,b\}} \cap K'_G = \emptyset$. Because H is doubly transitive, there is $\beta \in H$ such that $\beta(a) = x$ and $\beta(b) = y$. Hence $\beta \in \alpha G_{\{a,b\}}$. Then $\beta \in \lambda(K_G)$, but $\beta \notin \lambda(K'_G)$. Thus $\lambda(K_G) \neq \lambda(K'_G)$ and λ is one-to-one.

Definitions. Let (V,C) be a 2-t ec-graph. For every edge ab let e_{ab} be the number of vertices x such that $C(ab) = C(ax)$ and k_{ab} be the number of vertices x such that $C(ab) = C(xy)$ for some vertex y.

The following lemma lists some elementary combinatorial facts about 2-t ec-graphs.

Lemma 4. Let (V,C) be a doubly transitive edge colored graph and ab any edge. Then
$|C(E)|$ divides $|E| = \binom{|V|}{2}$,
e_{ab} divides $|V| - 1$ and
$e_{ab} + 1 \leq k_{ab}$.
For any edges ab and xy, $e_{ab} = e_{xy}$ and $k_{ab} = k_{xy}$, and
the number of edges of any color is $e = k_{ab} \cdot e_{ab}/2$.
If $e_{ab} = |V| - 1$, then (V,C) is monochromatic.
If $(V,C) \preceq (V,C')$, then $|C'(E)|$ divides $|C(E)|$.

Proof. See Isaac [13].

To simplify the proof of the classification in Theorem 6, we first consider in Theorem 5 the 2-t ec-graphs having at least two adjacent edges that are the same color. We summarize the results of Theorem 5 in the table below, abbreviating "q is a power of a prime" by "$q = p^m$" and "monochromatic" by "mono." The groups listed are the simple 2-transitive groups that are not 3-transitive.

Table II.

Group	Size of V	2-t ec-graphs, Th. 5
$PSL(2,11)$	11	mono
A_7	15	mono, BIBD
HS	176	mono
Co_3	276	mono
$Sz(q)$, $q = 2^{2a+1}$, $a \geq 1$	q^2+1	mono
$PSL(2,q)$, $q = p^m$	$q+1$	mono
$PSL(n+1,q)$, $q = p^m$	$\sum_{i=0}^{n} q^i$	mono, BIBD
$PSp(2n,2)$, $n \geq 3$	$2^{2n-1} \pm 2^{n-1}$	mono
$U_3(q)$, $q \neq 2$, $q = p^m$	q^3+1	mono, BIBD
${}^2G_2(q)$, $q > 3$, $q = 3^{2a+1}$	q^3+1	mono, BIBD

Theorem 5. Suppose (V,C) is a finite doubly transitive edge colored graph, G is a simple group

acting doubly transitively on V with $G \leq A(V,C)$ and there are adjacent edges ab and ax such that $C(ab) = C(ax)$. Then (V,C) is either monochromatic or it is derived from a BIBD with $\lambda = 1$.

Proof. Suppose that (V,C) is a 2-t ec-graph and $C(ab) = C(ax)$, where a, b and x are distinct vertices in V. We consider the different finite simple doubly transitive groups as separate cases. Kantor [14] gives a list of all of these individual groups and families of groups as well as information on the orbits of G_{ab}, the subgroup that fixes both a and b. The classification of triply transitive ec-graphs in Theorem 1 permits us to omit triply transitive groups from the list of groups we consider.

(i) Suppose that $G = PSL(2,11) \leq A(V,C)$ and $|V| = 11$. The orbit of x under G_{ab} has either 3 or 6 elements. For each y in this orbit $C(ab) = C(ay)$. This means that e_{ab} is either $1+3 = 4$, $1+6 = 7$ or, if there are x and x' in different orbits with $C(ab) = C(ax) = C(ax')$, $1+3+6 = 10$. (The vertex b contributes the 1 and the orbit of x contributes 3 or 6.) However, of these, only 10 divides $|V|-1 = 10$, so (V,C) is monochromatic.

(ii) Suppose that $G = A_7 \leq A(V,C)$ and $|V| = 15$. The orbit of x under G_{ab} has either 12 elements or is just $\{x\}$. If the orbit of x has twelve elements, then $e_{ab} = 13$, which does not divide 14. If there are x and x' in different orbits, (V,C) is monochromatic. So assume that the orbit of x is $\{x\}$ and without loss of generality $V \cup \{(0,0,0,0)\} = X$ is the 4-dimensional vector space over Z_2. Thus $(V,C_B) \preceq (V,C)$, where (V,C_B) is the 2-t ec-graph derived from the BIBD $PG(3,2)$. Since $e_{ab} = 2$, in (V,C) any two same colored lines (blocks) are disjoint. Because any same colored lines of (V,C) are disjoint and (V,B) has 35 colors, if $(V,C) \neq (V,C_B)$, (V,C) would have 7 colors with 5 lines per color. Suppose that $B(a,b) \neq B(i,j)$ and $C(ab) = C(ij)$. Then $\{a,b,i,j\}$ is a basis of X and the orbit of ij under G_{ab} determines the lines colored $C(ab)$. The orbit of these lines under G_a, the subgroup fixing a, determines all of the edges of the seven different colors of (V,C). However direct computation reveals that there is no automorphism of this (V,C) switching a and b, showing (V,C) is not a 2-t ec-graph. Thus (V,C) is either monochromatic or derived from the BIBD.

(iii) Suppose that $G = HS \leq A(V,C)$ and $|V| = 176$. The orbit of x under G_{ab} has either 12, 72 or 90 elements. As in (i), (V,C) is monochromatic.

(iv) Suppose that $G = Co_3 \leq A(V,C)$ and $|V| = 276$. The orbit of x under G_{ab} has either 112 or 162 elements. As in (i), (V,C) is monochromatic.

(v) Suppose that $G = Sz(q) \leq A(V,C)$, where $q = 2^{2a+1}$ and $a \geq 1$, and $|V| = q^2+1$. By direct computation $|G_{\{a,b\}}| = 2(q-1)$. Suzuki [19] shows that the only subgroups of $G = Sz(q)$ with a multiple of $2(q-1)$ elements are G and G_x, for some $x \in V$. However, no G_x can contain any $G_{\{a,b\}}$ as a subgroup: First, $G_{\{a,b\}}$ is not a subgroup of either G_a or G_b. Further, the nonidentity elements of $G_{ab} \leq G_{\{a,b\}}$ move every point of V except a and b, so $G_{\{a,b\}}$ is not a subgroup of G_c for any other $x \in V$. (See Dixon and Mortimer [11, p. 250].) Hence (V,C) is monochromatic.

(vi) Suppose that $G = PSL(2,q) \leq A(V,C)$, where q is a power of a prime, and $|V| = q+1$. The orbit of x under G_{ab} has at least $(q-1)/2$ elements. As in (i), (V,C) is monochromatic.

(vii) Suppose that $G = PSL(n+1,q) \leq A(V,C)$, where $n \geq 2$ and q is a power of a prime, and $|V| = \sum_{i=0}^{n} q^i$. Suppose first that x is not incident with $l = B(a,b)$, the line (block) on a and b. The orbit of x under G_{ab} is V/l, the vertices of V not on l. As in (i), (V,C) is monochromatic. Now suppose that x is incident with l. Because G is triply transitive on l, Theorem 1 forces l to be monochromatic. Thus $(V,C_B) \preceq (V,C)$, where (V,C_B) is derived from the projective BIBD $PG(n,q)$. If $(V,C) = (V,C_B)$, we are done. Otherwise there must be some line $B(u,v)$ with $l \cap B(u,v) = \emptyset$ such that $C(uv) = C(ab)$. Then v is not in the plane determined by a, b and u, so the orbit of v has at least q^n elements. As in (i), (V,C) is monochromatic.

(viii) Suppose that $G = PSp(2n,2) \leq A(V,C)$, where $n \geq 3$, and $|V| = 2^{2n-1} \pm 2^{n-1}$. The orbit of x under G_{ab} has either 2^{2n-2} or $2^{2n-2} \pm 2^{n-1} - 2$ elements. As in (i), (V,C) is monochromatic.

(ix) Suppose that $G = U_3(q) \leq A(V,C)$, where q is a power of a prime and $q > 2$, and $|V| = q^3+1$. We may consider V to be the vertices in a unital BIBD with $\lambda = 1$. For $q > 3$ the only maximal subgroup of G containing $G_{\{a,b\}}$ is the stabilizer of the block $B(a,b)$. (See Aschbacher [1] and Hartley [12].) Thus in this case (V,C) is either monochromatic or derived from a BIBD. Now assume that $q = 3$ and $|V| = 28$. Then G_{ab} has one orbit of length 2, namely $B(a,b)/\{a,b\}$, and three of length 8, giving $e_{ab} = 1 + 2i + 8j$, where $i \le 1$ and $j \le 3$. Further, e_{ab} divides 27, so $e_{ab} = 3$, 9 or 27. If $e_{ab} = 3$, we have the BIBD and if it is 27, (V,C) is monochromatic. For $e_{ab} = 9$, $k_{ab} \ge 10$, $e \ge 45$ and $|K(C(ab))| \ge 2 \cdot |G_{ab}| \cdot 45 > |G_a|$. But by Kleidman and Liebeck [16] no proper subgroup of G has more elements than G_a. Thus $K(C(ab)) = G$ and (V,C) is monochromatic.

(x) Suppose that $G = {}^2G_2(q) \le A(V,C)$, where $q = 3^{2a+1}$ and $q > 3$, and $|V| = q^3 + 1$. We may consider V to be the vertices in a unital BIBD for a Ree group and $\lambda = 1$. As in case (ix), for $q > 3$ the only maximal subgroup containing $G_{\{a,b\}}$ is the stabilizer of the block $B(a,b)$. So (V,C) is either monochromatic or derived from a BIBD. (See Kleidman [15]. When $q = 3$, ${}^2G_2(3) \cong P\Gamma L(2,8)$ is not simple and so is not part of this theorem. See Theorem 7.)

The general setting, considered below, includes the various edge colored graphs in Example 3 and Examples 5 through 11. We summarize all of the possibilities in the table below, abbreviating "trivial" by "tr", as well as the abbreviations for the previous table.

Table III.

Group	Size of V	All finite 2-t ec-graphs
$PSL(2,11)$	11	mono, tr
A_7	15	mono, tr, BIBD, Ex. 5
HS	176	mono, tr
Co_3	276	mono, tr
$Sz(q)$, $q = 2^{2a+1}$, $a \ge 1$	$q^2 + 1$	mono, tr
$PSL(2,q)$, $q = p^m$	$q + 1$	mono, tr, Ex. 3, 11
$PSL(n+1,q)$, $q = p^m$	$\sum_{i=0}^{n} q^i$	mono, tr, BIBD, Ex. 5, 6, 7
$PSp(2n,2)$, $n \ge 3$	$2^{2n-1} \pm 2^{n-1}$	mono, tr, Ex. 9
$U_3(q)$, $q \ne 2$, $q = p^m$	$q^3 + 1$	mono, tr, BIBD, Ex. 8, 10
${}^2G_2(q)$, $q > 3$, $q = 3^{2a+1}$	$q^3 + 1$	mono, tr, BIBD, Ex. 8

Theorem 6. If (V,C) is a finite doubly transitive edge colored graph and G is a simple group acting doubly transitively on V with $G \le A(V,C)$, then either $G = S_V$ and (V,C) is monochromatic or trivial or else one of the following cases occurs:

(i) G is a projective, unitary, or Ree group and (V,C) is the 2-t ec-graph resulting from the unique BIBD on V with $\lambda = 1$ determined by G;

(ii) (V,C) is a one-factorization and $G = PSL(2,p)$ for $p = 3$, 5, 7 or 11;

(iii) $G = PSL(n,p)$ for $p = 2$, 3 or 5 and (V,C) is a 2-t ec-graph in one of the families in Examples 5, 6 or 7;

(iv) $G = U_3(3)$, $G = {}^2G_2(3)$ or $G = U_3(5)$ and (V,C) is one of the 2-t ec-graphs in Example 8 or 10;

(v) $G = PSp(2m,2)$ and (V,C) is a 2-t ec-graph in one of the two families in Example 9;

(vi) $G = PSL(2,9)$ and (V,C) is the 2-t ec-graph in Example 11.

Proof. If for some distinct adjacent edges ab and ax we have $C(ab) = C(ax)$, then we can use the classification in Theorem 5. So assume that for any edge ab, $e_{ab} = 1$. Further assume that there are distinct edges ab and xy with $C(ab) = C(xy)$, since otherwise (V,C) is the trivial graph. Note that the number e of edges of any color is $k_{ab}/2$ because $e_{ab} = 1$. Also e divides $|E|$ and $e \le |V|/2$. Further $e = |V|/2$ iff (V,C) is a one-factorization. The number e depends on the size of the orbit of the edge xy under the group G_{ab}. If x and y are in the same orbit of size r, the size of the orbit of xy

is $r/2$ to ensure $e_{ab} = e_{xy} = 1$. If x and y are in different orbits, these orbits must be the same size r to ensure $e_{ab} = 1$.

(i) Suppose that $G = PSL(2,11) \leq A(V,C)$ and $|V| = 11$. The orbit of x under G_{ab} has 3 or 6 elements. The only possibility here is $r = 6$, but $e = 1 + r/2 = 4$ doesn't divide $|E|$. Hence (V,C) is either monochromatic or trivial.

(ii) Suppose that $G = A_7 \leq A(V,C)$ and $|V| = 15$. The orbit of x under G_{ab} has 1 or 12 elements. The only possibility is $r = 12$ and $e = 1 + r/2 = 7$. An examination of the orbit of xy shows that (V,C) is isomorphic to Example 5 with 15 elements. Hence (V,C) is monochromatic, trivial, a BIBD or Example 5.

(iii) Suppose that $G = HS \leq A(V,C)$ and $|V| = 176$. The orbit of x under G_{ab} has 12, 72 or 90 elements. Only $e = 7$ and $e = 88 = |V|/2$ divide $|E|$, and Cameron and Korchmaros [8] eliminate a one-factorization. For $e = 7$, $|K(ab)| = 7|G_{\{a,b\}}|$, but by Magliveras [17] there is no subgroup of HS of that order. Hence (V,C) is monochromatic or trivial.

(iv) Suppose that $G = Co_3 \leq A(V,C)$ and $|V| = 276$. The orbit of x under G_{ab} has 112 or 162 elements. As in (iii) only $e = 138 = |V|/2$ divides $|E|$, and Cameron and Korchmaros [8] eliminate a one-factorization. Hence (V,C) is monochromatic or trivial.

(v) Suppose that $G = Sz(q) \leq A(V,C)$ and $|V| = q^2 + 1$. Case (v) in Theorem 5 never used the additional hypothesis of Theorem 5, so here (V,C) is monochromatic or trivial.

(vi) Suppose that $G = PSL(2,q)$, where $q = p^i$, for some prime p, and $|V| = q + 1$. We may assume that $V = PG(1,q)$, a projective line. If $p = 2$, $G = PSL(2,q) = PGL(2,q)$ is triply transitive and $|V| = 2^i + 1$ is odd, contradicting Theorem 1. If (V,C) is a one-factorization, then p is either 3, 5, 7 or 11 by Cameron and Korchmaros [8]. Now suppose that $1 < e < |V|/2 = (q+1)/2$. The orbits of x and y both have size $(q-1)/2$. If x and y are in different orbits, $e = 1 + (q-1)/2 = (q+1)/2 = |V|/2$, giving a one-factorization. So assume that they are in the same orbit, $e = 1 + (q-1)/4 = (q+3)/4$ and so $q = 1(mod\ 4)$. Further $(q+3)/4$ evenly divides $|E|$ or, equivalently, $q+3$ divides $2q(q+1)$. Dividing $2q(q+1)$ by $q+3$ gives a remainder of 12. Thus $12 = 0(mod\ (q+3)/4)$, implying that $(q+3)/4 < 13$ or $q < 49$. Only $q = 9$ satisfies all of these conditions, fulfilled in Example 11. Because $PGL(2,9)$ is triply transitive on $PG(1,9)$, any two such graphs are isomorphic. Hence (V,C) is monochromatic, trivial, a one factorization or the 2-t ec-graph of Example 11.

(vii) Suppose that $G = PSL(n+1,q)$, where $q = p^i$, for some prime p, and $|V| = \sum_{i=0}^{n} q^i$. We may assume that V is the projective space $PG(n,q)$. Then G_{ab} leaves the line (block) $B(a,b)$ stable. The orbit of x under G_{ab} is either $V/B(a,b)$ or $B(a,b)/\{a,b\}$.

Suppose first that the orbit of x is $V/B(a,b)$ and so y is some other point in $V/B(a,b)$. By double transitivity, the orbit of a (and b) under G_{xy} is $V/B(x,y)$. Either $B(a,b)$ and $B(x,y)$ have a point in common, say z, or they don't. In the second case every vertex u in V will have an edge uv such that $C(ab) = C(xy) = C(uv)$ and (V,C) is a one-factorization, contradicting Cameron and Korchmaros [8]. So assume that z is the common point of $B(a,b)$ and $B(x,y)$. For any automorphism $\alpha \in G_{ab}$, z is also incident with $B(\alpha(x),\alpha(y))$. Now G is triply transitive on each line. So if $B(a,b)$ had more than three points, α could move z, contradicting the above reasoning. Hence, $G = PSL(n, Z_2)$ and we have Example 5.

Suppose now that the orbit of x under G_{ab} is $B(a,b)/\{a,b\}$. Hence we have a doubly transitive one-factorization of $B(a,b)$ and G is triply transitive on $B(a,b)$. By Cameron and Korchmaros [8] and Cameron [6] $|B(a,b)|$ is either 4, 6 or 8. The values of 4 and 6 correspond to Examples 6 and 7. For $|B(a,b)| = 8$, we would have $G = PSL(n,7)$ and G would act on $B(a,b)$ as $PGL(2,7)$. However, the triply transitive 2-t ec-graph on 8 vertices has instead $AGL(3,2)$ for its automorphism group. Hence (V,C) is monochromatic, trivial, a BIBD or one of Examples 5, 6 or 7.

(viii) Suppose that $G = PSp(2n,2) \leq A(V,C)$, $n \geq 3$ and $|V| = 2^{2n-1} \pm 2^{n-1}$. The orbit of x under G_{ab} has $2^{2n-2} \pm 2^{n-1} - 2$ or 2^{2n-2} elements. Hence the orbit of xy is $2^{2n-3} \pm 2^{n-2} - 1$ or 2^{2n-3}. Thus

either $e = |V|/2$, $e = 2^{2n-3} \pm 2^{n-2}$ or $e = 1 + 2^{2n-3}$. Cameron and Korchmaros [8] eliminate the first case $|V|/2$. Example 9 satisfies the second case so we show next that in this case there is no other (V, C'), where V is either Ω^+ or Ω^-. Because of the uniqueness of the orbit of the correct size, if there were such a C', we would have
$Y = \{u : \exists v \in V : C(ab) = C(uv)\} = \{u : \exists v \in V : C'(ab) = C'(uv)\}$. Suppose, for a contradiction, that $C \neq C'$. For some edge uv we would have $C(ab) = C(uv)$ but $C'(ab) \neq C'(uv)$; instead there are $w, z \in Y$ with $C'(ab) = C'(uw) = C'(vz)$. Now t_{a+b} is an automorphism for both (V, C) and (V, C') switching a with b and u with v. Then $C'(ab) = C'(ba)$ forces
$C'(uw) = C'(vt_{a+b}(w)) = C'(vz)$. So t_{a+b} switches w and z and $C(wz) = C(ab)$. Then
$G_{ab} \leq K(V, C, C(ab)) \cap K(V, C', C'(ab))$. Further, the orbit of u (and v) under G_{ab} is $Y/\{a, b\}$. For $p \in Y/\{a, b\}$ and $\kappa \in G_{ab}$ with $\kappa(u) = p$, κ must take the set $\{u, v, w, z\}$ to a set $\{p, q, r, s\}$ such that $C(ab) = C(pq) = C(rs)$ and $C'(ab) = C'(pr) = C'(qs)$. Clearly, $Y/\{a, b\}$ must be partitioned into subsets of size 4, implying $|Y| \equiv 2(mod\ 4)$. However, for $n \geq 3$, $|Y| = 2^{n-1}(2^{n-1} \pm 1) \equiv 0(mod\ 4)$.

Now consider the third case. Calculation shows that the greatest common divisor of $e = 1 + 2^{2n-3}$ and $|E|$ is less than e unless $n = 3$. For $n = 3$, $e = 9$ and $|V| = 28$ or $|V| = 36$. Suppose that there were a 2-t ec-graph $(V, C^\#)$ with $e = 9$ and $a, b \in V$. Let $W^\#$ be the set of 18 vertices at the ends of the 9 edges of color $C^\#(ab) = c^\#$ and $W = W^\#/\{a, b\}$. The elements of W differ from all u and v such that $C(ab) = C(uv)$. Thus the transvection t_{a+b} fixes each $w \in W$ and so
$t_{a+b} \in K(V, C^\#, c^\#)$. Thus on $W^\#$, t_{a+b} acts like the transposition (ab). For $w \in W$, let
$\kappa \in K(V, C^\#, c^\#)$ such that $\kappa(a) = w$ and let $\kappa(b) = w' \in W$. Then $p_w = \kappa \circ t_{a+b} \circ \kappa^{-1} \in K(V, C^\#, c^\#)$ switches w and w' and fixes the rest of $W^\#$. Hence $p_w \in G_{ab}$ and p_w acts on W like the transposition (ww'). This means that all of the p_x commute when considered as permutations on W. Because
$p_w = p_{w'}$ there are eight such elements of order two in G_{ab}, which, when acting on W, would generate a Boolean group of order 2^8. However, $|G_{ab}| = 2^7 \cdot 3^2$ or $|G_{ab}| = 2^7 \cdot 3 \cdot 5$, for $V = \Omega^+$ or $V = \Omega^-$, respectively. Thus in either case there is no subgroup of order 2^8. Hence (V, C) is either monochromatic, trivial or Example 9.

(ix) Suppose that $G = U_3(q) \leq A(V, C)$, where q is a power of a prime and $q \neq 2$, and $|V| = q^3 + 1$. As in Theorem 5 part (ix), for $q > 3$, the only maximal subgroup of G containing $G_{\{a,b\}}$ is the stabilizer of the line $B(a, b)$. Thus for $q > 3$, either $A(V, C)$ is monochromatic or $(V, C) \preceq (V, C_B)$, the 2-t ec-graph derived from the BIBD with $\lambda = 1$. Because G is triply transitive on $B(a, b)$, as in part (vii) the only new examples beyond those in Theorem 5 are when $q = 3$ or $q = 5$ as in Example 8. For $q = 5$, by Conway et al [10, 14] there are no other options. For $q = 3$, by Conway et al [10, 14] there is only one other maximal subgroup containing $G_{\{a,b\}}$, giving the 2-t ec-graph of Example 10. The last sentence in Example 10 implies that if $(V, C) \preceq (V, C')$ but (V, C) differs from (V, C') and (V, C_T), then (V, C) is the graph in Example 8. Hence, (V, C) is monochromatic, trivial, Example 8 or Example 10.

(x) Suppose that $G = {}^2G_2(q) \leq A(V, C)$, where $q = 3^{2a+1}$, $a \geq 0$, and $|V| = q^3 + 1$. As in Theorem 5 part (x), for $q > 3$, the only maximal subgroup of G containing $G_{\{a,b\}}$ is the stabilizer of $B(a, b)$. Thus for $q > 3$, either $A(V, C)$ is monochromatic or $(V, C) \preceq (V, C_B)$, the 2-t ec-graph derived from the BIBD with $\lambda = 1$. However, the only time this last possibility differs from (V, C_B) or (V, C_T) is when $q = 3$ as in Example 8, but then ${}^2G_2(3) = P\Gamma L(2, 8)$ is not simple. See Theorem 7 for this possibility. Hence (V, C) is monochromatic, trivial or the 2-t ec-graph derived from a BIBD.

In addition to the simple groups and the affine family of groups, the list of doubly transitive groups includes one other possibility, namely groups containing $P\Gamma L(2, 8) = {}^2G_2(3)$ and acting on a set of 28 unitals. We consider these groups in the following theorem.

Theorem 7. Suppose that V is the set of unitals for the group $G = P\Gamma L(2, 8) = {}^2G_2(3)$ and (V, C) is a doubly transitive edge colored graph with $G \leq A(V, C)$. There are seven possibilities:

(i) (V, C) is monochromatic;

(ii) (V,C) is trivial;
(iii) (V,C) is derived from the BIBD on V with $\lambda = 1$;
(iv) (V,C) is a one-factorization on 28 vertices;
(v) (V,C), for any color c, has $K(V,C,c) \cong PSL(2,8)$;
(vi) (V,C) is the meet of possibilities (iii) and (iv) (Example 8); or
(vii) (V,C) is the join of possibilities (iii) and (iv).

Proof. Let a and b be any two distinct elements of V. Because $|G| = 2^3 \cdot 3^3 \cdot 7 = 28 \cdot 27 \cdot 2$, $|G_{\{a,b\}}| = 4$. We find all subgroups K of G containing $G_{\{a,b\}}$. Let $J = PSL(2,8) = PGL(2,8)$, a simple normal subgroup of G with $|J| = 2^3 \cdot 3^2 \cdot 7 = 28 \cdot 9 \cdot 2$ and $G_{\{a,b\}} \leq J$. For $G_{\{a,b\}} \leq K \leq G$, $G_{\{a,b\}} \leq K \cap J$ and either $K \cap J = K$ or $3|K \cap J| = |K|$. Thus from the subgroups K such that $G_{\{a,b\}} \leq K \leq J$, we can find the remaining ones. Now J has only one maximal subgroup, $AGL(2,8)$, with 56 elements. (See Conway et al [10, 6].) The eight translations of $AGL(2,8)$ form a normal subgroup T of $AGL(2,8)$. It is easily checked that if K satisfies $G_{\{a,b\}} \leq K \leq J$, then K is one of four subgroups $G_{\{a,b\}}$, $AGL(2,8)$, $J = PSL(2,8)$ and T. These correspond, respectively, to the possibilities (ii), (iv), (v) and (vi) in the theorem. There are at most four more subgroups of $P\Gamma L(2,8)$ whose intersections with J are one of these four subgroups. Three of these potential subgroups actually exist: $P\Gamma L(2,8)$, $A\Gamma L(2,8)$ and the group B for the unique BIBD with $\lambda = 1$. (See Kantor [14].) These correspond, respectively, to the possibilities (i), (vii) and (iii) in the theorem. The fourth would have to be an index 2 subgroup of B. However, that would entail partitioning the six edges of each line of four points in the BIBD into two sets of three edges. By inspection no such partition is doubly transitive.

Remark. The 2-t ec-graph in part vii) of Theorem 7 is a 3-factorization of the complete graph on 28 vertices.

Section 2. The Affine Case

Suppose that V is an n-dimensional vector space over the finite field $\mathrm{GF}(p^k) = F$ with $|F| = p^k$ elements and $G = A(V,C) \leq A\Gamma L(n,p^k)$. Now $A\Gamma L(n,p^k) \leq AGL(nk,p)$, so $A(V,C) \leq AGL(d,p)$, $d = nk$. In the non-affine case, Lemma 3 allowed us to concentrate on the minimal 2-transitive subgroups, the simple groups and $P\Gamma L(2,8) = {}^2G_2(3)$ because all of the related groups contain these minimal groups. Unfortunately, in the affine case the situation is reversed: We know the large group, $AGL(d,p)$, and $A(V,C)$ is a subgroup of $AGL(d,p)$. Further, the often large number of non-isomorphic 2-t ec-graphs for such a group G makes a detailed classification such as in the previous section infeasible here. The first row of Table 1 (below) gives selected but representative sizes of $|V|$, the second row gives the number of non-isomorphic 2-t ec-graphs of size $|V|$ and the third gives the number of such graphs whose automorphism groups are neither an affine group nor the symmetric group $S_{|V|}$. (For $|V| > 2$, there are exactly two 2-t ec-graphs, the monochromatic and trivial ones, whose automorphism group is $S_{|V|}$.) Except for the cases $|V| = 28$, which includes the 2-t ec-graphs of Theorem 7, and the case $|V| = 31$, which has two different projective groups, the table clearly suggests that there are relatively few non-affine 2-t ec-graphs. However, the table suggests that for powers of primes there are many affine 2-t ec-graphs.

Size of V	7	8	9	13	16	25	27	28	31	49	64	81
non-isom.	6	4	8	10	10	≥ 18	8	10	12	≥ 20	≥ 19	≥ 28
not aff, **sym**	2	0	0	2	0	0	0	8	4	0	0	0

Table IV. The number of non-isomorphic 2-t ec-graphs
and non-affine, non-symmetric ones for selected sizes $|V|$.

Although a detailed classification for the affine case is unwieldy, we can describe the form of the chromomorphism subgroups. For $p > 2$ we show in Theorem 14 that any chromomorphism subgroup $K(c)$ can be written as the semidirect product of its subgroup T_K of "translations" with the subgroup $K(c)_0$ fixing 0. Because all chromomorphism subgroups $K(c)$ are conjugate, without loss of generality choose $c = C(-v, v)$ for some non-zero vector v and write K for $K(C(-vv))$. Then $G_{\{-v,v\}} \leq K_0 \leq G_0$. Further, Lemma 8 gives restrictions on the interactions of K_0, T_K and v. As a result the construction in Example 12 below gives all 2-t ec-graphs for $p > 2$, providing a suitable classification. Because of the symmetry of $-v$ and v, the "central symmetry" that switches each vector x with $-x$ plays a special role in the case $p > 2$, so in the notation below we reserve the letter σ for it.

For the other case $p = 2$ we have $v = -v$, which requires some changes–in particular we use $c = C(0, v)$, for some non-zero vector v. However, the changes are much deeper: There are 2-t ec-graphs besides those given in Example 12 and the analog to the conclusion of Theorem 14 is false for $p = 2$, as Example 13 illustrates. Nevertheless, as we will see, Example 12 and the generalization of the construction in Example 13 give all 2-t ec-graphs when $p = 2$.

Definitions and Notations. Let $G = A(V, C)$ be a doubly transitive subgroup of $AGL(d, p)$. Define $T = \{t_w : w \in V\}$, where for $w \in V$, t_w is the *translation* mapping x to $t_w(x) = x + w$. Denote the nonzero elements of V by V^*. For $p > 2$ and a fixed $v \in V^*$ suppose that $G_{\{-v,v\}} \leq K \leq G$. Define $T_K = T \cap K$, $V_K = \{w : t_w \in T_K\}$, $\overline{K}_0 = \{g \in G_0 : \exists t \in T : tg \in K\}$ and $V_{\overline{K}}$ to be the subspace generated by $\{g(v) : g \in \overline{K}_0\}$. For $p > 2$ define $\sigma \in AGL(d, p)$ by $\sigma(x) = -x$ for all $x \in V$. For $p = 2$, replace $G_{\{-v,v\}}$ with $G_{\{0,v\}}$.

Lemma 8. Given the preceding definitions,
T is normal in G and T_K is normal in K;
Every $\alpha \in G$ can be written uniquely as $\alpha = tg$, where $t \in T$ and $g \in G_0$;
V_K is a subspace of V;
$\overline{K}_0$ is a subgroup of G_0 with $K_0 \leq \overline{K}_0$;
If $v \in V_K$, then $V_{\overline{K}} \leq V_K$;
If $g \in \overline{K}_0$, then $g(V_K) = V_K$ and $g(V_{\overline{K}}) = V_{\overline{K}}$.

Proof. All of these results are well known or easily shown.

Example 12. Let $p > 2$, $v \in V^*$ and G a doubly transitive subgroup of $AGL(d, p)$. By Lemma 2 if $G_{\{-v,v\}} \leq K_0 \leq G_0$, then (V, C_{K_0}) is a 2-t ec-graph. Let $\overline{V}$ be the subspace generated by $\{g(v) : g \in K_0\}$. Let V'' be any subspace of V such that for all $g \in K_0$, $g(V'') = V''$ and $T'' = \{t_w : w \in V''\}$. Then $T''K_0 = K$ is a subgroup of G and (V, C_K) is a 2-t ec-graph. For example, $V'' = V$ and $V'' = \{0\}$ are always stable for any K_0. If $V'' = V$, then $K = TK_0$ gives a regular 2-t ec-graph; that is, each color of (V, C_K) is a regular graph. In general the groups G and K_0 determine which proper subspaces are stable. For any stable V'' one can easily show either $\overline{V} \leq V''$ or $\overline{V} \cap V'' = \{0\}$. In the first case, the graph is related to a BIBD where $\overline{V}$ is a block, illustrated in Figure 4a; in the second case, the subspaces $\overline{V}$ and V'' are effectively orthogonal in V, illustrated in Figure 4b.

For $p = 2$, we use $G_{\{0,v\}}$, since $-v = v$. Given K_0 such that $G_{0v} \leq K_0 \leq G_0$, let $\overline{V}$ be the subspace generated by $\{g(v) : g \in K_0\}$. Note that every chromomorphism group $K(V, C, C(0v))$ contains all translations t_{u+w}, where $C(uw) = C(0v)$. Thus, using the notation in the previous paragraph, $\overline{V} \leq V''$. If $T'' = \{t_u : u \in V''\}$ and for all $g \in K_0$, $g(V'') = V''$, then the semidirect product $K = T''K_0$ is a subgroup with $G_{\{0,v\}} \leq K$. Thus for any doubly transitive group G and $v \in V^*$ this construction gives a 2-t ec-graph.

Figure 5 illustrates the lattice of subgroups K with $G_{\{-v,v\}} \leq K \leq G$.

The following series of lemmas prepare the classification when $p > 2$. The lemmas 10 to13 consider separately the types of doubly transitive subgroups of $AGL(d, p)$, described in Kantor [14].

Lemma 9. Suppose that $p > 2$, $G = A(V, C)$ is doubly transitive, $G \leq AGL(d, p)$, $v \in V^*$, $G_{\{-v,v\}} \leq K \leq G$ and $\sigma \in G$, where σ is defined by $\sigma(v) = -v$ for all $v \in V$. Then $K = T_K K_0$.

Proof. Note that $\sigma \in G_{\{-v,v\}} \leq K$, σ commutes with all $g \in G_0$ and for $t_u \in T$, $\sigma t_u \sigma = t_{-u}$. For

$k \in K$, we can write $k = t_u g$ with $t_u \in T$ and $g \in G_0$. Then $\sigma(t_u g)\sigma(t_u g)^{-1} = t_{-2u} \in K$ and $t_u \in \langle t_{-2u} \rangle$, so $t_u \in K$. Thus $g = t_{-u}(t_u g) \in K$ and so $t_u \in T_K$ and $g \in K_0$.

Lemma 10. Suppose $G = A(V, C)$ is a doubly transitive group with $ASL(n, q) \leq G \leq A\Gamma L(n, q)$, $|V| = q^n$, $q = p^k$, $n \geq 2$ and $p > 2$. If $v \in V^*$ and $G_{\{-v,v\}} \leq K \leq G$, then $K = T_K K_0$.

Proof. If n is even, then $\sigma \in ASL(n, q) \leq G$ because the determinant of σ is 1. By Lemma 9, $K = T_K K_0$.

Suppose now that n is odd and κ is any element of $K = K(C(-vv))$. For $\kappa = tg \in K$, where $t \in T$ and $g \in G_0$ we know $g \in \overline{K}_0$. It suffices to show that $t \in T_K$, for then $g = t^{-1}(tg) \in K \cap \overline{K}_0 = K_0$. Let $u = \kappa(0) = t(0)$ and $w = g(v)$. Then $\kappa(\pm v) = u \pm w$. The dimension of the subspace $\langle u, v, w \rangle$ is at most 3. Suppose first that the dimension is 1 or 2 and pick a basis $\{e_1, \ldots, e_n\}$ of V such that u, v, $w \in < e_1, e_2 >$. Consider $\beta \in GL(n, q)$ such that $\beta(e_1) = -e_1$, $\beta(e_2) = -e_2$ and $\beta(e_i) = e_i$ for $i > 2$. Then $\beta \in SL(n, q) \leq G_0$. Further, $\beta(\pm v) = \mp v$ so $\beta \in K$. Also $\beta(u \pm w) = -u \mp w$ and $t_{-2u}(u \pm w) = -u \pm w$. Thus $t_{-2u} \in K$ and so $t = t_u \in K$.

Now assume that the dimension of $< u, v, w >$ is 3. Pick a basis $\{e_1, \ldots, e_n\}$ of V so that $u = e_1$, $v = e_2$ and $w = e_3$. Let $\gamma \in GL(n, q)$ such that $\gamma(u) = -u$, $\gamma(v) = v$, $\gamma(w) = -w$ and $\gamma(e_i) = e_i$ for $i \geq 4$. Then $\gamma \in SL(n, q) \leq G_0$ and $\gamma \in K$. Because $\gamma(u \pm w) = -u \mp w$, the argument above shows that $t = t_u \in K$.

Lemma 11. If $G = A(V, C)$ is a doubly transitive subgroup of $A\Gamma L(1, p^d)$ and $p > 2$, then $\sigma \in G$.

Proof. We can consider V as the field of order p^d. For $\Gamma_0 = A\Gamma L(1, p^d)_0$, $G_0 \leq \Gamma_0$ and Γ_0 is a semidirect product of the cyclic groups V^* and $Aut(V)$, of orders $p^d - 1$ and d, respectively. For a a generator of V^* and β the Frobenius map ($\beta(x) = x^p$), write the elements of Γ_0 as (a^i, β^j), where $(a^i, \beta^j)(a^k, \beta^m) = (a^i \beta^j(a^k), \beta^{j+m})$. Note that σ is the only element of order 2 in V^*. Because G is doubly transitive, some element of G_0 maps 1 to a and so is of the form (a, β^j). Now $(a, \beta^j)^d = (\prod_{i=0}^{d-1} \beta^{ij}(a), \beta^{jd}) = (\prod_{i=0}^{d-1} \beta^{ij}(a), 1) \in V^*$. Further, $\beta^{ij}(a) = a^{(p^{ij})}$, so $\prod_{i=0}^{d-1} \beta^{ij}(a) = a^z$, where $z = \sum_{i=0}^{d-1} p^{ij}$. I claim that z has even order in Z_{p^d-1} and so a^z generates σ, which will finish the lemma. Because $p^d \equiv 1 (\mathrm{mod}\, p^d - 1)$ we have $p^{ij} \equiv p^r (\mathrm{mod}\, p^d - 1)$, where r is the remainder of dividing ij by d. Thus, for h the greatest common divisor of j and d, rearranging terms gives $\sum_{i=0}^{d-1} p^{ij} \equiv \sum_{i=0}^{d-1} p^{ih} (\mathrm{mod}\, p^d - 1)$. We assume that j divides d, say $d = jk$. Then in the sum for z the terms repeat j times, giving $z = \sum_{i=0}^{d-1} p^{ij} = j \sum_{i=0}^{k-1} p^{ij} = j\left(\dfrac{p^d - 1}{p^j - 1}\right)$. To show z has even order we show that $p^j - 1$ has more factors of 2 than j does. For $j = 2^y m$, where m is odd, $p^j - 1 = [\prod_{x=0}^{y-1} (p^{2^x m} + 1)](p^m - 1)$. Each of these $y + 1$ factors is even because p is odd. So $p^j - 1$ has more factors of 2 than j does, z is of even order and a^z generates σ, which is thus in G.

Lemma 12. Suppose that $G = A(V, C)$ is a doubly transitive subgroup of $AGL(d, p)$, $p > 2$, $G_{\{-v,v\}} \leq K \leq G$ and $T_K = \{t_0\}$. Then $K = K_0$.

Proof. By the conjugacy of all chromomorphism subgroups, for all $a, b \in V$, $T \cap K(C(ab)) = \{t_0\}$. Let $Q_{ab} = \{x : \exists y : C(xy) = C(ab)\}$. We first show that $|Q_{ab}|$ divides $p^d - 1$. (Remark. From the definition before Lemma 4 $|Q_{ab}| = k_{ab}$.) To prove this claim, note that for any edge ab, the p^d translates of ab, which are the edges in $\{(a + w)(b + w) : w \in T\}$, must all have different colors because $T_K = \{t_0\}$. Further, if $C(ab) = C(xy)$, then $C((a + w)(b + w)) = C((x + w)(y + w))$. Hence the number of colors must be a multiple of p^d. Since $|V| = p^d$ and the number of edges is the product of the number of colors and the number of edges per color, the number of edges per color divides $(p^d - 1)/2$. Hence $|Q_{ab}|$ divides $p^d - 1$.

Let $S_{ab} = \sum_{x \in Q_{ab}} x$. Because $|Q_{ab}|$ is relatively prime to p, there is a unique $c_{ab} \in V$ such that

$|Q_{ab}|c_{ab} = S_{ab}$. (This c_{ab} acts like the center of gravity of Q_{ab}.) For $\kappa \in K(C(ab))$, $Q_{\kappa(a)\kappa(b)} = Q_{ab}$ and so $S_{\kappa(a)\kappa(b)} = S_{ab}$, showing that c_{ab} is fixed by $K(C(ab))$. Let $a' = t(a)$ and $b' = t(b)$, where t is the translation by $-c_{ab}$. Then $S_{a'b'} = \sum_{x\in Q_{ab}} t(x) = \sum_{x\in Q_{ab}} (x - c_{ab}) = S_{ab} - |Q_{ab}|c_{ab} = 0$. Hence $K(C(a'b')) \le G_0$. Further, $G_{\{a',b'\}} \le K(C(a'b'))$ and, for $c' = 2^{-1}(a' + b')$, $G_{\{a',b'\}} \le G_{c'}$. Thus $G_{\{a',b'\}} \le G_0 \cap G_{c'} = G_{0c'}$, which would give a contradiction unless $0 = c'$ and so $b' = -a'$. We now have $K(C(-a'a')) \le G_0$ and by conjugacy, for any $v \in V^*$, $K(C(-vv)) \le G_0$, whence $K(C(-vv)) = K_0$.

Lemma 13. Suppose that $G = A(V,C)$ is a doubly transitive subgroup of $AGL(d,p)$, where $p = 5, 7, 11, 19, 23, 29$ or 59, $d = 2$ and $SL(2,3) \triangleleft G_0$ or $SL(2,5) \triangleleft G_0$. For $v \in V^*$ if $G_{\{-v,v\}} \le K = K(C(-vv)) \le G$, then $K = T_K K_0$.

Proof. We may consider V as an affine plane. By Dixon and Mortimer [11, 239], each G_0 contains the automorphism μ represented by the matrix $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

Case 1: $t_v \in T_K \le K = K(-vv)$. Without loss of generality, $v = (1,-1)$. Because $d = 2$ either $T_K = T$ or $T_K =< t_v >$. If $T_K = T$ and $\kappa = tg \in K$, then $t \in T \le K$, $g \in K_0$ and $K = T_K K_0$. So assume that $T_K =< t_v >$, $V_K = \{(z,-z) : z \in Z_p\}$. Because $\overline{K}_0$ maps V_K to itself, if $C(ab) = C(-vv)$, then the lines (blocks) $B(a,b)$ and $B(-v,v)$ are parallel. Further, $B(\mu(a),\mu(b))$ is parallel to $B(a,b)$ and for $w = \mu(a) - a$, we have $t_w(a) = \mu(a)$ and $t_w(b) = \mu(b)$. Hence $t_w \in T_K =< t_v >$. Only $B(a,b) = B(-v,v)$ is compatible with this condition, forcing $K = T_K\overline{K}_0$ and so $K = T_K K_0$.

Case 2: $t_v \notin T_K$. If $T_K = \{t_0\}$, Lemma 12 shows that $K = K_0 = T_K K_0$. So suppose that $T_K =< t_u >$ and, without loss of generality, $v = (1,1)$. Note that $\mu(\pm v) = \mp v$, so $\mu \in K$. Because T is normal in G, $\mu t_u \mu$ is some translation t_w, and if $u = (a,b)$, then $w = (-b,-a)$. However, $t_w \in T_K =< t_u >$. These conditions imply $b = \pm a$. If $b = a$, then $u = v$, contradicting $t_v \notin T_K$. So $b = -a$. Note that μ commutes with t_u. Consider $K' = K(C(-uu))$. Because $\mu \in K'$ we must have $t_v \in K'$. Further, the conjugate to μ in K', say ν, must switch u and $-u$. Next ν commutes with t_v, forcing ν to have the matrix representation $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $\mu\nu = \sigma$ is in G and, by Lemma 9, $K = T_K K_0$.

Theorem 14. If (V,C) is a doubly transitive edge colored graph with $A(V,C) \le AGL(d,p)$ for $p > 2$, $v \in V^*$ and $K = K(C(-vv))$, then $K = T_K K_0$.

Proof. We use the classification of the doubly transitive groups contained in $AGL(d,p)$ together with the previous lemmas. (See Kantor [14] for a list of these groups.)

i) For $G \le A\Gamma L(1,p^d)$ use Lemmas 11 and 9.

ii) For $ASL(n,q) \le G$, where $n \ge 2$ and $q = p^d$, use Lemma 10.

iii) For $Sp(n,q) \le G$, where $q = p^d$, note that $\sigma \in Sp(n,q)$ and use Lemma 9.

iv) For $SL(2,3) \triangleleft G_0$ or $SL(2,5) \triangleleft G_0$ and $|V| = p^2$, $p = 5, 7, 11, 19, 23, 29$ or 59, use Lemma 13.

v) There are three other possible groups when $|V| = 3^d$, $d = 4$ or $d = 6$. By Aschbacher [3] all of these contain σ, so Lemma 9 applies.

The preceding theorem provides a means of constructing all possible 2-t ec-graphs with affine groups and $p > 2$ using Example 12. For V a d-dimensional vector space over the field of order p, one needs to determine which subgroups K leave a particular i-dimensional subspace V'' stable. The translations leaving V'' stable are obvious. So Theorem 14 allows us simply to check the subgroups of G_0 leaving V'' stable, a much more manageable task. In general when i and d are relatively prime there are relatively few choices of subgroups compared to when $(i,d) > 1$.

We turn now to the case $p = 2$. In addition to the construction discussed in Example 12 there is one other, more complicated construction illustrated in Example 13.

Example 13. Let V be the field of order 2^{kn} and F the subfield of order 2^k, where $1 < k,n$. The elements of $G = AGL(1,2^{kn})$ can be written as $g_{a,b}$, where $g_{a,b}(x) = ax + b$ and $a \in V^*$, $b \in V$. Note that $t_b = g_{1,b}$ and $G_{\{0,1\}} = \{t_0, t_1\}$. Let w generate V^*, $j = (2^{kn} - 1)/(2^k - 1)$ and $u = w^j$. Then u generates F^*. Let $K = K(V,C,C(01))$ be the subgroup generated by $g_{u,w}$ and all of the translations t_q, where $q \in F$. Then $G_{\{0,1\}} \le T_K = \{t_q : q \in F\} \le K$. Note that $(g_{u,w})^i = g_{y,z}$, where $y = u^i$ and $z = w\sum_{j=0}^{i-1} u^j$. For $i = 2^k - 1$, $\sum_{j=0}^{i-1} u^j = \sum_{q \in F^*} q = \sum_{q \in F} q = 0$ because F has at least 4 elements. Thus for $i = 2^k - 1$, $(g_{u,w})^i = g_{1,0} = t_0$. In general, the elements of K are of the form $g_{y,z+q}$, where $q \in F$, and for some i, $y = u^i$ and $z = w\sum_{j=0}^{i-1} u^j$. Thus, K is **not** the semidirect product of $T_K = \{t_q : q \in F\}$ and $K_0 = \{t_0\}$. Note that $|K| = |L|$, where $L = T_K\overline{K}_0$ and $\overline{K}_0 = \{g_{y,0} : y = u^i\}$. Further, (V,C_L) is derived from the BIBD in which F is one line (block). Both (V,C_K) and (V,C_L) are 2-t ec-graphs with the same translations in $K(C_K(ab))$ and $K(C_L(ab))$, but edges in one line (block) of (V,C_L) are split up among translates of those edges to form (V,C_K). Figures 6a and 6b illustrate this situation.

The existence of subgroups K that are not the semidirect products of Theorem 14 makes the classification of 2-t ec-graphs in the affine case with $p = 2$ more complicated. Fortunately, for any such subgroup K, as in Example 13, the subgroup $L = T_K\overline{K}_0$ is equinumerous to K and they have a common subgroup $J = T_K K_0$. The possible structure of (V,C_K) is restricted by the structures of (V,C_J) and (V,C_K). More explicitly, for $p = 2$ and V the d-dimensional vector space over Z_2, suppose that G is a doubly transitive subgroup of $AGL(d,2)$ and $v \in V^*$. For $G_{\{0,v\}} \le K \le G$, let $J = T_K K_0 \le K$ and $L = T_K\overline{K}_0$. Lemma 8 ensures that $G_{\{0,v\}} \le J \le L$. Clearly, (V,C_K), (V,C_L) and (V,C_J) are 2-t ec-graphs with $(V,C_J) \preceq (V,C_K)$ and $(V,C_J) \preceq (V,C_L)$. To show that $|K| = |L|$, first note that every $k \in K$ can be written uniquely as $k = tg$ for $t \in T$ and $g \in \overline{K_0}$. Further, if $t' \in T_K$, then $t'tg \in K$ as well. Thus $|K| \ge |T_K| \cdot |\overline{K_0}| = |L|$. Next suppose for $g \in \overline{K_0}$, tg and $t^*g \in K$. Then $(t^*g)(tg)^{-1} = t^*t^{-1} \in K$ and so $t^*t^{-1} \in T_K$ and $t^* \in T_K$. Then for a fixed g, $|\{t : tg \in K\}| \le |T_K|$ and so $|K| \le |T_K| \cdot |\overline{K_0}| = |L|$, showing $|K| = |L|$. Further, $T_K = T_L = T_J$. The set of edges of one color for either (V,C_K) or (V,C_L) is a union of $|\overline{K}_0|/|K_0|$ families of same colored edges of (V,C_J). As in Example 13, the families forming one color in (V,C_K) are translates of the families forming one color in (V,C_L). Thus the constructions of Example 12 and this natural generalization of Example 13 are the only ones possible when $p = 2$, although implementing Example 13 requires some attention. A more detailed classification for $p = 2$ would seem to require considerably more space.

The paragraph after Theorem 14 and the preceding paragraph give a classification of all finite 2-t ec-graphs whose automorphism groups are subgroups of an affine group.

Section 3. Regular Edge Colored Graphs

As mentioned in Example 3, a metric space becomes an edge colored graph with the coloring determined by the metric. Regular edge colored graphs correspond to metric spaces in which the configuration of distances from any given point to all other points is independent of the choice of point. A geometrically interesting and more restricted family of metric spaces are those with transitive isometry groups, as defined below. We classify the 2-t ec-graphs corresponding to both of these situations.

Definition. An edge colored graph (V,C) is *regular* iff for each color c, the edges ab such that $C(ab) = c$ form a regular graph on V.

Theorem 15. If (V,C) is a finite regular doubly transitive edge colored graph, then either (V,C) is monochromatic; (V,C) is a one-factorization; $|V| = 28$ and $A(V,C) = P\Gamma L(2,8)$; or $|V| = p^d$, $A(V,C) \le AGL(d,p)$ and $T \le K(V,C,ab)$.

Proof. From the classification of doubly transitive groups we need to consider three possibilities: first of all, the automorphism group $A(V,C)$ contains a simple doubly transitive group; secondly, it contains $P\Gamma L(2,8)$; and thirdly, it is a subgroup of some affine group $AGL(d,p)$. For the first possibility, we see from Theorem 6 that only the monochromatic 2-t ec-graphs and the one-factorizations are regular. Theorem 7 lists the four regular 2-t ec-graphs with 28 vertices and automorphism group $P\Gamma L(2,8)$. Using the numbering there, they are (i), (iv), (v) and (vii). Finally, for the third case suppose $A(V,C) \leq AGL(d,p)$ and $|V| = p^d$. First let $p > 2$. For $K = K(V,C,C(-vv))$, the number of edges of the color $C(-vv)$ is $|K|/|G_{\{-v,v\}}| = |T_K||K_0|/|G_{\{-v,v\}}|$. We show that $|K_0|/|G_{\{-v,v\}}|$ is relatively prime to p^d. Note that (V,C_{K_0}) is a 2-t ec-graph with $|K_0|/|G_{\{-v,v\}}|$ edges of color $C_{K_0}(-vv)$. In Lemma 12 we showed in this case that the number of edges of any given color is relatively prime to p^d. So all powers of p in $|T_K||K_0|/|G_{\{-v,v\}}|$ must be in $|T_K|$. In order that (V,C) be regular, the number of edges must be a multiple of p^d. Hence $T_K = T$. When $p = 2$, a similar argument holds, once we substitute $\overline{K}_0$ for K_0 and G_{0v} for $G_{\{-v,v\}}$.

Example 14. Let V be the d-dimensional vector space over Z_p, $G = AGL(d,p)$, ab any edge in V and $L = TG_{\{a,b\}}$ (or TG_{ab} if $p = 2$). Note that T is the smallest transitive subgroup of G, and so L is the smallest transitive subgroup containing $G_{\{a,b\}}$ (or G_{ab} if $p = 2$). Thus all regular 2-t ec-graphs (V,C) satisfy $(V,C_L) \preceq (V,C)$. Remark: T is isomorphic to V considered as a group, so (V,C_L) corresponds to the equidistance relation on V as a group defined in Sibley [18].

From Example 12 every subgroup K with $TG_{\{-v,v\}} \leq K \leq G$ (or $TG_{0v} \leq K \leq G$, if $p = 2$) gives a regular affine 2-t ec-graph. What more can we say about these graphs? For L as in Example 14, if $(V,C_L) \preceq (V,C)$, then the number of colors of (V,C) must divide the number of colors of (V,C_L), which is $\dfrac{p^d - 1}{2}$ if $p > 2$ and $p^d - 1$ if $p = 2$. Example 15 shows that all such divisors are possible. Unfortunately, Example 16 shows that there can be non-isomorphic 2-t ec-graphs for some divisors.

Example 15. Let V be the field of order p^d and $G = AGL(1,p^d)$. We construct a family of regular 2-t ec-graphs (V,C) with all possible numbers of colors. Now $G_0 = V^*$ is a cyclic group with $(|V| - 1)$ elements. For each divisor j of $(|V| - 1) = p^d - 1$ there is a unique subgroup H_j of V^* containing j elements. For $p > 2$, note that j is even iff $-1 \in H_j$. If $p > 2$, assume that j is an even divisor; if $p = 2$, j can be any divisor. Let ab be any edge in V. Define $K_j = TH_j = K(V,C_j,ab)$ from Lemma 2. Then (V,C_j) has $(|V| - 1)/j$ colors.

Example 16. Let V be the two-dimensional vector space over Z_5. There are two non-isomorphic colorings C and C' on V, such that $C(E) = C'(E) = \{0,1,3\}$, and both (V,C) and (V,C') are regular 2-t ec-graphs. Call the six classes of parallel lines B_m, for $m \in Z_5 \cup \{\infty\}$, where the lines $y = mx + c$, for m, $c \in Z_5$ are in B_m and the lines $x = c$ are in B_∞. Define

$$C(ab) = \left\{ \begin{array}{lll} 0 & \text{if} & B(a,b) \in B_0 \cup B_\infty \\ 1 & \text{if} & B(a,b) \in B_1 \cup B_4 \\ 3 & \text{if} & B(a,b) \in B_2 \cup B_3 \end{array} \right\} \text{ and } C'(ab) = \left\{ \begin{array}{lll} 0 & \text{if} & B(a,b) \in B_0 \cup B_\infty \\ 1 & \text{if} & B(a,b) \in B_1 \cup B_2 \\ 3 & \text{if} & B(a,b) \in B_3 \cup B_4 \end{array} \right\}.$$

Then $A(V,C)$ has 4800 elements and $K(V,C,C(ab))$ has 800 elements, whereas $A(V,C')$ has only 2400 elements and $K(V,C',C(ab))$ has 400 elements. Hence (V,C) and (V,C') are not isomorphic.

The regular 2-t ec-graphs over vector spaces have a number of nice properties that stem from the fact that the translations preserve all colors or, in the language of metric spaces, they are isometries. That is, for any edge ab, $T \leq K(V,C,C(ab))$.

Definition. An automorphism ρ of (V,C) is an *isometry* iff for all edges ab we have $C(ab) = C(\rho(a)\rho(b))$. Denote the group of isometries by $I(V,C)$. An edge colored graph (V,C) is *point color symmetric* iff $I(V,C)$ is transitive on V and $A(V,C)$ is transitive on the colors $C(E)$.

Remark. Chen and Teh [9] defined point color symmetric graphs more generally, to include graphs that were not complete graphs.

Theorem 16. A doubly transitive edge colored graph (V,C) is point color symmetric iff it is in the following list. This list also gives all doubly transitive symmetric association schemes.

(i) (V,C) is monochromatic,

(ii) (V,C) is regular and affine,

(iii) (V,C) is the example in part (v) of Theorem 7.

Proof. Double transitivity guarantees that $A(V,C)$ is transitive on $C(E)$. If $I(V,C)$ is transitive, then (V,C) is regular. An inspection of the groups $A(V,C)$ for the edge colored graphs in Theorem 15 reveals that only the ones listed in this theorem have transitive isometry groups, which are thus the point color symmetric ones. Similarly, the definition of a symmetric association scheme implies that it is a regular edge colored graph. An inspection of the edge colored graphs in Theorem 15 reveals that just the ones listed in this theorem satisfy the definition of a symmetric association scheme. (See Bannai and Ito [5, 52].)

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