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ABSOLUTE DIFFERENTIATION IN METRIC SPACES

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Abstract. In this article, we introduce a new notion of *(strong) absolute derivative*, for functions defined between metric spaces, and we investigate various properties and uses of this concept, especially regarding the geometry of abstract metric spaces carrying no other structure.

1. Introduction

1.1. **Overall Objectives.** The concept of a derivative was introduced in the context of the study of real-valued functions of a real variable, and has had significant impact on the development of Mathematics and its applications. Since then, this concept has been extended in various ways to complex-valued functions of real variables, or of complex variables, and to real and complex Banach spaces, to name a few cases. In most cases, an underlying arithmetic structure is used for the definition of a derivative, and the authors have previously (see [CIP]) extended the definition of derivative to functions from one topological field into another. For functions defined on differentiable manifolds, there is a concept of derivative that has been in use for many years, and the manifolds involved do not generally have an arithmetic structure (such as a topological field structure) defined on them; however, in this case, derivatives are defined using charts and atlases, which require the use of the arithmetic of a cartesian power of one of the classical topological fields, such as the real number field or the complex number field, to define derivatives of functions on the manifold in question.

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Nonsmooth calculus, as discussed in [H], and in articles referenced therein, considers metric spaces equipped with compatible measures in order to relax the notion of differentiability. But we do not equip our metric spaces with measures. Also, in both [H] and [AT, pg 55], the notion of a Metric Derivative is investigated, but in this case, the domains of the functions in question are closed intervals of real numbers. Our notion of derivative is more general, in that the domain need not be a subset of the reals. Note as well that discussion of Lipschitz functions leads to a result of Rademacher (presented, for example in [AT, pg 42]), that such functions are almost everywhere differentiable with respect to Lebesgue measure. Even when our spaces are equipped with Lebesgue measure, we have examples of functions that are Lipschitz, but not anywhere absolutely differentiable in our sense.

Here we will initiate the investigation of "derivatives" for functions between arbitrary metric spaces, in which the role of arithmetic is significantly diminished in comparison to the above-mentioned contexts. (In fact, the only use we make of arithmetic is to compute using the real numbers, because we use the metric to define our derivatives.) Thus the metric spaces we consider need not have any arithmetic structure defined on them at all, and they need not be locally homeomorphic to $\mathbb{R}^n$ or any other topological vector space.

1.2. **Organization of this Article.** In this article, we introduce the notions of absolute differentiability and strong absolute differentiability. We describe the relationships between these new concepts and classical notions of differentiability. A tool we will use to construct examples is the previously developed notion of a metric-preserving function (see [D]). Also, we will present various examples illustrating the connections between our new theory and the traditional ones. General results about absolute differentiation will be presented, some of which parallel results in the traditional setting, and some of which have no clear analogue. We will define a new class of metric spaces that we call rectifiably connected spaces, for which any two points can be connected by a segment of finite length. In such spaces, results such as the following hold: If the absolute derivative is zero then the function is constant. It is well known that if d is a metric on a space, then $\sqrt{d}$ also is a metric on that space. Thus, in some cases, a given metric d has the property that its square is also a metric, but it is also well known that in some cases, the square of a metric is not a metric. The notions we develop will help delineate when the square of a metric is not a metric.

This section of the article is organized as follows: First, we describe some general properties of absolute differentiability and strong absolute differentiability, then we discuss connections to classical notions of differentiability and derivatives. In particular, we have a result for absolute derivatives of functions from $\mathbb{R}^n$ into $\mathbb{R}^m$ that in a sense "parallels" the Cauchy-Riemann equations for functions of one complex variable.

Next, we give examples of spaces and functions for which zero absolute derivative does not imply that the function is constant, and then investigate when one can infer that all functions with a zero absolute derivative are constant; this leads naturally to a new class of metric spaces, namely *rectifiably connected* spaces, in which any two points can be joined by a segment of finite length. Specifically we show that if the domain is rectifiably connected, and the function in question has a zero absolute derivative, then the function is constant.

In section 4 we use the concept of a semi-rigid space to give sufficient conditions for continuity to imply absolute differentiability for all functions on a given metric space.

Finally, in section 5, we relate absolute differentiability to Hausdorff dimension. For example, this section culminates in a result which implies that, if a function f is continuously absolutely differentiable and its absolute derivative vanishes nowhere, then f preserves Hausdorff dimension for compact subsets of its domain.

1.3. **Rationale.** The uses of derivatives in classical mathematics and its applications are many and varied. However, many of them relate to the geometric properties of subsets of the domain of a function, or to the geometric properties of the codomain, or to geometric properties of the graph of the function itself. For this reason, it is natural to expect that if one could devise a definition of derivative that makes sense for metric spaces, it would be of use in the study of geometric properties of such spaces, and the geometric properties would likely translate into useful information about functions that are differentiable in this new sense. This is the fundamental reason, or rationale, for one to study some sort of differentiation in metric spaces. As we shall explain later, it also will make sense to call our new notion "absolute derivative", instead of derivative, essentially because these new "derivatives" can, by their very nature, never be negative.

2. Definitions and Notation

We investigate a new notion of "absolute derivative" of functions defined on metric spaces, which measures how the distance changes in the image, relative to distance in the domain. Consider metric spaces (X, d_X) and (Y, d_Y), and a function $f : X \to Y$. We say that f is *absolutely differentiable* at a point $p \in X$, provided that the following limit exists in $\mathbb{R}$:

$$\lim_{x \to p} \frac{d_Y(f(x), f(p))}{d_X(x, p)}.$$

In this case, the above limit is called the *absolute derivative of f at p*, and is denoted by

$$f^{|\prime|}(p).$$

A somewhat stronger notion of absolute differentiability is obtainable by taking an appropriate limit over X^2 and Y^2: f is said to be *strongly absolutely differentiable* at p if and only if the following limit exists:

$$\lim_{\substack{(x,y) \to (p,p) \\ x \neq y}} \frac{d_Y(f(x), f(y))}{d_X(x, y)}. \tag{1}$$

Note that this is the type of definition needed to extend the notion of differentiability to the metric space setting, since subtraction is in general not available for use in the "difference quotients". In fact, it is natural to call the fraction

$$\frac{d_Y(f(x), f(y))}{d_X(x, y)}$$

a *distance quotient*, in this setting. We will show that with these notions of differentiability, geometric properties of metric spaces and functions between them are naturally related to conditions of absolute differentiability and strong absolute differentiability. Observe that in our definition of strong absolute differentiability, it is important that in the distance quotients we require $x \neq y$, as indicated in (1).

Of course, a function f is *absolutely differentiable* (respectively *strongly absolutely differentiable*) provided that it is absolutely differentiable (respectively strongly absolutely differentiable) at every point of its domain.

It is clear that every strongly absolutely differentiable function is absolutely differentiable, but the absolute value function is an example that demonstrates that these notions do not coincide.

Recall from [D] that a function $f : [0, \infty) \to [0, \infty)$ is *metric preserving* provided that for every set X and every metric d on X, the function $f \circ d$ also is a metric on X. It is an easy observation that in this case, the metrics d and $f \circ d$ are equivalent metrics if and only if f is continuous at 0.

3. General Properties of Absolute Differentiability and of Strong Absolute Differentiability

In this section, we describe various general properties of our notions of absolute derivatives in metric spaces. Specifically, in subsection 3.1, we discuss elementary properties and examples, such as when the value of the absolute derivative at a point can indicate that the function is locally one-to-one, examples of classically differentiable functions that are not at all absolutely differentiable, and the chain rule for absolute differentiation. In subsection 3.2, we connect our new concept with classical partial differentiation, by showing, for instance, that for $\mathbb{R}^m$-valued functions of n real variables, a variant of the Cauchy-Riemann equations is available. In subsection 3.3, we relate absolute differentiability to classical notions of differentiability - we demonstrate that continuous differentiability of a real-valued function of one real variable implies strong absolute differentiability of the function, while mere differentiability implies only absolute differentiability (all at a point p), and that for such functions f, the formula $f^{|\prime|} = |f'|$ holds. We give examples of functions that are absolutely differentiable but not differentiable, and (on the complex plane), a function that is nowhere differentiable but is everywhere absolutely differentiable, with absolute derivative equal everywhere to 1. Finally, in 3.4 we consider a stronger notion of connectedness than mere path-connectedness or arc-wise connectedness, in order to find sufficient conditions for the traditional calculus implication between having zero (in our case, absolute) derivative and being a constant function.

3.1. **Elementary Properties and Examples.** Here we will state and prove some theorems that elucidate general properties of our new notion of an absolute derivative. The relationship to continuity is natural:

Theorem 3.1. *Let $f : X \to Y$ be absolutely differentiable at a point $x \in X$. Then f is continuous at x.*

Next, we explore what happens if the absolute derivative is nonzero. First, we show a generalization of the classical result that positive derivative implies monotone increasing behavior and negative derivative implies monotone decreasing behavior:

Theorem 3.2. *Let $f : X \to Y$ be strongly absolutely differentiable at a point $x \in X$, and suppose that $f^{|\prime|}(x)$ is nonzero. Then f is locally one-to-one at x.*

PROOF. Since f is strongly absolutely differentiable at x, let V be an open neighborhood of x such that for all $y, z \in V$ with $y \neq Z$, $\frac{d_Y(f(y),f(z))}{d_X(y,z)} > \frac{f^{|\prime|}(x)}{2} > 0$. Then it is clear that f is one-to-one on V. □

Similarly, we easily have the following:

Theorem 3.3. *If $f : X \to Y$ is absolutely differentiable at $x \in X$, and if the absolute derivative of f at x is nonzero, then there is a neighborhood V of x such that for all $y \in V \setminus \{x\}$, we have $f(y) \neq f(x)$.*

This is the best we can do when the function is not strongly absolutely differentiable: There exist functions, for example, the absolute value function on the reals, that are absolutely differentiable with nonzero absolute derivative at a point, but which are not locally one-to-one at that point.

To see the difference, in $\mathbb{R}^n$, between absolute differentiability and differentiability, consider the following example:

Example 3.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the formula $f(x, y) = (x, 2y)$. Note that*

$$\lim_{(x,0)\to(0,0)} \frac{|f(x,y) - f(0,0)|}{|x - 0|} = 1$$

and that

$$\lim_{(0,y)\to(0,0)} \frac{|f(x,y) - f(0,0)|}{|y - 0|} = 2$$

so that f is not absolutely differentiable; however, it is of course, differentiable, in the traditional sense.

We note that for our notion of absolute derivative and strong absolute derivative, a chain rule holds, and the proof of the corresponding theorem is completely analogous to the corresponding proof in traditional calculus:

Theorem 3.4. *Let X, Y, and Z be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be (strongly) absolutely differentiable functions. Then the composite function $g \circ f : X \to Z$ is also (strongly) absolutely differentiable, and*

$$(g \circ f)^{|\prime|}(x) = g^{|\prime|}(f(x))f^{|\prime|}(x).$$

It is interesting to note that strong absolute differentiability is closely related to continuity of the absolute derivative, as we see in the following. The authors are indebted to a gracious referee for furnishing corrected versions of the calculations in the following argument.

Theorem 3.5. *Let $f : X \to Y$ be absolutely differentiable. If $x_0 \in X$ and f is strongly absolutely differentiable at x_0, then the absolute derivative of f is continuous at x_0.*

Proof. We work in the extended real number system for now. Assume that the hypothesis holds, but not the conclusion. Then two cases arise: Either some sequence x_n that converges to x_0 satisfies $f^{|\prime|}(x_n) \to \infty$ or some sequence x_n that converges to x_0 satisfies $f^{|\prime|}(x_n) \to L < \infty$, where $L \neq f^{|\prime|}(x_0)$. We leave the first case to the reader, and treat the second case. Let $\varepsilon > 0$ with $|L - f^{|\prime|}(x_0)| > 3\varepsilon$, and let $\delta > 0$ be such that if $x, y \in \mathcal{B}(x_0, \delta)$ are distinct, then

$$\left|\frac{d_Y(f(x), f(y))}{d_X(x, y)} - f^{|\prime|}(x_0)\right| < \varepsilon.$$

Let n be such that $x_n \in \mathcal{B}(x_0, \delta)$ and $|L - f^{|\prime|}(x_n)| < \varepsilon$. Let $\delta_1 > 0$ be such that if y is any member of $\mathcal{B}(x_n, \delta_1) \setminus \{x_n\}$, then

$$\left|\frac{d_Y(f(x_n), f(y))}{d_X(x_n, y)} - f^{|\prime|}(x_n)\right| < \varepsilon.$$

Let $y \in [\mathcal{B}(x_0, \delta) \cap \mathcal{B}(x_n, \delta_1)] \setminus \{x_n\}$. Then

$$\begin{aligned} \left|L - f^{|\prime|}(x_0)\right| &\leq \left|L - f^{|\prime|}(x_n)\right| + \left|f^{|\prime|}(x_n) - \frac{d_Y(f(x_n), f(y))}{d_X(x_n, y)}\right| \\ &+ \left|\frac{d_Y(f(x_n), f(y))}{d_X(x_n, y)} - f^{|\prime|}(x_0)\right| \\ &< 3\varepsilon, \end{aligned}$$

a contradiction. □

The converse of the preceding result fails. For example, the absolute value function on the real line is absolutely differentiable on its domain, and its absolute derivative (the constant function 1) is continuous, but of course the absolute value function is not strongly absolutely differentiable at 0.

3.2. **Connections with Classical Partial Differentiation.** In the introduction, we mentioned that there is an analogue, for our absolutely differentiable functions, of the Cauchy-Riemann equations for complex analytic functions. We state and prove this result here, even though it is not a result about general metric spaces and their absolutely differentiable functions:

Theorem 3.6. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is absolutely differentiable at $\vec{x}_0 \in \mathbb{R}^n$, and $f = (u_1, ..., u_m)$, where each $u_j : \mathbb{R}^n \to \mathbb{R}$ has first-order partial derivatives at $\vec{x}_0$, then for each $k \leq n$,*

$$f^{|\prime|}(\vec{x}_0) = \left|\left|\sum_{j=1}^{m} \frac{\partial u_j}{\partial x_k}\left(\vec{x}_0\right) \vec{e}_j\right|\right|.$$

PROOF. Let $\vec{x}_0 = \left(x_1^{(0)}, ..., x_n^{(0)}\right)$, and let $S = \left\{x_1^{(0)}\right\} \times ... \times \left\{x_{k-1}^{(0)}\right\} \times \mathbb{R} \times \left\{x_{k+1}^{(0)}\right\} \times ... \times \left\{x_n^{(0)}\right\}$. We have

$$\begin{aligned} f^{|\prime|}(\vec{x}_0) &= \lim_{\vec{x}\to\vec{x}_0, \vec{x}\in S} \frac{\left|\left|\sum_{j=1}^{m} u_j(\vec{x})\vec{e}_j - \sum_{j=1}^{n} u_j(\vec{x}_0)\vec{e}_j\right|\right|}{||\vec{x}-\vec{x}_0||} \\ &= \lim_{\vec{x}\to\vec{x}_0, \vec{x}\in S} \frac{\left|\left|\sum_{j=1}^{m} \left[u_j(\vec{x}) - u_j(\vec{x}_0)\right]\vec{e}_j\right|\right|}{\left|x_k - x_k^{(0)}\right|} \\ &= \lim_{\vec{x}\to\vec{x}_0, \vec{x}\in S} \left|\left|\frac{\sum_{j=1}^{m} \left[u_j(\vec{x}) - u_j(\vec{x}_0)\right]}{\left|x_k - x_k^{(0)}\right|}\vec{e}_j\right|\right| \\ &= \left|\left|\sum_{j=1}^{m} \frac{\partial u_j}{\partial x_k}\left(\vec{x}_0\right) \vec{e}_j\right|\right|. \end{aligned}$$

Note that here we mean by $\vec{e}_j$ the j^{th} vector in the standard ordered basis for $\mathbb{R}^m$. □

The following example shows that absolute differentiability does not imply continuity of partial derivatives.

Example 3.2. *Define* $\varphi : \mathbb{R} \to \mathbb{R}$ *by*

$$\varphi(t) = \begin{cases} t^2 \sin\left(\frac{1}{t}\right) & \text{for } t \neq 0 \\ 0 & \text{for } t = 0 \end{cases}.$$

Then set $f(x, y) = (\varphi(x), \varphi(y)) = (u(x, y), v(x, y))$, *for* $(x, y) \in \mathbb{R}^2$. *The function* f *is absolutely differentiable at the origin of the plane, and the component functions* u *and* v *have first-order partial derivatives at the origin, but* u_x *and* v_y *are not continuous at* $(0, 0)$.

3.3. **Connections to Classical Notions of Differentiability and Derivatives.** In the real line and the complex plane, as well as in many other classically important settings, a metric is available, so we can consider absolute differentiability in that context. In fact, the last result in the previous section deals with one such case. Here we focus on these connections to classical notions of differentiability. The following two results relate differentiability and (strong) absolute differentiability on the real line, and in the complex plane, and help the reader understand why we call our notion "absolute" derivative:

Proposition 3.1. *If $X \subseteq \mathbb{R}$ and $p \in X$ is a limit point of X, then for any $f : X \to \mathbb{R}$, we have the following:*

(1) *If f is continuously differentiable at p then f is strongly absolutely differentiable at p, and in this case,*

$$f^{|\prime|}(p) = |f'(p)|.$$

(2) *If f is differentiable at p then f is absolutely differentiable at p, and in this case,*

$$f^{|\prime|}(p) = |f'(p)|.$$

The proof of the above is a straightforward argument, and a mild revision of it yields the next one, so both are left as exercises for the reader.

Proposition 3.2. *If $X \subseteq \mathbb{C}$ and $z_0 \in X$ is a limit point of X, and if $f : X \to \mathbb{C}$ is complex-analytic at z_0, then f is strongly absolutely differentiable at z_0, and $f^{|\prime|}(z_0) = |f'(z_0)|$.*

But there are absolutely differentiable functions on the real line that are not strongly absolutely differentiable. For example, recall that the absolute value function is such a function. On the other hand, on the complex plane, there are functions that are nowhere differentiable, but everywhere strongly absolutely differentiable. A canonical example of such a function is complex conjugation. (Note that complex conjugation is an isometry, and all isometries on metric spaces are everywhere strongly absolutely differentiable, with absolute derivative 1.)

Conversely, one might like to know when strong absolute differentiability implies differentiability. The following theorem and example address this issue. The referee's insightful observations led the authors to significantly revise the argument for the following result, for which we are quite appreciative.

Theorem 3.7. *Let $X \subseteq \mathbb{R}$ be connected and closed, and let $f : X \to \mathbb{R}$ be both strongly absolutely differentiable at a point $x_0 \in X$ and continuous sufficiently close to x_0. Then f is differentiable at x_0, and of course,*

$$f^{|\prime|}(x_0) = |f'(x_0)|.$$

Proof. Assume that f is not differentiable at x_0, say x is a sequence such that $x_n \to x_0$ strictly, but the sequence $\left\{\frac{f(x_n)-f(x_0)}{x_n-x_0}\right\}_{n\in\omega}$ does not converge. Since f is (strongly) absolutely differentiable at x_0, let $p \in (0,\infty)$ satisfy

$$p = \lim_{n\to\infty} \frac{|f(x_n)-f(x_0)|}{|x_n-x_0|}.$$

Let k, j be strictly increasing sequences of positive integers such that

(1) $x_{k_n} \to x_0$ and $x_{j_n} \to x_0$ monotonically and strictly,
(2) $\lim_{n\to\infty} \frac{f(x_{k_n})-f(x_0)}{x_{k_n}-x_0} = p$ and $\lim_{n\to\infty} \frac{f(x_{j_n})-f(x_0)}{x_{j_n}-x_0} = -p$.

We consider two cases:

Case 1: $x_{k_n}, x_{j_n} < x_0$ or $x_{k_n}, x_{j_n} > x_0$. Without loss of generality assume the latter. For each positive integer n, let I_n denote the interval $[x_{k_n}, x_{j_n}] \cup [x_{j_n}, x_{k_n}]$, and define $\varphi : X \to \mathbb{R}$ by $\varphi(t) = \frac{f(t)-f(x_0)}{t-x_0}$. We will apply φ on the intervals I_n. Let N be a positive integer such that

$$n > N \implies f(x_{j_n}) < f(x_0) < f(x_{k_n}).$$

By the Intermediate Value Theorem, let t be a sequence such that

(1) $n > N \implies t_n \in I_n$ and
(2) $n > N \implies \varphi(t_n) = 0$.

Then $\varphi \circ t \to 0$, contrary to the assumption that $p > 0$.

Case 2: $x_{k_n} < x_0 < x_{j_n}$ or $x_{j_n} < x_0 < x_{k_n}$. Again, we assume the latter. Let $I_n = [x_0, x_{k_n}]$. Without loss of generality we assume that $f(x_{j_n}) \le f(x_{k_n})$ for all n. Define $\varphi_n(t) = \frac{f(t)-f(x_{j_n})}{t-x_{j_n}}$. For t sufficiently close to x_0, $\varphi_n(t) < 0$, while $\varphi_n(x_{k_n}) \ge 0$, so by continuity of φ_n and the Intermediate Value Theorem, let $t_n \in I_n$ satisfy $\varphi_n(t_n) = 0$. Since f is strongly absolutely differentiable at x_0, we have $\lim_{n\to\infty} \varphi_n(t_n) = 0$, contrary to the assumption that $p > 0$. □

On the other hand, strong absolute differentiability does not imply differentiability, even on the real line, as the following example shows. Thus the connectedness assumption in the preceding theorem is essential.

Example 3.3. *There are a closed set $X \subseteq \mathbb{R}$, and a continuous function $f : X \to \mathbb{R}$ such that for some $x_0 \in X$, f is strongly absolutely differentiable at x_0, but is not differentiable at x_0.*

Proof. To see this, let $x_0 = 0$, let $p_0 = (0,0)$, let $p_1 \in \{(a,a)|a > 0\}$, and let $q_1 \in \{(a,-a)|a > 0\}$, be such that the slope of the line $\overline{p_1q_1}$ is $1 + \frac{1}{2} = \frac{3}{2}$. For each $n > 0$, let p_{n+1} be on the segment $\overline{p_0p_n}$ such that the slope of the segment $\overline{q_np_{n+1}}$ is $-1 - \frac{1}{2n+1}$, and then let q_{n+1} be on the segment $\overline{p_0q_n}$ so that the slope of the segment $\overline{p_{n+1}q_{n+1}}$ is $1 + \frac{1}{2n+2}$. Let $f = \{p_k | k \geq 0\} \cup \{q_k | k > 0\}$, and let X be the domain of f. Then X, $x_0 = 0$, and f possess the desired properties. □

However, for real-valued functions on an interval in the reals, strong absolute differentiability and continuous differentiability coincide:

Theorem 3.8. *Let I be an interval in the real line, and let $f : I \to \mathbb{R}$. Then f is strongly absolutely differentiable on I if and only if f is continuously differentiable on I.*

Proof. We have already seen that if f is continuously differentiable at a point, then it is strongly absolutely differentiable at that point. Thus only the converse remains to be shown. Thus suppose that f is strongly absolutely differentiable on I. Then f is continuous on I, since it is absolutely differentiable on I. Also, f is differentiable on I, and $f^{|\prime|} = |f'|$, and we know that $|f'|$ is continuous on I. Suppose that f' is not continuous at some $x_0 \in I$. Let x_n and y_n, $n > 0$, be sequences in I that converge to x_0, for which $f'(x_n) \to f'(x_0)$, and $f'(y_n) \to -f'(x_0) \neq f'(x_0)$. By the intermediate value property for derivatives of real-valued functions on an interval (Darboux's theorem), let z_n, $n > 0$ be a sequence such that for each n, z_n is between x_n and y_n, and $f'(z_n) = 0$. Then continuity of $|f'|$ implies that $f'(x_0) = 0$. But this is a contradiction. □

3.4. **When Does Zero Absolute Derivative Imply that the Function is Constant?** As is well known among students of calculus, any real-valued function defined on the real line for which the derivative is zero must be a constant function. However, as the authors have observed previously (in [CI]), more general settings, such as that for functions on an arbitrary topological field, admit the existence of functions one may refer to as "pseudo-constants" (because they have a zero derivative everywhere), which are nowhere locally constant, a term we will not explain in detail here. The same is true here: There are metric spaces X, and functions f defined on X, such that f is absolutely differentiable everywhere on X, with absolute derivative identically zero, but for which f is not constant in any neighborhood of any point of X. But we may prescribe a condition on the space X which guarantees that every such "pseudo-constant" is actually constant. This new property of a metric space is the geometric property of being *rectifiably connected*.

Definition 3.1. *Let X be a metric space. We say that X is rectifiably connected provided that for any points a and b of X, there is a path of finite length from a to b.*

Definition 3.2. *Let p be a point in a metric space X, and let $B \subseteq X$ be a ball centered at p in X. Let C be the component of B that contains p. Then we call C the central component of B.*

We can prove the following result. Our original argument only yielded the desired conclusion for strongly absolutely differentiable functions, and the referee noticed a gap in the argument. We are very grateful for this, as we then discovered a much better argument that applies, as we originally intended, to all absolutely differentiable functions on rectifiably connected spaces.

Theorem 3.9. *Let X be a rectifiably connected metric space, and let Y be any metric space, with $f : X \to Y$ absolutely differentiable. If $f^{|\prime|} = 0$, then f is constant.*

PROOF. Suppose not, and let $x_0, x_1 \in X$ with $f(x_0) \neq f(x_1)$. Let A be a rectifiable arc from x_0 to x_1 in X, with length $L > 0$, and let

$$\varepsilon = \frac{d_Y(f(x_0), f(x_1))}{L}.$$

Because f is absolutely differentiable, for each $x \in A$, let $r_x > 0$ be such that for all $y \in \mathcal{B}(x, r_x)$,

$$\frac{d_Y(f(x), f(y))}{d_X(x, y)} < \frac{\varepsilon}{2}.$$

Let $\mathcal{B} = \{\mathcal{B}(x, r_x) | x \in A\}$. Now, A is compact and locally connected (so that components of open subsets of A are open), and so the collection

$$\mathfrak{S} = \{C | C \text{ is a central component of some member of } \mathcal{B}\}$$

is an open cover of A, so let $\mathcal{C} = \{C_1, ..., C_m\} \subseteq \mathfrak{S}$ be a finite cover of A. Without loss of generality, we may assume that these central components C_1, ..., C_m are chosen so that their respective centers $c_1 \in C_1$, ..., $c_m \in C_m$ are ordered along the arc A in the direction from endpoint x_0 to endpoint x_1. Since each member of $\mathcal{C}$ is connected, it follows that for each $j \in \{1, ..., m\}$, $C_j \cap C_{j+1} \neq \emptyset$, i.e. $\mathcal{C}$ is a chain of the arc A.

Let $p_0, p_1, ..., p_m \in A$ satisfy the following:

(1) $p_0 = x_0$ and $p_m = x_1$
(2) for each $j \in \{1, 2, ..., m-1\}$, $p_j \in C_j \cap C_{j+1}$ and p_j separates c_j from c_{j+1}

Then for each j,

$$d_Y(f(p_j), f(c_{j+1})) + d_Y(f(c_{j+1}), f(p_{j+1})) < \varepsilon(d_X(p_j, c_{j+1}) + d_X(c_{j+1}, p_{j+1}),$$

so that

$$\begin{aligned} d_Y(f(x_0), f(x_1)) &= d_Y(f(p_0), f(p_m)) \\ &\leq \sum_{j=0}^{m-1} (d_Y(f(p_j), f(c_{j+1})) + d_Y(f(c_{j+1}), f(p_{j+1}))) \\ &< \sum_{j=0}^{m-1} \varepsilon(d_X(p_j, c_{j+1}) + d_X(c_{j+1}, p_{j+1}) \\ &\leq L\varepsilon = d_Y(f(x_0), f(x_1)), \end{aligned}$$

a contradiction. The desired result follows. □

To see that rectifiable connectivity of the domain is essential, recall first that for any positive real number $p < 1$, the p^{th} power function is a metric-preserving function, and then observe the following:

Example 3.4. *Let $p < 1$ be a positive real number, let (X, d) be a metric space, and let $f : (X, d^p) \to (X, d)$ be the identity function on the set X. Then f is absolutely differentiable, and $f^{|\prime|} = 0$.*

As a consequence, it follows that if $q > 1$ in $\mathbb{R}$, and if d is a metric on a set X that makes (X, d) rectifiably connected, then d^q is not a metric!

4. When does Continuity Imply Strong Absolute Differentiability?

We use here the terminology of [Tr]. Let X be a topological space, and let $p \in X$. Then p is a *rigid point* of X if every continuous $f : X \to X$ with $p \in f[X]$ is constant or the identity. The space X is *semi-rigid* if it has a rigid point. The space X is *rigid* if every point of X is rigid. Observe then that, trivially, every continuous self-map of a rigid space is strongly absolutely differentiable. The following result shows that this can occur when the space is not rigid.

Theorem 4.1. *There is a non-rigid, semi-rigid metric space (X, d) such that every continuous function $f : X \to X$ is strongly absolutely differentiable.*

Proof. Let $C \subseteq \mathbb{R}^3$ be the cone over $[0,1]^2 \times \{1\}$, with vertex $v = (0,0,0)$, and let R, S be rigid, arc-like (so one-dimensional and embeddable in $\mathbb{R}^2$) continua with endpoints (for the existence of such continua, see [Co] and [M]), and let $Y = R \uplus (S \setminus \{p,q\}) \uplus R$ be the disjoint union of R, $(S \setminus \{p,q\})$, and R, where p and q are endpoints of S, and compactified so that Y is an arc-like continuum (i.e. a copy of R replaces p and a copy of R replaces q). Let R_0 be a copy of R embedded in the square $[0,1]^2 \times \{1\}$, let R_1 be $\frac{1}{2}R_0 = \{\frac{1}{2}x | x \in R_0\}$, and let

$\varphi_{0,0} : R \to R_0$ and $\varphi_{0,1} : R \to R_1$ be homeomorphisms. Let $C_0 = \{(x_1, x_2, x_3) \in C \mid \frac{1}{2} \leq x_3 \leq 1\}$, and for each $k \in \mathbb{N}$, let $C_{k+1} = \frac{1}{2}C_k$. Let $\varphi_0 : Y \to C_0$ extend $\varphi_{0,0} \cup \varphi_{0,1}$ so that Y is embedded by φ_0 into C_0. Let $X_0 = \varphi_0[Y]$. For each $k \in \mathbb{N}$, let $X_{k+1} = \frac{1}{2}X_k$. Then let $X = \{v\} \cup \bigcup_{k=0}^{\infty} X_k$, with the subspace topology and metric it inherits from $\mathbb{R}^3$. The resulting metric space, (X, d), has the desired properties. (In fact, every continuous self-map f of X is defined by $f(x) = \frac{1}{2^k}x$, for some $k \in \mathbb{N}$.) □

5. Absolute differentiability and Hausdorff Dimension

In this section, we explore the relationship between strong absolute differentiability and the Hausdorff dimension of a space. In particular, we give conditions under which a strongly absolutely differentiable function preserves Hausdorff dimension.

5.1. **Preliminary notions.** We refer to [B] for the following definitions and notation.

Definition 5.1. *Let A be a subset of our metric space X, let ε be a positive real number, and let p be a nonnegative real number. Let $\mathcal{A} = \{\{A_i\}_{i=1}^{\infty} | A = \bigcup\{A_i\}_{i=1}^{\infty}\}$. Then*

$$\mathcal{M}(A, p, \varepsilon) = \inf\left\{ \sum_{i=1}^{\infty} (\mathrm{diam(A_i)})^p \,\middle|\, \{A_i\}_{i=1}^{\infty} \in \mathcal{A} \text{ and } i \in \mathbb{N} \setminus \{0\} \Rightarrow \mathrm{diam(A_i)} < \varepsilon \right\}.$$

Definition 5.2. *Let $A \subset X$, and $p \geq 0$. Then the p-dimensional Hausdorff measure of A is*

$$\mathcal{M}(A, p) = \sup\{\mathcal{M}(A, p, \varepsilon) | \varepsilon > 0\}.$$

Definition 5.3. *Let $A \subseteq X$. Then the Hausdorff dimension of A is (see page 198 of [B]) the number $D_H(A)$ such that*

$$\mathcal{M}(A, p) = \begin{cases} \infty & \text{if } p < D_H(A) \\ 0 & \text{if } p > D_H(A) \end{cases}.$$

Theorem 5.1. *Assume that the metric space X is compact, and let $C \in (0, \infty)$. Let $f : X \to Y$ be a function that is strongly absolutely differentiable on X, and* *satisfies* $f^{|\prime|} < C$. *Then there is* $\varepsilon > 0$ *such that for any* $p, q \in X$, *if* $d_X(p,q) < \varepsilon$, *then*

$$d_Y(f(p), f(q)) < Cd_X(p,q).$$

Proof. For each $p \in X$, let $\varepsilon_p > 0$ be such that for all $x, y \in X$ with $d_X(x,y) < \varepsilon_p$, $d_Y(f(x), f(y)) < Cd_X(x,y)$, and for each $p \in X$, let U_p be the ball of radius ε_p about p. The collection $\mathcal{C} = \{U_p | p \in X\}$ forms a covering of the space X, so let ε be the Lebesgue number of $\mathcal{C}$. Then ε is the desired positive number for which if $p, q \in X$ and $d(p,q) < \varepsilon$, then

$$d_Y(f(p), f(q)) < Cd_X(p,q).$$

□

In a similar manner, we may prove the following:

Theorem 5.2. *Assume that the metric space* X *is compact, and let* $C \in (0, \infty)$. *Let* $f : X \to Y$ *be a function that is strongly absolutely differentiable on* X, *and satisfies* $f^{|\prime|} > C$. *Then there is* $\varepsilon > 0$ *such that for any* $p, q \in X$, *if* $d_X(p,q) < \varepsilon$, *then*

$$d_Y(f(p), f(q)) > Cd_X(p,q).$$

As a consequence, we have the following result:

Theorem 5.3. *Let* $C \in (0, \infty)$, *and let* X *be any metric space, with* $f : X \to Y$ *strongly absolutely differentiable on some compact subset* A *of* X. *Then we have*

(1) *if* $f^{|\prime|} < C$ *on* A, *then* $\mathcal{M}(f[A], p) < C^p \mathcal{M}(A, p)$, *and*
(2) *if* $f^{|\prime|} > C$ *on* A, *then* $\mathcal{M}(f[A], p) > C^p \mathcal{M}(A, p)$.

Proof. For (1), it is enough to observe that by Theorem 5.1, we have that $P \subseteq X$ implies diam(f[P]) $\leq$ Cdiam(P) and use the definition of $\mathcal{M}(A, p)$. Similarly, for (2), we use Theorem 5.2. □

Corollary 5.1. *Let* X *be any metric space, with* $f : X \to Y$ *strongly absolutely differentiable on some compact subset* A *of* X.*Then we have*

(1) *if for some* $C \in (0, \infty)$, $f^{|\prime|} < C$ *on* A, *then* $D_H(f[A]) \leq D_H(A)$,
(2) *if for some* $C \in (0, \infty)$, $f^{|\prime|} > C$ *on* A, *then* $D_H(f[A]) \geq D_H(A)$,
(3) *if for some* $C_1, C_2 \in (0, \infty)$, $C_1 < f^{|\prime|} < C_2$ *on* A, *then* $D_H(f[A]) = D_H(A)$, *and*
(4) *if on* A, $f^{|\prime|}$ *is continuous and nowhere zero, then* $D_H(f[A]) = D_H(A)$.

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References

[AT] L. Ambrosio and P Tilli, *Topics on Analysis in Metric Spaces*, Oxford Lecture Series in Mathematics and its Applications, no. 25. Oxford University Press, 2004.

[B] M. F. Barnsley, *Fractals Everywhere*, second ed. Academic Press, Boston. 1993.

[C] P. Corazza, *Introduction to metric-preserving functions*, Amer. Math. Monthly **106**, (1999), no 4. pages 309-323.

[CIP] W. J. Charatonik, M. Insall, and B. Passer, *Differentiation in topological fields*, under review.

[Co] H. Cook, *Continua which admit only the identity mapping onto nondegenerate subcontinua*, Fund. Math. 60(1967), pages 241 - 249.

[D] J. Dobos, *A survey of metric preserving functions*, Quest. Answ. Gen. Top. 13(1995), no. 2. pages 129 - 134.

[H] J. Heinonen, *Nonsmooth calculus*, in Bull. Amer. Math. Soc. v. 44(2007), no. 2, pages. 163 - 232.

[M] T. Maćkowiak, *Singular arc-like continua*, Dissertationes Math. (Rozprawy Mat.) v. 257(1986), pages 1 - 40.

[P] C. C. Pugh, *Real Mathematical Analysis*, Springer, Berlin. 2003.

[Te] F. Terpe, *Metric preserving functions*, Proc. conf. on Top. and measure, 1983. pages 189 - 197.

[Tr] V. Trnkova, *Semi-rigid spaces*, in Trans. Amer. Math. Soc., v. 343, no. 1, May 1994. pages 305 - 325.

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