Assessing Understanding of Multiplication through Words, Pictures, and Numbers

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Abstract

The objective of this session is to engage mathematics teacher educators in a discussion of how to assess an understanding of the concept of multiplication as an operation and its relationships to other operations. The session will begin with a presentation of a previously published study assessing children's understanding of multiplication as grouping and the relationship between multiplication and addition. The assessment asked a series of problems involving words, pictures, and numbers. The results of the study indicate that the types of problems asked were successful in providing evidence of children's understanding of multiplication. The study also found that a group of third grade children had developed a better understanding of multiplication after just one multiplication unit from Investigations in Number, Data, and Space than a comparable group of fourth grade children had developed from an entire year of a traditional emphasis on memorizing multiplication facts. An interactive discussion of potential uses of this study and its assessment format in teacher education will follow the presentation

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Introduction

Although much research has shown various problems with curricular overemphasis on facts and skills, O'Brien and Casey (1983a, b) specifically demonstrated that many children who have experienced a "back to basics" curriculum "do not know what multiplication is. They have algorithmic skill but no mathematical knowledge of multiplication" (1983a, p. 250). Nonetheless, many parents and teachers continue to consider memorizing basic facts as the hallmark and primary goal of school mathematics, with a particular emphasis on memorizing multiplication facts during third grade. In contrast, the *Principles and Standards* (National Council of Teachers of Mathematics, 2000) argued that "learning mathematics with understanding is essential" and that research shows "the alliance of factual knowledge, procedural proficiency, and conceptual understanding makes all three components usable in powerful ways" (p. 20).

In a previously published article (Smith & Smith, 2006), from which this presentation is derived, we described an assessment of conceptual understanding of multiplication through words, pictures, and numbers used with two different groups of students at one school. These two groups of students were (a) a group of third-graders from a standards-based classroom immediately following their first unit on multiplication from *Investigations in Number, Data, and Space* (Tierney, Berle-Carman, & Akers, 1998; hereafter referred to as *Investigations*), and (b) a group of fourth graders at the same school who had received a traditional third-grade mathematics experience and had been "certified" by their teacher as having memorized all the traditional multiplication facts.

Conceptual Understanding of Multiplication

"Understand meanings of operations and how they relate to one another" is one of the *Principles and Standards'* three major themes for prekindergarten through Grade 12, and many researchers have explored the details of multiplication and how conceptual understandings of it develop (Greer, 1992; Harel & Confrey, 1994; Hiebert & Behr, 1988; Sowder et al., 1998). This research indicates that, although children typically develop additive reasoning quite naturally, multiplication is much more complex than addition and requires guidance to understand the new units and actions involved in multiplicative situations. Focusing on the operation of multiplying two numbers or memorizing multiplication facts before developing an understanding of multiplicative situations and their quantities prematurely narrows students' focus and gives students the wrong impression about the need to understand what it means to multiply and the situations in which multiplying is the appropriate thing to do.

To be able to assess students' understandings of multiplication concepts, one needs to consider the specific details of what it means to understand multiplication, how these understandings can be easily and effectively demonstrated, and how to interpret students' performances as evidence of understanding.

Elements of Understand Multiplication?

Building an understanding of the concept of multiplication requires developing a language for thinking about and describing multiplicative situations involving equal groups of quantities. In the most basic form, a multiplicative situation describes A equal groups with B things in each group, where the total number of things is equal to A x B. From our review of the literature on understanding multiplication, we have focused on five basic and interconnected concepts: (a) quantity, (b) multiplicative problem situations, (c) equal groups, (d) units relevant to discrete multiplication, and (e) how multiplicative situations differ from additive situations. Most of these understandings can develop from experiences using counting and grouping strategies to solve contextualized problems in the early grades.

Understanding quantity. The meaning of quantity often gets overlooked in addition situations, but a thorough understanding of quantity provides an important foundation for understanding multiplication. A quantity is a characteristic of objects that can be counted or measured, and the value of a quantity consists of a *number* and a *unit* (Center for Research in Mathematics and Science Education, 1998). Twelve pennies is an example of a quantity—it includes both a number (12) and a unit (pennies). Number names (e.g., twelve) are often used to describe the number portion of a quantity. Other representations for the number part of a quantity include pictures (e.g., 12 circles representing 12 pennies) and the numerals 0-9 arranged in a base-10 place-value system (e.g., 12 representing twelve or 109 representing one hundred nine). In addition to the numbers in these examples, a unit must be specified to know the complete quantity. Neither the number twelve nor the numeral 12 tells *what* is being quantified. Although a pictorial representation of a quantity explicitly shows one possible unit, such as 12 circles, this unit may be representing a different unit, such as 12 pennies or 12 round cans that contain some quantity of yet another item. In each case, both the number and the units must be clearly represented in words, numbers, or pictures to completely specify the quantity.

Understanding multiplicative problem situations. Students need to have sufficient experience figuring out the meaning of word problems describing multiplicative situations to make sense of those situations and to distinguish them from other situations suggesting addition, subtraction, or division operations. Students also need to understand the relationships between multiplication and division and be able to find each of the three possible unknown quantities in grouping/partitioning situations (e.g., given 24 cookies arranged in four bags of six cookies each, three different problems can be posed by providing any two of these three pieces of information and asking for the third). Carpenter, Fennema, Franke, Levi, & Empson (1999) refer to these three problem types as multiplication (the number of groups and the number in each group are known but the total is unknown), measurement division (the total and the number in the groups is known but the number of groups is unknown), and partitive division (the total and the number of groups is known but the number in each group is unknown). Meanings of conventional multiplication notation (e.g., $4 \times 6 = 24$) should be connected to the language and meanings of multiplicative situations and their units.

Understanding equal groups. Students need to have sufficient experience arranging objects into groups to understand the role of equal groups in multiplicative situations and to establish a motivation for multiplying equal groups instead of counting all of the objects in the problem. Number sense includes the ability to compose and decompose numbers. Multiplicative reasoning includes using factors and multiples as equal groups in composition and decomposition of numbers instead of using additive compositions. For example, six objects can be arranged into multiplicative groups (e.g., one group of six, two groups of three, three groups of two, or six groups of one) rather than additive groups (e.g., one and five, two and four, three and three, and six and zero). Visual images are particularly helpful in understanding grouping (e.g., the difference between a disorganized collection of 48 items and the same 48 items organized into 4 groups of 12 items or an array of 6 rows and 8 columns).

Understanding units relevant to multiplication. Students need to have sufficient experience with counting and arranging objects into groups to understand the differences between various kinds of units that are relevant to multiplication (as distinct from units that apply to additive situations), particularly the difference between singleton units (e.g., ones, donuts, or cents) and composite units (e.g., twos, fives, tens, dozens, or rows of x and columns of y in an x by y array).

Students need to understand that composite units can also be counted (e.g., the number 30 is three 10s as well as the counting number after 29).

Understanding the relationships between multiplication and addition. Addition most often involves the joining of unequal quantities of the same unit (e.g., adding 29 cents and 54 cents). However, the two numbers in a multiplication situation typically refer to different units (e.g., multiplying 29 cats by four legs for each cat) representing the number of equal groups and the number of items in each of the groups. Understanding the connections between multiplication and addition should include knowing how a product is related to a sum from the repeated addition of the number of items in the equal groups.

To develop flexibility in both additive and multiplicative reasoning, students need to have sufficient experience with counting, joining, and grouping processes to understand the differences involved in moving from addition to multiplication. Developing an understanding of the iterative process of multiplication can begin with skip counting or repeated addition (particularly with groups of 10), because these counting quantities themselves represent groupings that have equal numbers of the same units. Understanding the iterative process of multiplication at this informal level (number of groups and number in each group) can provide a foundation for understanding more formal definitions of various multiplicative structures.

Given these five elements of elementary multiplication, we can begin to think about assessing students' understanding of multiplication through developmentally appropriate tasks that are able to provide evidence of understanding of these interconnected concepts.

Assessment

For our assessment of children's understanding of multiplication, we started with a series of four different problems about 3 × 4 (see Table 1). These four problems were repeated three more times using increasing number combinations (5 × 8, 8 × 7, and 9 × 6). As shown in Table 1, these problems were intended to provide evidence of understanding in three different forms: words, pictures, and numbers.

Table 1. *Initial Series of Problems*

Problem #	Problem	Evidence Form
1.	3 × 4 = _____	Numbers
2.	Write a story problem for which 3 × 4 is the correct number sentence.	Words

3.	Draw a picture that shows 3 × 4.	Pictures
4.	Write an addition number sentence that shows 3 × 4.	Numbers

This series of problems addressed multiple goals for the assessment. We started with a conventional number sentence that we expected fourth-grade children would find familiar and for which they would quickly produce a correct answer. This problem attended to the goal of correct answers, whether through recall of facts, mental computations, or counting strategies. Next, we asked students to write a story problem for the number sentence to check their connections between multiplicative number sentences and problem situations, quantities, and units. These first two problems are similar to the approach taken by O'Brien and Casey (1983a), which was based on an earlier report by McIntosh (1979).

Next in this problem sequence, we asked students to draw a picture to allow us to examine their visual representations for multiplicative structures (most often shown as groupings with the same number of items in each group) and units, as distinct from those in additive relationships. We then asked students to write an addition number sentence that showed the same thing as the given multiplication number sentence to provide additional data about students' understandings of how multiplication and addition are related.

This collection of four problems created multiple opportunities for students to make connections between what they knew and the basic multiplication concepts (quantity, problem situations, grouping, units, and relationships with addition). Although the format of some of these problems might have been less familiar to students' whose curriculum consisted primarily of number facts, standard algorithms, and application word problems than to students experiencing the *Investigations* curriculum, this set of problems asked for a variety of representations to provide multiple opportunities to show understanding of basic multiplication concepts, particularly the most basic conception of multiplication as a number of equal groups. The absence of any evidence of these key ideas in students' responses would provide a compelling argument that they had not yet developed the desired basic understandings of multiplication.

We followed these 16 problems with 10 word problems to provide evidence of students' understanding of various multiplication and division situations and their functional facility in solving such word problems. We designed this collection of word problems using the various multiplication and division problem types identified in research on children's mathematical thinking (see Carpenter et al., 1999). We included multiplication, measurement division, and partitive division situations involving grouping/partitioning, rate, price, and multiplicative comparison. We used measurement division and partitive division problems in what was primarily a multiplication study to provide evidence of students' understandings of the structure and context of the problems and to illuminate careless decoding strategies that represent taking shortcuts in comprehending the words in the problem. The last four of these word problems used the same number combinations as the four multiplication number sentences we had given earlier in the interview (i.e., Items 1, 5, 9, and 13).

Data collected for this study included (a) students' written work created during the interview and (b) interviewer field notes about students' responses to probing questions. Probing questions included "How did you get that?" or "Can you tell me how your story problem [or picture, or number sentence] shows _______?" which the interviewer asked after students responded to each interview item.

Analysis of the data involved comparing correct answers within and across the two groups of students and analyzing students' written work and verbal comments during the interviews for evidence of understanding of the basic concepts of multiplication described above.

Results

Comparing Correct Answers

The comparative results for percent of correct answers on the interview problems are shown in Table 2. Also shown are the percents of immediate responses to the bare number sentences, defined as providing an answer within approximately 2 seconds (and attributed to recall of facts). Looking first at the results for the traditional group (Grade 4 students) we see that they immediately produced a correct answer 100% of the time only with 3×4. Only 70% of these students gave an immediate response to 5×8, and they provided fewer immediate responses and were less accurate as the products got larger (8×7 and 9×6), with only 20% immediate responses and 70% and 90% accuracy for the two largest products. In comparison, only 7% of the *Investigations* group (Grade 3 students) provided an immediate response to 3×4, and none of them gave an immediate response on the other three number sentences. However, this group was 100% accurate on all four of the bare number sentences.

Table 2. *Correct Answers by Group*

Bare Number Sentences and Conceptual Problems	**Grade 4 Traditional**		**Grade 3 *Investigations***	
	% Immediate Response	**% Correct Answer**	**% Immediate Responses**	**% Correct Answers**
1. 3 × 4 =	100	100	7	100
2. Write a story problem		10		73
3. Draw a picture		20		100
4. Write an addition number sentence		40		93
5. 5 × 8 =	70	100	0	100
6. Write a story problem		20		87
7. Draw a picture		40		100
8. Write an addition number sentence		40		100
9. 8 × 7 =	20	70	0	100
10. Write a story problem		20		93
11. Draw a picture		20		100
12. Write an addition number sentence		30		100
13. 9 × 6 =	20	90	0	100
14. Write a story problem		20		100
15. Draw a picture		20		100
16. Write an addition number sentence		40		100
Mean: Bare Number Sentences (Items 1, 5, 9, 13)	53	90	2	100
Mean: Conceptual Problems (Items 2-4, 6-8, 10-12, 14-16)		27		96
Mean: Word Problems (17-26)		84		96
Mean: All Items (1-26)		58		96

The traditional group struggled with the conceptual problems (write a story problem, draw a picture, and write an addition number sentence), and produced acceptable answers only 27% of the time. In contrast, the *Investigations* group produced acceptable answers on the conceptual problems 96% of the time. Interestingly, those students in the *Investigations* group who initially struggled with writing a story problem or an addition number sentence early in the interview performed better as the interview progressed. Generally, students in the traditional group who did not have a useful strategy for these conceptual problems early in the interview were unable to improve their performance as the interview progressed.

Table 2 also shows that the traditional students performed much better (84% correct) on the word problems (Items 17-26) than they did on the conceptual problems, although they still did not perform as well as the *Investigations* group (96% correct). Item 20 (a multiplicative comparison problem involving 3×6; *A giraffe is 3 times as tall as a man is. If the man is 6 feet tall, how tall is the giraffe?*) was the only item where the traditional group outperformed the *Investigations* group (100% versus 67%). Some of the students in the *Investigations* group struggled to make sense of the difficult structure of this problem, which asked them to identify the relation between two quantities where one quantity was a multiple of the other. This comparative relation is very different from the relation between a number of groups and the number of objects in each group (Carpenter et al., 1999). However, for the traditional group, this particular multiplicative comparison structure easily fit a key word strategy. All of the traditional students recognized the word *times* in the problem and successfully multiplied 3 times 6.

Table 3 compares means, modes, and standard deviations across the two groups of students for the three categories of items (bare number sentences, conceptual problems, and word problems), as well as for correct answers on all items. A one-tailed t-test of the differences in the means for the two groups of students in each of these categories shows that the correct answers for the *Investigations* group were significantly higher than for the traditional group only for the conceptual problems and that this difference in correct answers on the conceptual problems accounts for the significant difference in the total number of correct answers for these two groups.

Looking at scores and explanations for individual students, we noted that correct answers on this collection of word problems involved issues of understanding the multiplication and division situations as well as flexibility in problem solving strategies. Errors by students in the traditional group often resulted from not understanding the structure of the problem, not remembering the multiplication fact they needed, or retrieving an incorrect fact. When they could not correctly recall a particular multiplication fact, they did not attempt other, more reliable strategies for computing an answer to the problem. In particular, for Item 18 (a rate situation involving $21 \div 3$; *Sarah walks 3 miles an hour. How long will it take her to walk 21 miles?*), traditional students' typical strategy of looking for key words and searching for an appropriate operation produced many errors.

Table 3. *Correct Answers Statistical Comparison*

	4 Bare Number Sentences	**2 Concept Problem**	**10 Word Problems**	**Total Correct Answers**
Traditional Group				
Mean	3.60	3.20	8.40	15.20
Mode	4.00	0.00	10.00	7.00
Standard Deviation	0.70	4.08	2.22	5.53
Investigations **Group**				
Mean	4.00	11.47	9.60	25.07
Mode	4.00	12.00	10.00	26.00
Standard Deviation	0.00	0.92	0.51	1.10
***t*-Test Results**				
Mean Diff (*Invest*-Trad)	0.40	8.27	1.20	9.87
t Stat	1.8091	6.3068	1.6796	5.5656
P(T<=t) one-tail	0.0519	0.0000	0.0620	0.0002

Analyzing Conceptual Problems for Evidence of Understanding

Analyzing students' work involves looking for evidence of understanding and misconceptions in the products they produced and in their explanations of the work. This analysis of students' work is fundamentally different from counting the number of correct answers and computing percentages. Examining details of what students produce (or what they say during conversations) provides insights into their understandings of key concepts that cannot be inferred from percentages of correct answers alone.

Writing a story problem. First, consider some examples of the ways these students responded to the request to "Write a story problem for which __ × __ is the correct number sentence." Story problems were acceptable if they described a given number of groups with a given number of items in each group, totaling to the appropriate number.

Traditional students often wrote story problems using additive structures that had the same answer as the multiplication number sentence or simply followed its language. These typically used the same unit for both quantities, which is another indication of an additive structure. For example, traditional students wrote these story problems for 3 × 4:

- "Sue had 4 candles and Tamara had 8. How many did they have in all?"
- "Bobby had 4 baseball cards. He got 3 times as many as he had already. How many did he have in all?"

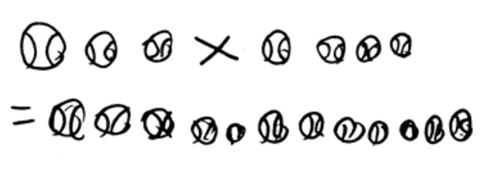
One traditional student wrote a story problem that began with an additive situation, recognized this was not correct, and then specified a multiplication operation: “Josh had 3 baseball cards and his friend had 4. ~~How many do they have in all?~~ How many would they have if they multiplided [sic] these numbers?”

Students in the *Investigations* group nearly always described a grouping situation, identified a number of groups, specified a number in each group, and asked for the total number of items. For example:

- “I had 4 boxes of doughnuts. Each had 3 doughnuts. How many doughnuts do I have?”
- “I have 5 fish. Each one gets 8 piecies [sic] of fish food. How many piecies [sic] do I have to have?”
- “I have 8 boxes of toys. Each box has 7 toys in it. How many toys do I have?”

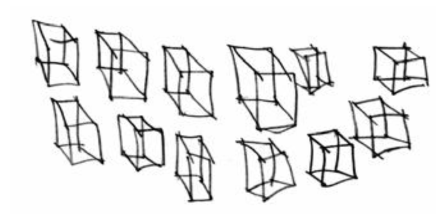
Drawing a picture. Traditional students’ responses to the request to “Draw a picture that shows __ × __” provided evidence of the same misconceptions about multiplication as their word problems: (a) multiplication is the structure of the multiplication number sentence, or (b) multiplication is the answer to a multiplication number sentence. Figure 1 shows four of these responses. Picture A shows a multiplication number sentence with a tree and grass added. Picture B shows baseballs in place of the numbers in a horizontal number sentence. Picture C shows stars in place of numbers in a vertical number sentence. Picture D shows the answer to 3×4 as 12 cubes without any grouping of the cubes.

Picture A



Picture B

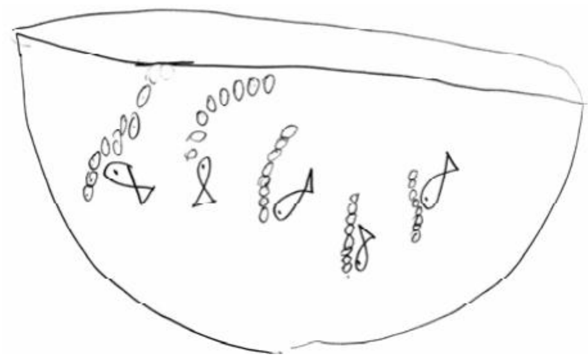
Picture C



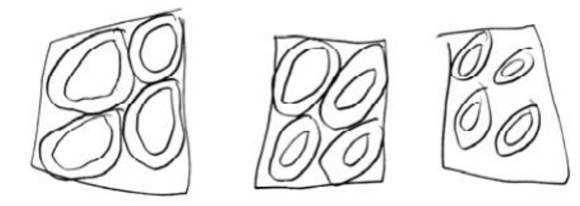
Picture D

Figure 1. *Pictures by Grade 4 Students in the Traditional Group*

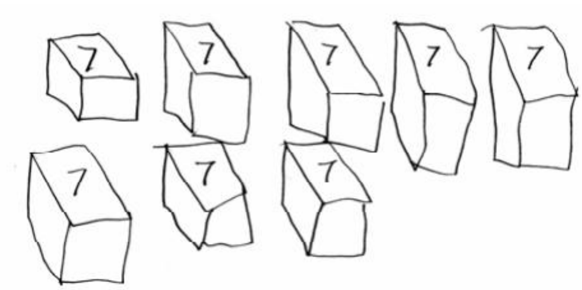
Figure 2 shows four examples that are typical of pictures by students in the *Investigations* group. Picture E shows five fish in a fish bowl, with each fish having eight pieces of fish food. Picture F shows 12 donuts arranged in three boxes of four donuts each. Picture G shows eight boxes labeled as having seven items in each box. Picture H shows 56 squares arranged in an array of eight rows and seven columns.



Picture E



Picture F



Picture G

Picture H

Figure 2. *Pictures by Grade 3 Students in the Investigations Group*

In every case, the pictures of students in the *Investigations* group showed understanding of multiplication as grouping, and their pictures represented the number of groups, the number in each group, and the product of the multiplication. With few exceptions, the pictures of students in the traditional group did not indicate an understanding of multiplication as grouping. Instead, their meaning of multiplication was limited to the number sentence and the answer. Those few who wrote word problems about equal groups also created pictures of equal groups.

Writing an addition number sentence. Many students in the traditional group wrote addition number sentences that totaled to the same sum as the product of the multiplication but did not include addends that were related in any way to a multiplication situation. For example, their responses to “Write an addition number sentence that shows 3 × 4” included the following:

- three times four   3 × 4 = 12
- 6 + 6 = 12
- 3 + 4 + 1 + 4 = 12
- 3 + 4 = 7

These students’ responses indicate that they did not understand enough about how multiplication is different from addition to see how the question was asking for more than the same answer. They knew how to write an addition number sentence, but did not indicate an understanding that both addition and multiplication number sentences can be used to show a number of groups with the same quantity in each group.

In contrast, nearly all of the students in the *Investigations* group wrote either 4 + 4 + 4 = 12 or 3 + 3 + 3 + 3 = 12 (or both), indicating a clear understanding of the applicability of an addition process to represent a situation involving equal groups. These responses most likely reflected these students’ experiences using repeated addition as a strategy for determining answers to multiplication problems. These differences in responses make clear that there can be differences between students’ understandings of the structure of a problem and their choices of strategies for solving the problem.

Conclusions

The Smith and Smith (1996) study shows that the collection of tasks used provided meaningful evidence for assessing children’s understandings of multiplication concepts, including understandings of the relationships between multiplication and addition. The study also provides evidence that memorizing multiplication facts produced much less understanding of the basic concepts of multiplication in a group of fourth-grade students receiving traditional instruction than a standards-based curriculum and instruction produced among a group of younger third-grade students. This is consistent with the broader claim that a focus on computational skills alone works against the development of the view that learning mathematics is a sense-making activity (Robinson, Robinson, & Maceli, 2000). These results also show under what curricular circumstances students have the opportunity to develop robust understandings of basic multiplication concepts, which contrasts with the findings of O'Brien and Casey (1983a, b) for students experiencing a traditional computation focus.

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