Area Based Fan Beam Projection Model for Computed Tomography

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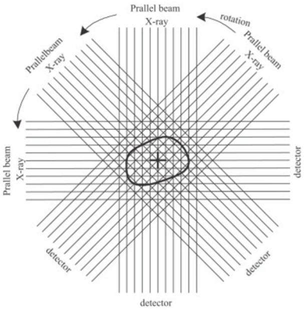
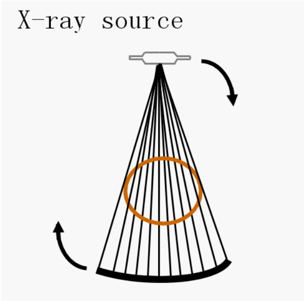
4. The emitter-detector pair are then rotated through an angular interval ϕ and the process is repeated.

Scanning methods

There are three main scanning methods in CT: parallel beam, fan beam, and cone beam.

Figure : Parallel and fan beams





Goal of CT

- The fundamental assumption of CT is that there exists an unknown function $f(x, y)$, which can be discrete or continuous, that describes the x-ray attenuation of an object through a particular plane.
- The goal of CT is to reconstruct this cross-sectional image accurately using as little projection data as possible.
- Different approaches are used to model the projection data and reconstruct the image.
- The most prevalent projection models treat the x-rays as infinitesimal lines, but in reality they have some finite width.
- Using an area based method, which takes into account the width of the x-rays, can increase the accuracy of the projection data.

Area Based Projection

- An area based projection representation was first mentioned in Kak's classical CT book [Kak 1988].
- Area based parallel beam projection models have been proposed by [Li & Zhu 2008, Zhu et al 2008].
- No papers have specifically dealt with an area based fan beam projection model.

Figure : Area based projection

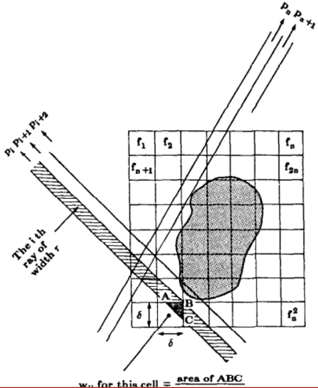


Image Reconstruction Algorithms

- There are various reconstruction algorithms in CT. The two major categories are analytical and algebraic approaches.
- Algebraic reconstruction involves solving linear systems of equations of the form:

$$Ax = b. \tag{1}$$

- Algebraic methods are superior to analytical when (1) posses a large amount of noise.
- The major drawback of algebraic reconstruction is the large computational load, however as computer performance has improved in the last few decades this method has become more widely utilized.

Algebraic Reconstruction in CT

Goal: solve the system (1), where:

- A$\in \mathbb{R}^{M\times N^2}$: each row of A corresponds to a beam of two x-rays at a particular rotation angle θ relative to the initial position. Each entry represents the fractional area that the beam covered of that square in the image.
- x: the image as a vector.
- b: the projection data vector.
- N: the size of the image.
- M: (# of beams) (# of x-ray resources).

ART and classical Cimmino

We will be using four algebraic reconstruction algorithms to test our model.

- The first is the algebraic reconstruction technique(ART) algorithm.

$$x^{(k+1)} = x^{(k)} + \lambda_k \frac{(b_i - a^i x^{(k)})}{||a^i||_2^2} (a^i)^T. \tag{2}$$

- The second is the classical Cimmino algorithm.

$$x^{(k+1)} = x^{(k)} + \lambda_k \sum_{i=1}^{M} w_i \frac{(b_i - a^i x^{(k)})}{||a^i||_2^2} (a^i)^T. \tag{3}$$

Compressed Sensing

- The theory of compressed sensing [Candes & Wakin 2008, Donoho 2008] has recently shown that signals and images that have sparse representations in some orthonormal basis can be reconstructed from much less data than what the Nyquist sampling theory requires [Shannon 1998].
- In many cases in tomography we can model the image as piecewise constant, such that the gradient,μ, is sparse. The image can then be reconstructed using total minimization of the gradient [Candes & Wakin 2008, Yu & Wang 2009].
- Then we can reconstruct the image by solving:

$$\min TV(|\mu|) \;\; \text{s.t.} \;\; Ax = b. \tag{4}$$

BCPCS and BCIMCS

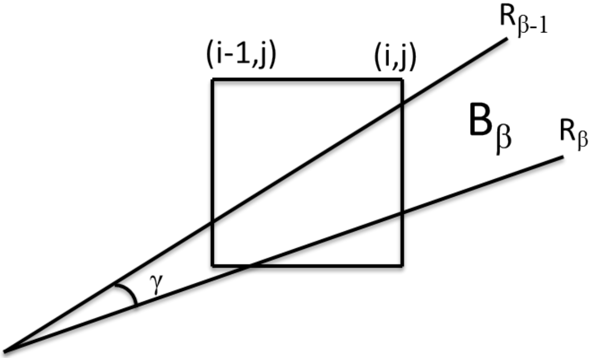
- The other two reconstruction algorithms we used here are block scheme compressed sensing based iterative algorithms proposed by [Li & Zhu 2010].
- The third algorithm is block cyclic projection for compressed sensing (BCPCS), which uses a block scheme based on ART.
- The last algorithm is block Cimmino for compressed sensing (BCIMCS), which uses a block scheme based on classical Cimmino.
- These last two algorithms apply block iterations and TV minimization alternatively to reconstruct the image.

Main Idea

- We start by creating an NxN grid with the object in the middle of the grid and coordinates (i, j) attached the top right corner of each pixel in the grid. This grid becomes the matrix Img.
- We assume that the emitter starts at a distance of **d** from the left hand corner of the grid along the 45° line, and let this situation correspond to θ=0.
- Now we need to calculate the area intersection formulas for this situation in order to find the entries of A.
- This calculation is eased by the fact that the fractional areas above the 45° line are a reflection of the ones below, hence we need only calculate the areas on one side.

We call the β^{th} beam below the 45° line, B_β. It has upper ray $R_{\beta-1}$ and lower ray R_β. Each ray is an angle γ away from the neighboring rays.

Figure : The β^{th} beam



Symmetry Theorem

Theorem: *If B_β covers area A within square (i, j), then $B_{-\beta}$ will cover the same area A within the square (j, i).*
Proof: The slope for each ray R_m is: $\tan(\frac{\pi}{4} - m\gamma)$.
So the slope for $\mathrm{R}_{-\beta}$ is $\tan(\frac{\pi}{4} + \beta\gamma)$
and the slope for R_β is $\tan(\frac{\pi}{4} - \beta\gamma) = \cot(\frac{\pi}{2} - (\frac{\pi}{4} - \beta\gamma)) = \cot(\frac{\pi}{4} + \beta\gamma) = 1/(\tan(\frac{\pi}{4} + \beta\gamma))$.
Thus R_β and $\mathrm{R}_{-\beta}$ have reciprocal slopes.

Proof Continued

Now R_β and $R_{-\beta}$ start from the same point, so if R_β goes through the point (x_1, y_1) then $R_{-\beta}$ must go through the point (y_1, x_1). Therefore if $R_{\beta-1}$ and R_β cover fractional area A in square (i, j), then $R_{-\beta+1}$ and $R_{-\beta}$ cover the same fractional area A of square (j, i).

$\square$

Thus calculating the fractional area formulas for the beams under the 45° line gives us the fractional area formulas for the beams above.

Image Rotation

- In reality the emitter-detector pair rotates around the object, however this means that new projection area equations will have to be formulated for each possible viewing angle.
- It is simpler, and equivalent, to consider that the object itself rotates while the emitter and detector remain stationary.
- This means that for the purposes of constructing A, we need only consider the area formulas calculated for the trivial case where the emitter is on the 45° line.

Intersection Cases

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- Suppose that we are looking that the i^{th} column of squares and that the beam in question has an upper ray $\mathrm{R}_{\beta-1}$ and a lower ray R_{β}.
- A and B are the intersection points of $\mathrm{R}_{\beta-1}$ with the vertical lines $i-1$ and i respectively.
- C and D are the intersection points of R_{β} with the vertical lines $i-1$ and i.
- E and F are where $\mathrm{R}_{\beta-1}$ and R_{β} intersect with one of the horizontal points of the grid.

The β^{th} beam has upper ray $R_{\beta-1}$ and lower ray R_β and γ is the angle between each ray.

$$A_y = (i-1+\frac{d}{\sqrt{2}}) \cdot \tan\left(\frac{\pi}{4} - (\beta-1)\cdot\gamma\right) - \frac{d}{\sqrt{2}}$$

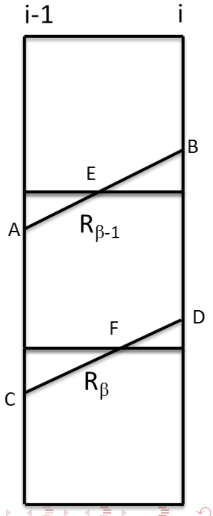
$$B_y = (i+\frac{d}{\sqrt{2}}) \cdot \tan\left(\frac{\pi}{4} - (\beta-1)\cdot\gamma\right) - \frac{d}{\sqrt{2}}$$

$$C_y = (i-1+\frac{d}{\sqrt{2}}) \cdot \tan\left(\frac{\pi}{4} - \beta\cdot\gamma\right) - \frac{d}{\sqrt{2}}$$

$$D_y = (i+\frac{d}{\sqrt{2}}) \cdot \tan\left(\frac{\pi}{4} - \beta\cdot\gamma\right) - \frac{d}{\sqrt{2}}$$

$$E_x = \frac{\lfloor B_y \rfloor + d/\sqrt{2}}{\tan\left(\frac{\pi}{4} - (\beta-1)\cdot\gamma\right)} - \frac{d}{\sqrt{2}}$$

$$F_x = \frac{\lfloor D_y \rfloor + d/\sqrt{2}}{\tan\left(\frac{\pi}{4} - \beta\cdot\gamma\right)} - \frac{d}{\sqrt{2}}$$



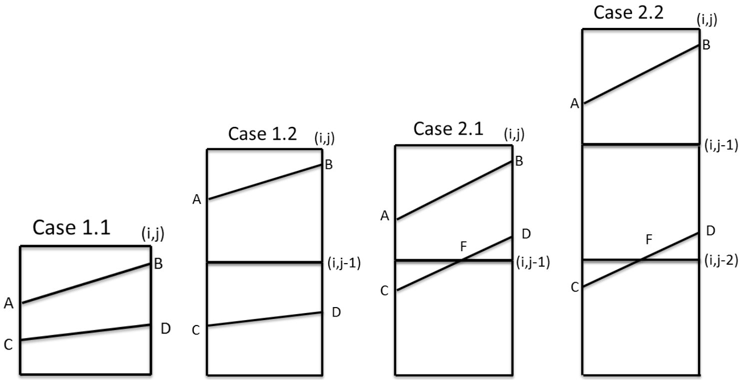


Figure : Cases 1 & 2

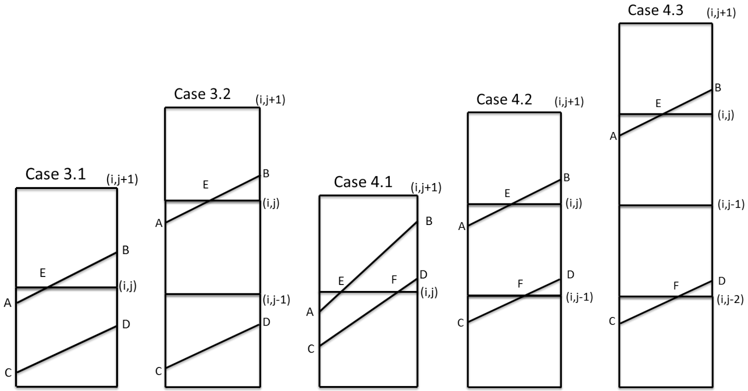


Figure : Cases 3 & 4

- Now we have the ability to build the first set of rows for the matrix A which will eventually be used to construct all of A, since the other rows of A will be linear transformations of the originals.
- Each row of A has a coordinate k, which is determined using total rotation angle θ, rotation interval ϕ, the number of beams(BS), and beam location β, which is positive below the 45°.
- The following formula determines k for each row a^k of A:

$$k = \begin{cases} \left(\frac{\theta}{\phi} + \frac{1}{2}\right)(BS) + \beta + 1 & \text{for} \quad \beta < 0 \\ \left(\frac{\theta}{\phi} + \frac{1}{2}\right)(BS) + \beta & \text{for} \quad \beta > 0 \end{cases},$$

  where $\beta = \pm 1, \ldots, \pm\frac{(BS)}{2}$.
- Note also that the original block corresponds to a θ=0 because we assume the detector and object always start in this orientation.

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- In pursuit of this end we model the rotation of Img as a linear transformation on x using a matrix Q.
- The rotated image vector is found using

$$x_\theta = Q_\theta x.$$

Rotating Rows

For any row k and the corresponding values of β and θ,

$$a^n x_\theta = a^n Q_\theta x = b_k,$$

$$\text{where} \quad n = \begin{cases} \frac{BS}{2} + \beta + 1 & \text{for} \quad \beta < 0 \\ \frac{BS}{2} + \beta & \text{for} \quad \beta > 0 \end{cases},$$

so a^n is one of the original rows created. Thus for our model, any row a^k, which is a permutation of original row a^n, can be found using the following formula

$$a^k = a^n Q_\theta.$$

Thus A can be fully computed from just the first block.

Introduction

- We tested four algorithms on system (1) created by our projection model, ART, Classical Cimmino, BCPCS, and BCIMCS using MATLAB coded programs.
- We reconstructed two test images using these algorithms, the Shepp-Logan head phantom [Kak 1988] and a real cardiac CT image [TEAM RADS].
- We used a PC (8GB, 2.5GHz CPU) for the numerical tests.

Imaging Parameters

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- $\#$ of detectors
- γ (ray separation)
- **d** (distance from emitter to bottom left corner of our grid)

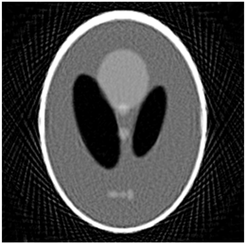
Table : **Parameters and Projection Data**

Experimental Parameters							
N	# of D's	ϕ	# of views	**d**	γ	k_{max}	ε
256	95	4	90	80	0.6383°	100	10^{-6}

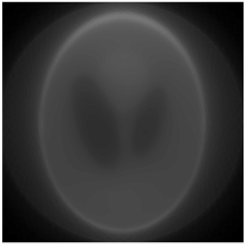
Projection Model Data	
Size of A	Time to compute A
8460x65536	1694.49 seconds

Shepp-Logan Phantom

ART Reconstruction

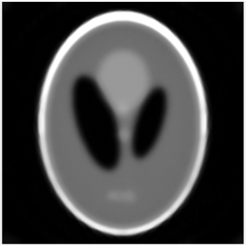


CIM Reconstruction



BCPCS Reconstruction

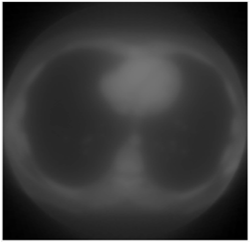
BCIMCS Reconstruction



Cardiac Phantom

ART Reconstruction

CIM Reconstruction



BCPCS Reconstruction

BCIMCS Reconstruction

Table : **Numerical Data**

Algorithm	Run Time(s)	k	$\|\|b - Ax^{(k)}\|\|_\infty$
ART Shepp-Logan	2998.72	100	0.9716
CIM Shepp-Logan	3326.63	100	259.7246
BCPCS Shepp-Logan	2060.70	100	1.2741
BCIMCS Shepp-Logan	2119.18	100	42.9673
ART Cardiac	2828.22	100	0.9513
CIM Cardiac	3298.62	100	249.0146
BCPCS Cardiac	1822.51	100	1.3103
BCIMCS Cardiac	1900.28	100	24.8333

Discussion

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- Our projection model is a success, with minor modifications needed.
- Possible future applications include the testing and development of new reconstruction algorithms.

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- We increase accuracy of projection data by taking into account the finite width of the beams, and thus improve the capability of testing reconstruction algorithms.
- This is the first model specifically dealing with area based fan beam projection, though similar models have been proposed for parallel beam.
- We reduce computation load by treating the detector as lying along the 45° line and using the rotation matrix Q to find the rest of A.
- In our numerical simulations the reconstruction algorithms were able to successfully reconstruct the image from b, the projection data.

Future Work

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- Allow the number of detectors to be even.
- Minimize memory storage and computation time for projection model programs.
- Test the proposed projection model with other phantoms and real CT images.
- Apply the proposed projection model to the research of other iterative reconstruction algorithms in CT.

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