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Classifying Regular Polytopes in Dimension 4 and Beyond

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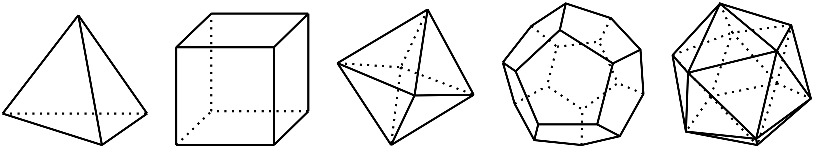
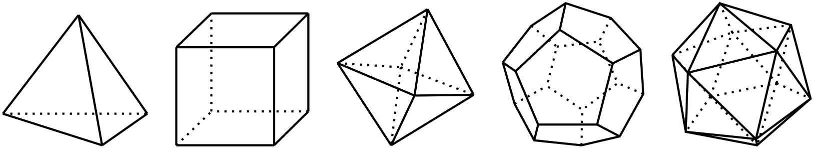
May 31, 2018

Motivating Questions

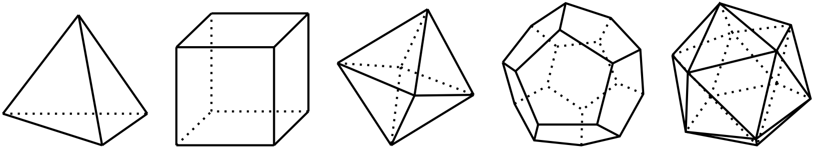


Motivating Questions



- How many regular convex polytopes are there in each dimension?

Motivating Questions



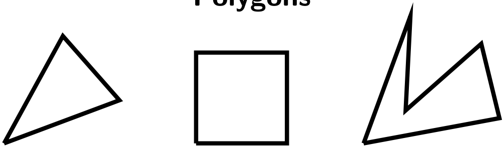
- How many regular convex polytopes are there in each dimension?
- How can we prove this?

Regular Convex Polygons

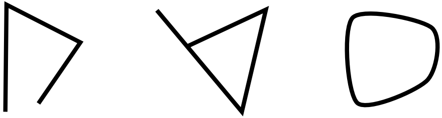
Definition

A **polygon** is a closed and connected shaped bounded by a finite number of lines.

Polygons



***Not* Polygons**

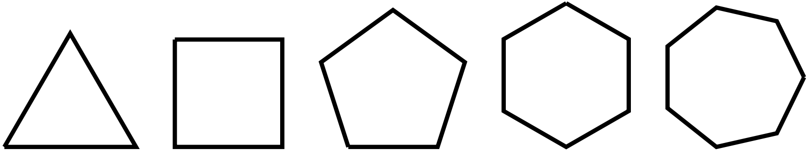


Regular Convex Polygons

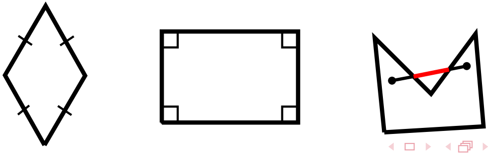
Definition

A **regular convex polygon** is a polygon that is equilateral, equiangular, and whose interior forms a convex set.

Regular Convex Polygons



Not **Regular Convex Polygons**



Regular Convex Polygons

Important things to notice:

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Regular Convex Polygons

Important things to notice:

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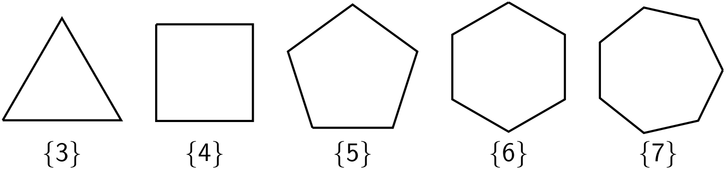
- The interior angles are all congruent
- They can be represented by a ***Schläfli symbol***
  - This is of the form $\{p\}$ where p represents the number of sides
  - It is unique



(we can keep going!)

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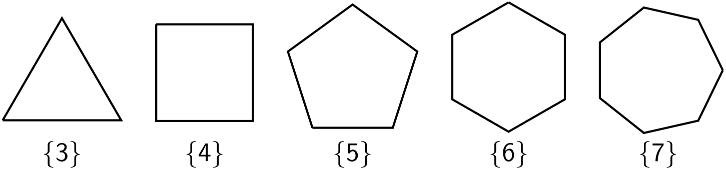

(we can keep going!)

- There are infinitely many!

Regular Convex Polyhedra

QUESTION: What is the 3-dimensional analog of the 2-dimensional regular convex polygons?

Regular Convex Polyhedra

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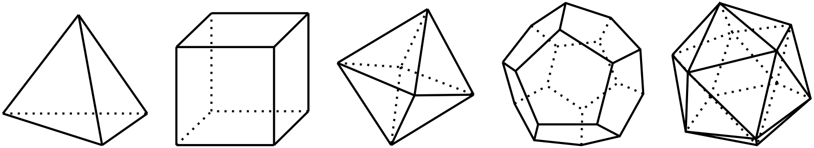
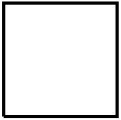
ANSWER: The regular convex polyhedra, or the **Platonic solids**

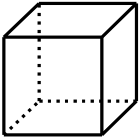


Regular Convex Polyhedra (The Platonic Solids)

Square



Cube



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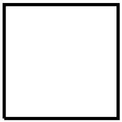

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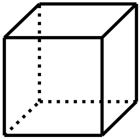

- All *faces* are congruent

Regular Convex Polyhedra (The Platonic Solids)

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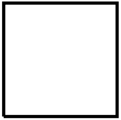

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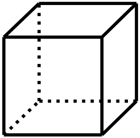

- All *faces* are congruent
  - Notice that the faces are a regular polygon!

Regular Convex Polyhedra (The Platonic Solids)

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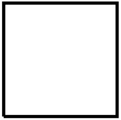
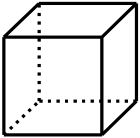


- All sides are the same lengths
- Interior angles are congruent

Cube



- All *faces* are congruent
  - Notice that the faces are a regular polygon!
- Angles formed by faces are congruent

Regular Convex Polyhedra (The Platonic Solids)

$$\{\, p \,,\, q \,\}$$

Regular Convex Polyhedra (The Platonic Solids)

$$\{\, p \,,\; q \,\}$$

$$\{\, 4 \,,\; 3 \,\}$$

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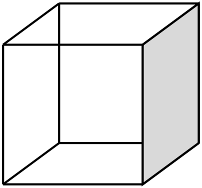


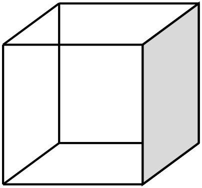
$\{4\}$

Square **facet**

Regular Convex Polyhedra (The Platonic Solids)

$$\{\ p\ ,\ \ q\ \}$$

$$\{4, 3\}$$



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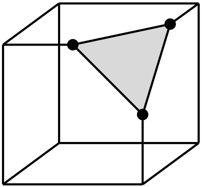
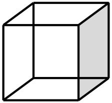


Triangle **vertex figure**

Regular Convex Polyhedra (The Platonic Solids)

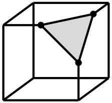
Definition

An n-polytope's **facets** are the type of $(n-1)$-polytopes that bound it.



Definition

The **vertex figure** of an n-polytope is the $(n-1)$-dimensional convex hull formed by connecting the center of each of the $(n-2)$-elements that are incident on a given vertex.



Regular Convex Polyhedra (The Platonic Solids)

Theorem

For a regular convex polyhedron $\{p, q\}$, $q\phi < 2\pi$ *where* ϕ *is the interior angle of a regular p-gon.*

Regular Convex Polyhedra (The Platonic Solids)

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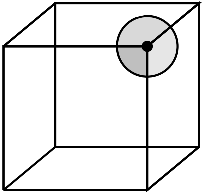
For a regular convex polyhedron $\{p, q\}$, $q\phi < 2\pi$ where ϕ is the interior angle of a regular p-gon.

In other words, the sum of the interior angles that meet at a given vertex must be less than 2π.



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The Platonic Solids have a Schläfli symbol of the form $\{p, q\}$.

Recall that $p, q \geq 3$.

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This will tell us which combinations of p and q will give us Platonic solids.

Regular Convex Polyhedra (The Platonic Solids)

Example:

Regular Convex Polyhedra (The Platonic Solids)

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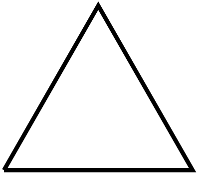
1. Suppose $p = 3$



Regular Convex Polyhedra (The Platonic Solids)

Example:

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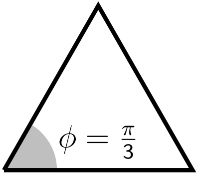

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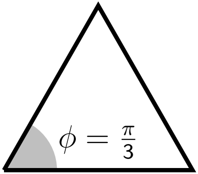

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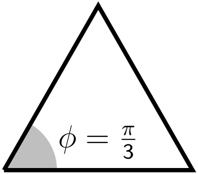

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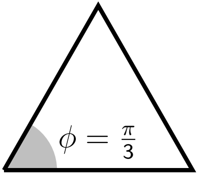

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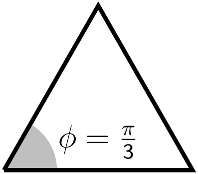

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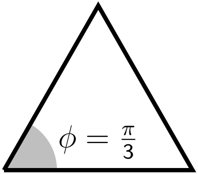

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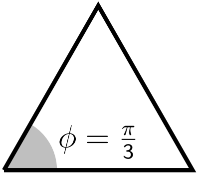
3. $q(\frac{\pi}{3}) < 2\pi$ is satisfied by $q \in \{3, 4, 5\}$



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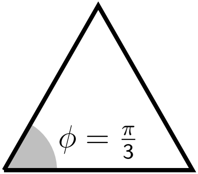

This means that
$\{3, 3\}$
$\{3, 4\}$
$\{3, 5\}$
are all Platonic solids!

Regular Convex Polyhedra (The Platonic Solids)

If we do this for $p = 4$, we get that q can only be 3.

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If we do this for $p \geq 6$, we don't get any $q \geq 3$ that work (this is because the interior angle of these polygons is so large).

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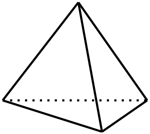
If we do this for $p \geq 6$, we don't get any $q \geq 3$ that work (this is because the interior angle of these polygons is so large).

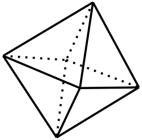
Therefore there are exactly five Platonic solids, given by

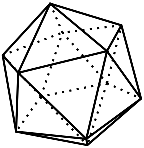
$$\{3,3\} \qquad \{3,4\} \qquad \{3,5\} \qquad \{4,3\} \qquad \{5,3\}$$

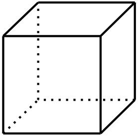
Regular Convex Polyhedra (The Platonic Solids)

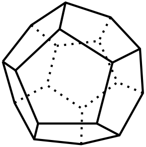
Name of Solid	Faces	Schläfli symbol
tetrahedron	4	$\{3,3\}$
octahedron	8	$\{3,4\}$
icosahedron	20	$\{3,5\}$
cube	6	$\{4,3\}$
dodecahedron	12	$\{5,3\}$


tetrahedron


octahedron


icosahedron


cube


dodecahedron

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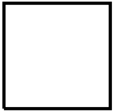
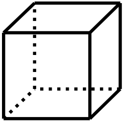
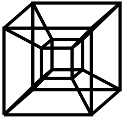


Cube



Tesseract



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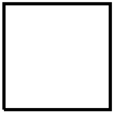
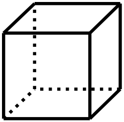


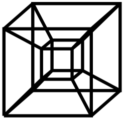
- Bounded by lines

Cube



- Bounded by regular polygons

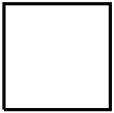
Tesseract



- Bounded by regular polyhedra

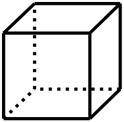
Regular Convex Polyhedra (The Platonic Solids)

Square



- Bounded by lines
- All sides are congruent

Cube



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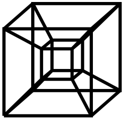
Tesseract



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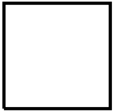
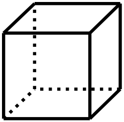


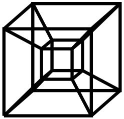
- Bounded by lines
- All sides are congruent
- Interior angles are congruent

Cube



- Bounded by regular polygons
- All faces are congruent
- Angles formed by faces are congruent

Tesseract



- Bounded by regular polyhedra
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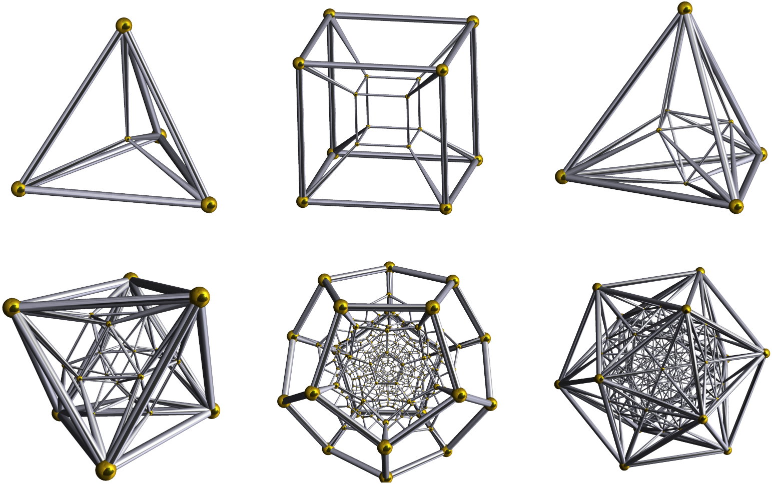
Regular Convex Polychora



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- There are only three regular convex polytopes in each dimension $n \geq 5$

Future Work

- Properties of n-polytopes ($n \geq 4$)
- Practical uses for this information
- Star polytopes

Thank you for coming!

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