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Extensions of Cayley-Sudoku Constructions

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By

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Abstract

A Cayley-Sudoku Table is a Cayley Table that also has smaller subrectangles that contain every element of the Group exactly once. This idea was explored in the "Cosets and Cayley Sudoku Tables" published in Mathematics Magazine. There are two main ways to build a Cayley-Sudoku Table. One restriction needed to form a "Cayley-Sudoku Table" leads to loop theory and a 1939 theorem of R. Baer.

1 INTRODUCTION

Sudoku puzzles have become extremely popular since they were first invented in the 1970's. The puzzle is a 9×9 array subdivided into nine smaller 3×3 boxes. The numbers 1-9 appear in only a few entries in the array. It is the job of the puzzler to fill in the entire array with the digits 1-9 in such a way that every digit appears exactly once in each row, once in each column and once in each of the smaller 3×3 arrays. Sudoku Puzzles have become so popular newspapers print them next to the daily crossword, and numerous books have been published both containing puzzles and about the mathematics behind the puzzles.

In April 2010, Mathematics Magazine published "Cosets and Cayley-Sudoku Tables," an article focused on using group theory to morph Cayley tables of groups into Sudoku-like puzzles [1]. That is, the article focused on creating blocks inside of a Cayley Table that contain each element of the group exactly one time. The authors found two different ways to construct a "Cayley-Sudoku" table, each formulated only using group theory principles. The first construction for these tables comes directly from Theories of Latin Squares [2]. Additionally, the second construction comes from Loop Theory [5]. Table 1 shows an example of a Sudoku puzzle:

3				5	7		8	
	2	4						
	7	8			1			9
7				6			9	
		6		7		8		
	3			4				6
8			9			1	5	
						9	4	
	4		3	1				8

Table 1: A typical Sudoku Table

Notice how some of the matrix is filled in, while most of the array is left empty. The puzzler must deduce which of the numbers, 1-9, should be placed in each box.
That same Sudoku puzzle is solved below with the original entries circled:

③	1	9	2	⑤	⑦	6	⑧	4
5	②	④	6	9	8	3	7	1
6	⑦	⑧	4	3	①	5	2	⑨
⑦	8	1	5	⑥	3	4	⑨	2
4	9	⑥	1	⑦	2	⑧	3	5
2	③	5	8	④	9	7	1	⑥
⑧	6	3	⑨	2	4	①	⑤	7
1	5	2	7	8	6	⑨	④	3
9	④	7	③	①	5	2	6	⑧

Table 2: A completed Sudoku table

For the remainder of this paper we will be talking about "Cayley Sudoku Tables" so it will be helpful to have a formal definition of this.

Definition 1.1. *A Cayley Sudoku Table is an arrangement of a Cayley table of a group G with composite order $|G| = nk$ that is comprised of smaller $n \times k$ blocks each containing every group element exactly once.*

Consider the Cayley table of the group $\mathbb{Z}_4 := \{1, 2, 3, 4 = 0\}$, its typical Cayley table would look like the below:

$+_4$	1	2	3	4
1	2	3	4	1
2	3	4	1	2
3	4	1	2	3
4	1	2	3	4

We can rearrange this table into a Cayley Sudoku Table. Because $|\mathbb{Z}_4| = 4 = 2 \times 2$ our blocks will be 2×2. One rearrangement is shown below:

$+_4$	1	2	3	4
4	1	2	3	4
2	3	4	1	2
3	4	1	2	3
1	2	3	4	1

Notice that each 2×2 block contains $1, 2, 3$ and 4 exactly once.

Another few things to define before we get started are cosets, conjugates and "complete sets of left coset representatives."

Definition 1.2. *A coset of H in G is a subset of G and is denoted $xH := \{xh : h \in H\}$ for left cosets and $Hx := \{hx : h \in H\}$ for right cosets for a fixed element $x \in G$*

Definition 1.3. *A conjugate is a subgroup of G denoted $H^g := g^{-1}Hg = \{g^{-1}hg : h \in H\}$ where $H \leq G$ and $\forall g \in G$.*

It is not obvious from the definition of a conjugate that it is a subgroup of G, so we prove it below:

Proof.

Assume G is a group and $H \leq G$. Because H is a subgroup of G we know:

- $H \subseteq G$
- $\varepsilon \in H$, where ε is the identity element of G
- $\forall h \in H$ we have $h^{-1} \in H$ and
- $\forall h_1, h_2 \in H$ we have $h_1 h_2 \in H$

We will need to prove the same four things for H^g, where $g \in G$ is fixed.

Subset:

Let $x \in H^g$ which means that $\exists h_1 \in H$ such that $x = g^{-1}h_1 g$. We know that g, g^{-1}, and h are all elements of G, and that G is closed so $g^{-1}h_1 g \in G$ which means that $x \in G$.

Identity:

Consider the following:

$$\begin{aligned}
g &= g \\
\varepsilon g &= g \\
g^{-1}\varepsilon g &= g^{-1}g = \varepsilon \\
g^{-1}\varepsilon g &\in H^g && \text{because } \varepsilon \in H \\
\therefore \varepsilon &\in H^g && \text{because} \varepsilon = g^{-1}\varepsilon g
\end{aligned}$$

Inverses:
Let $x \in H^g$ which means that $\exists h_1 \in H$ such that $x = g^{-1}h_1g$ as above. Consider:

$$\begin{aligned} x^{-1} &= (g^{-1}h_1g)^{-1} \\ &= (g)^{-1}(g^{-1}h_1)^{-1} && \text{shoes and socks} \\ &= g^{-1}h_1^{-1}(g^{-1})^{-1} && \text{shoes and socks} \\ &= g^{-1}h_1^{-1}g \end{aligned}$$

We know that $h_1^{-1} \in H$ because $H \leq G$ therefore, $g^{-1}h_1^{-1}g \in H^g$ implying that $x^{-1} \in H^g$.
Closure:
Let $x, y \in H^g$ which means $\exists\ h_1, h_2 \in H$ such that $x = g^{-1}h_1g$ and such that $y = g^{-1}h_2g$. Now consider $xy = (g^{-1}h_1g)(g^{-1}h_2g) = g^{-1}h_1(gg^{-1})h_2g = g^{-1}h_1h_2g$. We know that $h_1h_2 \in H$ because H is closed. So $g^{-1}h_1h_2g \in H^g$ making $xy \in H^g$.

□

Definition 1.4. *A Complete Set of Left (or right) Coset Representatives forever now denoted as C.S.L.C.R. of $K \leq G$ is a set $\{x_1, x_2, ..., x_n\}$ that satisfies*

- *$\forall x_i, x_j \in$ the set , ,$x_i \neq x_j$, $x_iK \neq x_jK$, and*

- *$\forall y \in G$, $yK = x_iK$ for some x_i in the set.*

2 Constructions of Cayley Sudoku Tables

Now we are ready to unpack the Constructions of Cayley Sudoku tables put forth in [1]. The authors found two unconventional ways to arrange Cayley Tables of groups of composite order so that the tables have subsquares that contain each element exactly once. The constructions are as follows:
Construction 1: Let G be a finite group. Assume H is a subgroup of G having order k and the number of distinct cosets is n (so that $|G| = nk$). If $Hg_1, Hg_2, \ldots, Hg_n$ are the n distinct right cosets of H in G, then arranging the Cayley table of G with columns labeled by the cosets $Hg_1, Hg_2, \ldots, Hg_n$ and the rows labeled by sets $T_1, T_2, \ldots, T_k$ yields a Cayley-Sudoku table of G with blocks of dimension $n \times k$ if and only if $T_1, T_2, \ldots, T_k$ partition G into complete sets of left coset representatives of H in G.

Construction 2: Assume H is a subgroup of G having order k and index n. Also suppose $t_1H, t_2H, \ldots, t_nH$ are the distinct left cosets of H in G. Arranging the Cayley table of G with columns labeled by the cosets $t_1H, t_2H, \ldots, t_nH$ and the rows by sets $L_1, L_2, \ldots, L_k$ yields a Cayley Sudoku table of G with blocks of dimension $n \times k$ if and only if $L_1, L_2, \ldots, L_k$ are complete sets of left coset representatives of H^g for all $g \in G$.

To the authors of [1] this second construction seemed very foreign and constrained. In the first construction you can take any group G and any subgroup H and make a Cayley Sudoku table. The second construction is unpredictable; not every group has a C.S.L.C.R. of a conjugate H^g. Thus we cannot use Construction 2 to build a Cayley-Sudoku Table for every group. It is not immediately apparent why the second construction has this funny restriction, but later on we will discover that this is related to Loop theory. First however, let's prove that these constructions actually make a Cayley Sudoku Table.

2.1 Construction One

Recall the statement: Let G with operation $\star$ be a finite group. Assume H is a subgroup of G having order k and the number of distinct cosets is n (so that $|G| = nk$). If $Hg_1, Hg_2, \ldots, Hg_n$ are the n distinct right cosets of H in G, then arranging the Cayley table of G with columns labeled by the cosets $Hg_1, Hg_2, \ldots, Hg_n$ and the rows labeled by sets $T_1, T_2, \ldots, T_k$ yields a Cayley-Sudoku table of G with blocks of dimension $n \times k$ if and only if $T_1, T_2, \ldots, T_k$ partition G into complete sets of left coset representatives of H in G.

Proof. According to the construction above we will have the distinct right cosets of our group G across the top as column labels. The rows will be labeled by complete sets of left coset representatives. The authors of [1] call this a cross-handed method because the construction utilizes both right and left cosets. The table below shows the general layout of Construction 1.

	Hg_1	Hg_2	$\ldots$	Hg_n
T_1				
T_2				
$\vdots$				
T_k				

To begin, let us look at one arbitrary block of the Cayley Sudoku Table indexed by the row labels $T_h = \{t_1, t_2, ..., t_n\}$ where $1 \leq h \leq n$ and the column labels Hg_i so that the particular block is as below:

	Hg_i
t_1	t_1Hg_i
t_2	t_2Hg_i
$\vdots$	$\vdots$
t_n	t_nHg_i

The elements in this block belong to the set $B := t_1Hg_i \cup t_2Hg_i \cup t_3Hg_i \cup ... \cup t_nHg_i = (t_1H \cup t_2H \cup t_3H \cup ... \cup t_nH)g_i$.

($\leftarrow$) Assume T_h is a C.S.L.C.R. this means that $t_1H \cup t_2H \cup ... \cup t_nH = G$ so $B = Gg_i$. Because G is a group, it is closed so $Gg_i = G = B$. This means that every element in G is in the block. There are n elements in T_h and there are k elements in Hg_i by construction so there are nk elements in the block. Additionally the order of G is nk, so every element is represented at exactly once.

($\rightarrow$) Assume that every element in G appears exactly on time in each block. This means that $B = G$. Again because G is closed $G = Gg_i^{-1} = Bg_i^{-1} = [(t_1H \cup t_2H \cup t_3H \cup ... \cup t_nH)g_i]g_i^{-1} = t_1H \cup t_2H \cup t_3H \cup ... \cup t_nH$. There are n right cosets. ❑

Let's look at an example of this construction:

Consider again the group $\mathbb{Z}_9 := \{1,2,3,4,5,6,7,8,9=0\}$ and the cyclic subgroup generated by 3 as $<3>:= \{9,3,6\}$.

The right and left cosets of $<3>$ in $\mathbb{Z}_9$ are as follows:

$$\begin{aligned} 9+<3> &= \{9,3,6\} = <3>+9 \\ 1+<3> = \{9+1,,3+1,6+1\} &= \{1,4,7\} = <3>+1 \\ 2+<3> = \{9+2,3+2,6+2\} &= \{2,5,8\} = <3>+2 \end{aligned}$$

We can now make complete sets of left coset representatives, $\{9,1,2\}, \{3,4,5\}$, and $\{6,7,8\}$: one element from each left coset. Now we are ready to make our table!

According to Construction 1, we must label the columns with the right cosets and label the rows with the left coset representatives, and our table is as follows:

	9	3	6	1	4	7	2	5	8
9	9	3	6	1	4	7	2	5	8
1	1	4	7	2	5	8	3	6	9
2	2	5	8	3	6	9	4	7	1
3	3	6	9	4	7	1	5	8	2
4	4	7	1	5	8	2	6	9	3
5	5	8	2	6	9	3	7	1	4
6	6	9	3	7	1	4	8	2	5
7	7	1	4	8	2	5	9	3	6
8	8	2	5	9	3	6	1	4	7

Table 3: a Cayley Table that preserves the Sudoku Properties

Construction 1 can also be applied using Left Cosets and Complete Sets of Right Coset Representatives. In this case, the rows of the table are labeled by the the left cosets and the columns are labeled by the C.S.R.C.R. $R_1, R_2, ...R_k$ as below:

	R_1	R_2	$\ldots$	R_k
y_1H				
y_2H				
$\vdots$				
y_nH				

Another example of Construction 1 is given without computations in Appendix 1.

2.2 Construction Two

Recall Construction 2: Assume H is a subgroup of G having order k and index n. Also suppose $t_1H, t_2H, \ldots, t_nH$ are the distinct left cosets of H in G. Arranging the Cayley table of G with columns labeled by the cosets $t_1H, t_2H, \ldots, t_nH$ and the rows by sets $L_1, L_2, \ldots, L_k$ yields a Cayley Sudoku table of G with blocks of dimension $n \times k$ if and only if $L_1, L_2, \ldots, L_k$ are complete sets of left coset representatives of H^g for all $g \in G$.

Proof. As before, let us consider how this Sudoku table will look. The columns will be labeled by the Left Cosets of H in G. The rows will be labeled by the C.S.L.C.R. of H^g as below:

	t_1H	t_2H	$\ldots$	t_nH
L_1				
L_2				
$\vdots$				
L_k				

Again we will look at an arbitrary block in this table with the column t_iH and the rows indexed by $L_h := \{l_1, l_2, ..., l_n\}$. The particular block will look like this:

	t_iH
l_1	l_1t_iH
l_2	l_2t_iH
$\vdots$	$\vdots$
l_n	l_nt_iH

It is our hypothesis that if $\{l_1, l_2, ..., l_n\}$ is a C.S.L.C.R. of H^g then the block would contain every element of G exactly one time.

Before we do the proof, let us first rewrite t_iH.

$$\begin{aligned} t_iH &= t_iHt_i^{-1}t_i && \text{multiplying by the "identity"} \\ &= [t_iHt_i^{-1}]t_i \\ &= H^{t_i^{-1}}t_i \end{aligned}$$

We know that $t_i^{-1} \in G$ which means that $H^{t_i^{-1}} \in H^g$. Suddenly our table of an arbitrary block looks like this:

	$H^{t_i^{-1}}t_i$
l_1	$l_1H^{t_i^{-1}}t_1$
l_2	$l_2H^{t_i^{-1}}t_2$
$\vdots$	$\vdots$
l_n	$l_nH^{t_i^{-1}}t_i$

Because if $\{l_1, l_2, ..., l_n\}$ are C.S.L.C.R of H^g they must also be a C.S.L.C.R of $H^{t_i^{-1}}$ by Universal Specialization. And our "new" columns are labeled by right cosets of $H^{t_i^{-1}}$. All of a sudden we have been able to "morph" Construction 2 into Construction 1, so the proof falls straight out.

❑

For an example of this, let us look at the group S_3 which is the group of all the permutations of 3 elements. The elements of the group are below:

$$\varepsilon = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$(12) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

$$(13) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$(23) = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}$$

$$(123) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$(132) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Now we need a subgroup H. Lets pick the subgroup $H := \{\varepsilon, (12)\}$. To build our Cayley Sudoku table using Construction 2 we will also need to calculate the left cosets of H and H^g in G, not to mention actually calculating H^g for every element in $g \in G$.

<u>Left Cosets of H:</u>

- $(12)H = \{(12)\varepsilon, (12)(12)\} = \{\varepsilon, (12)\} = \varepsilon H$
- $(13)H = \{(13)\varepsilon, (13)(12))\} = \{(13), \varepsilon\} = (123)H$
- $(23)H = \{(23)\varepsilon, (23)12)\} = \{(23), (132)\} = (132)H$

Now let us consider the subsets generated by $H^g \; \forall g \in S_3$

- $H^\varepsilon = \varepsilon^{-1} H \varepsilon = \{\varepsilon,\ (12)\} = H$
- $H^{(123)} = (123)^{-1} H (123) = (132) H (123) = \{(132)\varepsilon(123), (132)(12)(123)\} = \{\varepsilon, (13)\}$
- $H^{(13)} = (13)^{-1} H (13) = \{(13)e(13), (13)(12)(13)\} = \{\varepsilon, (32)\}$
- $H^{(12)} = (12)^{-1} H (12) = \{(12)\varepsilon(12), (12)(12)(12)\} = \{\varepsilon, (12)\} = H$
- $H^{(23)} = (23)^{-1} H (23) = \{(23)\varepsilon(23), (23)(12)(23)\} = \{\varepsilon, (13)\} = H^{(123)}$

- $H^{(132)} = (132)^{-1}H(132) = (123)H(132) = \{(123)\varepsilon(132), (123)(12)(132)\} = \{\varepsilon, (32)\} = H^{(13)}$

So we can see that, in fact, we have only 3 subsets generated by H^g
Now to make cosets with H^g!
We already know the cosets for $H^{(12)}$ because $H^{(12)} = H^\varepsilon$, those are the same as above.

- Left Cosets of H^ε
  - $(12)H = \{\varepsilon, (12)\} = \varepsilon H$
  - $(13)H = \{(123), (13)\} = (123)H$
  - $(23)H = \{(23), (132)\} = (132)H$
- Left Cosets of $H^{(123)}$
  - $(12)H^{(123)} = \{(12)\varepsilon, (12)(13)\} = \{(12), (132)\} = (132)H^{(123)}$
  - $(13)H^{(123)} = \{(13)\varepsilon, (13)(13)\} = \{\varepsilon, (13)\} = \varepsilon H^{(123)}$
  - $(23)H^{(123)} = \{(23)\varepsilon, (23)(13)\} = \{(23), (123)\} = (123)H^{(123)}$
- Left Cosets of $H^{(13)}$
  - $(12)H^{13} = \{(12)\varepsilon, (12)(23)\} = \{(12), (123)\} = (123)H^{(13)}$
  - $(13)H^{13} = \{(13)\varepsilon, (13)(23)\} = \{(13), (132)\} = (132)H^{(13)}$
  - $(23)H^{13} = \{(23)\varepsilon, (23)(23)\} = \{\varepsilon, (23)\} = \varepsilon H^{(13)}$

According to Construction 2, we need to find C.S.L.C.R. of H^g where these elements are left coset representatives for all elements $g \in S_3$. Clearly the set $\{(12), (13), (132)\}$ does not satisfy this condition because $(13)H^{(13)} = (132)H^{(13)}$, which means that it is not a complete set of left coset representatives of H^g <u>for all</u> $g \in S_3$.
A correct set of left coset representatives of H^g is given by $\{(12), (13), (23)\}$ or alternately $\{\varepsilon, (123), (132)\}$
So for our Cayley Sudoku Table using Construction 2 we will label the columns by the left cosets of H and the rows will be labeled by the C.S.L.C.R. of H^g. A complete table is giving below:

	ε	(12)	(123)	(13)	(23)	(132)
(12)	(12)	ε	(23)	(132)	(123)	(13)
(13)	(13)	(123)	(12)	ε	(132)	(23)
(23)	(23)	(132)	(13)	(123)	ε	(12)
ε	ε	(12)	(123)	(13)	(23)	(132)
(123)	(123)	(13)	(132)	(23)	(12)	ε
(132)	(132)	(23)	ε	(12)	(13)	(123)

Table 4: A Cayley Sudoku table of S_3

Using Construction 2 we have successfully made a Cayley Sudoku Table!

2.3 Conditions for Construction Two

As mentioned earlier, Construction 2 does not work for every group. There are three conditions for groups which have been discovered so far that ensure Construction 2 is applicable. The first two are found in [1], the third case comes from combinatorics.

The first condition is when H is a normal subgroup of G. In this case, it can be shown (and I will show it) that $H^g = H$ for all $g \in G$. Unfortunately, decomposing G into the left cosets and the complete set of left coset representatives only gives back to us the same Cayley Sudoku Tables as in Construction 1. This is because in normal subgroups the left cosets and the right cosets are identical.

Another circumstance when Construction 2 can be utilized was described in [1] with the following proposition:

Theorem 2.1. *Suppose the finite group G contains subgroups $T := \{t_1, t_2, ..., t_n\}$ and $H := \{h_1, h_2, ..., h_k\}$ such that $G = \{tk : t \in T, h \in H\} := TH$ and $T \cap H = \{\varepsilon\}$, then the elements of T form a complete set of left coset representatives of H and the cosets $Th_1, Th_2, ..., Th_k$ decompose G into complete sets of left coset representatives of H^g for every $g \in G$.*

Proof. Strategy: First we will see that T is a C.S.L.C.R. of H in G, and then show how T is a C.S.L.C.R. of H^g for all $g \in G$.
Finally we can prove that if T is C.S.L.C.R. of H^g for all $g \in G$ then $Th_1, Th_2, ..., Th_k$ will decompose G and each Th_j is a C.S.L.C.R. of H^g.
Let T and H be defined as above and fix $g \in G$.
First we want to show that T is a C.S.L.C.R. of H in G. Let's assume that $t_m \neq t_l$ and BWOC assume $t_m H = t_l H$. By the Left Coset Equality Test we see that $t_m^{-1} t_l \in H$. Alternately, we know because T is a subgroup of G that for $t_m, t_l \in T$ we also have $t_m^{-1} t_l \in T$ by closure and inverses. From our hypothesis we know that the only element in both H and T is ε so $t_m^{-1} t_l = \varepsilon$ which implies that $t_l = t_m$. So we have out contradiction. This means for all $t_m \neq t_l$, $t_m H \neq t_l H$.
Also let us make sure that every element in G is in a left coset of H. Let $x \in G$, by by hypothesis this means that $x = t_i h_j$ for some $t_i \in T$ and for some $h_j \in H$. This means that $x \in t_i H$. So we have that T is a C.S.L.C.R. of H in G.
Knowing this, we can show that because T is a C.S.L.C.R. of H in G that it is also a C.S.L.C.R. of H^g in G.
Finally, knowing that T is a C.S.L.C.R. of H^g in G, we can show that there exists disjoint sets $T_1, T_2, ..T_k$, such that $\cup_{i=1}^{k} T_i = G$ and each T_i is a C.S.L.C.R. of H^g for all $g \in G$.

Let's define H as $H = \{h_1 = \varepsilon, h_2, h_3, ..., h_k\}$ and then lets look at $T_j := \{t_1 h_j, t_2 H_j, ..., t_n h_j\}$ where $t_i \in T$. We need to show that for any $g \in G$ and any T_j that T_j is a C.S.L.C.R of H^g which means:

- $\forall x \in G, \exists\ t_l h_j$ such that $xH^g = t_l h_j H^g$ for some l, $1 \leq l \leq k$
- $\forall l, m$ if $t_l h_j H^g = t_m h_j H^g$ then $l = m$

We also must show that each of these T_j's is distinct from one another, which means that $\forall i \neq j$, $T_i \cap T_j = \emptyset$. Let's tackle this last part first.
Assume by way of contradiction that $T_i \cap T_j \neq \emptyset$ and let $x \in T_i \cap T_j$. Then

$$
\begin{aligned}
x =& t_m h_i = t_l h_j \quad \exists h_j, h_i \in H, h_i \neq h_j \\
\Rightarrow t_l^{-1} t_m =& h_j h_i^{-1} \in H \\
\Rightarrow t_l H =& t_m H && \text{LCET} \\
H =& H^{\varepsilon} \in H^g && \text{because } \varepsilon \in G \\
\therefore t_l =& t_m && T \text{ is a C.S.L.C.R. of } H \\
(\Rightarrow\Leftarrow)
\end{aligned}
$$

Because we reached a contradiction, we have proved that $T_i \cap T_j = \emptyset$.
Now we must prove that $\forall x \in G, \exists t_l h_j \in TH$ such that $x \in t_l h_j H^g$ for some l, $1 \leq l \leq k$.
Let's start by fixing $x \in G$, and $g \in G$.
We know from hypothesis that $gh_j^{-1} \in G$ for some $h_j \in H$. So we know that T is a C.S.L.C.R. of $H^{gh_j^{-1}}$. We also know that $xh_j \in G$ by closure of G so the following holds

$$
\begin{aligned}
xh_j &\in t_l H^{gh_j^{-1}} \text{ for some } t_l \in T \\
xh_j &= t_l (gh_j^{-1})^{-1} h (gh_j^{-1}) \text{ for some } h \in H \\
&= t_l h_j g^{-1} h g h_j^{-1} \\
x &= t_l h_j g^{-1} h g \\
x &\in t_l h_j H^g
\end{aligned}
$$

So we can see that for any $x \in G$ x is an element of a left coset of some $t_l h_j H^g$.

Because T is a C.S.L.C.R. of H in G it is also a C.S.L.C.R. of H^g in G. First we will prove an important lemma:
<u>Lemma 2.3.1:</u> Assume that $H \leq G$, and that $T \leq G$ if $G = TH$ then $\forall g \in G$, $G = TH^g$.
Assume the hypothesis and let $g \in G$ First we will show that $G = HT$:
($\supseteq$) Let $x \in HT$; $x = h_1 t_1$ for some $h_1 \in H$ and $t_1 \in T$. $\therefore x \in G$ by closure.
($\subseteq$) Let $x \in G$; $x^{-1} \in G = TH$ by inverses and definition of G. So $x^{-1} = h_2 t_2$ for some $h_2 \in H$ and $t_2 \in T$. This means that $x = h^{-1} t^{-1}$ by shoes and socks. Because $h^{-1} \in H$ and $t^{-1} \in T$, we can see that $x \in HT$.
Most importantly this means that we can write any element $g \in G$ as $g = ht$ for some $h \in H$ and $t \in T$.
Next step will be showing that $H^g = H^t$ where $g = ht$ as above.

We know $H^g = g^{-1}Hg = t^{-1}h^{-1}Hht$ by the definition of g. We also know from closure and inverses that $h^{-1}Hh = H$. So we now have that $H^g = t^{-1}Ht = H^t$.
Our final step for this lemma will be showing that $G = TH^g$
Like above, it is easy to show that any element in TH^g is also in G from closure because both T and H^g are subgroups.
($\subseteq$) Let $y \in G$, we also know that $t \in G$ from earlier. By closure and inverses, this means that $yt^{-1} \in G$. We can thus define $yt^{-1} = t_3h_3$ for some $h_3 \in H$ and $t_3 \in T$. We can multiply t_3h_3 by the identity of G in the form of tt^{-1} so we get $yt^{-1} = t_3h_3 = t_3tt^{-1}h_3$. Multiplying both sides by t gives $y = t_3t(t^{-1}h_3t)$. $t_3t \in T$ and $t^{-1}h_3t \in H^t$ so $y \in TH^t = TH^g$.
Having proved our Lemma, we can now see that $G = TH = TH^g$ and since T is a C.S.L.C.R. of H, it must also be a C.S.L.C.R. of H^g.
Let us first fix $g \in G$. And assume that $t_lh_jH^g = t_mh_jH^g$

$$
\begin{aligned}
t_lh_jH^g &= t_mh_jH^g && \\
(t_mh_j)^{-1}t_lh_j &\in H^g && \text{LCET} \\
h_j^{-1}t_m^{-1}t_lh_j &\in H^g && \text{Shoes and Socks} \\
h_j^{-1}t_m^{-1}t_lh_j &= g^{-1}h_sg && \text{definition of } H^g \\
t_m^{-1}t_l &= h_jg^{-1}h_sgh_j^{-1} && \\
t_m^{-1}t_l &= (gh_j^{-1})^{-1}h_sgh_j^{-1} && \text{Shoes and Socks} \\
t_m^{-1}t_l &\in H^{gh_j^{-1}} && \\
t_lH^{gh_j^{-1}} &= t_mH^{gh_j^{-1}} && \text{LCET} \\
\therefore t_l &= t_m && \\
\therefore l &= m && T \text{is a C.S.L.C.R. of H}
\end{aligned}
$$

Thus we have shown that $Th_1, Th_2, ..., Th_k$ decompose G into complete sets of left coset representatives of H^g for every $g \in G$ when $G = TH$.
❑

Let's look at an example of Construction 2 with a group that satisfies these conditions. Consider the group S_4, the permutation group of 4 elements. Let our L be defined as the cyclic subgroup generated by (123). So $H = <(123)> = \{\varepsilon, (123), (132)\}$. Now let $T := \{\varepsilon, (12)(34), (13)(24), (14)(23),$
$(24), (1234), (1432), (13)\}$. By exhaustively checking, which is an exercise left up to the reader, we see that this subgroups correctly decompose S_4 as is required by the statement of the con-

dition. So using construction 2 we can make the Cayley Sudoku table below:

	H	$(12)(34)H$	$(13)(24)H$	$(14)(23)H$	$(24)H$	(1234)	$(1432)H$	$(13)H$
T								
$T(123)$								
$T(132)$								

where every subsquare contains every element in S_4.

There is still another condition for a group that when satisfied means that we can use Construction 2 to make a Cayley-Sudoku table. This third instance occurs when a group G has exactly two conjugates. When a group has exactly two conjugates it satisfies Theorem 9.2.3 from [4], the proof for which is beyond the scope of this paper. However, from this theorem we get that if a group has exactly two conjugates, then there exists a C.S.L.C.R. of those conjugates so we can make a Cayley Sudoku table using Construction 2.
Interestingly enough, Mirsky's 9.2.3 has a corollary that states that if H is a subgroup of a finite group G then there exists one C.S.L.C.R. that is also a Complete Set of Right Coset Representatives. This corollary caused a bit of confusion when a group of students made a table using right cosets of H for column representatives and right coset representatives of H (not of H^g) as row representatives. What they had created was a Cayley Sudoku table that was neither Construction 1 nor Construction 2, so they thought. What actually happened was that the students had picked a C.S.R.C.R. that was also a C.S.L.C.R. so they had, in fact, used Construction 1. Principles from combinatorics guarantee that there always exists one C.S.L.C.R. that is also a C.S.R.C.R.
With any of these three conditions satisfied we can use Construction 2 to build a Cayley Sudoku Table. But what makes these three conditions necessary? Why must the row be labeled by C.S.L.C.R. of H^g? These questions will be explored in the remainder of the paper.

3 Latin Square Theory

One of the first introductions a math major has with Latin Squares is in Abstract Algebra where one proves that every Cayley table is a Latin Square. A Cayley table is the square matrix created by operating each element of a group with every other element.
A Sudoku Table satisfies the conditions of a Latin Square because in a Sudoku Table, the digits 1-9 can appear exactly one time in each row and each column. The only other quality that

Sudoku Tables possess that Latin Squares need not is that Sudoku Tables have subsquares that each contain every element exactly one time, looking at any of the smaller 3×3 squares from the introduction example will show this. Luckily, there is a property of Latin Squares that corresponds to this same idea.

Definition 3.1. *A latin rectangle is a rectangular array that can be completed to a latin square and is called (n, d) complete if it contains n different elements each of which appears exactly d times in the array.*

As you can see, some Latin Rectangles may contain more than each element more than one time, but our particular interest is in Latin Rectangles where each element occurs only one time. In our above notation they are $(n, 1)$ rectangles. If we have a Latin Square that is completely formed by Latin Rectangles we will have a construction very similar to a Sudoku Table. Latin Squares, however, can be made with any elements not just the numbers 1-9, so there exists constructions very similar to Sudoku tables, but any size and with almost any elements!
In 1967, J. Denes proved the following theorem about the relationship between Latin Rectangles and Latin Squares.

Theorem 3.2. *If L is the latin square representing the multiplication table of a group G of order n, where n is a complete number, then L can be split into a set of n $(n, 1)$ complete non-trivial latin rectangles (a latin rectangle is trivial if it consists of a single row or column).*

This is just the kind of theorem we are looking for! It says that we can take any group, make its Cayley Table and arrange it in such a way that we have a Cayley-Sudoku square! Not surprisingly, the proof of this theorem is the same as the proof of Construction 1. As we can see, in 1967, Latin Squares Theory had already proved our Construction 1 for creating Cayley Sudoku tables, with a different name of course. So we can look to Latin Squares theory for more insights into building Cayley Sudoku tables. It shouldn't surprise us that theories regarding the laying out of tables relate with one other. But the next section might be much more surprising.

4 Loop Theory

Cayley Sudoku tables have interesting roots in Loop Theory. We saw above that Construction 1 was closely related to a Latin Squares theory developed 40 years ago. In a similar manner

Construction 2 is directly related to Theories of Loops. Remember our strange H^g condition for Construction 2 to work? It seemed very strange, but in fact, it comes from Loop Theory. Before continuing, lets make sure that we have a few more definitions firmly in our grasp.

Definition 4.1. *A* <u>*quasigroup*</u> *is a set* Q *with a binary operation* $\star$ *such that* $\forall a, b \in Q\ \exists!\ x, y \in Q$ *s.t.* $a \star x = b$

$y \star a = b$

Definition 4.2. *A* <u>*loop*</u> *is a quasigroup with an identity element* ε *such that* $x \star \varepsilon = x = \varepsilon \star x$

A topic that is very important in Loops is the idea of a section defined below:

Definition 4.3. *Let* G *be a group,* H *a subgroup of* G *and* $\pi : G \rightarrow G/H$ *where* $x \in G$ *maps to* xH. *A mapping* $\sigma : G/H \rightarrow G$ *is a* <u>*section*</u> *if the composed map* $\pi \circ \sigma : G/H \rightarrow G/H$ *is the identity.*

It is important to note that a Section in Loop Theory is the equivalent of a C.S.L.C.R. so that is what we will continue to call them.
A section, or a C.S.L.C.R., satisfies two very important conditions which we will refer to for the remainder of this section on Loops.

Theorem 4.4.

1. *The smallest subgroup of* G *containing* $\{x_1, x_2, ..., x_n\}$ *is* G.
2. $\forall xH, yH \in G/H,\ \exists!\ x_i \in$ *C.S.L.C.R. s.t.* $x_i xH = yH$

In 1939 R. Baer developed a theory that directly relates C.S.L.C.R. and the conjugate of a Group. It is Baer's theorem that is of supreme interest to the building of Cayley-Sudoku tables.

Theorem 4.5. *Let* G *be a group,* H *a subgroup of* G. *A complete set of left coset representatives* $\{x_1, x_2, ..., x_n\}$ *of* H *in* G *satisfies Theorem 4.4.2 if and only if* $x_1, x_2, ..., x_n$ *is a complete set of Left Coset representatives of* H^g $\forall g \in G$ *where* $H^g := gHg^{-1}$

Proof. Assume that $\{x_1, x_2, ..., x_k\}$ is a complete set of left coset representatives of H in G. This means

- $\forall y \in G, \exists\ x_i$ such that $yH = x_iH$

- $\forall x_i, x_j$ if $x_iH = x_jH$ then $x_i = x_j$

Also assume that $\{x_1, x_2, ..., x_k\}$ satisfies Thm 4.4.2.
Now let $g \in G$ and $y \in G$, it will also be beneficial to note that $g^{-1} \in G$ because G is a group. We can thus build the Left Cosets $yg^{-1}H$ and $g^{-1}H$ which are both elements of G/H.
First we need to show that every element in G is in the coset of one of these coset representatives.

$$\begin{aligned}
\exists x_i \ s.t.\ yg^{-1}H &= x_ig^{-1}H && \text{by (4.4.2)}\\
(x_ig^{-1})^{-1}yg^{-1} &\in H && \text{LCET}\\
gx_i^{-1}yg^{-1} &\in H && \text{Shoes and Socks}\\
gx_i^{-1}yg^{-1} &= h_1 && \text{for some } h_1 \in H\\
x_i^{-1}y &= g^{-1}h_1g && \text{Group Operations}\\
x_i^{-1}y &\in H^g && \text{Definition of Conjugate}\\
yH^g &= x_iH^g && \text{LCET}
\end{aligned}$$

Now, we want to show that each of these coset representatives is unique in H^g.

$$\begin{aligned}
x_jH^g &= x_iH^g && \\
x_i^{-1}x_j &\in H^g && \text{LCET}\\
x_i^{-1}x_j &= g^{-1}h_2g && \text{for some } h_2 \in H\\
gx_i^{-1}x_jg^{-1} &= h_2 && \\
gx_i^{-1}x_jg^{-1} &\in H && \\
(x_ig^{-1})^{-1}x_jg^{-1} &\in H && \text{Shoes and Socks}\\
x_jg^{-1}H &= x_ig^{-1}H && \text{LCET}\\
x_j &= x_i && \text{Because } x_i, x_j \text{are elements of the C.S.L.C.R. of H in G}
\end{aligned}$$

Therefore, $\{x_1, x_2, ..., x_k\}$ is a C.S.L.C.R. of H^g in G.

Now assume that $\{x_1, x_2, ..., x_k\}$ is a C.S.L.C.R. H^g in G. We need to show that $\{x_1, x_2, ..., x_k\}$ satisfies 4.4.2, that is, we want to show that $\forall xH, yH \in G/H,\ \exists!\ x_i$ such that $x_ixH = yH$. Let

$x, y \in G$, then

$$\begin{aligned}
yxH^x &= x_iH^X & & x_1, x_2, ..., x_k \text{ is a complete set of LCR} \\
(yx)^{-1}x_i &\in H^x & & \text{LCET} \\
x^{-1}y^{-1}x_i &\in H & & \text{shoes and socks} \\
x^{-1}y^{-1}x_i &= x^{-1}h_3x & & \text{for some } h_3 \in H \\
y^{-1}x_ix &= h_3 & & \\
y^{-1}x_ix &\in H & & \\
x_ixH &= yH & & \text{LCET}
\end{aligned}$$

So we can see that for any generic element, $y \in G$, we can find a unique x_i that satisfies 4.4.2.

❑

We can use Complete Sets of Left Coset Representatives to create a Loop!
Indeed let $H \leq G$ and define $L := \{x_1, x_2, ..., x_k\}$ be a C.S.L.C.R. of H in G satisfying Theorem 4.4 where one of the x_i's is the identity element of G, we will call it ε.
Define $\star$ on L as $x_i \star x_j = x_l$ where $(x_ix_j)H = x_lH, \forall x_i, x_j \in L$.
Claim: L with $\star$ is a Loop.
For this to be the case, $\star$ must be a binary operation, L with $\star$ must have an identity element and $\forall a, b \in L$ $\exists! x_i, x_j \in L$ s.t.$a \star x_i = b$ and $x_j \star a = b$
Proof. Let us first start by showing that $\star$ is a binary operation.
Defined and closed: Let us consider the left coset x_ix_jH where $x_i, x_j \in L \subseteq G$. We know that $x_ix_j \in G$ because the group operation is closed. Because L is a complete set of left coset representatives we have that $\exists! x_l \in L$ such that $x_ix_jH = x_lH$. This means that $\exists x_l \in L$ such that $x_i \star x_j = x_l$ from the definition of $\star$, so $\star$ is defined and closed.
Well defined: Let $x_i, x_j, x_m, x_n \in L$ with $x_i = x_m$ and $x_j = x_n$. Consider $x_i \star x_j = x_l$ which equals the left coset equation $x_ix_jH = x_lH$ for some $x_l \in L$.

$$
\begin{aligned}
x_i x_j H &= x_l H \\
x_l^{-1} x_i x_j &\in H && \text{LCET} \\
x_l^{-1} x_i x_j &= h_1, && \text{for some } h_1 \in H \\
x_l^{-1} x_m x_n &= h_1 && \text{group operation is well defined} \\
x_l^{-1} x_m x_n &\in H \\
x_m x_n H &= x_l H && \text{LCET} \\
\therefore x_m \star x_n &= x_l
\end{aligned}
$$

This shows that $\star$ is well defined, and thus a binary operation.

Now let us consider an identity element for L with $\star$. Let $x_i \in L$, consider $x_i \varepsilon H = x_i H$ because ε is the identity element of G so $x_i \star \varepsilon = x_i$ by definition of $\star$. Similarly $\varepsilon x_i H = x_i H$ so $\varepsilon \star x_i = x_i$. This means that the identity element of G, which is in L by hypothesis, is also the identity element of L.

Finally we must show that $\forall a, b \in L\ \exists!\ x_i, x_j \in L$ s.t.$a \star x_i = b$ and $x_j \star a = b$.

Let us begin by letting $a, b \in L$. Consider the subgroup H and the left coset $a^{-1}bH$. We know from 4.4.2 that $\exists!\ x_i \in L$ such that

$$
\begin{aligned}
x_i H &= a^{-1} b H \\
(a^{-1}b)^{-1} x_i &\in H \\
(a^{-1}b)^{-1} x_i &= h_1, \text{for some } h_1 \in H \\
b^{-1} a x_i &= h_i \\
b^{-1} a x_i &\in H \\
a x_i H &= b^{-1} H \\
a \star x_i &= b \text{ by the definition of } \star.
\end{aligned}
$$

Similarly, consider the cosets aH and bH. Again from 4.4.2 we know $\exists!\ x_j \in L$ such that $x_j a H = bH$ which means that $x_j \star a = b$ by the definition of $\star$. ❑

We have now discovered the origin of our strange conjugate construction. Here we have found that the underlying structure of a Group, namely that it is a loop, causes this conjugate.

Typical Abstract Algebra courses do not cover Baer's theorem because it does not have very many applications to basic group theory. In fact, most abstract algebra courses do not even cover Loop Theory at all. But here we see that creating Cayley Sudoku tables of Groups relies very heavily on this idea of Loops.

What if there was a way to start with a generic Loop satisfying Baer's theorem and "build" a group from it that satisfied the hypotheses of Condition 2? In a sense we could be creating our own groups for the purposes of making a Cayley Sudoku table. What might this group look like?

Let's begin exploring with just a generic group L and its operation $\circ$. For every $\ell \in L$ define the function $\lambda_\ell : L \to L$ as $\lambda_\ell(x) = \ell \circ x$ for every $x \in L$. Our claim is that λ_ℓ is an element of S_L, to show this we will show that λ_ℓ is both injective and surjective.

Proof.

Injective:

$$\text{Assume that} \lambda_\ell(x) = \lambda_\ell(y) \text{ for some } x, y \in L$$
$$\ell \star x = \ell \star y \qquad \text{Def. of } \lambda_\ell$$
$$x = y \text{ Right cancelation property of Loops}$$

Therefore λ_l is injective.

Surjective: Let $x \in L$. We know there exists a solution y to $\ell \star y = x$ because ℓ, y, x are elements of a loop. By definition this means also that $\lambda_\ell(y) = x$, so the function λ is surjective.

❑

Having λ firmly established we can look at how to use λ to build a group.

Let $\Lambda = \{\lambda_\ell \ : \ \ell \in L\}$, a set of permutations of the Loop. Additionally consider $G =< \Lambda >$ which is the smallest subgroup of S_L containing every element in Λ. Lastly consider the stabilizer of G, denoted $G_e := \{\alpha \in G \ : \ \alpha(e) = \varepsilon\}$ where e is the identity of the Loop. We can see that G_e is the subgroup of G that contains all the permutations that leave ε unchanged. It is our hope that G with G_e will be a Group and Subgroup satisfying the necessary conditions for Construction 2.

To prove this we will first need to show that Λ is a C.S.L.C.R. of G_e in G.

Proof. We will need to show:

1. $\forall \lambda_i \neq \lambda_j \ \lambda_i G_e \neq \lambda_j G_e$
2. $\forall \beta \in G, \ \beta \in \lambda_i G_e$

First assume that $\lambda_i \neq \lambda_j$ and BWOC assume that $\lambda_i G_e = \lambda_j G_e$

$$
\begin{aligned}
\lambda_i G_e =& \lambda_j G_e \\
\lambda_j^{-1}\lambda_i \in& G_\varepsilon && \text{LCET} \\
\lambda_j^{-1}(\lambda_i(\varepsilon)) =& \varepsilon && \text{Definition of } G_\varepsilon \\
\lambda_i(\varepsilon) =& \lambda_j(\varepsilon) && \text{Operate } \lambda_j \text{ to both sides} \\
i \star \varepsilon =& j \star \varepsilon && \text{Definition of } \lambda \\
i =& j && \text{Right cancelation of Loops} \\
\lambda_i =& \lambda_j \\
\rightarrow \leftarrow
\end{aligned}
$$

Because we have reached a contradiction we must conclude that $\lambda_i G_e \neq \lambda_j G_e$.
Further let us assume that β is a permutation in G and that $\beta(e) = y$, which makes β not an element of G_ε. We want to show that $\beta \in \lambda_i G_e$. Additionally we know that $y \star e = y$ because e is the loop identity. This means that $\lambda_y(e) = y$ by the definition of λ.

Finally we can see that:

$$
\begin{aligned}
\lambda_y^{-1}(\beta(\varepsilon)) &= e \\
\lambda_y^{-1}\beta &\in G_e \\
\beta G_e &= \lambda_y G_e && \text{LCET} \\
\beta &\in \lambda_y G_e
\end{aligned}
$$

❑

So we can conclude that Λ is a C.S.L.C.R. of G_e in G. What we need now is for Theorem 4.4.2 to hold. That is:
$\forall \alpha G_e,\ \beta G_e \in G/G_e$ there exists a unique λ_a such that $\lambda_a \alpha G_e = \beta G_e$.
Proof. Consider the unique solution to $a \star \alpha(e) = \beta(e)$. This is unique because $a, \alpha, \beta \in L$. Our hope is that $\lambda_a \alpha G_e = \beta G_e$, which is equivalent to $\beta^{-1}(\lambda_a \alpha) \in G_e$ by the LCET. So let us look now at $\beta^{-1}(\lambda_a \alpha)(e)$, if it equals e then we will know it is an element of G_e.

$$\begin{aligned}\beta^{-1}(\lambda_a\alpha)(e) =& \beta^{-1}(\lambda_a(\alpha(e)))\\ =& \beta^{-1}(a \star \alpha(e)) && \text{Def of } \lambda_a\\ =& \beta^{-1}(\beta(e)) && \text{Def of } a \star \alpha(e)\\ =& e\end{aligned}$$

Thus we can conclude that $\lambda_a\alpha G_e = \beta G_e$. But is λ_a unique? Let's assume not, assume that $\lambda_z\alpha G_e = \beta G_e$.

$$\begin{aligned}\lambda_z\alpha G_e =& \beta G_e\\ \beta^{-1}(\lambda_z\alpha) \in& G_e\\ \beta^{-1}(\lambda_z\alpha)(e) =& e && \text{LCET}\\ (\lambda_z\alpha)(e) =& \beta(e)\\ z \star \alpha(e) =& \beta(e) && \text{Def of } \lambda_z\\ z \star \alpha(e) =& a \star \alpha(e) && \text{Def of } \beta(e)\\ z =& a && \text{Right cancelation of Loops}\end{aligned}$$

We can see that λ_a is unique which means that we have fully satisfied Theorem 4.4.2. ❑

Because we have satisfied the conditions for 4.4.2 we know that G will have a C.S.L.C.R. of a G_e^g, for all $g \in G$ (Theorem 4.5). We also know that Λ is that C.S.L.C.R. of G_e in G. Most importantly this means that we can make a Cayley-Sudoku Table of G using Construction 2!

Unfortunately, $< \lambda_\ell >$ does not always give us a new subgroup. Consider the Loop given by the table below:

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	1	5	3
3	3	5	2	1	4
4	4	1	5	3	2
5	5	3	4	2	1

This Loop has 5 elements so we expect each λ_ℓ to be in S_5 and our G to be a subgroup of S_5. Let's look at each λ_ℓ.

$\lambda_1 = 1 \circ x, \forall x \in L$. We can look then at the first row of the table and see that $\lambda_1 = \varepsilon$.
$\lambda_2 = 2 \circ x$, looking at the second row of the table (since that shows us every element operated with 2 of the left) we see that λ_2 gives us the permutation : $\lambda_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{bmatrix} = (12453)$.
Similarly $\lambda_3 = 3 \circ x = (13254)$, $\lambda_4 = 4 \circ x = (14352)$, and $\lambda_5 = 5 \circ x = (15)(234)$.
Now let us look at $< \varepsilon, (12453), (13254), (14352), (15)(234) >$. Unfortunately, the smallest such subgroup is not a true subgroup, but is in fact S_5. This loop did have the restriction that $\lambda_1 = \varepsilon$, and other loops do not have this restriction. It could then be possible to place some other restriction on L in order to assure that $< \lambda_\ell >$ is a proper subgroup, maybe then we could find a new instance where Construction 2 is applicable.
Consider the Loop table shown below

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	1	4	11	8	10	12	7	6	5	9	3
3	3	4	1	5	12	9	11	2	8	7	6	10
4	4	11	5	1	6	2	10	12	3	9	8	7
5	5	8	12	6	1	7	3	11	2	4	10	9
6	6	10	9	2	7	1	8	4	12	3	5	11
7	7	12	11	10	3	8	1	9	5	2	4	6
8	8	7	2	12	11	4	9	1	10	6	3	5
9	9	6	8	3	2	12	5	10	1	11	7	4
10	10	5	7	9	4	3	2	6	11	1	12	8
11	11	9	6	8	10	5	4	3	7	12	1	2
12	12	3	10	7	9	11	6	5	4	8	2	1

This loop has the following permutations:

$\lambda_1 = \varepsilon$
$\lambda_2 = (1\,2)(3\,4\,11\,9\,6\,10\,5\,8\,7\,12)$
$\lambda_3 = (1\,3)(2\,4\,5\,12\,10\,7\,11\,6\,9\,8)$
$\lambda_4 = (1\,4)(2\,11\,8\,10\,7\,10\,9\,3\,5\,6)$
$\lambda_5 = (1\,5)(2\,8\,11\,10\,4\,6\,7\,3\,12\,9)$
$\lambda_6 = (1\,6)(2\,10\,3\,9\,12\,11\,5\,7\,8\,4)$
$\lambda_7 = (1\,7)(2\,12\,6\,8\,9\,5\,3\,11\,4\,10)$
$\lambda_8 = (1\,8)(2\,7\,9\,10\,6\,4\,10\,5\,11\,3)$
$\lambda_9 = (1\,9)(2\,6\,12\,4\,3\,8\,10\,11\,7\,5)$

$\lambda_{10} = (1\,10)(2\,5\,4\,9\,11\,12\,8\,6\,3\,7)$
$\lambda_{11} = (1\,11)(2\,9\,7\,4\,8\,3\,6\,5\,10\,12)$
$\lambda_{12} = (1\,12)(2\,3\,10\,8\,5\,9\,4\,7\,6\,11)$

Using the Abstract Algebra software it was shown that $< \Lambda >$ as defined above has 95,040 elements in it. $< \Lambda >$ is actually isomorphic to M_{12} (the Mathieu twelve group). On the surface this may seem like a lot of elements, but it is significantly less than 12!(479,001,600). What we have created then is a Group that is smaller than S_{12} and A_{12} and thus have a new group that can be made into a Cayley Sudoku Table using Construction 2. Unfortunately M_{12} satisfies Thm 2.1 so it is not a new example.
As a final example consider the loop table given below:

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	5	6	4
3	3	1	2	6	4	5
4	4	6	5	3	2	1
5	5	4	6	1	3	2
6	6	5	4	2	1	3

which has the following permutations:
$\lambda_1 = \varepsilon$
$\lambda_2 = (123)(456)$
$\lambda_3 = (132)(465)$
$\lambda_4 = (143526)$
$\lambda_5 = (153624)$
$\lambda_6 = (163425)$
So $\Lambda = \{\varepsilon, (123)(456), (132)(465), (143526), (153624), (163425)\}$.
$G =< \Lambda >= \{\varepsilon, (456), (465), (123), (123)(456), (123)(465), (132), (132)(456), (132)(465),$
$(14)(26)(35), (142635), (143526), (153624), (15)(24)(36), (152436), (162534), (163425), (16)(25)(34)\}$
has 18 elements which again is significantly lower than 6! (720) elements. G_e is everything that leaves the identity unchanged so for this group $G_1 = \{\varepsilon, (456), (465)\}$. So again we can use Construction 2 to create Cayley Sudoku Table.

	G_e	$(123)\,G_e$	$(132)\,G_e$	$(14)(26)(35)\,G_e$	$(15)(24)(36)G_e$	$(16)(25)(34)\,G_e$
V						
$(456)V$						
$(465)V$						

Again it turns out that this group satisfies Thm 2.1. So it is not a new example of a group that can be made into a Cayley-Sudoku Table using Construction 2 as we had hoped.

5 EXTENSIONS AND A SURPRISING CONNECTIONS TO QUASIGROUPS.

In studying papers on Loop Theory and its relation to Latin squares I stumbled upon an article by Drury Wall, which pertained to "Sub-Quasigroups", an idea not unlike subgroups. But there is no theorem like LaGrange's theorem for Quasigroups, the order of a subquasigroup does not need to divide the order of the quasigroup [6]. In his paper Wall proves the following theorem:

Theorem 5.1. *Assume Q is a quasigroup of order n with two subquasigroups R, with order r, and S with order s. Assuming that R and S intersect that that intersection is P, then denote the order P as p. If $n = r + s + \max(r, s) - 2p$ then $r = s$ if and only if $T = P \cup [Q \backslash (R \cup S)]$ is a subquasigroup of Q.*

While the proof of this is beyond the scope of this paper, some applications are not.
Using an example from [6] consider the quasigroup below:

	a	b	c	d	e	f	g	h
a	a	b	c	d	f	e	g	h
b	b	a	d	c	e	f	h	g
c	c	d	a	b	g	h	f	e
d	d	c	b	a	h	g	e	f
e	f	e	h	g	b	a	d	c
f	e	f	g	h	a	b	c	d
g	h	g	e	f	c	d	a	b
h	g	h	f	e	d	c	b	a

For this table let $R = \{a, b, c, d\}$ and $S = \{a, b, e, f\}$ which means that $P = \{a, b\}$ and $T = \{a, b, g, h\}$. We have satisfied the hypothesis of the Wall's theorem from above which means that $r = s$ (which is easy to see) and that T is also a subquasigroup. Surprisingly enough, we can make a Cayley Sudoku Table of this subquasigroup! Because the order of any of these subquasigroups is half of the order of the quasigroup we can arrange the table with the columns labeled by any of the subquasigroups and then all the elements not in the subquasigroup. For example the table below has columns labeled by the subquasigroup R first and then all the elements not in R. Additionally the rows are labeled by complete sets of R and of $\backslash R$. And, as you can see below, we have a Cayley Sudoku Table for this quasigroup! We can also make Cayley Sudoku Tables using S or T instead of R but still labeling the columns with complete sets of elements from S and $\backslash S$ and T and $\backslash T$ respectively.

	a	b	c	d	e	f	g	h
a	a	b	c	d	f	e	g	h
e	f	e	h	g	b	a	d	c
b	b	a	d	c	e	f	h	g
f	e	f	g	h	a	b	c	d
c	c	d	a	b	g	h	f	e
g	h	g	e	f	c	d	a	b
d	d	c	b	a	h	g	e	f
h	g	h	f	e	d	c	b	a

Table 5: A Cayley Sudoku table made from a Quasigroup

6 CONCLUSION

Throughout this paper we have seen the interconnectedness of mathematics. One cannot simply look at Latin Squares theories apart from Loop Theory and Group Theory. What has been proved in the past, like Baer's theorem, may find to a new and different importance the more mathematics that is discovered and manipulated. What has come before becomes of critical importance to what we want to prove. Often "new" constructions in fact are not brand new, but are old theorems with newly done-up faces. Even when you think that you are simply dealing with arrays, you are not; you are dealing with all the underlying structures and ideas. So it has been especially in this paper. Using theorems of loops that are almost 80 years old (which is really not too old in the realm of mathematics) we have been able to explain some of the strange conditions for making Cayley Sudoku tables. Hopefully with further research and an epiphany here and there, we will be able to discover the underlying causes of all our separate conditions on Construction 2, or at least discover a few new ones. ❑

REFERENCES

1. Carmichael, Jennifer, Keith Schloman and Michael B. Ward. *Cosets and Cayley−Sudoku Tables*. Mathematics Magazine vol. 83, 130-139, 2010.

2. Denes, J and A.D. Keedwell. *Latin Squares and their Applications*. Academic Press Inc, New York, 1974.

3. Denes, J. *Algrbraic and Combinatoral Characterzations of Latin Squares*. Mathematica Slovaca, Vol. 17, No. 4, 249-265, 1967.

4. Mirsky, Leonid *Transversal Theory* Academic Press Inc, New York, 1971.

5. Nagy, Peter T. and Karl Strambach. *Loops in Group Theory and Lie Theory*. Walter De Gruyter and Co. Berlin, 2002.

6. Wall, D.W. *Sub−Quasigroups of Finite Quasigroups*. Pacific Journal of Mathematics Vol. 7, 1711-1714, 1957.

	(1)	(12)(34)	(13)(24)	(14)(23)	(123)	(243)	(142)	(134)	(132)	(143)	(234)	(124)
(1)	(1)	(12)(34)	(13)(24)	(14)(23)	(123)	(243)	(142)	(134)	(132)	(143)	(234)	(124)
(13)(24)	(13)(24)	(14)(23)	(1)	(12)(34)	(142)	(134)	(123)	(243)	(234)	(124)	(132)	(143)
(123)	(123)	(134)	(243)	(142)	(132)	(124)	(143)	(234)	(1)	(14)(23)	(12)(34)	(13)(24)
(243)	(243)	(142)	(123)	(134)	(143)	(234)	(132)	(124)	(12)(34)	(13)(24)	(1)	(14)(23)
(132)	(132)	(234)	(124)	(143)	(1)	(13)(24)	(14)(23)	(12)(34)	(123)	(142)	(134)	(243)
(143)	(143)	(124)	(234)	(132)	(12)(34)	(14)(23)	(13)(24)	(1)	(243)	(134)	(142)	(123)
(12)(34)	(12)(34)	(1)	(14)(23)	(13)(24)	(243)	(123)	(134)	(142)	(143)	(132)	(124)	(234)
(14)(23)	(14)(23)	(13)(24)	(12)(34)	(1)	(134)	(142)	(243)	(123)	(124)	(234)	(143)	(132)
(134)	(134)	(123)	(142)	(243)	(124)	(132)	(234)	(143)	(14)(23)	(1)	(13)(24)	(12)(34)
(142)	(142)	(243)	(134)	(123)	(234)	(143)	(124)	(132)	(13)(24)	(12)(34)	(14)(23)	(1)
(234)	(234)	(132)	(143)	(124)	(13)(24)	(1)	(12)(34)	(14)(23)	(142)	(123)	(243)	(134)
(124)	(124)	(143)	(132)	(234)	(14)(23)	(12)(34)	(1)	(13)(24)	(134)	(243)	(123)	(142)

Appendix 1: Construction 1 with A_3